Algebraic Geometry

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0. INTRODUCTION

In a very rough sketch we explain what algebraic geometry is about and what it can be used for. We stress the many correlations with other fields of research, such as complex analysis, topology, differential geometry, singularity theory, computer algebra, commutative algebra, number theory, enumerative geometry, and even theoretical physics. The goal of this section is just motivational; you will not find definitions or proofs here (and probably not even a mathematically precise statement).

0.1. **What is algebraic geometry?** To start from something that you probably know, we can say that algebraic geometry is the combination of *linear algebra* and *algebra*:

- In linear algebra, we study systems of linear equations in several variables.
- In algebra, we study (among other things) polynomial equations in one variable.

Algebraic geometry combines these two fields of mathematics by studying systems of polynomial equations in several variables.

Given such a system of polynomial equations, what sort of questions can we ask? Note that we cannot expect in general to write down explicitly all the solutions: we know from algebra that even a single complex polynomial equation of degree $d > 4$ in one variable can in general not be solved exactly. So we are more interested in statements about the geometric structure of the set of solutions. For example, in the case of a complex polynomial equation of degree d, even if we cannot compute the solutions we know that there are exactly d of them (if we count them with the correct multiplicities). Let us now see what sort of "geometric structure" we can find in polynomial equations in several variables.

Example 0.1.1. Probably the easiest example that is covered neither in linear algebra nor in algebra is that of a single polynomial equation in two variables. Let us consider the following example:

$$C_n = \{(x,y) \in \mathbb{C}^2 \; ; \; y^2 = (x-1)(x-2)\cdots(x-2n)\} \subset \mathbb{C}^2,$$

where $n \geq 1$. Note that in this case it is actually possible to write down all the solutions, because the equation is (almost) solved for y already: we can pick x to be any complex number, and then get two values for y — unless $x \in \{1,\dots,2n\}$, in which case we only get one value for y (namely 0).

So it seems that the set of equations looks like two copies of the complex plane with the two copies of each point $1,\dots,2n$ identified: the complex plane parametrizes the values for x, and the two copies of it correspond to the two possible values for y, i. e. the two roots of the number $(x-1)\cdots(x-2n)$.

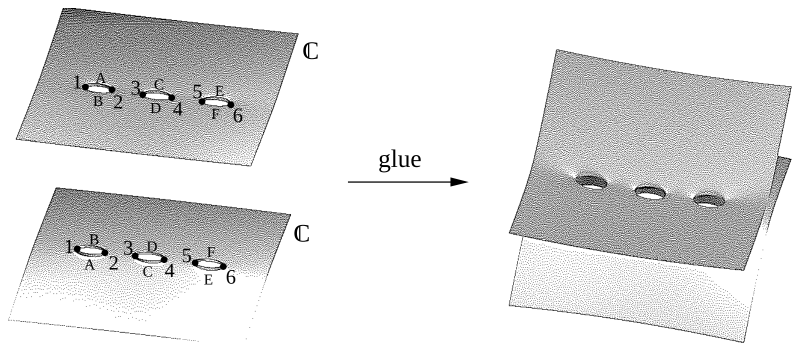
This is not quite true however, because a complex non-zero number does not have a distinguished first and second root that could correspond to the first and second copy of the complex plane. Rather, the two roots of a complex number get exchanged if you run around the origin once: if we consider a path

$$x = r\,e^{i\varphi} \qquad \text{for } 0 \leq \varphi \leq 2\pi \text{ and fixed } r > 0$$

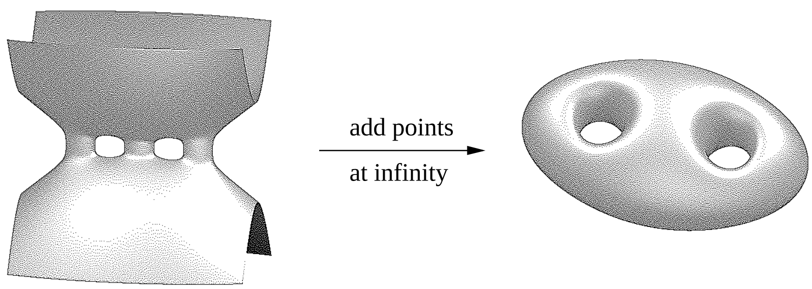
around the complex origin, the square root of this number would have to be defined by

$$\sqrt{x} = \sqrt{r}\,e^{\frac{i\varphi}{2}}$$

which gives opposite values at $\varphi = 0$ and $\varphi = 2\pi$. In other words, if in C_n we run around one of the points $1,\dots,2n$, we go from one copy of the plane to the other. The way to draw this topologically is to cut the two planes along the lines $[1,2],\dots,[2n-1,2n]$, and to glue the two planes along these lines as in this picture (lines marked with the same letter are to be identified):



To make the picture a little nicer, we can compactify our set by adding two points at infinity, in the same way as we go from $\mathbb{C}$ to $\mathbb{C}_\infty$ by adding a point ∞. If we do this here, we end up with a compact surface with $n-1$ handles:



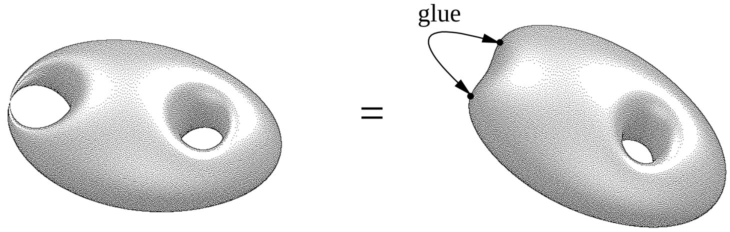
Such an object is called a surface *of genus* $n-1$; the example above shows a surface of genus 2.

Example 0.1.2. Example 0.1.1 is a little "cheated" because we said before that we want to figure out the geometric structure of equations that we cannot solve explicitly. In the example however, the polynomial equation was chosen so that we could solve it, and in fact we used this solution to construct the geometric picture. Let us see now what we can still do if we make the polynomial more complicated.

What happens if we consider

$$C_n = \{(x,y) \in \mathbb{C}^2 \; ; \; y^2 = f(x)\} \subset \mathbb{C}^2,$$

with f some polynomial in x of degree $2n$? Obviously, as long as the $2n$ roots of f are still distinct, the topological picture does not change. But if two of the roots approach each other and finally coincide, this has the effect of shrinking one of the tubes connecting the two planes until it finally reduces to a "singular point" (also called a *node*), as in the following picture on the left:



Obviously, we can view this as a surface with one handle less, where in addition we identify two of the points (as illustrated in the picture on the right). Note that we can still see the "handles" when we draw the surface like this, just that one of the handles results from the glueing of the two points.

Example 0.1.3. You have probably noticed that the polynomial equation of example 0.1.2 could be solved directly too. Let us now consider

$$C_d = \{(x,y) \in \mathbb{C}^2 \,;\, f(x,y) = 0\} \subset \mathbb{C}^2,$$

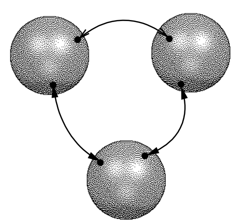
where f is an arbitrary polynomial of degree d. This is an equation that we certainly cannot solve directly if f is sufficiently general. Can we still deduce the geometric structure of C?

In fact, we can do this with the idea of example 0.1.2. We saw there that the genus of the surface does not change if we perturb the polynomial equation, even if the surface acquires singular points (provided that we know how to compute the genus of such a singular surface). So why not deform the polynomial f to something singular that is easier to analyze? Probably the easiest thing that comes into mind is to degenerate the polynomial f of degree d into a product of d linear equations $\ell_1, \dots, \ell_d$:

$$C'_d = \{(x,y) \in \mathbb{C}^2 \,;\, \ell_1(x,y) \cdots \ell_d(x,y) = 0\} \subset \mathbb{C}^2,$$

This surface should have the same "genus" as the original C_d.

It is easy to see what C'_d looks like: of course it is just a union of d lines. Any two of them intersect in a point, and we can certainly choose the lines so that no three of them intersect in a point. The picture below shows C'_d for $d = 3$ (note that every line is — after compactifying — just the complex sphere C_∞).



What is the genus of this surface? In the picture above it is obvious that we have one loop; so if $d = 3$ we get a surface of genus 1. What is the general formula? We have d spheres, and every two of them connect in a pair of points, so in total we have $\binom{d}{2}$ connections. But $d-1$ of them are needed to glue the d spheres to a connected chain without loops; only the remaining ones then add a handle each. So the genus of C'_d (and hence of C_d) is

$$\binom{d}{2} - (d-1) = \binom{d-1}{2}.$$

This is commonly called the *degree-genus formula* for plane curves.

Remark 0.1.4. One of the trivial but common sources for misunderstandings is whether we count dimensions over $\mathbb{C}$ or over $\mathbb{R}$. The examples considered above are *real surfaces* (the dimension over $\mathbb{R}$ is 2), but *complex curves* (the dimension over $\mathbb{C}$ is 1). We have used the word "surface" as this fitted best to the pictures that we have drawn. When looking at the theory however, it is usually best to call these objects curves. In what follows, we always mean the dimension over $\mathbb{C}$ unless stated otherwise.

Remark 0.1.5. What we should learn from the examples above:

- Algebraic geometry can make statements about the topological structure of objects defined by polynomial equations. It is therefore related to *topology* and *differential geometry* (where similar statements are deduced using analytic methods).

- The geometric objects considered in algebraic geometry need not be smooth (i. e. they need not be *manifolds*). Even if our primary interest is in smooth objects, degenerations to singular objects can greatly simplify a problem (as in example 0.1.3). This is a main point that distinguishes algebraic geometry from other "geometric" theories (e. g. differential or symplectic geometry). Of course, this comes at a price: our theory must be strong enough to include such singular objects and make statements how things vary when we degenerate from smooth to singular objects. In this regard, algebraic geometry is related to *singularity theory* which studies precisely these questions.

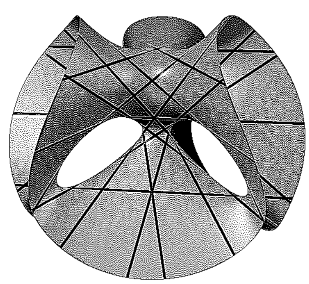
Remark 0.1.6. Maybe it looks a bit restrictive to allow only algebraic (polynomial) equations to describe our geometric objects. But in fact it is a deep theorem that for *compact* objects, we would not get anything different if we allowed *holomorphic* equations too. In this respect, algebraic geometry is very much related (and in certain cases identical) to *complex (analytic) geometry*. The easiest example of this correspondence is that a holomorphic map from the Riemann sphere $\mathbb{C}_\infty$ to itself must in fact be a rational map (i. e. the quotient of two polynomials).

Example 0.1.7. Let us now turn our attention to the next more complicated objects, namely complex surfaces in 3-space. We just want to give one example here. Let S be the *cubic surface*

$$S = \{(x,y,z)\,;\ 1+x^3+y^3+z^3-(1+x+y+z)^3 = 0\} \subset \mathbb{C}^3.$$

As this object has real dimension 4, it is impossible to draw pictures of it that reflect its topological properties correctly. Usually, we overcome this problem by just drawing the *real* part, i. e. we look for solutions of the equation over the real numbers. This then gives a real surface in $\mathbb{R}^3$ that we can draw. We should just be careful about which statements we can claim to "see" from this incomplete geometric picture.

The following picture shows the real part of the surface S:



In contrast to our previous examples, we have now used a *linear* projection to map the real 3-dimensional space onto the drawing plane.

We see that there are some lines contained in S. In fact, one can show that *every* smooth cubic surface has exactly 27 lines on it (see section 4.5 for details). This is another sort of question that one can ask about the solutions of polynomial equations, and that is not of topological nature: do they contain curves with special properties (in this case lines), and if so, how many? This branch of algebraic geometry is usually called *enumerative geometry*.

Remark 0.1.8. It is probably surprising that algebraic geometry, in particular enumerative geometry, is very much related to *theoretical physics*. In fact, many results in enumerative geometry have been found by physicists first.

Why are physicists interested e. g. in the number of lines on the cubic surface? We try to give a short answer to this (that is necessarily vague and incomplete): There is a branch of theoretical physics called *string theory* whose underlying idea is that the elementary particles (electrons, quarks,...) might not be point-like, but rather one-dimensional objects (the so-called strings), that are just so small that their one-dimensional structure cannot be observed directly by any sort of physical measurement. When these particles move in time, they sweep out a surface in space-time. For some reason this surface has a natural complex structure coming from the underlying physical theory.

Now the same idea applies to space-time in general: string theorists believe that space-time is not 4-dimensional as we observe it, but rather has some extra dimensions that are again so small in size that we cannot observe them directly. (Think e. g. of a long tube with a very small diameter — of course this is a two-dimensional object, but if you look at this tube from very far away you cannot see the small diameter any more, and the object looks like a one-dimensional line.) These extra dimensions are parametrized by a space that sometimes has a complex structure too; it might for example be the complex cubic surface that we looked at above.

So in this case we're in fact looking at complex curves in a complex surface. A priori, these curves can sit in the surface in any way. But there are *equations of motion* that tell you how these curves will sit in the ambient space, just as in classical mechanics it follows from the equations of motion that a particle will move on a straight line if no forces apply to it. In our case, the equations of motion say that the curve must map *holomorphically* to the ambient space. As we said in remark 0.1.6 above, this is equivalent to saying that we must have algebraic equations that describe the curve. So we are looking at exactly the same type of questions as we did in example 0.1.7 above.

Example 0.1.9. Let us now have a brief look at curves in 3-dimensional space. Consider the example

$$C = \{(x,y,z) = (t^3,t^4,t^5)\ ;\ t \in \mathbb{C}\} \subset \mathbb{C}^3.$$

We have given this curve parametrically, but it is in fact easy to see that we can give it equally well in terms of polynomial equations:

$$C = \{(x,y,z)\ ;\ x^3 = yz,\ y^2 = xz,\ z^2 = x^2y\}.$$

What is striking here is that we have *three* equations, although we would expect that a one-dimensional object in three-dimensional space should be given by two equations. But in fact, if you leave out any of the above three equations, you're changing the set that it describes: if you leave out e. g. the last equation $z^2 = x^2y$, you would get the whole z-axis $\{x = y = 0\}$ as additional points that do satisfy the first two equations, but not the last one.

So we see another important difference to linear algebra: it is not true that every object of codimension d can be given by d equations. Even worse, if you are given d equations, it is in general a very difficult task to figure out what dimension their solution has. There do exist algorithms to find this out for any given set of polynomials, but they are so complicated that you will in general want to use a computer program to do that for you. This is a simple example of an application of *computer algebra* to algebraic geometry.

Remark 0.1.10. Especially the previous example 0.1.9 is already very algebraic in nature: the question that we asked there does not depend at all on the ground field being the complex numbers. In fact, this is a general philosophy: even if algebraic geometry describes geometric objects (when viewed over the complex numbers), most methods do not rely on this, and therefore should be established in purely algebraic terms. For example, the genus of a curve (that we introduced topologically in example 0.1.1) can be defined in purely algebraic terms in such a way that all the statements from complex geometry (e. g. the degree-genus formula of example 0.1.3) extend to this more general setting. Many geometric questions then reduce to pure *commutative algebra*, which is in some sense the foundation of algebraic geometry.

Example 0.1.11. The most famous application of algebraic geometry to ground fields other than the complex numbers is certainly Fermat's Last Theorem: this is just the statement that the algebraic curve *over the rational numbers*

$$C = \{(x,y) \in \mathbb{Q}^2 \,;\, x^n + y^n = 1\} \subset \mathbb{Q}^2$$

contains only the trivial points where $x = 0$ or $y = 0$. Note that this is very different from the case of the ground field $\mathbb{C}$, where we have seen in example 0.1.3 that C is a curve of genus $\binom{n-1}{2}$. But a lot of the theory of algebraic geometry applies to the rational numbers (and related fields) as well, so if you look at the proof of Fermat's theorem (which you most probably will not understand) you will notice that it uses e. g. the concepts of algebraic curves and their genus all over the place, although the corresponding point set C contains only some trivial points. So, in some sense, we can view *(algebraic) number theory* as a part of algebraic geometry.

Remark 0.1.12. With this many relations to other fields of mathematics (and physics), it is obvious that we have to restrict our attention in this class to quite a small subset of the possible applications. Although we will develop the general theory of algebraic geometry, our focus will mainly be on geometric questions, neglecting number-theoretic aspects most of the time. So, for example, if we say "let k be an algebraically closed field", feel free to read this as "let k be the complex numbers" and think about geometry rather than algebra.

Every now and then we will quote results from or give applications to other fields of mathematics. This applies in particular to commutative algebra, which provides some of the basic foundations of algebraic geometry. So unless you want to take commutative algebra as a black box that spits out a useful theorem from time to time (which is possible but not recommended), you should get some background in commutative algebra while learning algebraic geometry. Some knowledge about geometric objects occurring in other fields of mathematics (manifolds, projective spaces, differential forms, vector bundles, ...) is helpful but not necessary. We will develop these concepts along the way as we need them.

0.2. **Exercises.** Note: As we have not developed any theory yet, you are not expected to be able to solve the following problems in a mathematically precise way. Rather, they are just meant as some "food for thought" if you want to think a little further about the examples considered in this section.

Exercise 0.2.1. What do we get in example 0.1.1 if we consider the equation

$$C'_n = \{(x,y) \in \mathbb{C}^2 \,;\, y^2 = (x-1)(x-2)\cdots(x-(2n-1))\} \subset \mathbb{C}^2$$

instead?

Exercise 0.2.2. (For those who know something about projective geometry:) In example 0.1.3, we argued that a polynomial of degree d in two complex variables gives rise to a surface of genus $\binom{d-1}{2}$. In example 0.1.1 however, a polynomial of degree $2n$ gave us a surface of genus $n-1$. Isn't that a contradiction?

Exercise 0.2.3.

(i) Show that the space of lines in $\mathbb{C}^n$ has dimension $2n-2$. (Hint: use that there is a unique line through any two given points in $\mathbb{C}^n$.)

(ii) Let $S \subset \mathbb{C}^3$ be a cubic surface, i. e. the zero locus of a polynomial of degree 3 in the three coordinates of $\mathbb{C}^3$. Find an argument why you would expect there to be finitely many lines in S (i. e. why you would expect the dimension of the space of lines in S to be 0-dimensional). What would you expect if the equation of S has degree less than or greater than 3?

Exercise 0.2.4. Let S be the specific cubic surface

$$S = \{(x,y,z)\ ;\ x^3+y^3+z^3 = (x+y+z)^3\} \subset \mathbb{C}^3.$$

(i) Show that there are exactly 3 lines contained in S.
(ii) Using the description of the space of lines of exercise 0.2.3, try to find an argument why these 3 lines should be counted with multiplicity 9 each (in the same way as e. g. double roots of a polynomial should be counted with multiplicity 2). We can then say that there are 27 lines on S, counted with their correct multiplicities.

(Remark: It is actually possible to prove that the number of lines on a cubic surface does not depend on the specific equation of the surface. This then shows, together with this exercise, that every cubic surface has 27 lines on it. You need quite a lot of theoretical background however to make this into a rigorous proof.)

Exercise 0.2.5. Show that if you replace the three equations defining the curve C in example 0.1.9 by

(i) $x^3 = y^2, x^5 = z^2, y^5 = z^4$, or
(ii) $x^3 = y^2, x^5 = z^2, y^5 = z^3 + \varepsilon$ for small but non-zero ε,

the resulting set of solutions is in fact 0-dimensional, as you would expect it from three equations in three-dimensional space. So we see that very small changes in the equations can make a very big difference in the result. In other words, we usually cannot apply numerical methods to our problems, as very small rounding errors can change the result completely.

Exercise 0.2.6. Let X be the set of all complex 2×3 matrices of rank at most 1, viewed as a subset of the $\mathbb{C}^6$ of all 2×3 matrices. Show that X has dimension 4, but that you need 3 equations to define X in the ambient 6-dimensional space $\mathbb{C}^6$.

1. Affine varieties

A subset of affine n-space $\mathbb{A}^n$ over a field k is called an algebraic set if it can be written as the zero locus of a set of polynomials. By the Hilbert basis theorem, this set of polynomials can be assumed to be finite. We define the Zariski topology on $\mathbb{A}^n$ (and hence on any subset of $\mathbb{A}^n$) by declaring the algebraic sets to be the closed sets.

Any algebraic set $X \subset \mathbb{A}^n$ has an associated radical ideal $I(X) \subset k[x_1,\dots,x_n]$ that consists of those functions that vanish on X. Conversely, for any radical ideal I there is an associated algebraic set $Z(I)$ which is the common zero locus of all functions in I. If k is algebraically closed, Hilbert's Nullstellensatz states that this gives in fact a one-to-one correspondence between algebraic sets in $\mathbb{A}^n$ and radical ideals in $k[x_1,\dots,x_n]$.

An algebraic set (or more generally any topological space) is called irreducible if it cannot be written as a union of two proper closed subsets. Irreducible algebraic sets in $\mathbb{A}^n$ are called affine varieties. Any algebraic set in $\mathbb{A}^n$ can be decomposed uniquely into a finite union of affine varieties. Under the correspondence mentioned above, affine varieties correspond to prime ideals. The dimension of an algebraic set (or more generally of a topological space) is defined to be the length of the longest chain of irreducible closed subsets minus one.

1.1. **Algebraic sets and the Zariski topology.** We have said in the introduction that we want to consider solutions of polynomial equations in several variables. So let us now make the obvious definitions.

Definition 1.1.1. Let k be a field (recall that you may think of the complex numbers if you wish). We define **affine *n*-space** over k, denoted $\mathbb{A}^n$, to be the set of all n-tuples of elements of k:

$$\mathbb{A}^n := \{(a_1,\dots,a_n)\ ;\ a_i \in k \text{ for } 1 \le i \le n\}.$$

The elements of the polynomial ring

$$\begin{aligned} k[x_1,\dots,x_n] :=& \{\text{polynomials in the variables } x_1,\dots,x_n \text{ over } k\} \\ =& \{\sum_I a_I x^I\ ;\ a_I \in k\} \end{aligned}$$

(with the sum taken over all multi-indices $I = (i_1,\dots,i_n)$ with $i_j \ge 0$ for all $1 \le j \le n$) define functions on $\mathbb{A}^n$ in the obvious way. For a given set $S \subset k[x_1,\dots,x_n]$ of polynomials, we call

$$\boldsymbol{Z(S)} := \{P \in \mathbb{A}^n\ ;\ f(P) = 0 \text{ for all } f \in S\} \subset \mathbb{A}^n$$

the **zero set** of S. Subsets of $\mathbb{A}^n$ that are of this form for some S are called **algebraic sets**. By abuse of notation, we also write $Z(f_1,\dots,f_i)$ for $Z(S)$ if $S = \{f_1,\dots,f_i\}$.

Example 1.1.2. Here are some simple examples of algebraic sets:

(i) Affine n-space itself is an algebraic set: $\mathbb{A}^n = Z(0)$.
(ii) The empty set is an algebraic set: $\emptyset = Z(1)$.
(iii) Any single point in $\mathbb{A}^n$ is an algebraic set: $(a_1,\dots,a_n) = Z(x_1 - a_1,\dots,x_n - a_n)$.
(iv) Linear subspaces of $\mathbb{A}^n$ are algebraic sets.
(v) All the examples from section 0 are algebraic sets: e. g. the curves of examples 0.1.1 and 0.1.3, and the cubic surface of example 0.1.7.

Remark 1.1.3. Of course, different subsets of $k[x_1,\dots,x_n]$ can give rise to the same algebraic set. Two trivial cases are:

(i) If two polynomials f and g are already in S, then we can also throw in $f+g$ without changing $Z(S)$.
(ii) If f is in S, and g is any polynomial, then we can also throw in $f \cdot g$ without changing $Z(S)$.

Recall that a subset S of a commutative ring R (in our case, $R = k[x_1,\dots,x_n]$) is called an **ideal** if it is closed both under addition and under multiplication with arbitrary ring elements. If $S \subset R$ is any subset, the set

$$(S) = \{f_1 g_1 + \cdots + f_m g_m \; ; \; f_i \in S, \, g_i \in R\}$$

is called the **ideal generated by** S; it is obviously an ideal. So what we have just said amounts to stating that $Z(S) = Z((S))$. It is therefore sufficient to only look at the cases where S is an ideal of $k[x_1,\dots,x_n]$.

There is a more serious issue though that we will deal with in section 1.2: a function f has the same zero set as any of its powers f^i; so e. g. $Z(x_1) = Z(x_1^2)$ (although the ideals (x_1) and (x_1^2) are different).

We will now address the question whether any algebraic set can be defined by a *finite* number of polynomials. Although this is entirely a question of commutative algebra about the polynomial ring $R = k[x_1,\dots,x_n]$, we will recall here the corresponding definition and proposition.

Lemma and Definition 1.1.4. *Let R be a ring. The following two conditions are equivalent:*

(i) *Every ideal in R can be generated by finitely many elements.*
(ii) *R satisfies the **ascending chain condition**: every (infinite) ascending chain of ideals $I_1 \subset I_2 \subset I_3 \subset \cdots$ is stationary, i. e. we must have $I_m = I_{m+1} = I_{m+2} = \cdots$ for some m.*

*If R satisfies these conditions, it is called **Noetherian**.*

Proof. (i) $\Rightarrow$ (ii): Let $I_1 \subset I_2 \subset \cdots$ be an infinite ascending chain of ideals in R. Then $I := \cup_i I_i$ is an ideal of R as well; so by assumption (i) it can be generated by finitely many elements. These elements must already be contained in one of the I_m, which means that $I_m = I_{m+1} = \cdots$.

(ii) $\Rightarrow$ (i): Assume that there is an ideal I that cannot be generated by finitely many elements. Then we can recursively construct elements f_i in I by picking $f_1 \in I$ arbitrary and $f_{i+1} \in I \backslash (f_1,\dots,f_i)$. It follows that the sequence of ideals

$$(f_1) \subset (f_1, f_2) \subset (f_1, f_2, f_3) \subset \cdots$$

is not stationary. $\square$

Proposition 1.1.5. ***(Hilbert basis theorem)*** *If R is a Noetherian ring then so is $R[x]$. In particular, $k[x_1,\dots,x_n]$ is Noetherian; so every algebraic set can be defined by finitely many polynomials.*

Proof. Assume that $I \subset R[x]$ is an ideal that is not finitely generated. Then we can define a sequence of elements $f_i \in I$ as follows: let f_0 be a non-zero element of I of minimal degree, and let f_{i+1} be an element of I of minimal degree in $I \backslash (f_0,\dots,f_i)$. Obviously, $\deg f_i \le \deg f_{i+1}$ for all i by construction.

For all i let $a_i \in R$ be the leading coefficient of f_i, and let $I_i = (a_0,\dots,a_i) \subset R$. As R is Noetherian, the chain of ideals $I_0 \subset I_1 \subset \cdots$ in R is stationary. Hence there is an m such that $a_{m+1} \in (a_0,\dots,a_m)$. Let $r_0,\dots,r_m \in R$ such that $a_{m+1} = \sum_{i=0}^m r_i a_i$, and consider the polynomial

$$f = f_{m+1} - \sum_{i=0}^{m} x^{\deg f_{m+1} - \deg f_i} r_i f_i.$$

We must have $f \in I \backslash (f_0,\dots,f_m)$, as otherwise the above equation would imply that $f_{m+1} \in (f_0,\dots,f_m)$. But by construction the coefficient of f of degree $\deg f_{m+1}$ is zero, so $\deg f < \deg f_{m+1}$, contradicting the choice of f_{m+1}. Hence $R[x]$ is Noetherian.

In particular, as k is trivially Noetherian, it follows by induction that $k[x_1,\dots,x_n]$ is. $\square$

We will now return to the study of algebraic sets and make them into topological spaces.

Lemma 1.1.6.

(i) *If* $S_1 \subset S_2 \subset k[x_1,\dots,x_n]$ *then* $Z(S_2) \subset Z(S_1) \subset \mathbb{A}^n$.
(ii) *If* $\{S_i\}$ *is a family of subsets of* $k[x_1,\dots,x_n]$ *then* $\bigcap_i Z(S_i) = Z(\bigcup_i S_i) \subset \mathbb{A}^n$.
(iii) *If* $S_1, S_2 \subset k[x_1,\dots,x_n]$ *then* $Z(S_1) \cup Z(S_2) = Z(S_1 S_2) \subset \mathbb{A}^n$.

In particular, arbitrary intersections and finite unions of algebraic sets are again algebraic sets.

Proof. (i) and (ii) are obvious, so let us prove (iii). "$\subset$": If $P \in Z(S_1) \cup Z(S_2)$ then $P \in Z(S_1)$ or $P \in Z(S_2)$. In particular, for any $f_1 \in S_1$, $f_2 \in S_2$, we have $f_1(P) = 0$ or $f_2(P) = 0$, so $f_1 f_2(P) = 0$. "$\supset$": If $P \notin Z(S_1) \cup Z(S_2)$ then $P \notin Z(S_1)$ and $P \notin Z(S_2)$. So there are functions $f_1 \in S_1$ and $f_2 \in S_2$ that do not vanish at P. Hence $f_1 f_2(P) \neq 0$, so $P \notin Z(S_1 S_2)$. $\square$

Remark 1.1.7. Recall that a **topology** on any set X can be defined by specifying which subsets of X are to be considered closed sets, provided that the following conditions hold:

(i) The empty set $\emptyset$ and the whole space X are closed.
(ii) Arbitrary intersections of closed sets are closed.
(iii) Finite unions of closed sets are closed.

Note that the standard definition of closed subsets of $\mathbb{R}^n$ that you know from real analysis satisfies these conditions.

A subset Y of X is then called open if its complement $X \backslash Y$ is closed. If X is a topological space and $Y \subset X$ any subset, Y inherits an **induced subspace topology** by declaring the sets of the form $Y \cap Z$ to be closed whenever Z is closed in X. A map $f : X \to Y$ is called *continuous* if inverse images of closed subsets are closed. (For the standard topology of $\mathbb{R}^n$ from real analysis and the standard definition of continuous functions, it is a theorem that a function is continuous if and only if inverse images of closed subsets are closed.)

Definition 1.1.8. We define the **Zariski topology** on $\mathbb{A}^n$ to be the topology whose closed sets are the algebraic sets (lemma 1.1.6 tells us that this gives in fact a topology). Moreover, any subset X of $\mathbb{A}^n$ will be equipped with the topology induced by the Zariski topology on $\mathbb{A}^n$. This will be called the Zariski topology on X.

Remark 1.1.9. In particular, using the induced subspace topology, this defines the Zariski topology on any algebraic set $X \subset \mathbb{A}^n$: the closed subsets of X are just the algebraic sets $Y \subset \mathbb{A}^n$ contained in X.

The Zariski topology is the standard topology in algebraic geometry. So whenever we use topological concepts in what follows we refer to this topology (unless we specify otherwise).

Remark 1.1.10. The Zariski topology is quite different from the usual ones. For example, on $\mathbb{A}^n$, a closed subset that is not equal to $\mathbb{A}^n$ satisfies at least one non-trivial polynomial equation and has therefore necessarily dimension less than n. So the closed subsets in the Zariski topology are in a sense "very small". It follows from this that any two non-empty open subsets of $\mathbb{A}^n$ have a non-empty intersection, which is also unfamiliar from the standard topology of real analysis.

Example 1.1.11. Here is another example that shows that the Zariski topology is "unusual". The closed subsets of $\mathbb{A}^1$ besides the whole space and the empty set are exactly the finite sets. In particular, if $f : \mathbb{A}^1 \to \mathbb{A}^1$ is *any* bijection, then f is a homeomorphism. (This last statement is essentially useless however, as we will not define morphisms between algebraic sets as just being continuous maps with respect to the Zariski topology. In fact, this example gives us a strong hint that we should not do so.)

1.2. **Hilbert's Nullstellensatz.** We now want to establish the precise connection between algebraic sets in $\mathbb{A}^n$ and ideals in $k[x_1,\dots,x_n]$, hence between geometry and algebra. We have already introduced the operation $Z(\cdot)$ that takes an ideal (or any subset of $k[x_1,\dots,x_n]$) to an algebraic set. Here is an operation that does the opposite job.

Definition 1.2.1. For a subset $X \subset \mathbb{A}^n$, we call

$$\boldsymbol{I(X)} := \{f \in k[x_1,\dots,x_n] \; ; \; f(P) = 0 \text{ for all } P \in X\} \subset k[x_1,\dots,x_n]$$

the **ideal** of X (note that this is in fact an ideal).

Remark 1.2.2. We have thus defined a two-way correspondence

$$\left\{ \begin{array}{c} \text{algebraic sets} \\ \text{in } \mathbb{A}^n \end{array} \right\} \begin{array}{c} \xrightarrow{I} \\ \xleftarrow{Z} \end{array} \left\{ \begin{array}{c} \text{ideals in} \\ k[x_1,\dots,x_n] \end{array} \right\}.$$

We will now study to what extent these two maps are inverses of each other.

Remark 1.2.3. Let us start with the easiest case of algebraic sets and look at points in $\mathbb{A}^n$. Points are minimal algebraic sets, so by lemma 1.1.6 (i) they should correspond to maximal ideals. In fact, the point $(a_1,\dots,a_n) \in \mathbb{A}^n$ is the zero locus of the ideal $I = (x_1 - a_1,\dots,x_n - a_n)$. Recall from commutative algebra that an ideal I of a ring R is maximal if and only if R/I is a field. So in our case I is indeed maximal, as $k[x_1,\dots,x_n]/I \cong k$. However, for general k there are also maximal ideals that are not of this form, e. g. $(x^2+1) \subset \mathbb{R}[x]$ (where $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$). The following proposition shows that this cannot happen if k is *algebraically closed*, i. e. if every non-constant polynomial in $k[x]$ has a zero.

Proposition 1.2.4. ***(Hilbert's Nullstellensatz*** *("theorem of the zeros")) Assume that k is algebraically closed (e. g. $k = \mathbb{C}$). Then the maximal ideals of $k[x_1,\dots,x_n]$ are exactly the ideals of the form $(x_1 - a_1,\dots,x_n - a_n)$ for some $a_i \in k$.*

Proof. Again this is entirely a statement of commutative algebra, so you can just take it on faith if you wish (in fact, many textbooks on algebraic geometry do so). For the sake of completeness we will give a short proof here in the case $k = \mathbb{C}$ that uses only some basic algebra; but feel free to ignore it if it uses concepts that you do not know. A proof of the general case can be found e. g. in [Ha] proposition 5.18.

So assume that $k = \mathbb{C}$. From the discussion above we see that it only remains to show that any maximal ideal $\mathfrak{m}$ is contained in an ideal of the form $(x_1 - a_1,\dots,x_n - a_n)$.

As $\mathbb{C}[x_1,\dots,x_n]$ is Noetherian, we can write $\mathfrak{m} = (f_1,\dots,f_r)$ for some $f_i \in \mathbb{C}[x_1,\dots,x_n]$. Let K be the subfield of $\mathbb{C}$ obtained by adjoining to $\mathbb{Q}$ all coefficients of the f_i. We will now restrict coefficients to this subfield K, so let $\mathfrak{m}_0 = \mathfrak{m} \cap K[x_1,\dots,x_n]$. Note that then $\mathfrak{m} = \mathfrak{m}_0 \cdot \mathbb{C}[x_1,\dots,x_n]$, as the generators f_i of $\mathfrak{m}$ lie in $\mathfrak{m}_0$.

Note that $\mathfrak{m}_0 \subset K[x_1,\dots,x_n]$ is a maximal ideal too, because if we had an inclusion $\mathfrak{m}_0 \subsetneq \mathfrak{m}_0' \subsetneq K[x_1,\dots,x_n]$ of ideals, this would give us an inclusion $\mathfrak{m} \subsetneq \mathfrak{m}' \subsetneq \mathbb{C}[x_1,\dots,x_n]$ by taking the product with $\mathbb{C}[x_1,\dots,x_n]$. (This last inclusion has to be strict as intersecting it with $K[x_1,\dots,x_n]$ gives the old ideals $\mathfrak{m}_0 \subsetneq \mathfrak{m}_0'$ back again.)

So $K[x_1,\dots,x_n]/\mathfrak{m}_0$ is a field. We claim that there is an embedding $K[x_1,\dots,x_n]/\mathfrak{m}_0 \hookrightarrow \mathbb{C}$. To see this, split the field extension $K[x_1,\dots,x_n]/\mathfrak{m}_0 : \mathbb{Q}$ into a purely transcendental part $L : \mathbb{Q}$ and an algebraic part $K[x_1,\dots,x_n]/\mathfrak{m}_0 : L$. As $K[x_1,\dots,x_n]/\mathfrak{m}_0$ and hence L is finitely generated over $\mathbb{Q}$ whereas $\mathbb{C}$ is of infinite transcendence degree over $\mathbb{Q}$, there is an embedding $L \subset \mathbb{C}$. Finally, as $K[x_1,\dots,x_n]/\mathfrak{m}_0 : L$ is algebraic and $\mathbb{C}$ algebraically closed, this embedding can be extended to give an embedding $K[x_1,\dots,x_n]/\mathfrak{m}_0 \subset \mathbb{C}$.

Let a_i be the images of the x_i under this embedding. Then $f_i(a_1,\dots,a_n) = 0$ for all i by construction, so $f_i \in (x_1 - a_1,\dots,x_n - a_n)$ and hence $\mathfrak{m} \subset (x_1 - a_1,\dots,x_n - a_n)$. □

Remark 1.2.5. The same method of proof can be used for any algebraically closed field k that has infinite transcendence degree over the prime field $\mathbb{Q}$ or $\mathbb{F}_p$.

Corollary 1.2.6. *Assume that k is algebraically closed.*

(i) *There is a 1:1 correspondence*

$$\{\textit{points in } \mathbb{A}^n\} \longleftrightarrow \{\textit{maximal ideals of } k[x_1,\dots,x_n]\}$$

given by $(a_1,\dots,a_n) \longleftrightarrow (x_1 - a_1,\dots,x_n - a_n)$.

(ii) *Every ideal* $I \subsetneq k[x_1,\dots,x_n]$ *has a zero in* $\mathbb{A}^n$.

Proof. (i) is obvious from the Nullstellensatz, and (ii) follows in conjunction with lemma 1.1.6 (i) as every ideal is contained in a maximal one. □

Example 1.2.7. We just found a correspondence between points of $\mathbb{A}^n$ and certain ideals of the polynomial ring. Now let us try to extend this correspondence to more complicated algebraic sets than just points. We start with the case of a collection of points in $\mathbb{A}^1$.

(i) Let $X = \{a_1,\dots,a_r\} \subset \mathbb{A}^1$ be a finite algebraic set. Obviously, $I(X)$ is then generated by the function $(x-a_1)\cdots(x-a_r)$, and $Z(I(X)) = X$ again. So Z is an inverse of I.

(ii) Conversely, let $I \subset k[x]$ be an ideal (not equal to (0) or (1)). As $k[x]$ is a principal ideal domain, we have $I = (f)$ for some non-constant monic function $f \in k[x]$. Now for the correspondence to work at all, we have to require that k be algebraically closed: for if f had no zeros, we would have $Z(I) = \emptyset$, and $I(Z(I)) = (1)$ would give us back no information about I at all. But if k is algebraically closed, we can write $f = (x-a_1)^{m_1}\cdots(x-a_r)^{m_r}$ with the a_i distinct and $m_i > 0$. Then $Z(I) = \{a_1,\dots,a_r\}$ and therefore $I(Z(I))$ is generated by $(x-a_1)\cdots(x-a_r)$, i. e. all exponents are reduced to 1. Another way to express this fact is that a function is in $I(Z(I))$ if and only if some power of it lies in I. We write this as $I(Z(I)) = \sqrt{I}$, where we use the following definition.

Definition 1.2.8. For an ideal $I \subset k[x_1,\dots,x_n]$, we define the **radical** of I to be

$$\sqrt{\boldsymbol{I}} := \{f \in k[x_1,\dots,x_n]\ ;\ f^r \in I \text{ for some } r > 0\}.$$

(In fact, this is easily seen to be an ideal.) An ideal I is called radical if $I = \sqrt{I}$. Note that the ideal of an algebraic set is always radical.

The following proposition says that essentially the same happens for $n > 1$. As it can be guessed from the example above, the case $Z(I(X))$ is more or less trivial, whereas the case $I(Z(I))$ is more difficult and needs the assumption that k be algebraically closed.

Proposition 1.2.9.

(i) *If* $X_1 \subset X_2$ *are subsets of* $\mathbb{A}^n$ *then* $I(X_2) \subset I(X_1)$.

(ii) *For any algebraic set* $X \subset \mathbb{A}^n$ *we have* $Z(I(X)) = X$.

(iii) *If k is algebraically closed, then for any ideal* $I \subset k[x_1,\dots,x_n]$ *we have* $I(Z(I)) = \sqrt{I}$.

Proof. (i) is obvious, as well as the "⊃" parts of (ii) and (iii).

(ii) "⊂": By definition $X = Z(I)$ for some I. Hence, by (iii) "⊃" we have $I \subset I(Z(I)) = I(X)$. By 1.1.6 (i) it then follows that $Z(I(X)) \subset Z(I) = X$.

(iii) "$\subset$": (This is sometimes also called Hilbert's Nullstellensatz, as it follows easily from proposition 1.2.4.) Let $f \in I(Z(I))$. Consider the ideal

$$J = I + (ft-1) \subset k[x_1,\dots,x_n,t].$$

This has empty zero locus in $\mathbb{A}^{n+1}$, as f vanishes on $Z(I)$, so if we require $ft = 1$ at the same time, we get no solutions. Hence $J = (1)$ by corollary 1.2.6 (i). In particular, there is a relation

$$1 = (ft-1)g_0 + \sum f_i g_i \in k[x_1,\dots,x_n,t]$$

for some $g_i \in k[x_1,\dots,x_n,t]$ and $f_i \in I$. If t^N is the highest power of t occurring in the g_i, then after multiplying with f^N we can write this as

$$f^N = (ft-1)G_0(x_1,\dots,x_n,ft) + \sum f_i G_i(x_1,\dots,x_n,ft)$$

where $G_i = f^N g_i$ is considered to be a polynomial in $x_1,\dots,x_n,ft$. Modulo $ft-1$ we get

$$f^N = \sum f_i G_i(x_1,\dots,x_n,1) \in k[x_1,\dots,x_n,t]/(ft-1).$$

But as the map $k[x_1,\dots,x_n] \to k[x_1,\dots,x_n,ft]/(ft-1)$ is injective, this equality holds in fact in $k[x_1,\dots,x_n]$, so $f^N \in I$. □

Corollary 1.2.10. *If k is algebraically closed, there is a one-to-one inclusion-reversing correspondence between algebraic sets in $\mathbb{A}^n$ and radical ideals in $k[x_1,\dots,x_n]$, given by the operations $Z(\cdot)$ and $I(\cdot)$. (This is also sometimes called the Nullstellensatz.)*

Proof. Immediately from proposition 1.2.9 and lemma 1.1.6 (i). □

From now on up to the end of section 4, *we will always assume that the ground field k is algebraically closed.*

Remark 1.2.11. Even though the radical $\sqrt{I}$ of an ideal I was easy to define, it is quite difficult to actually compute $\sqrt{I}$ for any given ideal I. Even worse, it is already quite difficult just to check whether I itself is radical or not. In general, you will need non-trivial methods of computer algebra to solve problems like this.

1.3. **Irreducibility and dimension.** The algebraic set $X = \{x_1x_2 = 0\} \subset \mathbb{A}^2$ can be written as the union of the two coordinate axes $X_1 = \{x_1 = 0\}$ and $X_2 = \{x_2 = 0\}$, which are themselves algebraic sets. However, X_1 and X_2 cannot be decomposed further into finite unions of smaller algebraic sets. We now want to generalize this idea. It turns out that this can be done completely in the language of topological spaces. This has the advantage that it applies to more general cases, i. e. open subsets of algebraic sets.

However, you will want to think only of the Zariski topology here, since the concept of irreducibility as introduced below does not make much sense in classical topologies.

Definition 1.3.1.

(i) A topological space X is said to be **reducible** if it can be written as a union $X = X_1 \cup X_2$, where X_1 and X_2 are (non-empty) closed subsets of X not equal to X. It is called **irreducible** otherwise. An irreducible algebraic set in $\mathbb{A}^n$ is called an **affine variety**.

(ii) A topological space X is called **disconnected** if it can be written as a *disjoint* union $X = X_1 \cup X_2$ of (non-empty) closed subsets of X not equal to X. It is called **connected** otherwise.

Remark 1.3.2. Although we have given this definition for arbitrary topological spaces, you will usually want to apply the notion of irreducibility only in the Zariski topology. For example, in the usual complex topology, the affine line $\mathbb{A}^1$ (i. e. the complex plane) is *reducible* because it can be written e. g. as the union of closed subsets

$$\mathbb{A}^1 = \{z \in \mathbb{C} \,;\, |z| \le 1\} \cup \{z \in \mathbb{C} \,;\, |z| \ge 1\}.$$

In the Zariski topology however, $\mathbb{A}^1$ is irreducible (as it should be).

In contrast, the notion of connectedness can be used in the "usual" topology too and does mean there what you think it should mean.

Remark 1.3.3. Note that there is a slight inconsistency in the existing literature: some authors call a variety what we call an algebraic set, and consequently an irreducible variety what we call an affine variety.

The algebraic characterization of affine varieties is the following.

Lemma 1.3.4. *An algebraic set $X \subset \mathbb{A}^n$ is an affine variety if and only if its ideal $I(X) \subset k[x_1,\dots,x_n]$ is a prime ideal.*

Proof. "$\Leftarrow$": Let $I(X)$ be a prime ideal, and suppose that $X = X_1 \cup X_2$. Then $I(X) = I(X_1) \cap I(X_2)$ by exercise 1.4.1 (i). As $I(X)$ is prime, we may assume $I(X) = I(X_1)$, so $X = X_1$ by proposition 1.2.9 (ii).

"$\Rightarrow$": Let X be irreducible, and let $fg \in I(X)$. Then $X \subset Z(fg) = Z(f) \cup Z(g)$, hence $X = (Z(f) \cap X) \cup (Z(g) \cap X)$ is a union of two algebraic sets. As X is irreducible, we may assume that $X = Z(f) \cap X$, so $f \in I(X)$. □

Example 1.3.5.

(i) $\mathbb{A}^n$ is an affine variety, as $I(\mathbb{A}^n) = (0)$ is prime. If $f \in k[x_1,\dots,x_n]$ is an irreducible polynomial, then $Z(f)$ is an affine variety. A collection of m points in $\mathbb{A}^n$ is irreducible if and only if $m = 1$.

(ii) Every affine variety is connected. The union of the n coordinate axes in $\mathbb{A}^n$ is always connected, although it is reducible for $n > 1$. A collection of m points in $\mathbb{A}^n$ is connected if and only if $m = 1$.

As it can be expected, any topological space that satisfies a reasonable finiteness condition can be decomposed uniquely into finitely many irreducible spaces. This is what we want to show next.

Definition 1.3.6. A topological space X is called **Noetherian** if every descending chain $X \supset X_1 \supset X_2 \supset \cdots$ of closed subsets of X is stationary.

Remark 1.3.7. By corollary 1.2.10 the fact that $k[x_1,\dots,x_n]$ is a Noetherian ring (see proposition 1.1.5) translates into the statement that any algebraic set is a Noetherian topological space.

Proposition 1.3.8. *Every Noetherian topological space X can be written as a finite union $X = X_1 \cup \cdots \cup X_r$ of irreducible closed subsets. If one assumes that $X_i \not\subset X_j$ for all $i \neq j$, then the X_i are unique (up to permutation). They are called the* ***irreducible components*** *of X.*

In particular, any algebraic set is a finite union of affine varieties in a unique way.

Proof. To prove existence, assume that there is a topological space X for which the statement is false. In particular, X is reducible, hence $X = X_1 \cup X_1'$. Moreover, the statement of the proposition must be false for at least one of these two subsets, say X_1. Continuing this construction, one arrives at an infinite chain $X \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$ of closed subsets, which is a contradiction as X is Noetherian.

To show uniqueness, assume that we have two decompositions $X = X_1 \cup \cdots \cup X_r = X_1' \cup \cdots \cup X_s'$. Then $X_1 \subset \bigcup_i X_i'$, so $X_1 = \bigcup (X_1 \cap X_i')$. But X_1 is irreducible, so we can assume $X_1 = X_1 \cap X_1'$, i. e. $X_1 \subset X_1'$. For the same reason, we must have $X_1' \subset X_i$ for some i. So $X_1 \subset X_1' \subset X_i$, which means by assumption that $i = 1$. Hence $X_1 = X_1'$ is contained in both decompositions. Now let $Y = \overline{X \backslash X_1}$. Then $Y = X_2 \cup \cdots \cup X_r = X_2' \cup \cdots \cup X_s'$; so proceeding by induction on r we obtain the uniqueness of the decomposition. □

Remark 1.3.9. It is probably time again for a warning: given an ideal I of the polynomial ring, it is in general not easy to find the irreducible components of $Z(I)$, or even to determine whether $Z(I)$ is irreducible or not. There are algorithms to figure this out, but they are computationally quite involved, so you will in most cases want to use a computer program for the actual calculation.

Remark 1.3.10. In the same way one can show that every algebraic set X is a (disjoint) finite union of *connected* algebraic sets, called the **connected components** of X.

Remark 1.3.11. We have now seen a few examples of the correspondence between geometry and algebra that forms the base of algebraic geometry: points in affine space correspond to maximal ideals in a polynomial ring, affine varieties to prime ideals, algebraic sets to radical ideals. Most concepts in algebraic geometry can be formulated and most proofs can be given both in geometric and in algebraic language. For example, the geometric statement that we have just shown that any algebraic set can be written as a finite union of irreducible components has the equivalent algebraic formulation that every radical ideal can be written uniquely as a finite intersection of prime ideals.

Remark 1.3.12. An application of the notion of irreducibility is the definition of the dimension of an affine variety (or more generally of a topological space; but as in the case of irreducibility above you will only want to apply it to the Zariski topology). Of course, in the case of complex varieties we have a geometric idea what the dimension of an affine variety should be: it is the number of complex coordinates that you need to describe X locally around any point. Although there are algebraic definitions of dimension that mimics this intuitive one, we will give a different definition here that uses only the language of topological spaces. Finally, all these definitions are of course equivalent and describe the intuitive notion of dimension (at least over $\mathbb{C}$), but it is actually quite hard to prove this rigorously.

The idea to define the dimension in algebraic geometry using the Zariski topology is the following: if X is an irreducible topological space, then any closed subset of X not equal to X must have dimension (at least) one smaller. (This is of course an idea that is not valid in the usual topology that you know from real analysis.)

Definition 1.3.13. Let X be a (non-empty) irreducible topological space. The **dimension** of X is the biggest integer n such that there is a chain $\emptyset \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X$ of irreducible closed subsets of X. If X is any Noetherian topological space, the dimension of X is defined to be the supremum of the dimensions of its irreducible components. A space of dimension 1 is called a **curve**, a space of dimension 2 a **surface**.

Remark 1.3.14. In this definition you should think of X_i as having dimension i. The content of the definition is just that there is "nothing between" varieties of dimension i and $i+1$.

Example 1.3.15. The dimension of $\mathbb{A}^1$ is 1, as single points are the only irreducible closed subsets of $\mathbb{A}^1$ not equal to $\mathbb{A}^1$. We will see in exercise 1.4.9 that the dimension of $\mathbb{A}^2$ is 2. Of course, the dimension of $\mathbb{A}^n$ is always n, but this is a fact from commutative algebra that we cannot prove at the moment. But we can at least see that the dimension of $\mathbb{A}^n$ is not less than n, because there are sequences of inclusions

$$\mathbb{A}^0 \subsetneq \mathbb{A}^1 \subsetneq \cdots \subsetneq \mathbb{A}^n$$

of linear subspaces of increasing dimension.

Remark 1.3.16. This definition of dimension has the advantage of being short and intuitive, but it has the disadvantage that it is very difficult to apply in actual computations. So for the moment we will continue to use the concept of dimension only in the informal way as we have used it so far. We will study the dimension of varieties rigorously in section 4, after we have developed more powerful techniques in algebraic geometry.

Remark 1.3.17. Here is another application of the notion of irreducibility (that is in fact not much more than a reformulation of the definition). Let X be an irreducible topological space (e. g. an affine variety). Let $U \subset X$ be a non-empty open subset, and let $Y \subsetneq X$ be a closed subset. The fact that X cannot be the union $(X\backslash U) \cup Y$ can be reformulated by saying that U cannot be a subset of Y. In other words, the **closure** of U (i. e. the smallest closed subset of X that contains U) is equal to X itself. Recall that an open subset of a topological space X is called **dense** if its closure is equal to the whole space X. With this wording, we have just shown that in an irreducible topological space every non-empty open subset is dense. Note that this is not true for reducible spaces: let $X = \{x_1x_2 = 0\} \subset \mathbb{A}^2$ be the union of the two coordinate axes, and let $U = \{x_1 \neq 0\} \cap X$ be the open subset of X consisting of the x_1-axis minus the origin. Then the closure of U in X is just the x_1-axis, and not all of X.

1.4. **Exercises.** In all exercises, the ground field k is assumed to be algebraically closed unless stated otherwise.

Exercise 1.4.1. Let $X_1, X_2 \subset \mathbb{A}^n$ be algebraic sets. Show that

(i) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$,
(ii) $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.

Show by example that taking the radical in (ii) is in general necessary, i. e. find algebraic sets X_1, X_2 such that $I(X_1 \cap X_2) \neq I(X_1) + I(X_2)$. Can you see geometrically what it means if we have inequality here?

Exercise 1.4.2. Let $X \subset \mathbb{A}^3$ be the union of the three coordinate axes. Determine generators for the ideal $I(X)$. Show that $I(X)$ cannot be generated by fewer than 3 elements, although X has codimension 2 in $\mathbb{A}^3$.

Exercise 1.4.3. In affine 4-dimensional space $\mathbb{A}^4$ with coordinates x, y, z, t let X be the union of the two planes

$$X' = \{x = y = 0\} \qquad \text{and} \qquad X'' = \{z = x - t = 0\}.$$

Compute the ideal $I = I(X) \subset k[x, y, z, t]$. For any $a \in k$ let $I_a \subset k[x, y, z]$ be the ideal obtained by substituting $t = a$ in I, and let $X_a = Z(I_a) \subset \mathbb{A}^3$.

Show that the family of algebraic sets X_a with $a \in k$ describes two skew lines in $\mathbb{A}^3$ approaching each other, until they finally intersect transversely for $a = 0$.

Moreover, show that the ideals I_a are radical for $a \neq 0$, but that I_0 is not. Find the elements in $\sqrt{I_0}\backslash I_0$ and interpret them geometrically.

Exercise 1.4.4. Let $X \subset \mathbb{A}^3$ be the algebraic set given by the equations $x_1^2 - x_2x_3 = x_1x_3 - x_1 = 0$. Find the irreducible components of X. What are their prime ideals? (Don't let the simplicity of this exercise fool you. As mentioned in remark 1.3.9, it is in general *very* difficult to compute the irreducible components of the zero locus of given equations, or even to determine if it is irreducible or not.)

Exercise 1.4.5. Let $\mathbb{A}^3$ be the 3-dimensional affine space over a field k with coordinates x, y, z. Find ideals describing the following algebraic sets and determine the minimal number of generators for these ideals.

(i) The union of the (x, y)-plane with the z-axis.
(ii) The union of the 3 coordinate axes.
(iii) The image of the map $\mathbb{A}^1 \to \mathbb{A}^3$ given by $t \mapsto (t^3, t^4, t^5)$.

Exercise 1.4.6. Let Y be a subspace of a topological space X. Show that Y is irreducible if and only if the closure of Y in X is irreducible.

Exercise 1.4.7. (For those of you who like pathological examples. You will need some knowledge on general topological spaces.) Find a Noetherian topological space with infinite dimension. Can you find an affine variety with infinite dimension?

Exercise 1.4.8. Let $X = \{(t,t^3,t^5) \; ; \; t \in k\} \subset \mathbb{A}^3$. Show that X is an affine variety of dimension 1 and compute $I(X)$.

Exercise 1.4.9. Let $X \subset \mathbb{A}^2$ be an irreducible algebraic set. Show that either

- $X = Z(0)$, i. e. X is the whole space $\mathbb{A}^2$, or
- $X = Z(f)$ for some irreducible polynomial $f \in k[x,y]$, or
- $X = Z(x-a, y-b)$ for some $a,b \in k$, i. e. X is a single point.

Deduce that $\dim(\mathbb{A}^2) = 2$. (Hint: Show that the common zero locus of two polynomials $f,g \in k[x,y]$ without common factor is finite.)

2. Functions, morphisms, and varieties

If $X \subset \mathbb{A}^n$ is an affine variety, we define the function field $K(X)$ of X to be the quotient field of the coordinate ring $A(X) = k[x_1,\dots,x_n]/I(X)$; this can be thought of as the field of rational functions on X. For a point $P \in X$ the local ring $\mathcal{O}_{X,P}$ is the subring of $K(X)$ of all functions that are regular (i. e. well-defined) at P, and for $U \subset X$ an open subset we let $\mathcal{O}_X(U)$ be the subring of $K(X)$ of all functions that are regular at every $P \in U$. The ring of functions that are regular on all of X is precisely $A(X)$.

Given two ringed spaces $(X,\mathcal{O}_X)$, $(Y,\mathcal{O}_Y)$ with the property that their structure sheaves are sheaves of k-valued functions, a set-theoretic map $f: X \to Y$ determines a pull-back map f^* from k-valued functions on Y to k-valued functions on X by composition. We say that f is a morphism if f is continuous and $f^*\mathcal{O}_Y(U) \subset \mathcal{O}_X(f^{-1}(U))$ for all open sets U in Y. In particular, this defines morphisms between affine varieties and their open subsets. Morphisms $X \to Y$ between affine varieties correspond exactly to k-algebra homomorphisms $A(Y) \to A(X)$.

In complete analogy to the theory of manifolds, we then define a prevariety to be a ringed space (whose structure sheaf is a sheaf of k-valued functions and) that is locally isomorphic to an affine variety. Correspondingly, there is a general way to construct prevarieties and morphisms between them by taking affine varieties (resp. morphisms between them) and patching them together. Affine varieties and their open subsets are simple examples of prevarieties, but we also get more complicated spaces as e. g. $\mathbb{P}^1$ and the affine line with a doubled origin. A prevariety X is called a variety if the diagonal $\Delta(X) \subset X \times X$ is closed, i. e. if X does not contain "doubled points".

2.1. **Functions on affine varieties.** After having defined affine varieties, our next goal must of course be to say what the maps between them should be. Let us first look at the easiest case: "regular functions", i. e. maps to the ground field $k = \mathbb{A}^1$. They should be thought of as the analogue of continuous functions in topology, or differentiable functions in real analysis, or holomorphic functions in complex analysis. Of course, in the case of algebraic geometry we want to have algebraic functions, i. e. (quotients of) polynomial functions.

Definition 2.1.1. Let $X \subset \mathbb{A}^n$ be an affine variety. We call

$$\boldsymbol{A(X)} := k[x_1,\dots,x_n]/I(X)$$

the **coordinate ring** of X.

Remark 2.1.2. The coordinate ring of X should be thought of as the ring of polynomial functions on X. In fact, for any $P \in X$ an element $f \in A(X)$ determines a polynomial map $X \to k$ (usually also denoted by f) given by $f \mapsto f(P)$:

- this is well-defined, because all functions in $I(X)$ vanish on X by definition,
- if the function $f: X \to k$ is identically zero then $f \in I(X)$ by definition, so $f = 0$ in $A(X)$.

Note that $I(X)$ is a prime ideal by lemma 1.3.4, so $A(X)$ is an integral domain. Hence we can make the following definition:

Definition 2.1.3. Let $X \subset \mathbb{A}^n$ be an affine variety. The quotient field $\boldsymbol{K(X)}$ of $A(X)$ is called the **field of rational functions** on X.

Remark 2.1.4. Recall that the quotient field K of an integral domain R is defined to be the set of pairs (f,g) with $f,g \in R$, $g \neq 0$, modulo the equivalence relation

$$(f,g) \sim (f',g') \iff fg' - gf' = 0.$$

An element (f,g) of K is usually written as $\frac{f}{g}$, and we think of it as the formal quotient of two ring elements. Addition of two such formal quotients is defined in the same way as you would expect to add fractions, namely

$$\frac{f}{g}+\frac{f'}{g'}:=\frac{fg'+gf'}{gg'},$$

and similarly for subtraction, multiplication, and division. This makes $K(X)$ into a field. In the case where $R=A(X)$ is the coordinate ring of an affine variety, we can therefore think of elements of $K(X)$ as being quotients of polynomial functions. We have to be very careful with this interpretation though, see example 2.1.7 and lemma 2.1.8.

Now let us define what we want to mean by a regular function on an open subset U of an affine variety X. This is more or less obvious: a regular function should be a rational function that is well-defined at all points of U:

Definition 2.1.5. Let $X\subset\mathbb{A}^n$ be an affine variety and let $P\in X$ be a point. We call

$$\boldsymbol{\mathcal{O}_{X,P}}:=\left\{\frac{f}{g}\ ;\ f,g\in A(X)\text{ and }g(P)\neq 0\right\}\subset K(X)$$

the **local ring** of X at the point P. Obviously, this should be thought of as the rational functions that are regular at P. If $U\subset X$ is a non-empty open subset, we set

$$\boldsymbol{\mathcal{O}_X(U)}:=\bigcap_{P\in U}\mathcal{O}_{X,P}.$$

This is a subring of $K(X)$. We call this the **ring of regular functions** on U.

Remark 2.1.6. The set $\boldsymbol{\mathfrak{m}_{X,P}}:=\{f\in A(X)\ ;\ f(P)=0\}$ of all functions that vanish at P is an ideal in $A(X)$. This is a maximal ideal, as $A(X)/\mathfrak{m}_{X,P}\cong k$, the isomorphism being evaluation of the polynomial at the point P. With this definition, $\mathcal{O}_{X,P}$ is just the localization of the ring $A(X)$ at the maximal ideal $\mathfrak{m}_{X,P}$. We will explain in lemma 2.2.10 where the name "local" (resp. "localization") comes from.

Example 2.1.7. We have just defined regular functions on an open subset of an affine variety $X\subset\mathbb{A}^n$ to be rational functions, i. e. elements in the quotient field $K(X)$, with certain properties. This means that every such function can be written as the "quotient" of two elements in $A(X)$. It does *not* mean however that we can always write a regular function as the quotient of two polynomials in $k[x_1,\dots,n_n]$. Here is an example showing this. Let $X\subset\mathbb{A}^4$ be the variety defined by the equation $x_1x_4=x_2x_3$, and let $U\subset X$ be the open subset of all points in X where $x_2\neq 0$ or $x_4\neq 0$. The function $\frac{x_1}{x_2}$ is defined at all points of X where $x_2\neq 0$, and the function $\frac{x_3}{x_4}$ is defined at points of X where $x_4\neq 0$. By the equation of X, these two functions coincide where they are both defined; in other words

$$\frac{x_1}{x_2}=\frac{x_3}{x_4}\in K(X)$$

by remark 2.1.4. So this gives rise to a regular function on U. But there is no representation of this function as a quotient of two polynomials in $k[x_1,x_2,x_3,x_4]$ that works on all of U — we have to use different representations at different points.

As we will usually want to write down regular functions as quotients of polynomials, we should prove a precise statement how regular functions can be patched together from different polynomial representations:

Lemma 2.1.8. *The following definition of regular functions is equivalent to the one of definition 2.1.5:*

Let U be an open subset of an affine variety $X\subset\mathbb{A}^n$. A set-theoretic map $\varphi:U\to k$ is called regular at the point $P\in U$ if there is a neighborhood V of P in U such that there are *polynomials $f,g \in k[x_1,\dots,x_n]$ with $g(Q) \neq 0$ and $\varphi(Q) = \frac{f(Q)}{g(Q)}$ for all $Q \in V$. It is called regular on U if it is regular at every point in U.*

Proof. It is obvious that an element of the ring of regular functions on U determines a regular function in the sense of the lemma.

Conversely, let $\varphi : U \to \mathbb{A}^1$ be a regular function in the sense of the lemma. Let $P \in U$ be any point, then there are polynomials f,g such that $g(Q) \neq 0$ and $\varphi(Q) = \frac{f(Q)}{g(Q)}$ for all points Q in some neighborhood V of P. We claim that $\frac{f}{g} \in K(X)$ is the element in the ring of regular functions that we seek.

In fact, all we have to show is that this element does not depend on the choices that we made. So let $P' \in U$ be another point (not necessarily distinct from P), and suppose that there are polynomials f',g' such that $\frac{f}{g} = \frac{f'}{g'}$ on some neighborhood V' of P'. Then $fg' = gf'$ on $V \cap V'$ and hence on X as $V \cap V'$ is dense in X by remark 1.3.17. In other words, $fg' - gf' \in I(X)$, so it is zero in $A(X)$, i. e. $\frac{f}{g} = \frac{f'}{g'} \in K(X)$. □

Remark 2.1.9. An almost trivial but remarkable consequence of our definition of regular functions is the following: let $U \subset V$ be non-empty open subsets of an affine variety X. If $\varphi_1, \varphi_2 : V \to k$ are two regular functions on V that agree on U, then they agree on all of V. This is obvious because the ring of regular functions (on any non-empty open subset) is a subring of the function field $K(X)$, so if two such regular functions agree this just means that they are the same element of $K(X)$. Of course, this is not surprising as open subsets are always dense, so if we know a regular function on an open subset it is intuitively clear that we know it almost everywhere anyway.

The interesting remark here is that the very same statement holds in complex analysis for holomorphic functions as well (or more generally, in real analysis for analytic functions): two holomorphic functions on a (connected) open subset $U \subset \mathbb{C}^n$ must be the same if they agree on any smaller open subset $V \subset U$. This is called the identity theorem for holomorphic functions — in complex analysis this is a real theorem because there the open subset V can be "very small", so the statement that the extension to U is unique is a lot more surprising than it is here in algebraic geometry. Still this is an example of a theorem that is true in literally the same way in both algebraic and complex geometry, although these two theories are quite different a priori.

Let us compute the rings $\mathcal{O}_X(U)$ explicitly in the cases where U is the complement of the zero locus of just a single polynomial.

Proposition 2.1.10. *Let $X \subset \mathbb{A}^n$ be an affine variety. Let $f \in A(X)$ and $X_f = \{P \in X \,;\, f(P) \neq 0\}$. (Open subsets of this form are called **distinguished open subsets**.) Then*

$$\mathcal{O}_X(X_f) = A(X)_f := \left\{ \frac{g}{f^r} \,;\, g \in A(X) \text{ and } r \geq 0 \right\}.$$

In particular, $\mathcal{O}_X(X) = A(X)$, i. e. any regular function on X is polynomial (take $f = 1$).

Proof. It is obvious that $A(X)_f \subset \mathcal{O}_X(X_f)$, so let us prove the converse. Let $\varphi \in \mathcal{O}_X(X_f) \subset K(X)$. Let $J = \{g \in A(X) \,;\, g\varphi \in A(X)\}$. This is an ideal in $A(X)$; we want to show that $f^r \in J$ for some r.

For any $P \in X_f$ we know that $\varphi \in \mathcal{O}_{X,P}$, so $\varphi = \frac{h}{g}$ with $g \neq 0$ in a neighborhood of P. In particular $g \in J$, so J contains an element not vanishing at P. This means that the zero locus of the ideal $I(X) + J \subset k[x_1,\dots,x_n]$ is contained in the set $\{P \in X \,;\, f(P) = 0\}$, or in other words that $Z(I(X)+J) \subset Z(f)$. By proposition 1.2.9 (i) it follows that $I(Z(f)) \subset I(Z(I(X)+J))$. So $f' \in I(Z(I(X)+J))$, where $f' \in k[x_1,\dots,x_n]$ is a representative of f. Therefore $f'^r \in I(X)+J$ for some r by the Nullstellensatz 1.2.9 (iii), and so $f^r \in J$. □

Remark 2.1.11. In the proof of proposition 2.1.10 we had to use the Nullstellensatz again. In fact, the statement is false if the ground field is not algebraically closed, as you can see from the example of the function $\frac{1}{x^2+1}$ that is regular on all of $\mathbb{A}^1(\mathbb{R})$, but not polynomial.

Example 2.1.12. Probably the easiest case of an open subset of an affine variety X that is not of the form X_f as in proposition 2.1.10 is the complement $U = \mathbb{C}^2 \backslash \{0\}$ of the origin in the affine plane. Let us compute $\mathcal{O}_{\mathbb{C}^2}(U)$. By definition 2.1.5 any element $\varphi \in \mathcal{O}_{\mathbb{C}^2}(U) \subset \mathbb{C}(x,y)$ is *globally* the quotient $\varphi = \frac{f}{g}$ of two polynomials $f,g \in \mathbb{C}[x,y]$. The condition that we have to satisfy is that $g(x,y) \neq 0$ for all $(x,y) \neq (0,0)$. We claim that this implies that g is constant. (In fact, this follows intuitively from the fact that a single equation can cut down the dimension of a space by only 1, so the zero locus of the polynomial g cannot only be the origin in $\mathbb{C}^2$. But we have not proved this rigorously yet.)

We know already by the Nullstellensatz that there is no non-constant polynomial that has empty zero locus in $\mathbb{C}^2$, so we can assume that g vanishes on $(0,0)$. If we write g as

$$g(x,y) = f_0(x) + f_1(x) \cdot y + f_2(x) \cdot y^2 + \cdots + f_n(x) \cdot y^n,$$

this means that $f_0(0) = 0$. We claim that $f_0(x)$ must be of the form x^m for some m. In fact:

- if f_0 is the zero polynomial, then $g(x,y)$ contains y as a factor and hence the whole x-axis in its zero locus,
- if f_0 contains more than one monomial, f_0 has a zero $x_0 \neq 0$, and hence $g(x_0,0) = 0$.

So $g(x,y)$ is of the form

$$g(x,y) = x^m + f_1(x) \cdot y + f_2(x) \cdot y^2 + \cdots + f_n(x) \cdot y^n.$$

Now set $y = \varepsilon$ for some small ε. As $g(x,0) = x^m$ and all f_i are continuous, the restriction $g(x,\varepsilon)$ cannot be the zero or a constant polynomial. Hence $g(x,\varepsilon)$ vanishes for some x, which is a contradiction.

So we see that we cannot have any denominators, i. e. $\mathcal{O}_{\mathbb{C}^2}(U) = \mathbb{C}[x,y]$. In other words, *a regular function on* $\mathbb{C}^2 \backslash \{0\}$ *is always regular on all of* $\mathbb{C}^2$. This is another example of a statement that is known from complex analysis for *holomorphic* functions, known as the removable singularity theorem.

2.2. **Sheaves.** We have seen in lemma 2.1.8 that regular functions on affine varieties are defined in terms of local properties: they are set-theoretic functions that can locally be written as quotients of polynomials. Local constructions of function-like objects occur in many places in algebraic geometry (and also in many other "topological" fields of mathematics), so we should formalize the idea of such objects. This will also give us an "automatic" definition of morphisms between affine varieties in section 2.3.

Definition 2.2.1. A **presheaf** $\mathcal{F}$ of rings on a topological space X consists of the data:

- for every open set $U \subset X$ a ring $\mathcal{F}(U)$ (think of this as the ring of functions on U),
- for every inclusion $U \subset V$ of open sets in X a ring homomorphism $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ called the restriction map (think of this as the usual restriction of functions to a subset),

such that

- $\mathcal{F}(\emptyset) = 0$,
- $\rho_{U,U}$ is the identity map for all U,
- for any inclusion $U \subset V \subset W$ of open sets in X we have $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$.

The elements of $\mathcal{F}(U)$ are usually called the **sections** of $\mathcal{F}$ over U, and the restriction maps $\rho_{V,U}$ are written as $f \mapsto f|_U$.

A presheaf $\mathcal{F}$ of rings is called a **sheaf** of rings if it satisfies the following glueing property: if $U \subset X$ is an open set, $\{U_i\}$ an open cover of U and $f_i \in \mathcal{F}(U_i)$ sections for all i such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j, then there is a unique $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$ for all i.

Remark 2.2.2. In the same way one can define (pre-)sheaves of Abelian groups / k-algebras etc., by requiring that all $\mathcal{F}(U)$ are objects and all $\rho_{V,U}$ are morphisms in the corresponding category.

Example 2.2.3. If $X \subset \mathbb{A}^n$ is an affine variety, then the rings $\mathcal{O}_X(U)$ of regular functions on open subsets of X (with the obvious restriction maps $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ for $U \subset V$) form a sheaf of rings $\mathcal{O}_X$, the **sheaf of regular functions** or **structure sheaf** on X. In fact, all defining properties of presheaves are obvious, and the glueing property of sheaves is easily seen from the description of regular functions in lemma 2.1.8.

Example 2.2.4. Here are some examples from other fields of mathematics: Let $X = \mathbb{R}^n$, and for any open subset $U \subset X$ let $\mathcal{F}(U)$ be the ring of continuous functions on U. Together with the obvious restriction maps, these rings $\mathcal{F}(U)$ form a sheaf, the *sheaf of continuous functions*. In the same way we can define the sheaf of k times differentiable functions, analytic functions, holomorphic functions on $\mathbb{C}^n$, and so on. The same definitions can be applied if X is a real or complex manifold instead of just $\mathbb{R}^n$ or $\mathbb{C}^n$.

In all these examples, the sheaves just defined "are" precisely the functions that are considered to be morphisms in the corresponding category (for example, in complex analysis the morphisms are just the holomorphic maps). This is usually expressed in the following way: a pair $(X, \mathcal{F})$ where X is a topological space and $\mathcal{F}$ is a sheaf of rings on X is called a **ringed space**. The sheaf $\mathcal{F}$ is then called the structure sheaf of this ringed space and usually written $\mathcal{O}_X$. Hence we have just given affine varieties the structure of a ringed space. (Although being general, this terminology will usually only be applied if $\mathcal{F}$ actually has an interpretation as the space of functions that are considered to be morphisms in the corresponding category.)

Remark 2.2.5. Intuitively speaking, any "function-like" object forms a presheaf; it is a sheaf if the conditions imposed on the "functions" are local. Here is an example illustrating this fact. Let $X = \mathbb{R}$ be the real line. For $U \subset X$ open and non-empty let $\mathcal{F}(U)$ be the ring of constant (real-valued) functions on U, i.e. $\mathcal{F}(U) \cong \mathbb{R}$ for all U. Let $\rho_{V,U}$ for $U \subset V$ be the obvious restriction maps. Then $\mathcal{F}$ is obviously a presheaf, but not a sheaf. This is because being constant is *not a local property*; it means that $f(P) = f(Q)$ for all P and Q that are possibly quite far away. For example, let $U = (0,1) \cup (2,3)$. Then U has an open cover $U = U_1 \cup U_2$ with $U_1 = (0,1)$ and $U_2 = (2,3)$. Let $f_1 : U_1 \to \mathbb{R}$ be the constant function 0, and let $f_2 : U_2 \to \mathbb{R}$ be the constant function 1. Then f_1 and f_2 trivially agree on the overlap $U_1 \cap U_2 = \emptyset$, but there is no *constant* function on U that restricts to both f_1 and f_2 on U_1 and U_2, respectively. There is however a uniquely defined *locally constant* function on U with that property. In fact, it is easy to see that the *locally constant* functions on X do form a sheaf.

Remark 2.2.6. If $\mathcal{F}$ is a sheaf on X and $U \subset X$ is an open subset, then one defines the restriction of $\mathcal{F}$ to U, denoted $\mathcal{F}|_U$, by $(\mathcal{F}|_U)(V) = \mathcal{F}(V)$ for all open subsets $V \subset U$. Obviously, this is again a sheaf.

Finally, let us see how the local rings of an affine variety appear in the language of sheaves.

Definition 2.2.7. Let X be a topological space, $P \in X$, and $\mathcal{F}$ a (pre-)sheaf on X. Consider pairs (U, φ) where U is an open neighborhood of P and $\varphi \in \mathcal{F}(U)$ a section of $\mathcal{F}$ over U. We call two such pairs (U, φ) and (U', φ') equivalent if there is an open neighborhood V of

P with $V \subset U \cap U'$ such that $\varphi|_V = \varphi'|_V$. (Note that this is in fact an equivalence relation.) The set of all such pairs modulo this equivalence relation is called the **stalk** $\mathcal{F}_P$ of $\mathcal{F}$ at P, its elements are called **germs** of $\mathcal{F}$.

Remark 2.2.8. If $\mathcal{F}$ is a (pre-)sheaf of rings (or k-algebras, Abelian groups, etc.) then the stalks of $\mathcal{F}$ are rings (or k-algebras, Abelian groups, etc.).

Remark 2.2.9. The interpretation of the stalk of a sheaf is obviously that its elements are sections of $\mathcal{F}$ that are defined in an (arbitrarily small) neighborhood around P. Hence e. g. on the real line the germ of a differentiable function at a point P allows you to compute the derivative of this function at P, but none of the actual values of the function at any point besides P. On the other hand, we have seen in remark 2.1.9 that holomorphic functions on a (connected) complex manifold are already determined by their values on any open set, so germs of holomorphic functions carry "much more information" than germs of differentiable functions. In algebraic geometry, this is similar: it is already quite obvious that germs of regular functions must carry much information, as the open subsets in the Zariski topology are so big. We will now show that the stalk of $\mathcal{O}_X$ at a point P is exactly the local ring $\mathcal{O}_{X,P}$, which finally gives a good motivation for the name "local ring".

Lemma 2.2.10. *Let X be an affine variety and $P \in X$. The stalk of $\mathcal{O}_X$ at P is $\mathcal{O}_{X,P}$.*

Proof. Recall that $\mathcal{O}_X(U) \subset \mathcal{O}_{X,P} \subset K(X)$ for all $P \in U$ by definition.

Therefore, if we are given a pair (U, φ) with $P \in U$ and $\varphi \in \mathcal{O}_X(U)$, we see that $\varphi \in \mathcal{O}_{X,P}$ determines an element in the local ring. If we have another equivalent pair (U', φ'), then φ and φ' agree on some V with $P \in V \subset U \cap U'$ by definition, so they determine the same element in $\mathcal{O}_X(V)$ and hence in $\mathcal{O}_{X,P}$.

Conversely, if $\varphi \in \mathcal{O}_{X,P}$ is an element in the local ring, we can write it as $\varphi = \frac{f}{g}$ with polynomials f, g such that $g(P) \neq 0$. Then there must be a neighborhood U of P on which g is non-zero, and therefore the (U, φ) defines an element in the stalk of $\mathcal{O}_X$ at P. □

2.3. **Morphisms between affine varieties.** Having given the structure of ringed spaces to affine varieties, there is a natural way to define morphisms between them. In this section we will allow ourselves to view morphisms as set-theoretic maps on the underlying topological spaces with additional properties (see lemma 2.1.8).

Definition 2.3.1. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces whose structure sheaves $\mathcal{O}_X$ and $\mathcal{O}_Y$ are sheaves of k-valued functions (in the case we are considering right now X and Y will be affine varieties or open subsets of affine varieties). Let $f : X \to Y$ be a set-theoretic map.

(i) If $\varphi : U \to k$ is a k-valued (set-theoretic) function on an open subset U of Y, the composition $\varphi \circ f : f^{-1}(U) \to k$ is again a set-theoretic function. It is denoted by $f^*\varphi$ and is called the **pull-back** of φ.

(ii) The map f is called a **morphism** if it is continuous, and if it pulls back regular functions to regular functions, i. e. if $f^*\mathcal{O}_Y(U) \subset \mathcal{O}_X(f^{-1}(U))$ for all open $U \subset Y$.

Remark 2.3.2. Recall that a function $f : X \to Y$ between topological spaces is called **continuous** if inverse images of open subsets are always open. In the above definition (ii), the requirement that f be continuous is therefore necessary to formulate the second condition, as it ensures that $f^{-1}(U)$ is open, so that $\mathcal{O}_X(f^{-1}(U))$ is well-defined.

Remark 2.3.3. In our context of algebraic geometry $\mathcal{O}_X$ and $\mathcal{O}_Y$ will always be the sheaves of regular maps constructed in definition 2.1.5. But in fact, this definition of morphisms is used in many other categories as well, e. g. one can say that a set-theoretic map between complex manifolds is holomorphic if it pulls back holomorphic functions to holomorphic functions. In fact, it is almost the general definition of morphisms between ringed spaces —

the only additional twist in the general case is that if $f : X \to Y$ is a continuous map between arbitrary ringed spaces $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$, there is no a priori definition of the pull-back map $\mathscr{O}_Y(U) \to \mathscr{O}_X(f^{-1}(U))$. In the case above we solved this problem by applying the set-theoretic viewpoint that gave us a notion of pull-back in our special case. In more general cases (e. g. for schemes that we will discuss later in section 5) one will have to include these pull-back maps in the data needed to define a morphism.

We now want to show that for affine varieties the situation is a lot easier: we actually do not have to deal with open subsets, but it suffices to check the pull-back property on *global* functions only:

Lemma 2.3.4. *Let $f : X \to Y$ be a continuous map between affine varieties. Then the following are equivalent:*

(i) *f is a morphism (i. e. f pulls back regular functions on open subsets to regular functions on open subsets).*
(ii) *For every $\varphi \in \mathscr{O}_Y(Y)$ we have $f^*\varphi \in \mathscr{O}_X(X)$, i. e. f pulls back* global *regular functions to* global *regular functions.*
(iii) *For every $P \in X$ and every $\varphi \in \mathscr{O}_{Y,f(P)}$ we have $f^*\varphi \in \mathscr{O}_{X,P}$, i. e. f pulls back* germs *of regular functions to* germs *of regular functions.*

Proof. (i) $\Rightarrow$ (ii) is trivial, and (iii) $\Rightarrow$ (i) follows immediately from the definition of $\mathscr{O}_Y(U)$ and $\mathscr{O}_X(f^{-1}(U))$ as intersections of local rings. To prove (ii) $\Rightarrow$ (iii) let $\varphi \in \mathscr{O}_{Y,f(P)}$ be the germ of a regular function on Y. We write $\varphi = \frac{g}{h}$ with $g, h \in A(Y) = \mathscr{O}_Y(Y)$ and $h(f(P)) \neq 0$. By (ii), f^*g and f^*h are global regular functions in $A(X) = \mathscr{O}_X(X)$, hence $f^*\varphi = \frac{f^*g}{f^*h} \in \mathscr{O}_{X,P}$, since we have $h(f(P)) \neq 0$. $\square$

Example 2.3.5. Let $X = \mathbb{A}^1$ be the affine line with coordinate x, and let $Y = \mathbb{A}^1$ be the affine line with coordinate y. Consider the set-theoretic map

$$f : X \to Y, \quad x \mapsto y = x^2.$$

We claim that this is a morphism. In fact, by lemma 2.3.4 (ii) we just have to show that f pulls back polynomials in $k[y]$ to polynomials in $k[x]$. But this is obvious, as the pull-back of a polynomial $\varphi(y) \in k[y]$ is just $\varphi(x^2)$ (i. e. we substitute x^2 for y in φ). This is still a polynomial, so it is in $k[x]$.

Example 2.3.6. For the very same reason, every polynomial map is a morphism. More precisely, let $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ be affine varieties, and let $f : X \to Y$ be a polynomial map, i. e. a map that can be written as $f(P) = (f_1(P), \dots, f_n(P))$ with $f_i \in k[x_1, \dots, x_m]$. As f then pulls back polynomials to polynomials, we conclude first of all that f is continuous. Moreover, it then follows from lemma 2.3.4 (ii) that f is a morphism. In fact, we will show now that all morphisms between affine varieties are of this form.

Lemma 2.3.7. *Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be affine varieties. There is a one-to-one correspondence between morphisms $f : X \to Y$ and k-algebra homomorphisms $f^* : A(Y) \to A(X)$.*

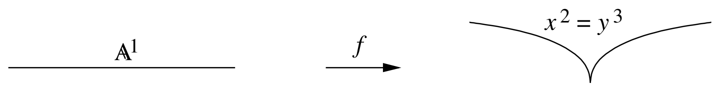
Proof. Any morphism $f : X \to Y$ determines a k-algebra homomorphism $f^* : \mathscr{O}_Y(Y) = A(Y) \to \mathscr{O}_X(X) = A(X)$ by definition. Conversely, if

$$g : k[y_1, \dots, y_m]/I(Y) \to k[x_1, \dots, x_n]/I(X)$$

is any k-algebra homomorphism then it determines a polynomial map $f = (f_1, \dots, f_m) : X \to Y$ as in example 2.3.6 by $f_i = g(y_i)$, and hence a morphism. $\square$

Example 2.3.8. Of course, an **isomorphism** is defined to be a morphism $f: X \to Y$ that has an inverse (i. e. a morphism such that there is a morphism $g: Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$). A warning is in place here that an isomorphism of affine varieties is *not* the same as a bijective morphism (in contrast e. g. to the case of vector spaces). For example, let $X \subset \mathbb{A}^2$ be the curve given by the equation $x^2 = y^3$, and consider the map

$$f: \mathbb{A}^1 \to X, \quad t \mapsto (x = t^3, y = t^2).$$



This is a morphism as it is given by polynomials, and it is bijective as the inverse is given by

$$f^{-1}: X \to \mathbb{A}^1, \quad (x,y) \mapsto \begin{cases} \frac{x}{y} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

But if f was an isomorphism, the corresponding k-algebra homomorphism

$$k[x,y]/(x^2 - y^3) \to k[t], \quad x \mapsto t^3 \text{ and } y \mapsto t^2$$

would have to be an isomorphism by lemma 2.3.7. This is obviously not the case, as the image of this algebra homomorphism contains no linear polynomials.

Example 2.3.9. As an application of morphisms, let us consider products of affine varieties. Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be affine varieties with ideals $I(X) \subset k[x_1, \dots, x_n]$ and $I(Y) \subset k[y_1, \dots, y_m]$. As usual, we define the **product** $X \times Y$ of X and Y to be the set

$$X \times Y = \{(P,Q) \in \mathbb{A}^n \times \mathbb{A}^m \,;\, P \in X \text{ and } Q \in Y\} \subset \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}.$$

Obviously, this is an algebraic set in $\mathbb{A}^{n+m}$ with ideal

$$I(X \times Y) = I(X) + I(Y) \subset k[x_1, \dots, x_n, y_1, \dots, y_m]$$

where we consider $k[x_1, \dots, x_n]$ and $k[y_1, \dots, y_m]$ as subalgebras of $k[x_1, \dots, x_n, y_1, \dots, y_m]$ in the obvious way. Let us show that it is in fact a variety, i. e. irreducible:

Proposition 2.3.10. *If X and Y are affine varieties, then so is $X \times Y$.*

Proof. For simplicity, let us just write x for the collection of the x_i, and y for the collection of the y_i. By the above discussion it only remains to show that $I(X \times Y)$ is prime. So let $f, g \in k[x,y]$ be polynomial functions such that $fg \in I(X \times Y)$; we have to show that either f or g vanishes on all of $X \times Y$, i. e. that $X \times Y \subset Z(f)$ or $X \times Y \subset Z(g)$.

So let us assume that $X \times Y \not\subset Z(f)$, i. e. there is a point $(P,Q) \in X \times Y \backslash Z(f)$ (where $P \in X$ and $Q \in Y$). Denote by $f(\cdot, Q) \in k[x]$ the polynomial obtained from $f \in k[x,y]$ by plugging in the coordinates of Q for y. For all $P' \in X \backslash Z(f(\cdot, Q))$ (e. g. for $P' = P$) we must have

$$Y \subset Z(f(P', \cdot) \cdot g(P', \cdot)) = Z(f(P', \cdot)) \cup Z(g(P', \cdot)).$$

As Y is irreducible and $Y \not\subset Z(f(P', \cdot))$ by the choice of P', it follows that $Y \subset Z(g(P', \cdot))$.

This is true for all $P' \in X \backslash Z(f(\cdot, Q))$, so we conclude that $(X \backslash Z(f(\cdot, Q)) \times Y \subset Z(g)$. But as $Z(g)$ is closed, it must in fact contain the closure of $(X \backslash Z(f(\cdot, Q)) \times Y$ as well, which is just $X \times Y$ as X is irreducible and $X \backslash Z(f(\cdot, Q))$ a non-empty open subset of X (see remark 1.3.17). $\square$

The obvious projection maps

$$\pi_X : X \times Y \to X, \ (P,Q) \mapsto P \quad \text{and} \quad \pi_Y : X \times Y \to Y, \ (P,Q) \mapsto Q$$

are given by (trivial) polynomial maps and are therefore morphisms. The important main property of the product $X \times Y$ is the following:

Lemma 2.3.11. *Let X and Y be affine varieties. Then the product $X \times Y$ satisfies the following universal property: for every affine variety Z and morphisms $f : Z \to X$ and $g : Z \to Y$, there is a unique morphism $h : Z \to X \times Y$ such that $f = \pi_X \circ h$ and $g = \pi_Y \circ h$, i. e. such that the following diagram commutes:*

$$\begin{tikzcd} Z \arrow[dr, dotted, "h"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\ & X \times Y \arrow[r, "\pi_Y"'] \arrow[d, "\pi_X"] & Y \\ & X & \end{tikzcd}$$

In other words, giving a morphism $Z \to X \times Y$ "is the same" as giving two morphisms $Z \to X$ and $Z \to Y$.

Proof. Let A be the coordinate ring of Z. Then by lemma 2.3.7 the morphism $f : Z \to X$ is given by a k-algebra homomorphism $\tilde{f} : k[x_1,\dots,x_n]/I(X) \to A$. This in turn is determined by giving the images $\tilde{f}_i := \tilde{f}(x_i) \in A$ of the generators x_i, satisfying the relations of I (i. e. $F(\tilde{f}_1,\dots,\tilde{f}_n) = 0$ for all $F(x_1,\dots,x_n) \in I(X)$). The same is true for g, which is determined by the images $\tilde{g}_i := \tilde{g}(y_i) \in A$.

Now it is obvious that the elements $\tilde{f}_i$ and $\tilde{g}_i$ determine a k-algebra homomorphism

$$k[x_1,\dots,x_n,y_1,\dots,y_m]/(I(X)+I(Y)) \to A,$$

which determines a morphism $h : Z \to X \times Y$ by lemma 2.3.7.

To show uniqueness, just note that the relations $f = \pi_X \circ h$ and $g = \pi_Y \circ h$ imply immediately that h must be given set-theoretically by $h(P) = (f(P), g(P))$ for all $P \in Z$. $\square$

Remark 2.3.12. It is easy to see that the property of lemma 2.3.11 determines the product $X \times Y$ uniquely up to isomorphism. It is therefore often taken to be the defining property for products.

Remark 2.3.13. If you have heard about tensor products before, you will have noticed that the coordinate ring of $X \times Y$ is just the tensor product $A(X) \otimes A(Y)$ of the coordinate rings of the factors (where the tensor product is taken as k-algebras). See also section 5.4 for more details.

Remark 2.3.14. Lemma 2.3.7 allows us to associate an affine variety up to isomorphism to any finitely generated k-algebra that is a domain: if A is such an algebra, let $x_1,\dots,x_n$ be generators of A, so that $A = k[x_1,\dots,x_n]/I$ for some ideal I. Let X be the affine variety in $\mathbb{A}^n$ defined by the ideal I; by the lemma it is defined up to isomorphism. Therefore we should make a (very minor) redefinition of the term "affine variety" to allow for objects that are isomorphic to an affine variety in the old sense, but that do not come with an intrinsic description as the zero locus of some polynomials in affine space:

Definition 2.3.15. A ringed space $(X, \mathcal{O}_X)$ is called an **affine variety** over k if

(i) X is irreducible,
(ii) $\mathcal{O}_X$ is a sheaf of k-valued functions,
(iii) X is isomorphic to an affine variety in the sense of definition 1.3.1.

Here is an example of an affine variety in this new sense although it is not a priori given as the zero locus of some polynomials in affine space:

Lemma 2.3.16. *Let X be an affine variety and $f \in A(X)$, and let $X_f = X \backslash Z(f)$ be a distinguished open subset as in proposition 2.1.10. Then the ringed space $(X_f, \mathcal{O}_X|_{X_f})$ is isomorphic to an affine variety with coordinate ring $A(X)_f$.*

Proof. Let $X \subset \mathbb{A}^n$ be an affine variety, and let $f' \in k[x_1,\dots,x_n]$ be a representative of f. Let $J \subset k[x_1,\dots,x_n,t]$ be the ideal generated by $I(X)$ and the function $1-tf'$. We claim that the ringed space $(X_f, \mathcal{O}_X|_{X_f})$ is isomorphic to the affine variety

$$Z(J) = \{(P,\lambda)\ ;\ P \in X \text{ and } \lambda = \tfrac{1}{f'(P)}\} \subset \mathbb{A}^{n+1}.$$

Consider the projection map $\pi : Z(J) \to X$ given by $\pi(P,\lambda) = P$. This is a morphism with image X_f and inverse morphism $\pi^{-1}(P) = (P, \frac{1}{f'(P)})$, hence π is an isomorphism. It is obvious that $A(Z(J)) = A(X)_f$. □

Remark 2.3.17. So we have just shown that even open subsets of affine varieties are themselves affine varieties, provided that the open subset is the complement of the zero locus of a single polynomial equation.

Example 2.1.12 shows however that not all open subsets of affine varieties are themselves isomorphic to affine varieties: if $U \subset \mathbb{C}^2\backslash\{0\}$ we have seen that $\mathcal{O}_U(U) = k[x,y]$. So if U was an affine variety, its coordinate ring must be $k[x,y]$, which is the same as the coordinate ring of $\mathbb{C}^2$. By lemma 2.3.7 this means that U and $\mathbb{C}^2$ would have to be isomorphic, with the isomorphism given by the identity map. Obviously, this is not true. Hence U is not an affine variety. It can however be covered by two open subsets $\{x \neq 0\}$ and $\{y \neq 0\}$ which are both affine by lemma 2.3.16. This leads us to the idea of *patching* affine varieties together, which we will do in the next section.

2.4. **Prevarieties.** Now we want to extend our category of objects to more general things than just affine varieties. Remark 2.3.17 showed us that not all open subsets of affine varieties are themselves isomorphic to affine varieties. But note that every open subset of an affine variety can be written as a finite union of distinguished open subsets (as this is equivalent to the statement that every closed subset of an affine variety is the zero locus of finitely many polynomials). Hence any such open subset can be *covered* by affine varieties. This leads us to the idea that we should study objects that are not affine varieties themselves, but rather can be covered by (finitely many) affine varieties. Note that the following definition is completely parallel to the definition 2.3.15 of affine varieties (in the new sense).

Definition 2.4.1. A **prevariety** is a ringed space $(X, \mathcal{O}_X)$ such that

(i) X is irreducible,
(ii) $\mathcal{O}_X$ is a sheaf of k-valued functions,
(iii) there is a finite open cover $\{U_i\}$ of X such that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine variety for all i.

As before, a morphism of prevarieties is just a morphism as ringed spaces (see definition 2.3.1).

An open subset $U \subset X$ of a prevariety such that $(U, \mathcal{O}_X|_U)$ is isomorphic to an affine variety is called an **affine open set**.

Example 2.4.2. Affine varieties and open subsets of affine varieties are prevarieties (the irreducibility of open subsets follows from exercise 1.4.6).

Remark 2.4.3. The above definition is completely analogous to the definition of manifolds. Recall how manifolds are defined: first you look at open subsets of $\mathbb{R}^n$ that are supposed to form the patches of your space, and then you define a manifold to be a topological space that looks locally like these patches. In the algebraic case now we can say that the affine varieties form the basic patches of the spaces that we want to consider, and that e. g. open subsets of affine varieties are spaces that look locally like affine varieties.

As we defined a prevariety to be a space that can be covered by affine opens, the most general way to construct prevarieties is of course to take some affine varieties (or prevarieties that we have already constructed) and patch them together:

Example 2.4.4. Let X_1, X_2 be prevarieties, let $U_1 \subset X_1$ and $U_2 \subset X_2$ be non-empty open subsets, and let $f : (U_1, \mathscr{O}_{X_1}|_{U_1}) \to (U_2, \mathscr{O}_{X_2}|_{U_2})$ be an isomorphism. Then we can define a prevariety X, obtained by **glueing** X_1 and X_2 along U_1 and U_2 via the isomorphism f:

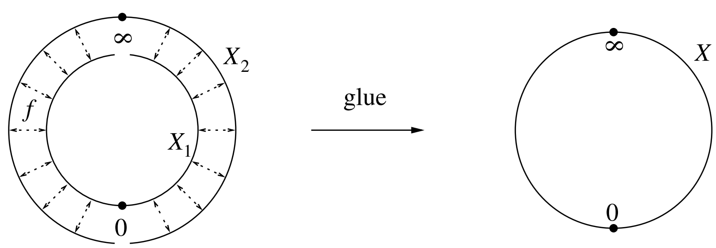
- As a set, the space X is just the disjoint union $X_1 \cup X_2$ modulo the equivalence relation $P \sim f(P)$ for all $P \in U_1$.
- As a topological space, we endow X with the so-called *quotient topology* induced by the above equivalence relation, i. e. we say that a subset $U \subset X$ is open if and only if $i_1^{-1}(U) \subset X_1$ and $i_2^{-1}(U) \subset X_2$ are both open, with $i_1 : X_1 \to X$ and $i_2 : X_2 \to X$ being the obvious inclusion maps.
- As a ringed space, we define the structure sheaf $\mathscr{O}_X$ by

$$\begin{aligned}\mathscr{O}_X(U) = \{(\varphi_1, \varphi_2)\,;\, &\varphi_1 \in \mathscr{O}_{X_1}(i_1^{-1}(U)), \varphi_2 \in \mathscr{O}_{X_2}(i_2^{-1}(U)),\\ &\varphi_1 = \varphi_2 \text{ on the overlap (i. e. } f^*(\varphi_2|_{i_2^{-1}(U)\cap U_2}) = \varphi_1|_{i_1^{-1}(U)\cap U_1})\}.\end{aligned}$$

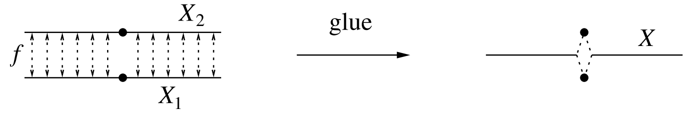
It is easy to check that this defines a sheaf of k-valued functions on X and that X is irreducible. Of course, every point of X has an affine neighborhood, so X is in fact a prevariety.

Example 2.4.5. As an example of the above glueing construction, let $X_1 = X_2 = \mathbb{A}^1$, $U_1 = U_2 = \mathbb{A}^1 \backslash \{0\}$.

- Let $f : U_1 \to U_2$ be the isomorphism $x \mapsto \frac{1}{x}$. The space X can be thought of as $\mathbb{A}^1 \cup \{\infty\}$: of course the affine line $X_1 = \mathbb{A}^1 \subset X$ sits in X. The complement $X \backslash X_1$ is a single point that corresponds to the zero point in $X_2 \cong \mathbb{A}^1$ and hence to "$\infty = \frac{1}{0}$" in the coordinate of X_1. In the case $k = \mathbb{C}$, the space X is just the Riemann sphere $\mathbb{C}_\infty$.



We denote this space by $\mathbb{P}^1$. (This is a special case of a projective space; see section 3.1 and remark 3.3.7 for more details.)
- Let $f : U_1 \to U_2$ be the identity map. Then the space X obtained by glueing along f is "the affine line with the zero point doubled":



Obviously this is a somewhat weird space. Speaking in classical terms (and thinking of the complex numbers), if we have a sequence of points tending to the zero, this sequence would have two possible limits, namely the two zero points. Usually we want to exclude such spaces from the objects we consider. In the theory of manifolds, this is simply done by requiring that a manifold satisfies the so-called *Hausdorff property*, i. e. that every two distinct points have disjoint open neighborhoods. This is obviously not satisfied for our space X. But the analogous definition does not make sense in the Zariski topology, as non-empty open subsets are never disjoint. Hence we need a different characterization of the geometric concept of "doubled points". We will do this in section 2.5.

Example 2.4.6. Let X be the complex affine curve

$$X = \{(x,y) \in \mathbb{C}^2 \ ; \ y^2 = (x-1)(x-2)\cdots(x-2n)\}.$$

We have already seen in example 0.1.1 that X can (and should) be "compactified" by adding two points at infinity, corresponding to the limit $x \to \infty$ and the two possible values for y. Let us now construct this compactified space rigorously as a prevariety.

To be able to add a limit point "$x = \infty$" to our space, let us make a coordinate change $\tilde{x} = \frac{1}{x}$, so that the equation of the curve becomes

$$y^2 \tilde{x}^{2n} = (1-\tilde{x})(1-2\tilde{x})\cdots(1-2n\tilde{x}).$$

If we make an additional coordinate change $\tilde{y} = \frac{y}{x^n}$, this becomes

$$\tilde{y}^2 = (1-\tilde{x})(1-2\tilde{x})\cdots(1-2n\tilde{x}).$$

In these coordinates we can add our two points at infinity, as they now correspond to $\tilde{x} = 0$ (and therefore $\tilde{y} = \pm 1$).

Summarizing, our "compactified curve" of example 0.1.1 is just the prevariety obtained by glueing the two affine varieties

$$\begin{aligned} X &= \{(x,y) \in \mathbb{C}^2 \ ; \ y^2 = (x-1)(x-2)\cdots(x-2n)\} \\ \text{and} \quad \tilde{X} &= \{(\tilde{x},\tilde{y}) \in \mathbb{C}^2 \ ; \ \tilde{y}^2 = (1-\tilde{x})(1-2\tilde{x})\cdots(1-2n\tilde{x})\} \end{aligned}$$

along the isomorphism

$$\begin{aligned} f : U \to \tilde{U}, \quad (x,y) &\mapsto (\tilde{x},\tilde{y}) = \left(\frac{1}{x}, \frac{y}{x^n}\right), \\ f^{-1} : \tilde{U} \to U, \quad (\tilde{x},\tilde{y}) &\mapsto (x,y) = \left(\frac{1}{\tilde{x}}, \frac{\tilde{y}}{\tilde{x}^n}\right), \end{aligned}$$

where $U = \{x \neq 0\} \subset X$ and $\tilde{U} = \{\tilde{x} \neq 0\} \subset \tilde{X}$.

Of course one can also glue together more than two prevarieties. As the construction is the same as in the case above, we will just give the statement and leave its proof as an exercise:

Lemma 2.4.7. *Let $X_1, \dots, X_r$ be prevarieties, and let $U_{i,j} \subset X_i$ be non-empty open subsets for $i, j = 1, \dots, r$. Let $f_{i,j} : U_{i,j} \to U_{j,i}$ be isomorphisms such that*

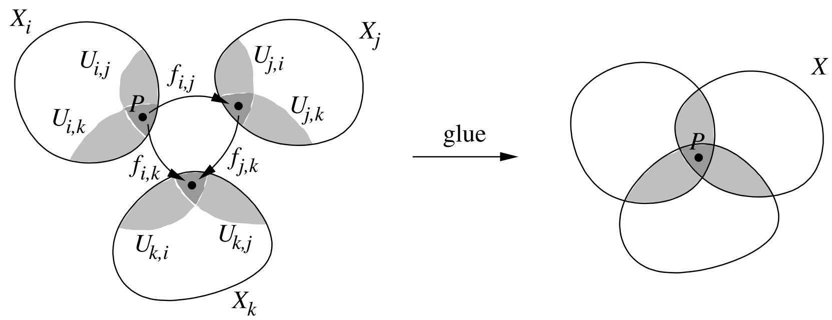
(i) $f_{i,j} = f_{j,i}^{-1}$,
(ii) $f_{i,k} = f_{j,k} \circ f_{i,j}$ *where defined.*

Then there is a prevariety X, obtained by glueing the X_i along the morphisms $f_{i,j}$ as in example 2.4.4 (see below).

Remark 2.4.8. The prevariety X in the lemma 2.4.7 can be described as follows:

- As a set, X is the disjoint union of the X_i, modulo the equivalence relation $P \sim f_{i,j}(P)$ for all $P \in U_{i,j}$.
- To define X as a topological space, we say that a subset $Y \subset X$ is closed if and only if all restrictions $Y \cap X_i$ are closed.
- A regular function on an open set $U \subset X$ is a collection of regular functions $\varphi_i \in \mathcal{O}_{X_i}(X_i \cap U)$ that agree on the overlaps.

Condition (ii) of the lemma gives a compatibility condition for triple overlaps: consider three patches X_i, X_j , X_k that have a common intersection. Then we want to identify every point $P \in U_{i,j}$ with $f_{i,j}(P) \in U_{j,k}$, and the point $f_{i,j}(P)$ with $f_{j,k}(f_{i,j}(P))$ (if it lies in $U_{j,k}$). So the glueing map $f_{i,k}$ must map P to the same point $f_{j,k}(f_{i,j}(P))$ to get a consistent glueing. This is illustrated in the following picture:



Let us now consider some examples of morphisms between prevarieties.

Example 2.4.9. Let $f : \mathbb{P}^1 \to \mathbb{A}^1$ be a morphism. We claim that f must be constant.

In fact, consider the restriction $f|_{\mathbb{A}^1}$ of f to the open affine subset $\mathbb{A}^1 \subset \mathbb{P}^1$. By definition the restriction of a morphism is again a morphism, so $f|_{\mathbb{A}^1}$ must be of the form $x \mapsto y = p(x)$ for some polynomial $p \in k[x]$. Now consider the second patch of $\mathbb{P}^1$ with coordinate $\tilde{x} = \frac{1}{x}$. Applying this coordinate change, we see that $f|_{\mathbb{P}^1\backslash\{0\}}$ is given by $\tilde{x} \mapsto p(\frac{1}{\tilde{x}})$. But this must be a morphism too, i. e. $p(\frac{1}{\tilde{x}})$ must be a polynomial in $\tilde{x}$. This is only true if p is a constant.

In the same way as prevarieties can be glued, we can patch together morphisms too. Of course, the statement is essentially that we can check the property of being a morphism on affine open covers:

Lemma 2.4.10. *Let X,Y be prevarieties and let $f : X \to Y$ be a set-theoretic map. Let $\{U_1,\dots,U_r\}$ be an open cover of X and $\{V_1,\dots,V_r\}$ an* affine *open cover of Y such that $f(U_i) \subset V_i$ and $(f|_{U_i})^*\mathcal{O}_Y(V_i) \subset \mathcal{O}_X(U_i)$. Then f is a morphism.*

Proof. We may assume that the U_i are affine, as otherwise we can replace the U_i by a set of affines that cover U_i. Consider the restrictions $f_i : U_i \to V_i$. The homomorphism $f_i^* : \mathcal{O}_Y(V_i) = A(V_i) \to \mathcal{O}_X(U_i) = A(U_i)$ is induced by some morphism $U_i \to V_i$ by lemma 2.3.7 which is easily seen to coincide with f_i. In particular, the f_i are continuous, and therefore so is f. It remains to show that f^* takes sections of $\mathcal{O}_Y$ to sections of $\mathcal{O}_X$. But if $V \subset Y$ is open and $\varphi \in \mathcal{O}_Y(V)$, then $f^*\varphi$ is at least a section of $\mathcal{O}_X$ on the sets $f^{-1}(V) \cap U_i$. Since $\mathcal{O}_X$ is a sheaf and the U_i cover X, these sections glue to give a section in $\mathcal{O}_X(f^{-1}(V))$. □

Example 2.4.11. Let $f : \mathbb{A}^1 \to \mathbb{A}^1, x \mapsto y = f(x)$ be a morphism given by a polynomial $f \in k[x]$. We claim that there is a unique extension morphism $\tilde{f} : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\tilde{f}|_{\mathbb{A}^1} = f$. We can assume that $f = \sum_{i=1}^n a_i x^i$ is not constant, as otherwise the result is trivial. It is then obvious that the extension should be given by $\tilde{f}(\infty) = \infty$. Let us check that this defines in fact a morphism.

We want to apply lemma 2.4.10. Consider the standard open affine cover of the domain $\mathbb{P}^1$ by the two affine lines $V_1 = \mathbb{P}^1\backslash\{\infty\}$ and $V_2 = \mathbb{P}^1\backslash\{0\}$. Then $U_1 := \tilde{f}^{-1}(V_1) = \mathbb{A}^1$, and $\tilde{f}|_{\mathbb{A}^1} = f$ is a morphism. On the other hand, let $U_2 := \tilde{f}^{-1}(V_2)\backslash\{0\}$. Consider the coordinates $\tilde{x} = \frac{1}{x}$ and $\tilde{y} = \frac{1}{y}$ on U_2 and V_2, respectively. In these coordinates, the map $\tilde{f}$ is given by

$$\tilde{y} = \frac{\tilde{x}^n}{\sum_{i=1}^n a_i \tilde{x}^{n-i}};$$

in particular $\tilde{x} = 0$ maps to $\tilde{y} = 0$. So by defining $\tilde{f}(\infty) = \infty$, we get a morphism $\tilde{f} : \mathbb{P}^1 \to \mathbb{P}^1$ that extends f by lemma 2.4.10.

2.5. **Varieties.** Recall example 2.4.5 (ii) where we constructed a prevariety that was "not Hausdorff" in the classical sense: take two copies of the affine line $\mathbb{A}^1$ and glue them together on the open set $\mathbb{A}^1 \backslash \{0\}$ along the identity map. The prevariety X thus obtained is the "affine line with the origin doubled"; its strange property is that even in the classical topology (for $k = \mathbb{C}$) the two origins do not have disjoint neighborhoods. We will now define an algebro-geometric analogue of the Hausdorff property and say that a prevariety is a variety if it has this property.

Definition 2.5.1. Let X be a prevariety. We say that X is a **variety** if for every prevariety Y and every two morphisms $f_1, f_2 : Y \to X$, the set $\{P \in Y \,;\, f_1(P) = f_2(P)\}$ is closed in Y.

Remark 2.5.2. Let X be the affine line with the origin doubled. Then X is not a variety — if we take $Y = \mathbb{A}^1$ and let $f_1, f_2 : Y \to X$ be the two obvious inclusions that map the origin in Y to the two different origins in X, then f_1 and f_2 agree on $\mathbb{A}^1 \backslash \{0\}$, which is not closed in $\mathbb{A}^1$.

On the other hand, if X has the Hausdorff property that we want to characterize, then two morphisms $Y \to X$ that agree on an open subset of Y should also agree on Y.

One can make this definition somewhat easier and eliminate the need for an arbitrary second prevariety Y. To do so note that we can define products of prevarieties in the same way as we have defined products of affine varieties (see example 2.3.9 and exercise 2.6.13). For any prevariety X, consider the **diagonal morphism**

$$\Delta : X \to X \times X, \quad P \mapsto (P,P).$$

The image $\Delta(X) \subset X \times X$ is called the **diagonal** of X. Of course, the morphism $\Delta : X \to \Delta(X)$ is an isomorphism onto its image (with inverse morphism being given by $(P,Q) \mapsto P$). So the space $\Delta(X)$ is not really interesting as a new prevariety; instead the main question is how $\Delta(X)$ is embedded in $X \times X$:

Lemma 2.5.3. *A prevariety X is a variety if and only if the diagonal $\Delta(X)$ is closed in $X \times X$.*

Proof. It is obvious that a variety has this property (take $Y = X \times X$ and f_1, f_2 the two projections to X). Conversely, if the diagonal $\Delta(X)$ is closed and $f_1, f_2 : Y \to X$ are two morphisms, then they induce a morphism $(f_1, f_2) : Y \to X \times X$ by the universal property of exercise 2.6.13, and

$$\{P \in Y \mid f_1(P) = f_2(P)\} = (f_1, f_2)^{-1}(\Delta(X))$$

is closed. □

Lemma 2.5.4. *Every affine variety is a variety. Any open or closed subprevariety of a variety is a variety.*

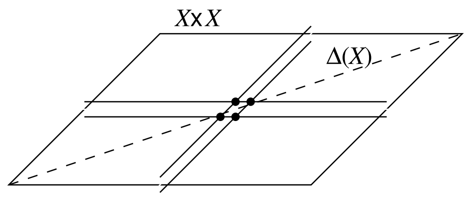
Proof. If $X \subset \mathbb{A}^n$ is an affine variety with ideal $I(X) = (f_1, \dots, f_r)$, the diagonal $\Delta(X) \subset \mathbb{A}^{2n}$ is an affine variety given by the equations $f_i(x_1, \dots, x_n) = 0$ for $i = 1, \dots, r$ and $x_i = y_i$ for $i = 1, \dots, n$, where $x_1, \dots, x_n, y_1, \dots, y_n$ are the coordinates on $\mathbb{A}^{2n}$. This is obviously closed, so X is a variety by lemma 2.5.3.

If $Y \subset X$ is open or closed, then $\Delta(Y) = \Delta(X) \cap (Y \times Y)$; i. e. if $\Delta(X)$ is closed in $X \times X$ then so is $\Delta(Y)$ in $Y \times Y$. □

Example 2.5.5. Let us illustrate lemma 2.5.3 in the case of the affine line with a doubled origin. So let $X_1 = X_2 = \mathbb{A}^1$, and let X be the prevariety obtained by glueing X_1 to X_2 along the identity on $\mathbb{A} \backslash \{0\}$. Then $X \times X$ is covered by the four affine varieties $X_1 \times X_1$, $X_1 \times X_2$,

$X_2 \times X_1$, and $X_2 \times X_2$ by exercise 2.6.13. As we glue along $\mathbb{A}^1 \backslash \{0\}$ to obtain X, it follows that the space $X \times X$ contains the point $(P,Q) \in \mathbb{A}^1 \times \mathbb{A}^1$

- once if $P \neq 0$ and $Q \neq 0$,
- twice if $P = 0$ and $Q \neq 0$, or if $P \neq 0$ and $Q = 0$,
- four times if $P = 0$ and $Q = 0$.



In particular, $X \times X$ contains four origins now. But the diagonal $\Delta(X)$ contains only two of them (by definition of the diagonal we have to take the same origin in both factors). So on the patch $X_1 \times X_2$, the diagonal is given by $\{(P,P)\ ;\ P \neq 0\} \subset X_1 \times X_2 = \mathbb{A}^1 \times \mathbb{A}^1$, which is not closed. So we see again that X is not a variety.

2.6. **Exercises.**

Exercise 2.6.1. An algebraic set $X \subset \mathbb{A}^2$ defined by a polynomial of degree 2 is called a *conic*. Show that any irreducible conic is isomorphic either to $Z(y-x^2)$ or to $Z(xy-1)$.

Exercise 2.6.2. Let $X, Y \subset \mathbb{A}^2$ be irreducible conics as in exercise 2.6.1, and assume that $X \neq Y$. Show that X and Y intersect in at most 4 points. For all $n \in \{0,1,2,3,4\}$, find an example of two conics that intersect in exactly n points. (For a generalization see theorem 6.2.1.)

Exercise 2.6.3. Which of the following algebraic sets are isomorphic over the complex numbers?

(a) $\mathbb{A}^1$	(b) $Z(xy) \subset \mathbb{A}^2$
(c) $Z(x^2+y^2) \subset \mathbb{A}^2$	(d) $Z(y^2-x^3-x^2) \subset \mathbb{A}^2$
(e) $Z(x^2-y^3) \subset \mathbb{A}^2$	(f) $Z(y-x^2, z-x^3) \subset \mathbb{A}^3$

Exercise 2.6.4. Let X be an affine variety, and let G be a finite group. Assume that G **acts on** X, i. e. that for every $g \in G$ we are given a morphism $g : X \to X$ (denoted by the same letter for simplicity of notation), such that $(g \circ h)(P) = g(h(P))$ for all $g, h \in G$ and $P \in X$.

(i) Let $A(X)^G$ be the subalgebra of $A(X)$ consisting of all G-invariant functions on X, i. e. of all $f : X \to k$ such that $f(g(P)) = f(P)$ for all $P \in X$. Show that $A(X)^G$ is a finitely generated k-algebra.

(ii) By (i), there is an affine variety Y with coordinate ring $A(X)^G$, together with a morphism $\pi : X \to Y$ determined by the inclusion $A(X)^G \subset A(X)$. Show that Y can be considered as the **quotient** of X by G, denoted X/G, in the following sense:
  (a) π is surjective.
  (b) If $P, Q \in X$ then $\pi(P) = \pi(Q)$ if and only if there is a $g \in G$ such that $g(P) = Q$.

(iii) For a given group action, is an affine variety with the properties (ii)(a) and (ii)(b) uniquely determined?

(iv) Let $\mathbb{Z}_n = \{\exp(\frac{2\pi i k}{n})\ ;\ k \in \mathbb{Z}\} \subset \mathbb{C}$ be the group of n-th roots of unity. Let $\mathbb{Z}_n$ act on $\mathbb{C}^m$ by multiplication in each coordinate. Show that $\mathbb{C}/\mathbb{Z}_n$ is isomorphic to $\mathbb{C}$ for all n, but that $\mathbb{C}^2/\mathbb{Z}_n$ is not isomorphic to $\mathbb{C}^2$ for $n \geq 2$.

Exercise 2.6.5. Are the following statements true or false: if $f : \mathbb{A}^n \to \mathbb{A}^m$ is a polynomial map (i. e. $f(P) = (f_1(P), \dots, f_m(P))$ with $f_i \in k[x_1, \dots, x_n]$), and...

(i) $X \subset \mathbb{A}^n$ is an algebraic set, then the image $f(X) \subset \mathbb{A}^m$ is an algebraic set.
(ii) $X \subset \mathbb{A}^m$ is an algebraic set, then the inverse image $f^{-1}(X) \subset \mathbb{A}^n$ is an algebraic set.
(iii) $X \subset \mathbb{A}^n$ is an algebraic set, then the graph $\Gamma = \{(x, f(x)) \mid x \in X\} \subset \mathbb{A}^{n+m}$ is an algebraic set.

Exercise 2.6.6. Let $f : X \to Y$ be a morphism between affine varieties, and let $f^* : A(Y) \to A(X)$ be the corresponding map of k-algebras. Which of the following statements are true?

(i) If $P \in X$ and $Q \in Y$, then $f(P) = Q$ if and only if $(f^*)^{-1}(I(P)) = I(Q)$.
(ii) f^* is injective if and only if f is surjective.
(iii) f^* is surjective if and only if f is injective.
(iv) f is an isomorphism if and only if f^* is an isomorphism.

If a statement is false, is there maybe a weaker form of it which is true?

Exercise 2.6.7. Let X be a prevariety. Show that:

(i) X is a Noetherian topological space,
(ii) any open subset of X is a prevariety.

Exercise 2.6.8. Let $(X, \mathcal{O}_X)$ be a prevariety, and let $Y \subset X$ be an irreducible closed subset. For every open subset $U \subset Y$ define $\mathcal{O}_Y(U)$ to be the ring of k-valued functions f on U with the following property: for every point $P \in Y$ there is a neighborhood V of P in X and a section $F \in \mathcal{O}_X(V)$ such that f coincides with F on U.

(i) Show that the rings $\mathcal{O}_Y(U)$ together with the obvious restriction maps define a sheaf $\mathcal{O}_Y$ on Y.
(ii) Show that $(Y, \mathcal{O}_Y)$ is a prevariety.

Exercise 2.6.9. Let X be a prevariety. Consider pairs (U, f) where U is an open subset of X and $f \in \mathcal{O}_X(U)$ a regular function on U. We call two such pairs (U, f) and (U', f') equivalent if there is an open subset V in X with $V \subset U \cap U'$ such that $f|_U = f|_{U'}$.

(i) Show that this defines an equivalence relation.
(ii) Show that the set of all such pairs modulo this equivalence relation is a field. It is called the **field of rational functions** on X and denoted $K(X)$.
(iii) If X is an affine variety, show that $K(X)$ is just the field of rational functions as defined in definition 2.1.3.
(iv) If $U \subset X$ is any non-empty open subset, show that $K(U) = K(X)$.

Exercise 2.6.10. If U is an open subset of a prevariety X and $f : U \to \mathbb{P}^1$ a morphism, is it always true that f can be extended to a morphism $\tilde{f} : X \to \mathbb{P}^1$?

Exercise 2.6.11. Show that the prevariety $\mathbb{P}^1$ is a variety.

Exercise 2.6.12.

(i) Show that every isomorphism $f : \mathbb{A}^1 \to \mathbb{A}^1$ is of the form $f(x) = ax + b$ for some $a, b \in k$, where x is the coordinate on $\mathbb{A}^1$.
(ii) Show that every isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ is of the form $f(x) = \frac{ax+b}{cx+d}$ for some $a, b, c, d \in k$, where x is an affine coordinate on $\mathbb{A}^1 \subset \mathbb{P}^1$. (For a generalization see corollary 6.2.10.)
(iii) Given three distinct points $P_1, P_2, P_3 \in \mathbb{P}^1$ and three distinct points $Q_1, Q_2, Q_3 \in \mathbb{P}^1$, show that there is a unique isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f(P_i) = Q_i$ for $i = 1, 2, 3$.

(Remark: If the ground field is $\mathbb{C}$, the very same statements are true in the complex analytic category as well, i. e. if "morphisms" are understood as "holomorphic maps" (and $\mathbb{P}^1$ is the Riemann sphere $\mathbb{C}_\infty$). If you know some complex analysis and have some time to kill, you may try to prove this too.)

Exercise 2.6.13. Let X and Y be prevarieties with affine open covers $\{U_i\}$ and $\{V_j\}$, respectively. Construct the product prevariety $X \times Y$ by glueing the affine varieties $U_i \times V_j$ together. Moreover, show that there are projection morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ satisfying the "usual" universal property for products: given morphisms $f : Z \to X$ and $g : Z \to Y$ from any prevariety Z, there is a unique morphism $h : Z \to X \times Y$ such that $f = \pi_X \circ h$ and $g = \pi_Y \circ h$.

3. Projective varieties

Similarly to the affine case, a subset of projective n-space $\mathbb{P}^n$ over k is called a projective algebraic set if it can be written as the zero locus of a (finite) set of homogeneous polynomials. The Zariski topology on $\mathbb{P}^n$ is the topology whose closed sets are the projective algebraic sets. The concepts of irreducibility and dimension are purely topological and extend therefore immediately to subsets of projective space. We prove a projective version of the Nullstellensatz and make projective varieties into ringed spaces that are varieties.

The main property of projective varieties distinguishing them from affine varieties is that (over $\mathbb{C}$ in the classical topology) they are compact. In terms of algebraic geometry this translates into the statement that if $f : X \to Y$ is a morphism between projective varieties then $f(X)$ is closed in Y.

3.1. **Projective spaces and projective varieties.** In the last section we have studied varieties, i. e. topological spaces that are locally isomorphic to affine varieties. In particular, the ability to glue affine varieties together allowed us to construct *compact* spaces (over the ground field $\mathbb{C}$) like e. g. $\mathbb{P}^1$, whereas affine varieties themselves are never compact unless they are a single point (see exercise 3.5.6). Unfortunately, the description of a variety in terms of its affine patches is often quite inconvenient in practice, as we have seen already in the calculations in the last section. It would be desirable to have a global description of the spaces that does not refer to glueing methods.

Projective varieties form a large class of "compact" varieties that do admit such a unified global description. In fact, the class of projective varieties is so large that it is not easy to construct a variety that is *not* (an open subset of) a projective variety.

To construct projective varieties, we need to define projective spaces first. Projective spaces are "compactifications" of affine spaces. We have seen $\mathbb{P}^1$ already as a compactification of $\mathbb{A}^1$; general projective spaces are an extension of this construction to higher dimensions.

Definition 3.1.1. We define **projective *n*-space** over k, denoted $\mathbb{P}^n$, to be the set of all one-dimensional linear subspaces of the vector space k^{n+1}.

Remark 3.1.2. Obviously, a one-dimensional linear subspace of k^{n+1} is uniquely determined by a non-zero vector in k^{n+1}. Conversely, two such vectors $a = (a_0, \dots, a_n)$ and $b = (b_0, \dots, b_n)$ in k^{n+1} span the same linear subspace if and only if they differ only by a common scalar, i. e. if $b = \lambda a$ for some non-zero $\lambda \in k$. In other words,

$$\mathbb{P}^n = \{(a_0, \dots, a_n) \,;\, a_i \in k, \text{ not all } a_i = 0\}/\sim$$

with the equivalence relation

$$(a_0, \dots, a_n) \sim (b_0, \dots, b_n) \quad \text{if} \quad a_i = \lambda b_i \text{ for some } \lambda \in k\backslash\{0\} \text{ and all } i.$$

This is often written as

$$\mathbb{P}^n = (k^{n+1}\backslash\{0\})/(k\backslash\{0\}),$$

and the point P in $\mathbb{P}^n$ determined by $(a_0, \dots, a_n)$ is written as $P = (a_0 : \cdots : a_n)$ (the notation $[a_0, \dots, a_n]$ is also common in the literature). So the notation $(a_0 : \cdots : a_n)$ means that the a_i are not all zero, and that they are defined only up to a common scalar multiple. The a_i are called the **homogeneous coordinates** of the point P (the motivation for this name will become obvious in the course of this section).

Example 3.1.3. Consider the one-dimensional projective space $\mathbb{P}^1$. Let $(a_0 : a_1) \in \mathbb{P}^1$ be a point. Then we have one of the following cases:

(i) $a_0 \neq 0$. Then P can be written as $P = (1:a)$ with $a = \frac{a_1}{a_0} \in k$. Obviously $(1:a) = (1:b)$ if and only if $a = b$, i. e. the ambiguity in the homogeneous coordinates is gone if we fix one of them to be 1. So the set of these points is just $\mathbb{A}^1$. We call $a = \frac{a_1}{a_0}$ the **affine coordinate** of the point P; it is uniquely determined by P (and not just up to a multiple as for the homogeneous coordinates).

(ii) $a_0 = 0$, and therefore $a_1 \neq 0$. There is just one such point that we can write as $(0:1)$.

So $\mathbb{P}^1$ is just $\mathbb{A}^1$ with one point added. This additional point $(0:1)$ can be thought of as a "point at infinity", as you can see from the fact that its affine coordinate is formally $\frac{1}{0}$. So we arrive at the same description of $\mathbb{P}^1$ as in example 2.4.5 (i).

Remark 3.1.4. There is a completely analogous description of $\mathbb{P}^n$ as $\mathbb{A}^n$ with some points added "at infinity": let $P = (a_0 : \cdots : a_n) \in \mathbb{P}^n$ be a point. Then we have one of the following cases:

(i) $a_0 \neq 0$. Then $P = (1 : \alpha_1 : \cdots : \alpha_n)$ with $\alpha_i = \frac{a_i}{a_0}$ for all i. The α_i are the affine coordinates of P; they are uniquely determined by P and are obtained by setting $a_0 = 1$. So the set of all P with $a_0 \neq 0$ is just $\mathbb{A}^n$.

(ii) $a_0 = 0$, i. e. $P = (0 : a_1 : \cdots : a_n)$, with the a_i still defined only up to a common scalar. Obviously, the set of such points is $\mathbb{P}^{n-1}$; the set of all one-dimensional linear subspaces of $\mathbb{A}^n$. We think of these points as points at infinity; the new twist compared to $\mathbb{P}^1$ is just that we have *a point at infinity for every one-dimensional linear subspace of* $\mathbb{A}^n$, i. e. for every "direction" in $\mathbb{A}^n$. So, for example, two lines in $\mathbb{A}^n$ will meet at infinity (when compactified in $\mathbb{P}^n$) if and only if they are parallel, i. e. point in the same direction. (This is good as it implies that two distinct lines always intersect in exactly one point.)

Usually, it is more helpful to think of the projective space $\mathbb{P}^n$ as the affine space $\mathbb{A}^n$ compactified by adding some points (parametrized by $\mathbb{P}^{n-1}$) at infinity, rather than as the set of lines in $\mathbb{A}^{n+1}$.

Remark 3.1.5. In the case $k = \mathbb{C}$, we claim that $\mathbb{P}^n$ is a compact space (in the classical topology). In fact, let

$$S^{2n+1} = \{(a_0, \dots, a_n) \in \mathbb{C}^{n+1} \; ; \; |a_0|^2 + \cdots + |a_n|^2 = 1\}$$

be the unit sphere in $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$. This is a compact space as it is closed and bounded, and there is an obvious surjective map

$$S^{2n+1} \to \mathbb{P}^n, \; (a_0, \cdots, a_n) \mapsto (a_0 : \cdots : a_n).$$

As images of compact sets under continuous maps are compact, it follows that $\mathbb{P}^n$ is also compact.

Remark 3.1.6. In complete analogy to affine algebraic sets, we now want to define projective algebraic sets to be subsets of $\mathbb{P}^n$ that can be described as the zero locus of some polynomials in the homogeneous coordinates. Note however that if $f \in k[x_0, \dots, x_n]$ is an arbitrary polynomial, it does not make sense to write down a definition like

$$Z(f) = \{(a_0 : \cdots : a_n) \; ; \; f(a_0, \dots, a_n) = 0\},$$

because the a_i are only defined up to a common scalar. For example, if $f(x_0, x_1) = x_1^2 - x_0$ then $f(1,1) = 0$ but $f(-1,-1) \neq 0$, although $(1:1)$ and $(-1:-1)$ are the same point in $\mathbb{P}^1$. To get rid of this problem we have to require that f be **homogeneous**, i. e. that all of its monomials have the same (total) degree d. This is equivalent to the requirement

$$f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n) \text{ for all } \lambda,$$

so in particular we see that

$$f(\lambda x_0,\dots,\lambda x_n)=0 \iff f(x_0,\dots,x_n)=0,$$

i. e. the condition that a *homogeneous* polynomial in the homogeneous coordinates vanishes is indeed well-defined.

Definition 3.1.7. For every $f\in k[x_0,\dots,x_n]$ let $f^{(d)}$ denote the degree-d part of f, i. e. $f=\sum f^{(d)}$ with $f^{(d)}$ homogeneous of degree d for all d.

Lemma 3.1.8. *Let $I\subset k[x_0,\dots,x_n]$ be an ideal. The following are equivalent:*

(i) *I can be generated by homogeneous polynomials.*
(ii) *For every $f\in I$ we have $f^{(d)}\in I$ for all d.*

An ideal that satisfies these conditions is called ***homogeneous****.*

Proof. (i) $\Rightarrow$ (ii): Let $I=(f_1,\dots,f_m)$ with all f_i homogeneous. Then every $f\in I$ can be written as $f=\sum_i a_i f_i$ for some $a_i\in k[x_0,\dots,x_n]$ (which need not be homogeneous). Restricting this equation to the degree-d part, we get $f^{(d)}=\sum_i (a_i)^{(d-\deg f_i)} f_i\in I$.

(ii) $\Rightarrow$ (i): Any ideal can be written as $I=(f_1,\dots,f_m)$ with the f_i possibly not being homogeneous. But by (ii) we know that all $f_i^{(d)}$ are in I too, so it follows that I is generated by the homogeneous polynomials $f_i^{(d)}$. $\square$

Remark 3.1.9. Note that it is *not* true that every element of a homogeneous ideal I is a homogeneous polynomial: we can always add two polynomials of I to get another element of I, even if they do not have the same degree.

With the exception of the homogeneity requirement, the following constructions are now completely analogous to the affine case:

Definition 3.1.10. Let $I\subset k[x_0,\dots,x_n]$ be a homogeneous ideal (or a set of homogeneous polynomials). The set

$$\boldsymbol{Z(I)} := \{(a_0:\dots:a_n)\in\mathbb{P}^n \,;\, f(a_0,\dots,a_n)=0 \text{ for all } f\in I\}$$

is called the **zero locus** of I; this is well-defined by remark 3.1.6. Subsets of $\mathbb{P}^n$ that are of the form $Z(I)$ are called **algebraic sets**. If $X\subset\mathbb{P}^n$ is any subset, we call

$$\begin{aligned}\boldsymbol{I(X)} :=&\text{the ideal generated by}\\ &\{f\in k[x_0,\dots,x_n] \text{ homogeneous} \,;\, f(a_0,\dots,a_n)=0 \text{ for all } (a_0:\dots:a_n)\in X\}\\ &\subset k[x_0,\dots,x_n]\end{aligned}$$

the **ideal** of X; by definition this is a homogeneous ideal.

If we want to distinguish between the affine zero locus $Z(I)\subset\mathbb{A}^{n+1}$ and the projective zero locus $Z(I)\subset\mathbb{P}^n$ of the same (homogeneous) ideal, we denote the former by $Z_a(I)$ and the latter by $Z_p(I)$.

Remark 3.1.11. A remark that is sometimes useful is that every projective algebraic set can be written as the zero locus of finitely many homogeneous polynomials *of the same degree*. This follows easily from the fact that $Z(f)=Z(x_0^d f,\dots,x_n^d f)$ for all homogeneous polynomials f and every $d\ge 0$.

Example 3.1.12. Let $L\subset\mathbb{A}^{n+1}$ be a linear subspace of dimension $k+1$; it can be given by $n-k$ linear equations in the coordinates of $\mathbb{A}^{n+1}$. The image of L under the quotient map $(\mathbb{A}^{n+1}\backslash\{0\})/(k\backslash\{0\})=\mathbb{P}^n$, i. e. the subspace of $\mathbb{P}^n$ given by the same $n-k$ equations (now considered as equations in the homogeneous coordinates on $\mathbb{P}^n$) is called a **linear subspace** of $\mathbb{P}^n$ of dimension k. Once we have given projective varieties the structure of varieties, we will see that a linear subspace of $\mathbb{P}^n$ of dimension k is isomorphic to $\mathbb{P}^k$. For example, a line in $\mathbb{P}^3$ (with homogeneous coordinates x_0,x_1,x_2,x_3) is given by two linearly independent equations in the x_i. One example is the line

$$\{x_2 = x_3 = 0\} = \{(a_0 : a_1 : 0 : 0)\ ;\ a_0,a_1 \in k\} \subset \mathbb{P}^3,$$

which is "obviously isomorphic" to $\mathbb{P}^1$.

Example 3.1.13. Consider the conics in $\mathbb{A}^2$

$$X_1 = \{x_2 = x_1^2\} \quad \text{and} \quad X_2 = \{x_1x_2 = 1\}$$

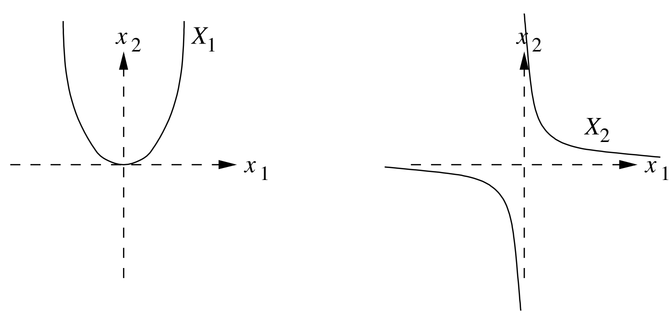
of exercise 2.6.1. We want to "compactify" these conics to projective algebraic sets $\tilde{X}_1$, $\tilde{X}_2$ in $\mathbb{P}^2$. Note that for a projective algebraic set we need the defining polynomials to be homogeneous, which is not yet the case here. On the other hand, we have an additional coordinate x_0 that you can think of as being 1 on the affine space $\mathbb{A}^2 \subset \mathbb{P}^2$. So it is obvious that we should make the defining equations homogeneous by adding suitable powers of x_0: consider

$$\tilde{X}_1 = \{x_0x_2 = x_1^2\} \quad \text{and} \quad \tilde{X}_2 = \{x_1x_2 = x_0^2\}$$

in $\mathbb{P}^2$. Then the restriction of $\tilde{X}_i$ to the affine space $\mathbb{A}^2 \subset \mathbb{P}^2$ is just given by X_i for $i = 1,2$. We call $\tilde{X}_i$ the **projective completion** of X_i; it can be done in the same way for all affine varieties (see exercise 3.5.3).

Let us consider $\tilde{X}_1$ first. The points that we add at infinity correspond to those where $x_0 = 0$. It follows from the defining equation that $x_1 = 0$ as well; and then we must necessarily have $x_2 \neq 0$ as the coordinates cannot be simultaneously zero. So there is only one point added at infinity, namely $(0:0:1)$. It corresponds to the "vertical direction" in $\mathbb{A}^2$, which is the direction in which the parabola $x_2 = x_1^2$ goes off to infinity (at both ends actually).

For $\tilde{X}_2$, the added points have again $x_0 = 0$. This means that $x_1x_2 = 0$, which yields the two points $(0:1:0)$ and $(0:0:1)$ in $\mathbb{P}^2$: we added two points at infinity, one corresponding to the "horizontal" and one to the "vertical" direction in $\mathbb{A}^2$. This is clear from the picture below as the hyperbola $x_1x_2 = 1$ extends to infinity both along the x_1 and the x_2 axis.



Note that the equations of $\tilde{X}_1$ and $\tilde{X}_2$ are exactly the same, up to a permutation of the coordinates. Even if we have not given projective varieties the structure of varieties yet, it should be obvious that $\tilde{X}_1$ and $\tilde{X}_2$ will be isomorphic varieties, with the isomorphism being given by exchanging x_0 and x_1. Hence we see that the two distinct types of conics in $\mathbb{A}^2$ become the same in projective space: there is only one projective conic in $\mathbb{P}^2$ up to isomorphism. The difference in the affine case comes from the fact that some conics "meet infinity" in one point (like X_1), and some in two (like X_2).

Proposition 3.1.14.

(i) *If $I_1 \subset I_2$ are homogeneous ideals in $k[x_0,\dots,x_n]$ then $Z(I_2) \subset Z(I_1)$.*
(ii) *If $\{I_i\}$ is a family of homogeneous ideals in $k[x_0,\dots,x_n]$ then $\bigcap_i Z(I_i) = Z(\bigcup_i I_i) \subset \mathbb{P}^n$.*
(iii) *If $I_1,I_2 \subset k[x_0,\dots,x_n]$ are homogeneous ideals then $Z(I_1) \cup Z(I_2) = Z(I_1I_2) \subset \mathbb{P}^n$.*

In particular, arbitrary intersections and finite unions of algebraic sets are again algebraic sets.

Proof. The proof is the same as in the affine case (proposition 1.1.6). □

Definition 3.1.15. We define the **Zariski topology** on $\mathbb{P}^n$ to be the topology whose closed sets are the algebraic sets (proposition 3.1.14 tells us that this gives in fact a topology). Moreover, any subset X of $\mathbb{P}^n$ (in particular any algebraic set) will be equipped with the topology induced by the Zariski topology on $\mathbb{P}^n$. This will be called the Zariski topology on X.

Remark 3.1.16. The concepts of irreducibility and dimension introduced in section 1.3 are purely topological ones, so they apply to projective algebraic sets (or more generally to any subset of $\mathbb{P}^n$) as well. They have the same geometric interpretation as in the affine case. Irreducible algebraic sets in $\mathbb{P}^n$ are called **projective varieties**. As in the affine case (see lemma 1.3.4) a projective algebraic set X is irreducible if and only if its ideal $I(X)$ is a prime ideal. In particular, $\mathbb{P}^n$ itself is irreducible.

3.2. **Cones and the projective Nullstellensatz.** We will now establish a correspondence between algebraic sets in $\mathbb{P}^n$ and homogeneous radical ideals in $k[x_0,\dots,x_n]$, similar to the affine case. This is quite straightforward; the only twist is that there is no zero point $(0:\cdots:0)$ in $\mathbb{P}^n$, and so the zero locus of the radical homogeneous ideal $(x_0,\dots,x_n)$ is empty although the ideal is not equal to (1). So we will have to exclude this ideal from our correspondence, which is why it is sometimes called the **irrelevant ideal**.

As we want to use the results of the affine case for the proof of this statement, let us first establish a connection between projective algebraic sets in $\mathbb{P}^n$ and certain affine algebraic sets in $\mathbb{A}^{n+1}$.

Definition 3.2.1. An affine algebraic set $X \subset \mathbb{A}^{n+1}$ is called a **cone** if it is not empty, and if we have for all $\lambda \in k$

$$(x_0,\dots,x_n) \in X \quad \Rightarrow \quad (\lambda x_0,\dots,\lambda x_n) \in X.$$

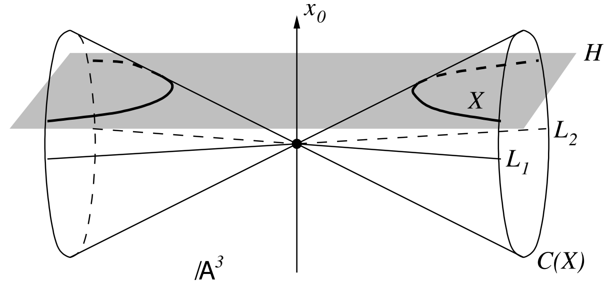
If $X \subset \mathbb{P}^n$ is a projective algebraic set, then

$$\boldsymbol{C(X)} := \{(x_0,\dots,x_n) \,|\, (x_0:\cdots:x_n) \in X\} \cup \{0\}$$

is called the cone over X (obviously this is a cone).

Remark 3.2.2. In other words, a cone is an algebraic set in $\mathbb{A}^{n+1}$ that can be written as a (usually infinite) union of lines through the origin. The cone over a projective algebraic set $X \subset \mathbb{P}^n$ is the inverse image of X under the projection map $\mathbb{A}^{n+1}\backslash\{0\} \to (\mathbb{A}^{n+1}\backslash\{0\})/(k\backslash\{0\}) = \mathbb{P}^n$, together with the origin.

Example 3.2.3. The following picture shows an example of a (two-dimensional) cone $C(X)$ in $\mathbb{A}^3$ over the (one-dimensional) projective variety X in $H = \mathbb{P}^2$:



($C(X)$ consists only of the "boundary" of the cone, not of the "interior".) Note that $C(X)$ contains the two lines L_1 and L_2, which correspond to "points at infinity" of the projective space $\mathbb{P}^2$.

Lemma 3.2.4.

(i) *Let $I \subsetneq k[x_0,\dots,x_n]$ be a homogeneous ideal. If $X = Z_p(I) \subset \mathbb{P}^n$, then $C(X) = Z_a(I) \subset \mathbb{A}^{n+1}$.*
(ii) *Conversely, if $X \subset \mathbb{P}^n$ is a projective algebraic set and $I(X) \subset k[x_0,\dots,x_n]$ is its homogeneous ideal, then $I(C(X)) = I(X)$.*

In other words, there is a one-to-one correspondence between projective algebraic sets in $\mathbb{P}^n$ and affine cones in $\mathbb{A}^{n+1}$, given by taking the zero locus of the same homogeneous ideal (not equal to (1)) either in $\mathbb{P}^n$ or in $\mathbb{A}^{n+1}$.

Proof. This is obvious from the definitions. □

Using this lemma, it is now very simple to derive a projective version of the Nullstellensatz:

Proposition 3.2.5. *("The projective Nullstellensatz")*

(i) *If $X_1 \subset X_2$ are algebraic sets in $\mathbb{P}^n$ then $I(X_2) \subset I(X_1)$.*
(ii) *For any algebraic set $X \subset \mathbb{P}^n$ we have $Z_p(I(X)) = X$.*
(iii) *For any homogeneous ideal $I \subset k[x_0,\dots,x_n]$ such that $Z_p(I)$ is not empty we have $I(Z_p(I)) = \sqrt{I}$.*
(iv) *For any homogeneous ideal $I \subset k[x_0,\dots,x_n]$ such that $Z_p(I)$ is empty we have either $I = (1)$ or $\sqrt{I} = (x_0,\dots,x_n)$. In other words, $Z_p(I)$ is empty if and only if $(x_0,\dots,x_n)^r \subset I$ for some r.*

Proof. The proofs of (i) and (ii) are literally the same as in the affine case, see proposition 1.2.9.

(iii): Let $X = Z_p(I)$. Then

$$I(Z_p(I)) = I(X) = I(C(X)) = I(Z_a(I)) = \sqrt{I}$$

by lemma 3.2.4 and the affine Nullstellensatz of proposition 1.2.9 (iii).

(iv): If $Z_p(I)$ is empty, then $Z_a(I)$ is either empty or just the origin. So corollary 1.2.10 tells us that $I = (1)$ or $\sqrt{I} = (x_0,\dots,x_n)$. In any case, this means that $x_i^{k_i} \in I$ for some k_i, so $(x_0,\dots,x_n)^{k_0+\cdots+k_n} \subset I$. □

Theorem 3.2.6. *There is a one-to-one inclusion-reversing correspondence between algebraic sets in $\mathbb{P}^n$ and homogeneous radical ideals in $k[x_0,\dots,x_n]$ not equal to $(x_0,\dots,x_n)$, given by the operations $Z(\cdot)$ and $I(\cdot)$.*

Proof. Immediately from proposition 3.2.5. □

3.3. **Projective varieties as ringed spaces.** So far we have defined projective varieties as topological spaces. Of course we want to make them into ringed spaces and finally show that they are varieties in the sense of definitions 2.4.1 and 2.5.1. So let $X \subset \mathbb{P}^n$ be a projective variety. First of all we have to make X into a ringed space whose structure sheaf is a sheaf of k-valued functions. The construction is completely analogous to the affine case discussed in section 2.1.

Definition 3.3.1. The ring

$$\boldsymbol{S(X)} := k[x_0,\dots,x_n]/I(X)$$

is called the **homogeneous coordinate ring** of X.

Remark 3.3.2. In contrast to the affine case, the elements of $S(X)$ do *not* define functions on X, because the homogeneous coordinates are only determined up to a common scalar. Rather, to get well-defined functions, we have to take quotients of two homogeneous polynomials *of the same degree* d, because then a rescaling of the homogeneous coordinates by a factor $\lambda \in k\backslash\{0\}$ gives a factor of λ^d in both the numerator and denominator, so that it cancels out:

Definition 3.3.3. Let

$$S(X)^{(d)} := \{f^{(d)} \; ; \; f \in S(X)\}$$

be the degree-d part of $S(X)$. Note that this is well-defined: if $f \in I(X)$ then $f^{(d)} = 0$ by lemma 3.1.8. We define the **field of rational functions** to be

$$\boldsymbol{K(X)} := \left\{ \frac{f}{g} \; ; \; f,g \in S(X)^{(d)} \text{ and } g \neq 0 \right\}.$$

By remark 3.3.2, the elements of $K(X)$ give set-theoretic functions to the ground field k wherever the denominator is non-zero. Now as in the affine case set

$$\mathcal{O}_{X,P} := \left\{ \frac{f}{g} \in K(X) \; ; \; g(P) \neq 0 \right\} \quad \text{and} \quad \mathcal{O}_X(U) := \bigcap_{P \in U} \mathcal{O}_{X,P}$$

for $P \in X$ and $U \subset X$ open. It is easily seen that this is a sheaf of k-valued functions.

Remark 3.3.4. In the same way as for affine varieties (see exercise 2.6.9) one can show that the function field $K(X)$ defined above agrees with the definition for general varieties.

Remark 3.3.5. Note that $\mathcal{O}_X(X) = k$, i. e. *every regular function on all of X is constant*. This follows trivially from the description of $K(X)$: if the function is to be defined everywhere g must be a constant. But then f has to be a constant too as it must have the same degree as g. A (slight) generalization of this will be proved in corollary 3.4.10.

Proposition 3.3.6. *Let X be a projective variety. Then* $(X, \mathcal{O}_X)$ *is a prevariety.*

Proof. We need to find an open affine cover of X. Consider the open subset

$$X_0 = \{(a_0 : \cdots : a_n) \in X \; ; \; a_0 \neq 0\} = X \cap \mathbb{A}^n$$

(where $\mathbb{A}^n \subset \mathbb{P}^n$ as in remark 3.1.4). If $X = Z(f_1, \dots, f_r)$ with $f_i \in k[x_0, \dots, x_n]$ homogeneous, set $g_i(x_1, \dots, x_n) = f_i(1, x_1, \dots, x_n) \in k[x_1, \dots, x_n]$ and define $Y = Z(g_1, \dots, g_r) \subset \mathbb{A}^n$. We claim that there is an isomorphism

$$F : X \cap \mathbb{A}^n \to Y, \; (a_0 : \cdots : a_n) \mapsto \left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0} \right).$$

In fact, it is obvious that a set-theoretic inverse is given by

$$F^{-1} : Y \to X \cap \mathbb{A}^n, \; (a_1, \dots, a_n) \mapsto (1 : a_1 : \cdots : a_n).$$

Moreover, F is a morphism because it pulls back a regular function on (an open subset of) Y of the form

$$\frac{p(a_1, \dots, a_n)}{q(a_1, \dots, a_n)} \quad \text{to} \quad \frac{p(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0})}{q(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0})},$$

which is a regular function on $X \cap \mathbb{A}^n$ as it can be rewritten as a quotient of two homogeneous polynomials of the same degree (by canceling the fractions in the numerator and denominator). In the same way, F^{-1} pulls back a regular function on (an open subset of) $X \cap \mathbb{A}^n$

$$\frac{p(a_0, \dots, a_n)}{q(a_0, \dots, a_n)} \quad \text{to} \quad \frac{p(1, a_1, \dots, a_n)}{q(1, a_1, \dots, a_n)},$$

which is a regular function on Y. So F is an isomorphism.

In the same way we can do this for the open sets $X_i = \{(x_0 : \cdots : x_n) \in X \,;\, x_i \neq 0\}$ for $i = 0, \dots, n$. As the x_i cannot be simultaneously zero, it follows that the X_i form an affine cover of X. So X is a prevariety. □

Remark 3.3.7. Following the proof of proposition 3.3.6, it is easy to see that our "new" definition of $\mathbb{P}^1$ agrees with the "old" definition of example 2.4.5 (i) by glueing two affine lines $\mathbb{A}^1$.

Remark 3.3.8. Proposition 3.3.6 implies that all our constructions and results for prevarieties apply to projective varieties as well. For example, we know what morphisms are, and have defined products of projective varieties. We have also defined the field of rational functions for prevarieties in exercise 2.6.9; it is easy to check that this definition agrees with the one in definition 3.3.3.

Although this gives us the definition of morphisms and products, we would still have to apply our glueing techniques to write down a morphism or a product. So we should find a better description for morphisms and products involving projective varieties:

Lemma 3.3.9. *Let $X \subset \mathbb{P}^n$ be a projective variety (or an open subset of a projective variety). Let $f_1, \dots, f_m \in k[x_0, \dots, x_n]$ be homogeneous polynomials of the same degree in the homogeneous coordinates of $\mathbb{P}^n$, and assume that for every $P \in X$ at least one of the f_i does not vanish at P. Then the f_i define a morphism*

$$f : X \to \mathbb{P}^m,\ P \in X \mapsto (f_0(P) : \cdots : f_m(P)).$$

Proof. First of all note that f is well-defined set-theoretically: we have assumed that the image point can never be $(0 : \cdots : 0)$; and if we rescale the homogeneous coordinates x_i we get

$$\begin{aligned}
&(f_0(\lambda x_0 : \cdots : \lambda x_n) : \cdots : f_m(\lambda x_0 : \cdots : \lambda x_n)) \\
&\quad = (\lambda^d f_0(x_0 : \cdots : x_n) : \cdots : \lambda^d f_m(x_0 : \cdots : x_n)) \\
&\quad = (f_0(x_0 : \cdots : x_n) : \cdots : f_m(x_0 : \cdots : x_n)),
\end{aligned}$$

where d is the common degree of the f_i. To check that f is a morphism, we want to use lemma 2.4.10, i.e. check the condition on an affine open cover. So let $\{V_i\}$ be the affine open cover of $\mathbb{P}^m$ with $V_i = \{(y_0 : \cdots : y_m) \,;\, y_i \neq 0\}$, and let $U_i = f^{-1}(V_i)$. Then in the affine coordinates on V_i the map $f|_{U_i}$ is given by the quotients of polynomials $\frac{f_j}{f_i}$ for $j = 0, \dots, m$ with $j \neq i$, hence gives a morphism as $f_i(P) \neq 0$ on U_i. So f is a morphism by lemma 2.4.10. □

Remark 3.3.10. It should be noted however that not every morphism between projective varieties can be written in this form. The following example shows that this occurs already in quite simple cases. For a more precise statement see lemma 7.5.14.

Example 3.3.11. By lemma 3.3.9, the map

$$f : \mathbb{P}^1 \mapsto \mathbb{P}^2,\ (s : t) \mapsto (x : y : z) = (s^2 : st : t^2)$$

is a morphism (as we must have $s \neq 0$ or $t \neq 0$ for every point of $\mathbb{P}^1$, it follows that $s^2 \neq 0$ or $t^2 \neq 0$; hence the image point is always well-defined).

Let $X = f(\mathbb{P}^1)$ be the image of f. We claim that X is a projective variety with ideal $I = (xz - y^2)$. In fact, it is obvious that $f(\mathbb{P}^1) \subset Z(I)$. Conversely, let $P = (x : y : z) \in Z(I)$. As $xz - y^2 = 0$ we must have $x \neq 0$ or $z \neq 0$; let us assume without loss of generality that $x \neq 0$. Then $(x : y) \in \mathbb{P}^1$ is a point that maps to $(x^2 : xy : y^2) = (x^2 : xy : xz) = (x : y : z)$.

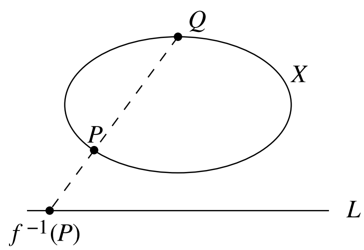
It is now easy to show that $f : \mathbb{P}^1 \to X$ is in fact an isomorphism: the inverse image $f^{-1} : X \to \mathbb{P}^1$ is given by

$$f^{-1}(x : y : z) = (x : y) \quad \text{and} \quad f^{-1}(x : y : z) = (y : z).$$

Note that at least one of the two points $(x:y)$ and $(y:z)$ is always well-defined; and if they are both defined they agree because of the equation $xz=y^2$. By lemma 3.3.9 both equations determine a morphism where they are well-defined; so by lemma 2.4.10 they glue to give an inverse morphism f^{-1}. Note that f^{-1} is a (quite simple) morphism between projective varieties that cannot be written globally in the form of lemma 3.3.9.

Summarizing, we have shown that f is an isomorphism: the curve $\{xz=y^2\}\subset\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$. This example should be compared to exercise 2.6.1 and example 3.1.13. It is a special case of the Veronese embedding of 3.4.11.

Finally, let us analyze the isomorphism f geometrically. Let $Q=(1:0:0)\in X$, and let $L\subset\mathbb{P}^2$ be the line $\{x=0\}$. For any point $P=(a:b:c)\neq Q$ there is a unique line $\overline{PQ}$ through P and Q with equation $yc=zb$. This line has a unique intersection point $\overline{PQ}\cap L$ with the line L, namely $(0:b:c)$. If we identify L with $\mathbb{P}^1$ in the obvious way, we see that the above geometric construction gives us exactly $f^{-1}(P)=\overline{PQ}\cap L$. We say that f^{-1} is the **projection from** Q to L.



Example 3.3.12. Consider $\mathbb{P}^n$ with homogeneous coordinates $x_0,\dots,x_n$, and $\mathbb{P}^m$ with homogeneous coordinates $y_0,\dots,y_m$. We want to find an easy description of the product $\mathbb{P}^n\times\mathbb{P}^m$.

Let $\mathbb{P}^N=\mathbb{P}^{(n+1)(m+1)-1}$ be projective space with homogeneous coordinates $z_{i,j}$, $0\le i\le n$, $0\le j\le m$. There is an obviously well-defined set-theoretic map $f:\mathbb{P}^n\times\mathbb{P}^m\to\mathbb{P}^N$ given by $z_{i,j}=x_iy_j$.

Lemma 3.3.13. *Let $f:\mathbb{P}^n\times\mathbb{P}^m\to\mathbb{P}^N$ be the set-theoretic map as above. Then:*

(i) *The image $X=f(\mathbb{P}^n\times\mathbb{P}^m)$ is a projective variety in $\mathbb{P}^N$, with ideal generated by the homogeneous polynomials $z_{i,j}z_{i',j'}-z_{i,j'}z_{i',j}$ for all $0\le i,i'\le n$ and $0\le j,j'\le m$.*

(ii) *The map $f:\mathbb{P}^n\times\mathbb{P}^m\to X$ is an isomorphism. In particular, $\mathbb{P}^n\times\mathbb{P}^m$ is a projective variety.*

(iii) *The closed subsets of $\mathbb{P}^n\times\mathbb{P}^m$ are exactly those subsets that can be written as the zero locus of polynomials in $k[x_0,\dots,x_n,y_0,\dots,y_m]$ that are bihomogeneous in the x_i and y_i.*

The map f is called the ***Segre embedding****.*

Proof. (i): It is obvious that the points of $f(\mathbb{P}^n\times\mathbb{P}^m)$ satisfy the given equations. Conversely, let P be a point in $\mathbb{P}^N$ with coordinates $z_{i,j}$ that satisfy the given equations. At least one of these coordinates must be non-zero; we can assume without loss of generality that it is $z_{0,0}$. Let us pass to affine coordinates by setting $z_{0,0}=1$. Then we have $z_{i,j}=z_{i,0}z_{0,j}$; so by setting $x_i=z_{i,0}$ and $y_j=z_{0,j}$ we obtain a point of $\mathbb{P}^n\times\mathbb{P}^m$ that is mapped to P by f.

(ii): Continuing the above notation, let $P\in f(\mathbb{P}^n\times\mathbb{P}^m)$ be a point with $z_{0,0}=1$. If $f(x_i,y_j)=P$, it follows that $x_0\neq0$ and $y_0\neq0$, so we can assume $x_0=1$ and $y_0=1$ as the x_i and y_j are only determined up to a common scalar. But then it follows that $x_i=z_{i,0}$ and $y_j=z_{0,j}$; i. e. f is bijective.

The same calculation shows that f and f^{-1} are given (locally in affine coordinates) by polynomial maps; so f is an isomorphism.

(iii): It follows by the isomorphism of (ii) that any closed subset of $\mathbb{P}^n \times \mathbb{P}^m$ is the zero locus of homogeneous polynomials in the $z_{i,j}$, i. e. of bihomogeneous polynomials in the x_i and y_j (of the same degree). Conversely, a zero locus of bihomogeneous polynomials can always be rewritten as a zero locus of bihomogeneous polynomials of the same degree in the x_i and y_i by remark 3.1.11. But such a polynomial is obviously a polynomial in the $z_{i,j}$, so it determines an algebraic set in $X \cong \mathbb{P}^n \times \mathbb{P}^m$. $\square$

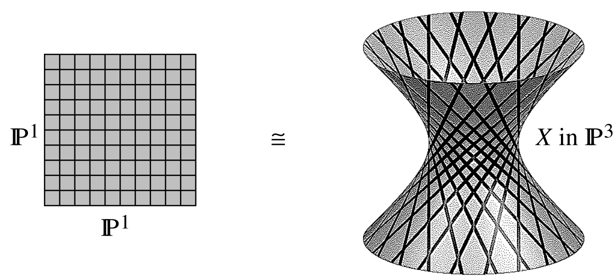
Example 3.3.14. By lemma 3.3.13, $\mathbb{P}^1 \times \mathbb{P}^1$ is (isomorphic to) the quadric surface

$$X = \{(z_{0,0} : z_{0,1} : z_{1,0} : z_{1,1})\ ;\ z_{0,0}z_{1,1} = z_{1,0}z_{0,1}\} \subset \mathbb{P}^3.$$

by the isomorphism

$$\mathbb{P}^1 \times \mathbb{P}^1 \to X,\ ((x_0 : x_1), (y_0 : y_1)) \mapsto (x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1).$$

In particular, the "lines" $\mathbb{P}^1 \times P$ and $P \times \mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$ where the first or second factor is constant are mapped to lines in $X \subset \mathbb{P}^3$. We can see these two families of lines on the quadric surface X:



Corollary 3.3.15. *Every projective variety is a variety.*

Proof. We have already seen in proposition 3.3.6 that every projective variety is a prevariety, so by lemma 2.5.3 and lemma 2.5.4 it only remains to be shown that the diagonal $\Delta(\mathbb{P}^n) \subset \mathbb{P}^n \times \mathbb{P}^n$ is closed. We can describe this diagonal as

$$\Delta(\mathbb{P}^n) = \{((x_0 : \cdots : x_n), (y_0 : \cdots : y_n))\ ;\ x_iy_j - x_jy_i = 0 \text{ for all } i, j\},$$

because these equations mean exactly that the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ y_0 & y_1 & \cdots & y_n \end{pmatrix}$$

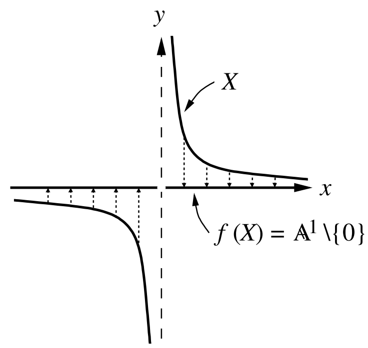
has rank (at most 1), i. e. that $(x_0 : \cdots : x_n) = (y_0 : \cdots : y_n)$.

In particular, it follows by lemma 3.3.13 (iii) that $\Delta(\mathbb{P}^n) \subset \mathbb{P}^n \times \mathbb{P}^n$ is closed. $\square$

3.4. **The main theorem on projective varieties.** The most important property of projective varieties is that they are compact in the classical topology (if the ground field is $k = \mathbb{C}$). We have seen this already for projective spaces in remark 3.1.5, and it then follows for projective algebraic sets as well as they are closed subsets (even in the classical topology) of the compact projective spaces. Unfortunately, the standard definition of compactness does not make sense at all in the Zariski topology, so we need to find an alternative description.

One property of compact sets is that they are mapped to compact sets under continuous maps. In our language, this would mean that images of projective varieties under a morphism should be closed. This is what we want to prove.

Remark 3.4.1. Note first that this property definitely does not hold for affine varieties: consider e. g. the affine variety $X = \{(x,y)\ ;\ xy = 1\} \subset \mathbb{A}^2$ and the projection morphism $f : X \to \mathbb{A}^1$, $(x,y) \mapsto x$. The image of f is $\mathbb{A}^1 \backslash \{0\}$, which is not closed in $\mathbb{A}^1$. In fact, we can see from example 3.1.13 why it is not closed: the "vertical point at infinity", which would map to $x = 0 \in \mathbb{A}^1$ and make the image closed, is missing in the affine variety X.



To prove the above mentioned statement we start with a special case (from which the general one will follow easily).

Theorem 3.4.2. *The projection map* $\pi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n$ *is closed, i. e. if* $X \subset \mathbb{P}^n \times \mathbb{P}^m$ *is closed then so is* $\pi(X)$.

Proof. Let $X \subset \mathbb{P}^n \times \mathbb{P}^m$ be an algebraic set. By lemma 3.3.13 (iii) we can write X as the zero locus of polynomials $f_1(x,y), \dots, f_r(x,y)$ bihomogeneous in the coordinates x_i of $\mathbb{P}^n$ and y_i of $\mathbb{P}^m$ (where we use the short-hand notation $f_i(x,y)$ for $f_i(x_0,\dots,x_n,y_0,\dots,y_m)$). By remark 3.1.11 we may assume that all f_i have the same degree d in the y_i.

Let $P \in \mathbb{P}^n$ be a fixed point. Then $P \in \pi(X)$ if and only if the common zero locus of the polynomials $f_i(P,y)$ in y is non-empty in $\mathbb{P}^m$, which by proposition 3.2.5 is the case if and only if

$$(y_0,\dots,y_m)^s \not\subset (f_1(P,y),\dots,f_r(P,y)) \qquad (*)$$

for all $s \geq 0$. As $(*)$ is obvious for $s < d$, it suffices to show that for any $s \geq d$, the set of all $P \in \mathbb{P}^n$ satisfying $(*)$ is closed, as $\pi(X)$ will then be the intersection of all these sets and therefore closed as well.

Note that the ideal $(y_0,\dots,y_m)^s$ is generated by the $\binom{m+s}{m}$ monomials of degree s in the y_i, which we denote by $M_i(y)$ (in any order). Hence $(*)$ is *not* satisfied if and only if there are polynomials $g_{i,j}(y)$ such that $M_i(y) = \sum_j g_{i,j}(y) f_j(P,y)$ for all i. As the M_i and f_j are homogeneous of degree s and d, respectively, this is the same as saying that such relations exist with the $g_{i,j}$ homogeneous of degree $s-d$. But if we let $N_i(y)$ be the collection of all monomials in the y_i of degree $s-d$, this is in turn equivalent to saying that the collection of polynomials $\{N_i(y) f_j(P,y)\ ;\ 1 \leq i \leq \binom{m+s-d}{m}, 1 \leq j \leq r\}$ generates the whole vector space of polynomials of degree s. Writing the coefficients of these polynomials in a matrix $A = A_s(P)$, this amounts to saying that A has rank (at least) $\binom{m+s}{m}$. Hence $(*)$ is satisfied if and only if all minors of A of size $\binom{m+s}{m}$ vanish. But as the entries of the matrix A are homogeneous polynomials in the coefficients of P, it follows that the set of all P satisfying $(*)$ is closed. □

Remark 3.4.3. Let us look at theorem 3.4.2 from an algebraic viewpoint. We start with some equations $f_i(x,y)$ and ask for the image of the projection map $(x,y) \mapsto x$, which can be written as

$$\{x\ ;\ \text{there is a } y \text{ such that } f_i(x,y) = 0 \text{ for all } i\}.$$

In other words, we are trying to *eliminate* the variables y from the system of equations $f_i(x,y) = 0$. The statement of the theorem is that the set of all such x can itself be written as the solution set of some polynomial equations. This is sometimes called the *main theorem of elimination theory*.

Corollary 3.4.4. *The projection map $\pi : \mathbb{P}^n \times Y \to Y$ is closed for any variety Y.*

Proof. Let us first show the statement for $Y \subset \mathbb{A}^m$ being an affine variety. Then we can regard Y as a subspace of $\mathbb{P}^m$ via the embedding $\mathbb{A}^m \subset \mathbb{P}^m$ (Y is neither open nor closed in $\mathbb{P}^m$, but that does not matter). Now if $Z \subset \mathbb{P}^n \times Y$ is closed, let $\tilde{Z} \subset \mathbb{P}^n \times \mathbb{P}^m$ be the projective closure. By theorem 3.4.2, $\pi(\tilde{Z})$ is closed in $\mathbb{P}^m$, where π is the projection morphism. Therefore

$$\pi(Z) = \pi(\tilde{Z} \cap (\mathbb{P}^n \times Y)) = \pi(\tilde{Z}) \cap Y$$

is closed in Y.

If Y is any variety we can cover it by affine open subsets. As the condition that a subset is closed can be checked by restricting it to the elements of an open cover, the statement follows from the corresponding one for the affine open patches that we have just shown. $\square$

Remark 3.4.5. Corollary 3.4.4 is in fact the property of $\mathbb{P}^n$ that captures the idea of compactness (as we will see in corollary 3.4.7). Let us therefore give it a name: we say that a variety X is **complete** if the projection map $\pi : X \times Y \to Y$ is closed for every variety Y. (You can think of the name "complete" as coming from the geometric idea that it contains all the "points at infinity" that were missing in affine varieties.) So corollary 3.4.4 says that $\mathbb{P}^n$ is complete. Moreover, any projective variety $Z \subset \mathbb{P}^n$ is complete, because any closed set in $Z \times Y$ is also closed in $\mathbb{P}^n \times Y$, so its image under the projection morphism to Y will be closed as well.

Remark 3.4.6. We have just seen that every projective variety is complete. In fact, whereas the converse of this statement is not true, it is quite hard to write down an example of a complete variety that is not projective. We will certainly not meet such an example in the near future. So for practical purposes you can usually assume that the terms "projective variety" and "complete variety" are synonymous.

Corollary 3.4.7. *Let $f : X \to Y$ be a morphism of varieties, and assume that X is complete. Then the image $f(X) \subset Y$ is closed.*

Proof. We factor f as $f : X \xrightarrow{\Gamma} X \times Y \xrightarrow{\pi} Y$, where $\Gamma = (\mathrm{id}_X, f)$ (the so-called *graph morphism*), and π is the projection to Y.

We claim that $\Gamma(X) = \{(P, f(P))\ ;\ P \in X\} \subset X \times Y$ is closed. To see this, note first that the diagonal $\Delta(Y) \subset Y \times Y$ is closed as Y is a variety. Now $\Gamma(X)$ is just the inverse image of $\Delta(Y)$ under the morphism $(f, \mathrm{id}_Y) : X \times Y \to Y \times Y$, and is therefore also closed.

As X is complete, it follows that $f(X) = \pi(\Gamma(X))$ is closed. $\square$

Corollary 3.4.8. *Let $X \subset \mathbb{P}^n$ be a projective variety that contains more than one point, and let $f \in k[x_0, \dots, x_n]$ be a non-constant homogeneous polynomial. Then $X \cap Z(f) \neq \emptyset$.*

Proof. Assume that the statement is false, i. e. that f is non-zero on all of X. Let $P, Q \in X$ be two distinct points of X and choose a homogeneous polynomial $g \in k[x_0, \dots, x_n]$ of the same degree as f such that $g(P) = 0$ and $g(Q) \neq 0$. Let $F : X \to \mathbb{P}^1$ be the morphism defined by $R \mapsto (f(R) : g(R))$; this is well-defined as f is non-zero on X by assumption.

By corollary 3.4.7 the image $F(X)$ is closed in $\mathbb{P}^1$. Moreover, $F(X)$ is irreducible as X is. Therefore, $F(X)$ is either a point or all of $\mathbb{P}^1$. But by assumption $(0 : 1) \notin F(X)$, so $F(X)$ must be a single point. But this is a contradiction, as $F(P) = (f(P) : g(P)) = (1 : 0)$ and $F(Q) = (f(Q) : g(Q)) \neq (1 : 0)$ by the choice of g. $\square$

Remark 3.4.9. Again this statement is false for affine varieties: consider e. g. $X = \{x = 0\} \subset \mathbb{A}^2$ and $f = x-1$, then $X \cap Z(f) = \emptyset$ although X is a line (and therefore contains more than one point). This example worked because in $\mathbb{A}^2$ we can have parallel lines. In $\mathbb{P}^2$ such lines would meet at infinity, so the intersection would be non-empty then.

Corollary 3.4.10. *Every regular function on a complete variety is constant.*

Proof. Let $f : X \to \mathbb{A}^1$ be a regular function on a complete variety X. Consider f as a morphism to $\mathbb{P}^1$ that does not assume the value ∞. In particular, $f(X) \subsetneq \mathbb{P}^1$, hence it is a single point by corollary 3.4.7. $\square$

Example 3.4.11. (This is a generalization of example 3.3.11 and exercise 3.5.2.) Let $f_i(x_0,\dots,x_n), 0 \le i \le N = \binom{n+d}{n} - 1$ be the set of all monomials in $k[x_0,\dots,x_n]$ of degree d, i. e. of the monomials of the form $x_0^{i_0} \cdots x_n^{i_n}$ with $i_0 + \cdots + i_n = d$. Consider the map

$$F : \mathbb{P}^n \to \mathbb{P}^N,\ (x_0 : \cdots : x_n) \mapsto (f_0 : \cdots : f_N).$$

By lemma 3.3.9 this is a morphism (note that the monomials $x_0^d,\dots,x_n^d$, which cannot be simultaneously zero, are among the f_i). So by corollary 3.4.7 the image $X = F(\mathbb{P}^n)$ is a projective variety.

We claim that $F : X \to F(X)$ is an isomorphism. All we have to do to prove this is to find an inverse morphism. This is not hard: we can do this on an affine open cover, so let us consider the open subset where $x_0 \ne 0$ (and therefore $x_0^d \ne 0$). We can then pass to affine coordinates and set $x_0 = 1$. The inverse morphism is then given by $x_i = \frac{x_i x_0^{d-1}}{x_0^d}$ for $1 \le 1 \le n$.

The morphism F is therefore an isomorphism and thus realizes $\mathbb{P}^n$ as a subvariety of $\mathbb{P}^N$. This is usually called the degree-d **Veronese embedding**. Its importance lies in the fact that degree-d polynomials in the coordinates of $\mathbb{P}^n$ are translated into *linear* polynomials when viewing $\mathbb{P}^n$ as a subvariety of $\mathbb{P}^N$. An example of this application will be given in corollary 3.4.12.

The easiest examples are the degree-d embeddings of $\mathbb{P}^1$, given by

$$\mathbb{P}^1 \to \mathbb{P}^d,\ (s:t) \mapsto (s^d : s^{d-1}t : s^{d-2}t^2 : \cdots : t^d).$$

The special cases $d = 2$ and $d = 3$ are considered in example 3.3.11 and exercise 3.5.2.

Note that by applying corollary 3.4.7 we could conclude that $F(X)$ is a projective variety without writing down its equations. Of course, in theory we could also write down the equations, but this is quite messy in this case.

Corollary 3.4.12. *Let $X \subset \mathbb{P}^n$ be a projective variety, and let $f \in k[x_0,\dots,x_n]$ be a non-constant homogeneous polynomial. Then $X \backslash Z(f)$ is an affine variety.*

Proof. We know this already if f is a linear polynomial (see the proof of proposition 3.3.6). But by applying a Veronese embedding of degree d, we can always assume this. $\square$

3.5. **Exercises.**

Exercise 3.5.1. Let L_1 and L_2 be two disjoint lines in $\mathbb{P}^3$, and let $P \in \mathbb{P}^3 \backslash (L_1 \cup L_2)$ be a point. Show that there is a unique line $L \subset \mathbb{P}^3$ meeting L_1, L_2, and P (i. e. such that $P \in L$ and $L \cap L_i \ne \emptyset$ for $i = 1,2$).

Exercise 3.5.2. Let $C \subset \mathbb{P}^3$ be the "twisted cubic curve" given by the parametrization

$$\mathbb{P}^1 \to \mathbb{P}^3 \quad (s:t) \mapsto (x:y:z:w) = (s^3 : s^2t : st^2 : t^3).$$

Let $P = (0:0:1:0) \in \mathbb{P}^3$, and let H be the hyperplane defined by $z = 0$. Let φ be the projection from P to H, i. e. the map associating to a point Q of C the intersection point of the unique line through P and Q with H.

(i) Show that φ is a morphism.
(ii) Determine the equation of the curve $\varphi(C)$ in $H \cong \mathbb{P}^2$.
(iii) Is $\varphi : C \to \varphi(C)$ an isomorphism onto its image?

Exercise 3.5.3. Let $I \subset k[x_1,\dots,x_n]$ be an ideal. Define I^h to be the ideal generated by $\{f^h \,;\, f \in I\} \subset k[x_0,\dots,x_n]$, where

$$f^h(x_0,\dots,x_n) := x_0^{\deg(f)} \cdot f\left(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right)$$

denotes the homogenization of f with respect to x_0. Show that:

(i) I^h is a homogeneous ideal.
(ii) $Z(I^h) \subset \mathbb{P}^n$ is the closure of $Z(I) \subset \mathbb{A}^n$ in $\mathbb{P}^n$. We call $Z(I^h)$ the **projective closure** of $Z(I)$.
(iii) Let $I = (f_1,\dots,f_k)$. Show by an example that $I^h \neq (f_1^h,\dots,f_k^h)$ in general. (Hint: You may consider (again) the twisted cubic curve of exercise 3.5.2.)

Exercise 3.5.4. In this exercise we will make the space of all lines in $\mathbb{P}^n$ into a projective variety.

Fix $n \geq 1$. We define a set-theoretic map

$$\varphi : \{\text{lines in } \mathbb{P}^n\} \to \mathbb{P}^N$$

with $N = \binom{n+1}{2} - 1$ as follows. For every line $L \subset \mathbb{P}^n$ choose two distinct points $P = (a_0 : \cdots : a_n)$ and $Q = (b_0 : \cdots : b_n)$ on L and define $\varphi(L)$ to be the point in $\mathbb{P}^N$ whose homogeneous coordinates are the $\binom{n+1}{2}$ maximal minors of the matrix

$$\begin{pmatrix} a_0 & \cdots & a_n \\ b_0 & \cdots & b_n \end{pmatrix},$$

in any fixed order. Show that:

(i) The map φ is well-defined and injective.
(ii) The image of φ is a projective variety that has a finite cover by affine spaces $\mathbb{A}^{2(n-1)}$ (in particular, its dimension is $2(n-1)$). It is called the *Grassmannian* $G(1,n)$. Hint: recall that by the Gaussian algorithm most matrices (what does this mean?) are equivalent to one of the form

$$\begin{pmatrix} 1 & 0 & a_2' & \cdots & a_n' \\ 0 & 1 & b_2' & \cdots & b_n' \end{pmatrix}$$

for some a_i', b_i'.
(iii) $G(1,1)$ is a point, $G(1,2) \cong \mathbb{P}^2$, and $G(1,3)$ is the zero locus of a quadratic equation in $\mathbb{P}^5$.

Exercise 3.5.5. Let V be the vector space over k of homogeneous degree-2 polynomials in three variables x_0, x_1, x_2, and let $\mathbb{P}(V) \cong \mathbb{P}^5$ be its projectivization.

(i) Show that the space of conics in $\mathbb{P}^2$ can be identified with an open subset U of $\mathbb{P}^5$. (One says that U is a "moduli space" for conics in $\mathbb{P}^2$ and that $\mathbb{P}^5$ is a "compactified moduli space".) What geometric objects can be associated to the points in $\mathbb{P}^5 \backslash U$?
(ii) Show that it is a linear condition in $\mathbb{P}^5$ for the conics to pass through a given point in $\mathbb{P}^2$. More precisely, if $P \in \mathbb{P}^2$ is a point, show that there is a linear subspace $L \subset \mathbb{P}^5$ such that the conics passing through P are exactly those in $U \cap L$. What happens in $\mathbb{P}^5 \backslash U$, i. e. what do the points in $(\mathbb{P}^5 \backslash U) \cap L$ correspond to?
(iii) Prove that there is a unique conic through any five given points in $\mathbb{P}^2$, as long as no three of them lie on a line. What happens if three of them do lie on a line?

Exercise 3.5.6. Show that an affine variety over $\mathbb{C}$ is never compact in the classical topology unless it is a single point. (Hint: Given an affine variety $X \subset \mathbb{A}^n$, show that the image of X under the projection map $\mathbb{A}^n \to \mathbb{A}^1$ onto the first coordinate is either a point or an open subset (in the Zariski topology) of $\mathbb{A}^1$. Conclude that an affine variety with more than one point is never *bounded*, i. e. is never contained in a ball $\{(z_1,\dots,z_n)\ ;\ |z_1|^2+\cdots+|z_n|^2 \le R^2\} \subset \mathbb{C}^n$, and therefore not compact.)

Exercise 3.5.7. Let $G(1,n)$ be the Grassmannian of lines in $\mathbb{P}^n$ as in exercise 3.5.4. Show that:

(i) The set $\{(L,P)\ ;\ P \in L\} \subset G(1,n) \times \mathbb{P}^n$ is closed.
(ii) If $Z \subset G(1,n)$ is any closed subset then the union of all lines $L \subset \mathbb{P}^n$ such that $L \in Z$ is closed in $\mathbb{P}^n$.
(iii) Let $X,Y \subset \mathbb{P}^n$ be disjoint projective varieties. Then the union of all lines in $\mathbb{P}^n$ intersecting X and Y is a closed subset of $\mathbb{P}^n$. It is called the **join** $J(X,Y)$ of X and Y.

Exercise 3.5.8. Recall that a *conic* is a curve in $\mathbb{P}^2$ that can be given as the zero locus of an irreducible homogeneous polynomial $f \in k[x_0,x_1,x_2]$ of degree 2. Show that for any 5 given points $P_1,\dots,P_5 \in \mathbb{P}^2$ *in general position*, there is a unique conic passing through all the P_i. This means: there is a non-empty open subset $U \subset \mathbb{P}^2 \times \cdots \times \mathbb{P}^2$ such that there is a unique conic through the P_i whenever $(P_1,\dots,P_5) \in U$. (Hint: By mapping a conic $\{a_0x_0^2+a_1x_1^2+a_2x_2^2+a_3x_0x_1+a_4x_0x_2+a_5x_1x_2=0\}$ to the point $(a_0:\cdots:a_5) \in \mathbb{P}^5$, you can think of "the space of all conics" as an open subset of $\mathbb{P}^5$.)

4. DIMENSION

We have already introduced the concept of dimension of a variety. Now we develop some methods that allow to compute the dimension of most varieties rigorously. We show that the dimension of $\mathbb{A}^n$ and $\mathbb{P}^n$ is n. The dimension of a variety equals the dimension of any of its non-empty open subsets. Every irreducible component of the zero locus of a single function on an affine or projective variety X has dimension $\dim X - 1$.

Two varieties are called birational if they contain isomorphic open subsets. As a large class of examples of birational varieties we construct the blow-up of an affine variety in a subvariety or an ideal. We study in detail the case of blowing up a single point P in a variety X. In this case, the exceptional hypersurface is the tangent cone $C_{X,P}$.

For any point P in a variety X, the tangent space $T_{X,P}$ is the linear space dual to M/M^2, where $M \subset \mathcal{O}_{X,P}$ is the maximal ideal. The point P is called a smooth point of X if $T_{X,P} = C_{X,P}$, i. e. if X "can be approximated linearly" around P. Smoothness can easily be checked by the Jacobi criterion.

As an application of the theory developed so far, we show that every smooth cubic surface X has exactly 27 lines on it. We study the configuration of these lines, and show that X is isomorphic to $\mathbb{P}^2$ blown up in 6 suitably chosen points.

4.1. **The dimension of projective varieties.** Recall that in section 1.3 we have introduced the notion of dimension for every (Noetherian) topological space, in particular for every variety X: the dimension $\dim X$ of X is the largest integer n such that there is a chain of irreducible closed subsets of X

$$\emptyset \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = X.$$

For simplicity of notation, in what follows we will call this a *longest chain* in X.

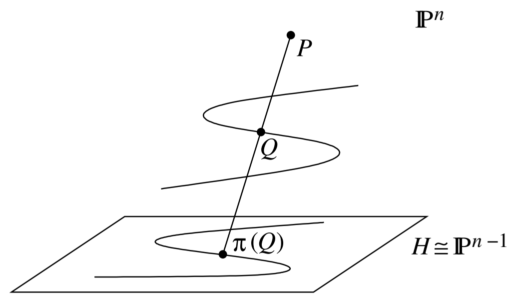
While this definition is quite simple to write down, it is very difficult to use in practice. In fact, we have not even been able yet to compute the dimensions of quite simple varieties like $\mathbb{A}^n$ or $\mathbb{P}^n$ (although it is intuitively clear that these spaces should have dimension n). In this section, we will develop techniques that allow us to compute the dimensions of varieties rigorously.

Remark 4.1.1. We will start our dimension computations by considering projective varieties. It should be said clearly that the theory of dimension is in no way special or easier for projective varieties than it is for other varieties — in fact, it should be intuitively clear that the dimension of a variety is essentially a *local* concept that can be computed in the neighborhood of any point. The reason for us to start with projective varieties is simply that we know more about them: the main theorem on projective varieties and its corollaries of section 3.4 are so strong that they allow for quite efficient applications in dimension theory. One could as well start by looking at the dimensions of affine varieties (and most textbooks will do so), but this requires quite some background in (commutative) algebra that we do not have yet.

Remark 4.1.2. The main idea for our dimension computations is to compare the dimensions of varieties that are related by morphisms with various properties. For example, if $f : X \to Y$ is a *surjective* morphism, we would expect that $\dim X \geq \dim Y$. If $f : X \to Y$ is a morphism with finite fibers, i. e. such that $f^{-1}(P)$ is a finite set for all $P \in Y$, we would expect that $\dim X \leq \dim Y$. In particular, if a morphism both is surjective and has finite fibers, we expect that $\dim X = \dim Y$.

Example 4.1.3. The standard case in which we will prove and apply the idea of comparing dimensions is the case of projections from a point. We have already seen such projections in example 3.3.11 and exercise 3.5.2; let us now consider the general case.

Let $X \subsetneq \mathbb{P}^n$ be a projective variety, and let $P \in \mathbb{P}^n$ be a point that is not in X. By a change of coordinates we can assume that $P = (0 : \cdots : 0 : 1)$. Let $H \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n$ be a linear subspace of codimension 1 that does not contain P; again by a change of coordinates we can assume that $H = \{x_n = 0\}$. We define a **projection map** $\pi : X \to H$ from P as follows: for every point $Q \in X$ let $\pi(Q)$ be the intersection point of the line $\overline{PQ}$ with H. (Note that this is well-defined as $Q \neq P$ by assumption.)



This is in fact a morphism: if $Q = (a_0 : \cdots : a_n) \in X$, the line $\overline{PQ}$ is given parametrically by

$$\overline{PQ} = \{(\lambda a_0 : \cdots : \lambda a_{n-1} : \lambda a_n + \mu) \in \mathbb{P}^n \ ; \ (\lambda : \mu) \in \mathbb{P}^1\}.$$

The intersection point of this line with H is obviously the point $(a_0 : \cdots : a_{n-1} : 0)$, which is well-defined by the assumption that $Q \neq P$. Hence the projection π is given in coordinates by

$$\pi : X \to \mathbb{P}^{n-1}, \ (a_0 : \cdots : a_n) \mapsto (a_0 : \cdots : a_{n-1}).$$

In particular, this is a polynomial map and therefore a morphism.

Note that projections always have finite fibers: by construction, the inverse image $\pi^{-1}(Q)$ of a point $Q \in H$ must be contained in the line $\overline{PQ} \cong \mathbb{P}^1$, but it must also be an algebraic set and cannot contain the point P, hence it must be a finite set.

Note also that we can repeat this process if the image of X is not all of $\mathbb{P}^{n-1}$: we can then project $\pi(X)$ from a point in $\mathbb{P}^{n-1}$ to $\mathbb{P}^{n-2}$, and so on. After a finite number of such projections, we arrive at a *surjective* morphism $X \to \mathbb{P}^m$ for some m that is the composition of projections as above. In particular, as this morphism is surjective and has finite fibers, we expect $\dim X = m$. This is the idea that we will use for our dimension computations.

Let us start with some statements about dimensions that are not only intuitively clear but actually also easy to prove.

Lemma 4.1.4.

(i) *If* $\emptyset \neq X_0 \subsetneq \cdots \subsetneq X_n = X$ *is a longest chain in* X *then* $\dim X_i = i$ *for all* i.

(ii) *If* $Y \subsetneq X$ *is a closed subvariety of the variety* X *then* $\dim Y < \dim X$.

(iii) *Let* $f : X \to Y$ *be a surjective morphism of projective varieties. Then every longest chain* $\emptyset \neq Y_0 \subsetneq \cdots \subsetneq Y_n$ *in* Y *can be lifted to a chain* $\emptyset \neq X_0 \subsetneq \cdots \subsetneq X_n$ *in* X *(i. e. the* X_i *are closed and irreducible with* $f(X_i) = Y_i$ *for all* i*). In particular,* $\dim X \geq \dim Y$.

Proof. (i): It is obvious that $\dim X_i \geq i$. If we had $\dim X_i > i$ there would be a longer chain in X_i than $\emptyset \neq X_0 \subsetneq \cdots \subsetneq X_i$. This chain could then be extended by the X_j for $j > i$ to a chain in X that is longer than the given one.

(ii): We can extend a longest chain $\emptyset \neq Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n = Y$ in Y to a chain $\emptyset \neq Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n = Y \subsetneq X$ in X which is one element longer.

(iii): We prove the statement by induction on $n = \dim Y$; there is nothing to show if $n = 0$. Otherwise let $Z_1, \ldots, Z_r \subset X$ be the irreducible components of $f^{-1}(Y_{n-1})$, so that $f(Z_1) \cup \cdots \cup f(Z_r) = Y_{n-1}$. Note that Y_{n-1} is irreducible and the $f(Z_i)$ are closed by corollary

3.4.7, so one Z_i must map surjectively to Y_{n-1}. Applying the induction hypothesis to the restriction $f|_{Z_i} : Z_i \to Y_{n-1}$ we get $\dim Z_i \geq \dim Y_{n-1} = n-1$, so there is a chain $\emptyset \neq X_0 \subsetneq \cdots \subsetneq X_{n-1} = Z_i$. Extending this chain by X at the end, we thus obtain a chain in X of length n lying over the given chain in Y. □

Lemma 4.1.5. *Let $X \subsetneq \mathbb{P}^n$ be a projective variety, and assume without loss of generality that $P = (0 : \cdots : 0 : 1) \notin X$.*

(i) *Any homogeneous polynomial $f \in k[x_0, \dots, x_n]$ satisfies a relation of the form*
$$f^D + a_1 f^{D-1} + a_2 f^{D-2} + \cdots + a_D = 0 \quad \textit{in } S(X) = k[x_0, \dots, x_n]/I(X)$$
for some $D > 0$ and some homogeneous polynomials $a_i \in k[x_0, \dots, x_{n-1}]$ that do not depend on the last variable x_n.

(ii) *Let $\pi : X \to \mathbb{P}^{n-1}$ be the projection from P as in example 4.1.3. If $Y \subset X$ is a closed subvariety such that $\pi(Y) = \pi(X)$ then $Y = X$.*

Remark 4.1.6. Before we prove this lemma let us give the idea behind these statements. In (i), you should think of f as being a polynomial containing the variable x_n, while the a_i do not. So for given values of $x_0, \dots, x_{n-1}$ the relation in (i) is a non-zero polynomial equation in x_n that therefore allows only finitely many values for x_n on X. As the projection from P is just given by dropping the last coordinate x_n, the statement of (i) is just that this projection map has finite fibers.

We have argued in remark 4.1.1 that we then expect the dimension of $\pi(X)$ to be less than or equal to the dimension of X. To show this we will want to take a longest chain in X and project it down to $\pi(X)$. It is obvious that the images of the elements of such a chain in X are again closed subvarieties in $\pi(X)$, but it is not a priori obvious that a strict inclusion $X_i \subsetneq X_{i+1}$ translates into a strict inclusion $\pi(X_i) \subsetneq \pi(X_{i+1})$. This is exactly the statement of (ii).

Proof. (i): Let d be the degree of f. Consider the morphism
$$\tilde{\pi} : X \to \mathbb{P}^n, \ (x_0 : \cdots : x_n) \mapsto (y_0 : \cdots : y_n) := (x_0^d : \cdots : x_{n-1}^d : f(x_0, \dots, x_n))$$
(which is well-defined since $P \notin X$). The image of $\tilde{\pi}$ is closed by corollary 3.4.7 and is therefore the zero locus of some homogeneous polynomials $F_1, \dots, F_r \in k[y_0, \dots, y_n]$. Note that
$$Z(y_0, \dots, y_{n-1}, F_1, \dots, F_r) = \emptyset \subset \mathbb{P}^n$$
because the F_i require the point to be in the image $\tilde{\pi}(X)$, while the $x_0, \dots, x_{n-1}$ do not vanish simultaneously on X. So by the projective Nullstellensatz of proposition 3.2.5 (iv) it follows that some power of y_n is in the ideal generated by $y_0, \dots, y_{n-1}, F_1, \dots, F_r$. In other words,
$$y_n^D = \sum_{i=0}^{n-1} g_i(y_0, \dots, y_n) \cdot y_i \quad \text{in } S(\tilde{\pi}(X)) = k[y_0, \dots, y_n]/(F_1, \dots, F_r)$$
for some D. Substituting the definition of $\tilde{\pi}$ for the y_i thus shows that there is a relation
$$f^D + a_1 f^{D-1} + a_2 f^{D-2} + \cdots + a_D = 0 \quad \text{in } S(X)$$
for some homogeneous $a_i \in k[x_0, \dots, x_{n-1}]$.

(ii): Assume that the statement is false, i. e. that $Y \subsetneq X$. Then we can pick a homogeneous polynomial $f \in I(Y) \backslash I(X) \subset k[x_0, \dots, x_n]$ of some degree d that vanishes on Y but not on X.

Now pick a relation as in (i) for the smallest possible value of D. In particular we then have $a_D \neq 0$ in $S(X)$, i. e. $a_D \notin I(X)$. But we have chosen f such that $f \in I(Y)$, therefore the relation (i) tells us that $a_D \in I(Y)$ as well.

It follows that $a_D \in I(Y) \backslash I(X)$. But note that $a_D \in k[x_0, \dots, x_{n-1}]$, so a_D is a function on $\mathbb{P}^{n-1}$ that vanishes on $\pi(Y)$ but not on $\pi(X)$, in contradiction to the assumption. $\square$

Corollary 4.1.7. *Let $X \subsetneq \mathbb{P}^n$ be a projective variety, and assume without loss of generality that $P = (0 : \cdots : 0 : 1) \notin X$. Let $\pi : X \to \mathbb{P}^{n-1}$ be the projection from P as in example 4.1.3. Then* $\dim X = \dim \pi(X)$.

Proof. Let $\emptyset \neq X_0 \subsetneq \cdots \subsetneq X_r = X$ be a longest chain in X. Then $\emptyset \neq Y_0 \subsetneq \cdots \subsetneq Y_r = Y$ with $Y_i = \pi(X_i)$ is a chain in $\pi(X)$: note that the Y_i are closed by corollary 3.4.7, irreducible as they are the images of irreducible sets, and no two of them can coincide by lemma 4.1.5. It follows that $\dim \pi(X) \geq \dim X$. But also $\dim \pi(X) \leq \dim X$ by lemma 4.1.4 (iii), so the statement follows. $\square$

Corollary 4.1.8. *The dimension of $\mathbb{P}^n$ is n.*

Proof. By lemma 4.1.4 (ii) we know that

$$\dim \mathbb{P}^0 < \dim \mathbb{P}^1 < \dim \mathbb{P}^2 < \dim \mathbb{P}^3 < \cdots . \qquad (*)$$

Moreover, we have seen in example 4.1.3 that every projective variety X can be mapped surjectively to some $\mathbb{P}^n$ by a sequence of projections from points; it then follows that $\dim X = \dim \mathbb{P}^n$ by corollary 4.1.7. In other words, every dimension that occurs as the dimension of some projective variety must occur already as the dimension of some projective space. But combining $(*)$ with lemma 4.1.4 (i) we see that every non-negative integer occurs as the dimension of some projective variety — and therefore as the dimension of some projective space. So in $(*)$ we must have $\dim \mathbb{P}^n = n$ for all n. $\square$

Proposition 4.1.9. *Let $X \subset \mathbb{P}^n$ be a projective variety, and let $f \in k[x_0, \dots, x_n]$ be a non-constant homogeneous polynomial that does not vanish identically on X. Then* $\dim(X \cap Z(f)) = \dim X - 1$.

Remark 4.1.10. Note that in the statement of this proposition $X \cap Z(f)$ may well be reducible; the statement is then that there is at least one component that has dimension $\dim X - 1$ (and that no component has bigger dimension). We will prove a stronger statement, namely a statement about *every* component of $X \cap Z(f)$, in corollary 4.2.5.

Proof. Let $m = \dim X$. After applying a Veronese embedding of degree $\deg f$ as in example 3.4.11 we can assume that f is linear. Now construct linear functions $f_0, \dots, f_m$ and algebraic sets $X_0, \dots, X_{m+1} \subset X$ inductively as follows: Let $X_0 = X$ and $f_0 = f$. For $i \geq 0$ let $X_{i+1} = X_i \cap Z(f_i)$, and let f_{i+1} be any linear form such that

(i) f_{i+1} does not vanish identically on any component of X_{i+1}, and
(ii) f_{i+1} is linearly independent from the $f_1, \dots, f_i$.

It is obvious that (i) can always be satisfied. Moreover, (ii) is automatic if X_{i+1} is not empty (as $f_1, \dots, f_i$ vanish on X_{i+1}), and easy to satisfy otherwise (as then (i) is no condition).

Applying lemma 4.1.4 (ii) inductively, we see that no component of X_i has dimension bigger than $m - i$. In particular, X_{m+1} must be empty. Hence the linear forms $f_0, \dots, f_m$ do not vanish simultaneously on X; so they define a morphism $\pi : X \to \mathbb{P}^m$. As the f_i are linear and linearly independent, π is up to a change of coordinates the same as $f_i = x_i$ for $0 \leq i \leq m$, so it is just a special case of a continued projection from points as in example 4.1.3. In particular, $\dim \pi(X) = \dim X = m$ by corollary 4.1.7. By lemma 4.1.4 (ii) it then follows that $\pi(X) = \mathbb{P}^m$, i. e. π is surjective.

Now suppose that every component of $X_1 = X \cap Z(f)$ has already dimension at most $m-2$, then by the above inductive argument already X_m is empty and the forms $f_0, \dots, f_{m-1}$ do not vanish simultaneously on X. But this means that $(0 : \cdots : 0 : 1) \notin \pi(X)$, which contradicts the surjectivity of π. $\square$

4.2. **The dimension of varieties.** After having exploited the main theorem on projective varieties as far as possible, let us now study the dimension of more general varieties. We have already remarked that the dimension of a variety should be a local concept; in particular the dimension of any open subvariety U of a variety X should be the same as that of X. This is what we want to prove first.

Proposition 4.2.1. *Let X be a variety, and let $U \subset X$ be a non-empty open subset of X. Then* $\dim U = \dim X$.

Proof. "$\le$": Let $\emptyset \neq U_0 \subsetneq U_1 \subsetneq \cdots \subsetneq U_n = U$ be a longest chain in U. If X_i denotes the closure of U_i in X for all i, then $\emptyset \neq X_0 \subsetneq \cdots \subsetneq X_n = X$ is a chain in X.

"$\ge$": We will prove this in several steps.

Step 1: Let $\emptyset \neq X_0 \subsetneq \cdots \subsetneq X_n = X$ be a longest chain in X, and assume that $X_0 \subset U$. Then set $U_i = X_i \cap U$ for all i; we claim that $\emptyset \neq U_0 \subsetneq \cdots \subsetneq U_n = U$ is a chain in U (from which it then follows that $\dim U \ge \dim X$). In fact, the only statement that is not obvious here is that $U_i \neq U_{i+1}$ for all i. So assume that $U_i = U_{i+1}$ for some i. Then

$$\begin{aligned} X_{i+1} &= (X_{i+1} \cap U) \cup (X_{i+1} \cap (X \backslash U)) \\ &= (X_i \cap U) \cup (X_{i+1} \cap (X \backslash U)) \\ &= X_i \cup (X_{i+1} \cap (X \backslash U)), \end{aligned}$$

where the last equality follows from $X_i \cap (X\backslash U) \subset X_{i+1} \cap (X \backslash U)$. But this is a contradiction to X_{i+1} being irreducible, as X_i is neither empty nor all of X_{i+1}. So we have now proven the proposition in the case where the element X_0 of a longest chain in X lies in U.

Step 2: Let X be a projective variety. Then we claim that we can always find a longest chain $\emptyset \neq X_0 \subsetneq \cdots \subsetneq X_n$ (with $n = \dim X$) such that $X_0 \subset U$. We will construct this chain by descending recursion on n, starting by setting $X_n = X$. So assume that $X_i \subsetneq X_{i+1} \subsetneq \cdots \subsetneq X_n = X$ has already been constructed such that $X_i \cap U \neq \emptyset$. Pick any non-constant homogeneous polynomial f that does not vanish identically on any irreducible component of $X_i \backslash U$. By proposition 4.1.9 there is a component of $X_i \cap Z(f)$ of dimension $i-1$; call this X_{i-1}. We have to show that $X_{i-1} \cap U \neq \emptyset$. Assume the contrary; then X_{i-1} must be contained in $X_i \backslash U$. But by the choice of f we know that X_{i-1} is not a whole component of $X_i \backslash U$, so it can only be a proper subset of a component of $X_i \backslash U$. But by lemma 4.1.4 (ii) the components of $X_i \backslash U$ have dimension at most $i-1$, and therefore proper subsets of them have dimension at most $i-2$. This is a contradiction to $\dim X_{i-1} = i-1$.

Combining steps 1 and 2, we have now proven the proposition if X is a projective variety. Of course the statement then also follows if X is an affine variety: let $\bar{X}$ be the projective closure of X as in exercise 3.5.3, then by applying our result twice we get $\dim U = \dim \bar{X} = \dim X$.

Step 3: Let X be any variety, and let $\emptyset \neq X_0 \subsetneq \cdots \subsetneq X_n = X$ be a longest chain in X. Let $V \subset X$ be an affine open neighborhood of the point X_0; then $\dim V = \dim X$ by step 1. In the same way we can find an affine open subset W of U such that $\dim W = \dim U$. As $V \cap W \neq \emptyset$, it finally follows from steps 1 and 2 that

$$\dim X = \dim V = \dim(V \cap W) = \dim W = \dim U.$$

□

In particular, as every variety can be covered by affine varieties, this proposition implies that it is sufficient to study the dimensions of affine varieties. Let us first prove the affine equivalent of proposition 4.1.9.

Example 4.2.2.

(i) As $\mathbb{A}^n$ is an open subset of $\mathbb{P}^n$, it follows by corollary 4.1.8 that $\dim \mathbb{A}^n = n$.

(ii) As $\mathbb{A}^{m+n}$ is an open subset of $\mathbb{P}^n \times \mathbb{P}^m$, it follows by (i) that $\dim(\mathbb{P}^n \times \mathbb{P}^m) = n+m$.

(iii) Let $f \in k[x_1,\dots,x_n]$ be a non-constant polynomial. We claim that $Z(f) \subset \mathbb{A}^n$ has dimension $n-1$. In fact, let $\bar{X} \subset \mathbb{P}^n$ be the projective closure of $Z(f)$; by proposition 4.1.9 there is a component Y of $\bar{X}$ of dimension $n-1$. As the homogenized polynomial f does not contain x_0 as a factor, $\bar{X}$ cannot contain the whole "infinity locus" $\mathbb{P}^n \backslash \mathbb{A}^n \cong \mathbb{P}^{n-1}$. So the part of $\bar{X}$ in the infinity locus has dimension at most $n-2$; in particular the component Y of $\bar{X}$ has non-empty intersection with $\mathbb{A}^n$. In other words, $Z(f) \subset \mathbb{A}^n$ has dimension $n-1$.

(iv) Let $f \in k[x_1,\dots,x_n]$ be as in (iii); we claim that in fact the dimension of *every* irreducible component of $Z(f) \subset \mathbb{A}^n$ is $n-1$: in fact, as $k[x_1,\dots,x_n]$ is a unique factorization domain, we can write f as a product $f_1 \cdots f_r$ of irreducible polynomials, so that the decomposition of $Z(f)$ into its irreducible components is $Z(f_1) \cup \cdots \cup Z(f_r)$. Now we can apply (iii) to the f_i separately to get the desired result.

(v) The corresponding statements to (iii) and (iv) are true for the zero locus of a homogeneous polynomial in $\mathbb{P}^n$ as well (the proof is the same).

By (iv) and (v), there is a one-to-one correspondence between closed subvarieties of $\mathbb{A}^n$ (resp. $\mathbb{P}^n$) of dimension $n-1$ and non-constant irreducible polynomials in $k[x_1,\dots,x_n]$ (resp. non-constant homogeneous polynomials in $k[x_0,\dots,x_n]$). Varieties that are of this form are called **hypersurfaces**; if the degree of the polynomial is 1 they are called **hyperplanes**.

Remark 4.2.3. Next we want to prove for general affine varieties $X \subset \mathbb{A}^n$ that the dimension of (every component of) $X \cap Z(f)$ is $\dim X - 1$. Note that this does *not* follow immediately from the projective case as it did for $X = \mathbb{A}^n$ in example 4.2.2 (iii) or (iv):

(i) As for example 4.2.2 (iii), of course we can still consider the projective closure $\bar{X}$ of X in $\mathbb{P}^n$ and intersect it with the zero locus of the homogenization of f; but proposition 4.1.9 only gives us the existence of one component of dimension $\dim X - 1$ in $\bar{X} \cap Z(f)$. It may well be that there is a component of $\bar{X} \cap Z(f)$ that is contained in the "hyperplane at infinity" $\mathbb{P}^n \backslash \mathbb{A}^n$, in which case we get no information about the affine zero locus $X \cap Z(f)$. As an example you may consider the projective variety $X = \{x_0 x_2 = x_1^2\} \subset \mathbb{P}^2$ and $f = x_1$: then $X \cap Z(f) = (1:0:0) \cup (0:0:1)$ contains a point $(0:0:1)$ at infinity as an irreducible component.

(ii) As for example 4.2.2 (iv), note that a factorization of f as for $\mathbb{A}^n$ is simply not possible in general. For example, in the case just considered in (i), $Z(f)$ intersects X in two points, but there is no decomposition of the linear function f into two factors that vanish on only one of the points.

Nevertheless the idea of the proof is still to use projections from points:

Proposition 4.2.4. *Let $X \subset \mathbb{A}^n$ be an affine variety, and let $f \in k[x_1,\dots,x_n]$ be a non-constant polynomial that does not vanish identically on X. Then* $\dim(X \cap Z(f)) = \dim X - 1$ *(unless $X \cap Z(f) = \emptyset$).*

Proof. We prove the statement by induction on n (not on $\dim X$!); there is nothing to show for $n = 0$. If $X = \mathbb{A}^n$ the statement follows from example 4.2.2 (iv), so we can assume that $X \subsetneq \mathbb{A}^n$.

Let $\bar{X}$ be the projective closure in $\mathbb{P}^n$; we can assume by an affine change of coordinates that $P = (0:\cdots:0:1) \notin \bar{X}$. Consider the projection $\bar{\pi} : \bar{X} \to \mathbb{P}^{n-1}$ from P as in example 4.1.3. Obviously, we can restrict this projection map to the affine space $\mathbb{A}^n \subset \mathbb{P}^n$ given by $x_0 \neq 0$; we thus obtain a morphism $\pi : X \to \pi(X)$ that is given in coordinates by $(a_1,\dots,a_n) \mapsto (a_1,\dots,a_{n-1})$. Note that $\pi(X)$ is closed in $\mathbb{A}^n$, as $\pi(X) = \bar{\pi}(\bar{X}) \cap \mathbb{A}^n$.

By lemma 4.1.5 (i) applied to the function x_n we see that there is a relation

$$p(x_n) := x_n^D + a_1 x_n^{D-1} + \cdots a_D = 0 \quad \text{in } A(X) \tag{$*$}$$

for some $D > 0$ and some $a_i \in k[x_1,\dots,x_{n-1}]$ that do not depend on x_n. Let K be the field $k(x_1,\dots,x_{n-1})$ of rational functions in $n-1$ variables. Set $V = K[x_n]/p(x_n)$; by $(*)$ this is a D-dimensional vector space over K (with basis $1, x_n, \dots, x_n^{D-1}$). Obviously, every polynomial $g \in k[x_1,\dots,x_n]$ defines a vector space homomorphism $g : V \to V$ (by polynomial multiplication), so we can talk about its determinant $\det g \in K$. Moreover, it is easy to see that $\det g \in k[x_1,\dots,x_{n-1}]$, as the definition of the determinant does not use divisions. Note also that $\det g = g^D$ if $g \in k[x_1,\dots,x_{n-1}]$.

Now go back to our original problem: describing the zero locus of the given polynomial f on X. We claim that

$$\pi(X \cap Z(f)) = \pi(X) \cap Z((f) \cap k[x_1,\dots,x_{n-1}]) \supset \pi(X) \cap Z(\det f)$$

(in fact there is equality, but we do not need this). The first equality is obvious from the definition of π. To prove the second inclusion, note that by the Nullstellensatz it suffices to show that $(f) \cap k[x_1,\dots,x_{n-1}] \subset \sqrt{(\det f)}$. So let $g \in (f) \cap k[x_1,\dots,x_{n-1}]$; in particular $g = f \cdot b$ for some $b \in k[x_1,\dots,x_n]$. It follows that

$$g^D = \det g = \det f \cdot \det b \in (\det f),$$

i. e. $g \in \sqrt{(\det f)}$, as we have claimed.

The rest is now easy:

$$\begin{aligned}
\dim(X \cap Z(f)) &= \dim \pi(X \cap Z(f)) && \text{by corollary 4.1.7 and proposition 4.2.1} \\
&\geq \dim(\pi(X) \cap Z(\det f)) && \text{by the inclusion just proven} \\
&= \dim \pi(X) - 1 && \text{by the induction hypothesis} \\
&= \dim X - 1 && \text{by corollary 4.1.7 and proposition 4.2.1 again.}
\end{aligned}$$

The opposite inequality follows trivially from lemma 4.1.4 (ii). □

It is now quite easy to extend this result to a statement about *every* component of $X \cap Z(f)$:

Corollary 4.2.5. *Let $X \subset \mathbb{A}^n$ be an affine variety, and let $f \in k[x_1,\dots,x_n]$ be a non-constant polynomial that does not vanish identically on X. Then every irreducible component of $X \cap Z(f)$ has dimension* $\dim X - 1$.

Proof. Let $X \cap Z(f) = Z_1 \cup \cdots \cup Z_r$ be the decomposition into irreducible components; we want to show that $\dim Z_1 = \dim X - 1$. Let $g \in k[x_1,\dots,x_n]$ be a polynomial that vanishes on $Z_2,\dots,Z_r$ but not on Z_1, and let $U = X_g = X \backslash Z(g)$. Then U is an affine variety by lemma 2.3.16, and $U \cap Z(f)$ has only one component $Z_1 \cap U$. So the statement follows from proposition 4.2.4 together with proposition 4.2.1. □

Remark 4.2.6. Proposition 4.2.1 and especially corollary 4.2.5 are the main properties of the dimension of varieties. Together they allow to compute the dimension of almost any variety without the need to go back to the cumbersome definition. Here are two examples:

Corollary 4.2.7. *Let $f : X \to Y$ be a morphism of varieties, and assume that the dimension of all fibers $n = \dim f^{-1}(P)$ is the same for all $P \in Y$. Then $\dim X = \dim Y + n$.*

Proof. We prove the statement by induction on $\dim Y$; there is nothing to show for $n = 0$ (i. e. if Y is a point).

By proposition 4.2.1 we can assume that $Y \subset \mathbb{A}^m$ is an affine variety. Let $f \in k[x_1,\dots,x_m]$ be any non-zero polynomial in the coordinates of $\mathbb{A}^m$ that vanishes somewhere, but not everywhere on Y, let $Y' \subset Y$ be an irreducible component of $Y \cap Z(f)$, and let $X' = f^{-1}(Y')$. Then it follows by corollary 4.2.5 and the induction hypothesis that

$$\dim X = \dim X' + 1 = \dim Y' + n + 1 = \dim Y + n.$$

$\square$

Example 4.2.8.

(i) For any varieties X, Y we have $\dim(X \times Y) = \dim X + \dim Y$ (apply corollary 4.2.7 to the projection morphism $X \times Y \to X$).
(ii) Combining corollary 4.2.7 with proposition 4.2.1 again, we see that it is actually sufficient that $f^{-1}(P)$ is non-empty and of the same dimension for all P in a non-empty open subset U of Y.

Corollary 4.2.9. *Let X and Y be affine varieties in $\mathbb{A}^n$. Then every irreducible component of $X \cap Y \subset \mathbb{A}^n$ has dimension at least* $\dim X + \dim Y - n$.

Proof. Rewrite $X \cap Y$ as the intersection of $X \times Y$ with the diagonal $\Delta(\mathbb{A}^n)$ in $\mathbb{A}^n \times \mathbb{A}^n$. The diagonal is given by the zero locus of the n functions $x_i - y_i$ for $1 \le i \le n$, where $x_1,\dots,x_n,y_1,\dots,y_n$ are the coordinates of $\mathbb{A}^n \times \mathbb{A}^n$. By corollary 4.2.5, every component of the intersection of an affine variety Z with the zero locus of a non-constant function has dimension at least equal to $\dim Z - 1$ (it is $\dim Z$ if f vanishes identically on Z, and $\dim Z - 1$ otherwise). Applying this statement n times to the functions $x_i - y_i$ on $X \times Y$ in $\mathbb{A}^n \times \mathbb{A}^n$ we conclude that every component of $X \cap Y$ has dimension at least $\dim(X \times Y) - n = \dim X + \dim Y - n$. $\square$

Remark 4.2.10. (For commutative algebra experts) There is another more algebraic way of defining the dimension of varieties that is found in many textbooks: the dimension of a variety X is the transcendence degree over k of the field of rational functions $K(X)$ on X. Morally speaking, this definition captures the idea that the dimension of a variety is the number of independent coordinates on X. We have not used this definition here as most propositions concerning dimensions would then have required methods of (commutative) algebra that we have not developed yet.

Here are some ideas that can be used to show that this algebraic definition of dimension is equivalent to our geometric one:

- If $U \subset X$ is a non-empty open subset we have $K(U) = K(X)$, so with the algebraic definition of dimension it is actually trivial that $\dim U = \dim X$.
- It is then also obvious that $\dim \mathbb{A}^n = \operatorname{tr}\deg k(x_1,\dots,x_n) = n$.
- Let $\pi : X \to \pi(X)$ be a projection map as in the proof of proposition 4.2.4. The relation $(*)$ in the proof can be translated into the fact that $K(X)$ is an algebraic field extension of $K(\pi(X))$ (we add one variable x_n, but this variable satisfies a polynomial relation). In particular, these two fields have the same transcendence degree, translating into the fact that $\dim \pi(X) = \dim X$.

4.3. **Blowing up.** We have just seen in 4.2.1 that two varieties have the same dimension if they contain an isomorphic (non-empty) open subset. In this section we want to study this relation in greater detail and construct a large and important class of examples of varieties that are not isomorphic but contain an isomorphic open subset. Let us first make some definitions concerning varieties containing isomorphic open subsets. We will probably not use them very much, but they are often found in the literature.

Definition 4.3.1. Let X and Y be varieties. A **rational map** f from X to Y, written $f : X \dashrightarrow Y$, is a morphism $f : U \to Y$ (denoted by the same letter) from a non-empty open subset $U \subset X$ to Y. We say that two such rational maps $f : U \to Y$ and $g : V \to Y$ with $U,V \subset X$ are the same if $f = g$ on $U \cap V$.

A rational map $f : X \dashrightarrow Y$ is called **dominant** if its image is dense in Y, i. e. if f is given by a morphism $f : U \to Y$ such that $f(U)$ contains a non-empty open subset of Y. If $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ are rational maps, and if f is dominant, then the composition $g \circ f : X \dashrightarrow Z$ is a well-defined rational map.

A **birational map** from X to Y is a rational map with an inverse, i. e. it is a (dominant) rational map $f : X \dashrightarrow Y$ such that there is a (dominant) rational map $g : Y \dashrightarrow X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$ as rational maps. Two varieties X and Y are called **birational** if there is a birational map between them. In other words, X and Y are birational if they contain an isomorphic non-empty open subset.

We will now construct the most important examples of birational morphisms (resp. birational varieties), namely blow-ups.

Construction 4.3.2. Let $X \subset \mathbb{A}^n$ be an affine variety, and let $f_0,\dots,f_r \in k[x_1,\dots,x_n]$ be polynomial functions that do not vanish identically on X. Then $U = X \backslash Z(f_0,\dots,f_r)$ is a non-empty open subset of X, and there is a well-defined morphism

$$f : U \to \mathbb{P}^r, P \mapsto (f_0(P) : \cdots : f_r(P)).$$

Now consider the graph

$$\Gamma = \{(P, f(P)) \,;\, P \in U\} \subset X \times \mathbb{P}^r$$

which is isomorphic to U (with inverse morphism $(P,Q) \mapsto P$). Note that Γ is in general not closed in $X \times \mathbb{P}^r$, because the points in $X \backslash U$ where $(f_0 : \cdots : f_r)$ is ill-defined as a point in $\mathbb{P}^r$ are "missing".

The closure of Γ in $X \times \mathbb{P}^r$ is called the **blow-up** of X in $(f_0,\dots,f_r)$; we denote it by $\tilde{X}$. It is a closed subset of $X \times \mathbb{P}^r$, and it is irreducible as Γ is; so it is a closed subvariety of $X \times \mathbb{P}^r$. In particular, there are projection morphisms $\pi : \tilde{X} \to X$ and $p : \tilde{X} \to \mathbb{P}^r$. Note that X and $\tilde{X}$ both contain U as a dense open subset, so X and the blow-up $\tilde{X}$ have the same dimension.

Let us now investigate the geometric meaning of blow-ups.

Example 4.3.3. If $r = 0$ in the above notation, i. e. if there is only one function f_0, the blow-up $\tilde{X}$ is isomorphic to X. In fact, we then have $\tilde{X} \subset X \times \mathbb{P}^0 \cong X$, so $\tilde{X}$ is the smallest closed subvariety containing U.

Example 4.3.4. Let $X = \mathbb{A}^2$ with coordinates x_0, x_1, and let $f_0 = x_0$, $f_1 = x_1$. Then the blow-up of X in (f_0, f_1) is a subvariety of $\mathbb{A}^2 \times \mathbb{P}^1$. The morphism $(x_0,x_1) \mapsto (x_0 : x_1)$ is well-defined on $U = X \backslash \{(0,0)\}$; so on this open subset the graph is given by

$$\Gamma = \{((x_0,x_1),(y_0 : y_1)) \,;\, x_0 y_1 = x_1 y_0\} \subset U \times \mathbb{P}^1.$$

The closure of Γ is now obviously given by the same equation, considered in $\mathbb{A}^2 \times \mathbb{P}^1$:

$$\tilde{X} = \{((x_0,x_1),(y_0 : y_1)) \,;\, x_0 y_1 = x_1 y_0\} \subset \mathbb{A}^2 \times \mathbb{P}^1.$$

The projection morphisms to $X = \mathbb{A}^2$ and $\mathbb{P}^1$ are obvious.

Note that the inverse image of a point $P = (x_0,x_1) \in X \backslash \{(0,0)\}$ under π is just the single point $((x_0,x_1),(x_0 : x_1))$ — we knew this before. The inverse image of $(0,0) \in X$ however is $\mathbb{P}^1$, as the equation $x_0 y_1 = x_1 y_0$ imposes no conditions on y_0 and y_1 if $(x_0,x_1) = (0,0)$.

To give a geometric interpretation of the points in $\pi^{-1}(0,0)$ let us first introduce one more piece of notation. Let $Y \subset X$ be a closed subvariety that has non-empty intersection with U. As U is also a subset of $\tilde{X}$, we can consider the closure of $Y \cap U$ in $\tilde{X}$. We call this the **strict transform** of Y. Note that by definition the strict transform of Y is just the blow-up of Y at $(f_0,\dots,f_r)$; so we denote it by $\tilde{Y}$.

Now let $C \subset X = \mathbb{A}^2$ be a curve, given by the equation

$$g(x_0,x_1) = \sum_{i,j} a_{i,j} x_0^i x_1^j = a_{0,0} + a_{1,0}x_0 + a_{0,1}x_1 + a_{1,1}x_0x_1 + \cdots .$$

Assume that $a_{0,0} = 0$, i. e. that C passes through the origin in $\mathbb{A}^2$, and that $(a_{1,0}, a_{0,1}) \neq (0,0)$, so that C has a well-defined *tangent line* at the origin, given by the linearization $a_{1,0}x_0 + a_{0,1}x_1 = 0$ of g. Let us compute the strict transform $\tilde{C}$. Of course, the points $((x_0,x_1),(y_0:y_1))$ of $\tilde{C}$ satisfy the equation

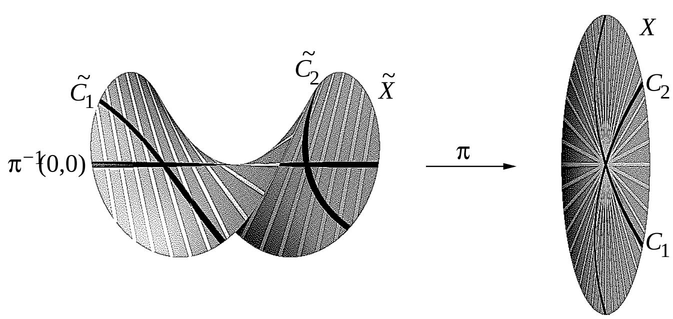
$$a_{1,0}x_0 + a_{0,1}x_1 + a_{1,1}x_0x_1 + a_{2,0}x_0^2 + a_{0,2}x_1^2 + \cdots = 0. \qquad (*)$$

But it is *not* true that $\tilde{C}$ is just the common zero locus in $\mathbb{A}^2 \times \mathbb{P}^1$ of this equation together with $x_0y_1 = x_1y_0$, because this common zero locus contains the whole fiber $\pi^{-1}(0,0) \cong \mathbb{P}^1$ — but $\tilde{C}$ has to be irreducible of dimension 1, so it cannot contain this $\mathbb{P}^1$. In fact, we have forgotten another relation: on the open set where $x_0 \neq 0$ and $x_1 \neq 0$ we can multiply $(*)$ with $\frac{y_0}{x_0}$; using the relation $\frac{y_0}{x_0} = \frac{y_1}{x_1}$ we get

$$a_{1,0}y_0 + a_{0,1}y_1 + a_{1,1}y_0x_1 + a_{2,0}x_0y_0 + a_{0,2}x_1y_1 + \cdots = 0.$$

This equation must then necessarily hold on the closure $\tilde{C}$ too. Restricting it to the origin $(x_0,x_1) = (0,0)$ we get $a_{1,0}y_0 + a_{0,1}y_1 = 0$, which is precisely the equation of the tangent line to C at $(0,0)$. In other words, *the strict transform $\tilde{C}$ of C intersects the fiber $\pi^{-1}(0,0)$ precisely in the point of $\mathbb{P}^1$ corresponding to the tangent line of C in* $(0,0)$. In this sense we can say that the points of $\pi^{-1}(0,0)$ correspond to tangent directions in X at $(0,0)$.

The following picture illustrates this: we have two curves C_1, C_2 that intersect at the origin with different tangent directions. The strict transforms $\tilde{C}_1$ and $\tilde{C}_2$ are then disjoint on the blow-up $\tilde{X}$.



Let us now generalize the results of this example to general blow-ups. Note that in the example we would intuitively say that we have "blown up the origin", i. e. the zero locus of the functions $f_0,\dots,f_r$. In fact, the blow-up construction depends only on the ideal generated by the f_i:

Lemma 4.3.5. *The blow-up of an affine variety X at $(f_0,\dots,f_r)$ depends only on the ideal $I \subset A(X)$ generated by $f_0,\dots,f_r$. We will therefore usually call it the blow-up of X at the ideal I. If $I = I(Y)$ for a closed subset $Y \subset X$, we will also call it the blow-up of X in Y.*

Proof. Let $(f_0,\dots,f_r)$ and $(f'_0,\dots,f'_s)$ be two sets of generators of the same ideal $I \subset A(X)$, and let $\tilde{X}$ and $\tilde{X}'$ be the blow-ups of X at these sets of generators. By assumption we have relations in $A(X)$

$$f_i = \sum_j g_{i,j} f'_j \quad \text{and} \quad f'_j = \sum_k g'_{j,k} f_k.$$

We want to define a morphism $\tilde{X} \to \tilde{X}'$ by sending $(P,(y_0:\cdots:y_r))$ to $(P,(y'_0:\cdots:y'_s))$, where $y'_j = \sum_k g'_{j,k}(P) y_k$. First of all we show that this defines a morphism to $X \times \mathbb{P}^s$, i. e. that the y'_j cannot be simultaneously zero. To do this, note that by construction we have the relation $(y_0 : \cdots : y_r) = (f_0 : \cdots : f_r)$ on $X \backslash Z(I) \subset \tilde{X} \subset X \times \mathbb{P}^r$, i.e. these two vectors are linearly dependent (and non-zero) at each point in this set. Hence the linear relations $f_i = \sum_{j,k} g_{i,j} g'_{j,k} f_k$ in $f_0, \dots, f_r$ imply the corresponding relations $y_i = \sum_{j,k} g_{i,j} g'_{j,k} y_k$ in $y_0, \dots, y_r$ on this set, and thus also on its closure, which is by definition $\tilde{X}$. So if we had $y'_j = \sum_k g'_{j,k} y_k = 0$ for all j then we would also have $y_i = \sum_j g_{i,j} y'_j = 0$ for all i, which is a contradiction.

Hence we have defined a morphism $\tilde{X} \to X \times \mathbb{P}^s$. By construction it maps the open subset $X \backslash Z(f_0, \dots, f_r) \subset \tilde{X}$ to $X \backslash Z(f'_0, \dots, f'_s) \subset \tilde{X}'$, so it must map its closure $\tilde{X}$ to $\tilde{X}'$ as well. By the same arguments we get an inverse morphism $\tilde{X}' \to \tilde{X}$, so $\tilde{X}$ and $\tilde{X}'$ are isomorphic. □

Let us now study the variety $\tilde{X}$ itself, in particular over the locus $Z(f_0, \dots, f_r)$ where $\pi : \tilde{X} \to X$ is not an isomorphism.

Lemma 4.3.6. *Let $X \subset \mathbb{A}^n$ be an affine variety, and let $\tilde{X}$ be the blow-up of X at the ideal $I = (f_0, \dots, f_r)$. Then:*

(i) *The blow-up $\tilde{X}$ is contained in the set*
$$\{(P, (y_0 : \cdots : y_r))\ ;\ y_i f_j(P) = y_j f_i(P) \text{ for all } i, j = 0, \dots, r\} \subset X \times \mathbb{P}^r.$$

(ii) *The inverse image $\pi^{-1}(Z(f_0, \dots, f_r))$ is of pure dimension $\dim X - 1$. It is called the **exceptional hypersurface**.*

Proof. (i): By definition we must have $(y_0 : \cdots : y_r) = (f_0(P) : \cdots : f_r(P))$ on the non-empty open subset $X \backslash Z(I) \subset \tilde{X}$. So these equations must be true as well on the closure of this open subset, which is $\tilde{X}$ by definition.

(ii): It is enough to prove the statement on the open subset where $y_i \neq 0$, as these open subsets for all i cover $\tilde{X}$. Note that on this open subset the condition $f_i(P) = 0$ implies $f_j(P) = 0$ for all j by the equations of (i). So the inverse image $\pi^{-1}(Z(f_0, \dots, f_r))$ is given by one equation $f_j = 0$, and is therefore of pure dimension $\dim \tilde{X} - 1 = \dim X - 1$ by corollary 4.2.5. □

Example 4.3.7. In example 4.3.4, $X = \mathbb{A}^2$ has dimension 2, and the exceptional hypersurface was isomorphic to $\mathbb{P}^1$, which has dimension 1.

Remark 4.3.8. The equations in lemma 4.3.6 (i) are in general not the only ones for $\tilde{X}$. Note that they do not impose any conditions over the zero locus $Z(f_0, \dots, f_r)$ at all, so that it would seem from these equations that the exceptional hypersurface is always $\mathbb{P}^r$. This must of course be false in general just for dimensional reasons (see lemma 4.3.6 (ii)).

In fact, we can write down explicitly the equations for the exceptional hypersurface. We will do this here only in the case of the blow-up of (the ideal of) a point P, which is the most important case. By change of coordinates, we can then assume that P is the origin in $\mathbb{A}^n$.

For any $f \in k[x_1, \dots, x_n]$ we let f^{in} be the "initial polynomial" of f, i.e. if $f = \sum_i f^{(i)}$ is the splitting of f such that $f^{(i)}$ is homogeneous of degree i, then f^{in} is by definition equal to the smallest non-zero $f^{(i)}$. If $I \subset k[x_1, \dots, x_n]$, we let I^{in} be the ideal generated by the initial polynomials f^{in} for all $f \in I$. Note that I^{in} is by definition a homogeneous ideal. So its affine zero locus $Z_a(I^{in}) \subset \mathbb{A}^n$ is a cone, and there is also a well-defined projective zero locus $Z_p(I^{in})$. By exercise 4.6.8, the exceptional hypersurface of the blowup of an affine variety $X \subset \mathbb{A}^n$ in the origin is precisely $Z_p(I(X)^{in})$. (The proof of this statement is very similar to the computation of $\tilde{C}$ in example 4.3.4.)

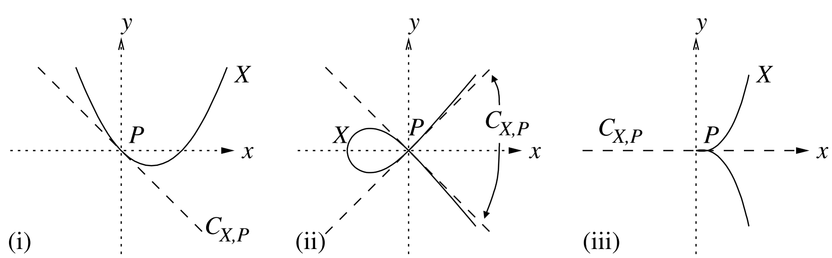
Let us figure out how this can be interpreted geometrically. By construction, $I(X)^{in}$ is obtained from $I(X)$ by only keeping the terms of lowest degree, so it can be interpreted as an "approximation" of $I(X)$ around zero, just in the same way as the Taylor polynomial approximates a function around a given point. Note also that $Z_a(I(X)^{in})$ has the same dimension as X by lemma 4.3.6 (ii). Hence we can regard $Z_a(I(X)^{in}) \subset \mathbb{A}^n$ as *the cone that approximates X best around the point P*. It is called the **tangent cone** of X in P and denoted $C_{X,P}$. The exceptional locus of the blow-up $\tilde{X}$ of X in P is then the "projectivized tangent cone", i. e. it corresponds to "tangent directions" in X through P, just as in example 4.3.4.

Example 4.3.9. Here are some examples of tangent cones.

(i) Let $X = \{(x,y)\ ;\ y = x(x-1)\} \subset \mathbb{A}^2$. The tangent cone of X in $P = (0,0)$ is given by keeping only the linear terms of the equation $y = x(x-1)$, i. e. $C_{X,P} = \{(x,y)\ ;\ y = -x\}$ is the tangent line to X in P. Consequently, the exceptional hypersurface of the blow-up of X in P contains only one point. In fact, $\tilde{X}$ is *isomorphic* to X in this case: note that on X, the ideal of P is just given by the single function x, as $(y-x(x-1),x) = (x,y)$. So we are blowing up at $f_0 = x$ only. It follows then by example 4.3.3 that $\tilde{X} = X$.

(ii) Let $X = \{(x,y)\ ;\ y^2 = x^2 + x^3\} \subset \mathbb{A}^2$. This time there are no linear terms in the equation of X, so the tangent cone in $P = (0,0)$ is given by the quadratic terms $C_{X,P} = \{(x,y)\ ;\ y^2 = x^2\}$, i. e. it is the union of the two tangent lines $y = x$ and $y = -x$ to X in P (see the picture below). The exceptional hypersurface of the blow-up of X in P therefore contains exactly two points, one for every tangent direction in P. In other words, the two local branches of X around P get separated in the blow-up. Note that we cannot apply the argument of (i) here that $\tilde{X}$ should be isomorphic to X: the ideal of P cannot be generated on X by one function only. While it is true that the zero locus of $(x, y^2 - x^2 - x^3)$ is P, the *ideal* $(x, y^2 - x^2 - x^3) = (x, y^2)$ is not equal to $I(P) = (x,y)$ — and this is the important point. In particular, we see that the blow-up of X in an ideal I really does depend on the ideal I and not just on its zero locus, i. e. on the radical of I.

(iii) Let $X = \{(x,y)\ ;\ y^2 = x^3\} \subset \mathbb{A}^2$. This time the tangent cone is $C_{X,P} = \{y^2 = 0\}$, i. e. it is only one line. So for $\tilde{X}$ the point $P \in X$ is replaced by only one single point again, as in (i). But in this case X and $\tilde{X}$ are *not* isomorphic, as we will see in 4.4.7.



Remark 4.3.10. Let X be any variety, and let $Y \subset X$ be a closed subset. For an affine open cover $\{U_i\}$ of X, let $\tilde{U}_i$ be the blow-up of U_i in $U_i \cap Y$. It is then easy to see that the $\tilde{U}_i$ can be glued together to give a blow-up variety $\tilde{X}$.

In what follows, we will only need this in the case of the blow-up of a point, where the construction is even easier as it is local around the blown-up point: let X be a variety, and let $P \in X$ be a point. Choose an affine open neighborhood $U \subset X$ of P, and let $\tilde{U}$ be the blow-up of U in P. Then we obtain $\tilde{X}$ by glueing $X \backslash P$ to $\tilde{U}$ along the common open subset $U \backslash P$. In particular, this defines the tangent cone $C_{X,P}$ to X at P for any variety X: it is the affine cone over the exceptional hypersurface of the blow-up of X in P.

This sort of glueing currently works only for blow-ups at subvarieties, i. e. for blow-ups at radical ideals. For the general construction we would need to patch ideals, which we do not know how to do at the moment.

Note however that it is easy to see that for projective varieties, the blow-up at a homogeneous ideal can be defined in essentially the same way as for affine varieties: let $X \subset \mathbb{P}^n$ be a projective variety, and let $Y \subset X$ be a closed subset. If $f_0, \dots, f_r$ are homogeneous generators of $I(Y)$ of the same degree, the blow-up of X in Y is precisely the closure of

$$\Gamma = \{(P, (f_0(P) : \cdots : f_r(P)) \,;\, P \in U\} \subset X \times \mathbb{P}^r$$

in $X \times \mathbb{P}^r$ (this is easily checked on the affine patches $f_i \neq 0$).

Example 4.3.11. The following property of blow-ups follows trivially from the definitions, yet it is one of their most important properties.

Let $X \subset \mathbb{A}^n$ be an affine variety, and let $f_0, \dots, f_r$ be polynomials that do not vanish identically on X. Note that the morphism $f : P \mapsto (f_0(P) : \cdots : f_r(P))$ to $\mathbb{P}^r$ is only well-defined on the open subset $U = X \backslash Z(f_0, \dots, f_r)$ of X. In general, we can not expect that this morphism can be extended to a morphism on all of X. But we can always extend it "after blowing up the ideal $(f_0, \dots, f_r)$ of the indeterminacy locus", i. e. there is an extension $\tilde{f} : \tilde{X} \to \mathbb{P}^r$ (that agrees with f on the open subset U), namely just the projection from $\tilde{X} \subset X \times \mathbb{P}^r \to \mathbb{P}^r$. So blowing up is a way to extend morphisms to bigger sets on which they would otherwise be ill-defined. The same is true for projective varieties and the construction at the end of remark 4.3.10. Let us consider a concrete example of this idea in the next lemma and the following remark:

Lemma 4.3.12. *$\mathbb{P}^1 \times \mathbb{P}^1$ blown up in one point is isomorphic to $\mathbb{P}^2$ blown up in two points.*

Proof. We know from example 3.3.14 that $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to the quadric surface

$$Q = \{(x_0 : x_1 : x_2 : x_3) \,;\, x_0 x_3 = x_1 x_2\} \subset \mathbb{P}^3.$$

Let $P = (0 : 0 : 0 : 1) \in Q$, and let $\tilde{Q} \subset \mathbb{P}^3 \times \mathbb{P}^2$ be the blow-up of Q in the ideal $I(P) = (x_0, x_1, x_2)$.

On the other hand, let $R_1 = (0 : 1 : 0), R_2 = (0 : 0 : 1) \in \mathbb{P}^2$, and let $\tilde{\mathbb{P}}^2 \subset \mathbb{P}^2 \times \mathbb{P}^3$ be the blow-up of $\mathbb{P}^2$ in the ideal $I = (y_0^2, y_0 y_1, y_0 y_2, y_1 y_2)$. Note that this is not quite the ideal $I(R_1 \cup R_2) = (y_0, y_1 y_2)$, but this does not matter: the blow-up is a local construction, so let us check that we are doing the right thing around R_1. There is an open affine neighborhood around R_1 given by $y_1 \neq 0$, and on this neighborhood the ideal I is just $(y_0^2, y_0, y_0 y_2, y_2) = (y_0, y_2)$, which is precisely the ideal of R_1. The same is true for R_2, so the blow-up of $\mathbb{P}^2$ in I is actually the blow-up of $\mathbb{P}^2$ in the two points R_1 and R_2.

Now we claim that an isomorphism is given by

$$f : \tilde{Q} \mapsto \tilde{\mathbb{P}}^2,\ ((x_0 : x_1 : x_2 : x_3), (y_0 : y_1 : y_2)) \mapsto ((y_0 : y_1 : y_2), (x_0 : x_1 : x_2 : x_3)).$$

In fact, this is easy to check: obviously, f is an isomorphism $\mathbb{P}^2 \times \mathbb{P}^3 \to \mathbb{P}^3 \times \mathbb{P}^2$, so we only have to check that f maps $\tilde{Q}$ to $\tilde{\mathbb{P}}^2$, and that f^{-1} maps $\tilde{\mathbb{P}}^2$ to $\tilde{Q}$. Note that it suffices to check this *away from the blown-up points*: $f^{-1}(\tilde{\mathbb{P}}^2)$ is a closed subset of $\mathbb{P}^3 \times \mathbb{P}^2$, so if it contains a non-empty open subset $U \subset Q$ (e. g. $\tilde{Q}$ minus the exceptional hypersurface), it must contain all of Q.

But this is now easy to check: on $\tilde{Q}$ we have $x_0 x_3 = x_1 x_2$ and $(y_0 : y_1 : y_2) = (x_0 : x_1 : x_2)$ (where this is well-defined), so in the image of f we get the correct equations

$$(x_0 : x_1 : x_2 : x_3) = (x_0^2 : x_0 x_1 : x_0 x_2 : x_0 x_3) = (x_0^2 : x_0 x_1 : x_0 x_2 : x_1 x_2) = (y_0^2 : y_0 y_1 : y_0 y_2 : y_1 y_2)$$

for the image point to lie in $\tilde{\mathbb{P}}^2$. Conversely, on $\tilde{\mathbb{P}}^2$ we have $(x_0 : x_1 : x_2 : x_3) = (y_0^2 : y_0 y_1 : y_0 y_2 : y_1 y_2)$ where defined, so we conclude $x_0 x_3 = x_1 x_2$ and $(y_0 : y_1 : y_2) = (x_0 : x_1 : x_2)$. □

Remark 4.3.13. The proof of lemma 4.3.12 is short and elegant, but not very insightful. Let us try to understand geometrically what is going on.

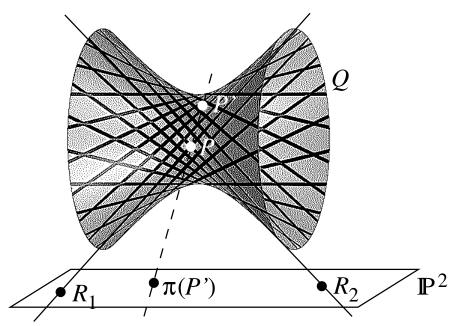
As in the proof, we think of $\mathbb{P}^1 \times \mathbb{P}^1$ as the quadric

$$Q = \{(x_0 : x_1 : x_2 : x_3) \,;\, x_0 x_3 = x_1 x_2\} \subset \mathbb{P}^3.$$

Consider the projection π from P to $\mathbb{P}^2$, given in coordinates by $\pi(x_0 : x_1 : x_2 : x_3) = (x_0 : x_1 : x_2)$. We have considered projections from points before, but so far the projection point P was always assumed not to lie on the given variety Q. This is not the case here, and consequently π is only well-defined on $Q \backslash P$. To construct $\pi(P)$ we would have to take "the line through P and P" and intersect it with a given $\mathbb{P}^2 \subset \mathbb{P}^3$ that does not contain P. Of course this is ill-defined. But there is a well-defined line through P and any point P' near P which we can intersect with $\mathbb{P}^2$. It is obvious that $\pi(P)$ should be the limit of these projection points when P' tends to P. The line $\overline{P'P}$ will then become a tangent line to Q. But Q, being two-dimensional, has a one-parameter family of tangent lines. This is why $\pi(P)$ is ill-defined. But we also see from this discussion that blowing up P on Q, i. e. replacing it by the set of tangent lines through P, will exactly resolve the indeterminacy.

We have thus constructed a morphism $\tilde{Q} = \widetilde{\mathbb{P}^1 \times \mathbb{P}^1} \to \mathbb{P}^2$ by projection from P. If there is an inverse morphism, it is easy to see what it would have to look like: pick a point $R \in \mathbb{P}^2 \subset \mathbb{P}^3$. The points mapped to R by π are exactly those on the line $\overline{PR}$ not equal to P. In general, this line intersects the quadric Q in two points, one of which is P. So there is exactly one point on Q which maps to R. This reasoning is false however if the whole line $\overline{PR} = \mathbb{P}^1$ lies in Q. This whole line would then be mapped to R, so that we cannot have an isomorphism. But of course we expect again that this problem can be taken care of by blowing up R in $\mathbb{P}^2$, so that it is replaced by a $\mathbb{P}^1$ that can then be mapped one-to-one to $\overline{PR}$.

There are obviously two such lines $\overline{PR_1}$ and $\overline{PR_2}$, given by $R_1 = (0:1:0)$ and $R_2 = (0:0:1)$. If you think of Q as $\mathbb{P}^1 \times \mathbb{P}^1$ again, these lines are precisely the "horizontal" and "vertical" lines $\mathbb{P}^1 \times \{\text{point}\}$ and $\{\text{point}\} \times \mathbb{P}^1$ passing through P. So we would expect that $\tilde{\pi}$ can be made into an isomorphism after blowing up R_1 and R_2, which is what we have shown in lemma 4.3.12.



4.4. **Smooth varieties.** Let $X \subset \mathbb{A}^n$ be an affine variety, and let $P \in X$ be a point. By a change of coordinates let us assume that $P = (0, \dots, 0)$ is the origin. In remark 4.3.8 we have defined the tangent cone of X in P to be the closed subset of $\mathbb{A}^n$ given by the initial ideal of X, i. e. the "local approximation" of X around P given by keeping only the terms of the defining equations of X of minimal degree. Let us now make a similar definition, but where we only keep the *linear* terms of the defining equations.

Definition 4.4.1. For any polynomial $f \in k[x_1, \dots, x_n]$ denote by $f^{(1)}$ the linear part of f. For an ideal $I \subset k[x_1, \dots, x_n]$ denote by $I^{(1)} = \{f^{(1)} \, ; \, f \in I\}$ the vector space of all linear parts of the elements of I; this is by definition a vector subspace of the n-dimensional space $k[x_1, \dots, x_n]^{(1)}$ of all linear forms

$$\{a_1 x_1 + \dots + a_n x_n \, ; \, a_i \in k\}.$$

The zero locus $Z(I^{(1)})$ is then a linear subspace of $\mathbb{A}^n$. It is canonically dual (as a vector space) to $k[x_1,\dots,x_n]^{(1)}/I^{(1)}$, since the pairing

$$k[x_1,\dots,x_n]^{(1)}/I^{(1)} \times Z(I^{(1)}) \to k, \qquad (f,P) \mapsto f(P)$$

is obviously non-degenerate.

Now let $X \subset \mathbb{A}^n$ be a variety. By a linear change of coordinates, assume that $P = (0,\dots,0) \in X$. Then the linear space $Z(I(X)^{(1)})$ is called the **tangent space** to X at P and denoted $T_{X,P}$.

Remark 4.4.2. Let us make explicit the linear change of coordinates mentioned in the definition. If $P = (a_1,\dots,a_n) \in X$, we need to change coordinates from the x_i to $y_i = x_i - a_i$. By a (purely formal) Taylor expansion we can rewrite any polynomial $f \in k[x_1,\dots,x_n]$ as

$$f(x_1,\dots,x_n) = f(P) + \sum_i \frac{\partial f}{\partial x_i}(P)\cdot y_i + (\text{terms at least quadratic in the } y_i),$$

so we see that the tangent space $T_{X,P}$ to any point $P = (a_1,\dots,a_n) \in X$ is given by the equations

$$\sum_i \frac{\partial f}{\partial x_i}(P)\cdot (x_i - a_i) = 0$$

for all $f \in I(X)$.

Here is an alternative description of the tangent space. For simplicity, we will assume again that the coordinates have been chosen such that $P = (0,\dots,0)$.

Lemma 4.4.3. *Let $X \subset \mathbb{A}^n$ be a variety, and assume that $P = (0,\dots,0) \in X$. Then*

$$k[x_1,\dots,x_n]^{(1)}/I(X)^{(1)} = M/M^2,$$

where $M = \{\varphi\,;\ \varphi(P) = 0\} \subset \mathscr{O}_{X,P}$ is the maximal ideal in the local ring of X at P.

Proof. Recall that

$$\mathscr{O}_{X,P} = \Big\{ \frac{f}{g}\,;\ f,g \in A(X), g(P) \neq 0 \Big\},$$

and therefore

$$M = \Big\{ \frac{f}{g}\,;\ f,g \in A(X), f(P) = 0, g(P) \neq 0 \Big\}.$$

There is an obvious homomorphism $k[x_1,\dots,x_n]^{(1)}/I(X)^{(1)} \to M/M^2$ of k-vector spaces. We will show that it is bijective.

Injectivity: Let $f \in k[x_1,\dots,x_n]^{(1)}$ be a linear function. Then $\frac{f}{1}$ is zero in $\mathscr{O}_{X,P}$ if and only if it is zero in $A(X)$, i. e. if and only if $f \in I(X)$.

Surjectivity: Let $\varphi = \frac{f}{g} \in M$. Without loss of generality we can assume that $g(P) = 1$. Set

$$\varphi' = \sum_i \frac{\partial \varphi}{\partial x_i}(P)\cdot x_i,$$

which is obviously an element of $k[x_1,\dots,x_n]^{(1)}$. We claim that $\varphi-\varphi' \in M^2$. In fact,

$$\begin{aligned}
g(\varphi-\varphi') &= f - g\sum_i \frac{\frac{\partial f}{\partial x_i}(P)g(P)-\frac{\partial g}{\partial x_i}(P)f(P)}{g(P)^2}\, x_i \\
&= f - g\sum_i \frac{\partial f}{\partial x_i}(P)\, x_i \\
&\equiv f - g(P)\sum_i \frac{\partial f}{\partial x_i}(P)\, x_i \qquad (\text{mod } M^2) \qquad (\text{as } g-g(P) \text{ and } x_i \text{ are in } M) \\
&= f - \sum_i \frac{\partial f}{\partial x_i}(P)\, x_i \\
&\equiv 0 \qquad (\text{mod } M^2) \qquad (\text{as this is the linear Taylor expression for } f).
\end{aligned}$$

So $\varphi = \varphi'$ in M/M^2. □

Remark 4.4.4. In particular, this lemma gives us a more intrinsic definition of the tangent space $T_{X,P}$: we can say that $T_{X,P}$ is the dual of the k-vector space M/M^2, where M is the maximal ideal in the local ring $\mathcal{O}_{X,P}$. This alternative definition shows that the tangent space $T_{X,P}$ (as an abstract vector space) is independent of the chosen embedding of X in affine space. It also allows us to define the tangent space $T_{X,P}$ for any variety X (that is not necessarily affine).

Let us now compare tangent spaces to tangent cones.

Remark 4.4.5. Let X be an affine variety, and assume for simplicity that $P=(0,\dots,0)\in X$. For all polynomials $f\in k[x_1,\dots,x_n]$ vanishing at P, linear terms are always initial. Hence the ideal generated by $I(X)^{(1)}$ is contained in the ideal $I(X)^{in}$ defining the tangent cone (see remark 4.3.8). So the tangent cone $C_{X,P}\subset \mathbb{A}^n$ is contained in the tangent space $T_{X,P}\subset\mathbb{A}^n$. In particular, we always have $\dim T_{X,P}\geq \dim C_{X,P}=\dim X$. Summarizing, we can say that, in studying the local properties of X around P, the tangent cone has the advantage that it always has the "correct" dimension $\dim X$, whereas the tangent space has the advantage that it is always a linear space. We should give special attention to those cases when both notions agree, i. e. when X "can be approximated linearly" around P.

Definition 4.4.6. A variety X is called **smooth** at the point $P\in X$ if $T_{X,P}=C_{X,P}$, or equivalently, if the tangent space $T_{X,P}$ to X at P has dimension (at most) $\dim X$. It is called **singular** at P otherwise. We say that X is smooth if it is smooth at all points $P\in X$; otherwise X is singular.

Example 4.4.7. Consider again the curves of example 4.3.9:

(i) $X=\{y=x(x-1)\}\subset\mathbb{A}^2$,
(ii) $X=\{y^2=x^2+x^3\}\subset\mathbb{A}^2$,
(iii) $X=\{y^2=x^3\}\subset\mathbb{A}^2$.

In case (i), the tangent space is $\{y=-x\}\subset\mathbb{A}^2$ and coincides with the tangent cone: X is smooth at $P=(0,0)$. In the cases (ii) and (iii), there are no linear terms in the defining equations of X. So the tangent space of X at P is all of $\mathbb{A}^2$, whereas the tangent cone is one-dimensional. Hence in these cases X is singular at P.

In case (iii) let us now consider the blow-up of X in $P=(0,0)$. Let us first blow up the ambient space $\mathbb{A}^2$ in P; we know already that this is given by

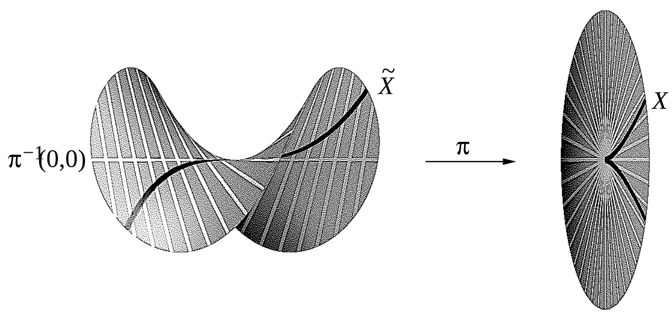
$$\tilde{\mathbb{A}}^2=\{((x,y),(x':y'))\ ;\ xy'=x'y\}\subset\mathbb{A}^2\times\mathbb{P}^1.$$

So local affine coordinates of $\tilde{\mathbb{A}}^2$ around the point $((0,0),(1:0))$ are $(u,v)\in\mathbb{A}^2$, where

$$u=\frac{y'}{x'} \quad\text{and}\quad v=x$$

so that $((x,y),(x':y')) = ((v,uv),(1:u))$. In these local coordinates, the equation $y^2 = x^3$ of the curve X is given by $(uv)^2 = v^3$. The exceptional hypersurface has the local equation $v = 0$, so away from this hypersurface the curve X is given by the equation $v = u^2$. By definition, this is then also the equation of the blow-up $\tilde{X}$.

So we conclude first of all that *the blow-up $\tilde{X}$ is smooth*, although X was not. We say that the singularity $P \in X$ got "resolved" by blowing up. We can also see that the blow-up of the curve (with local equation $v = u^2$) is tangent to the exceptional hypersurface (with local equation $v = 0$). All this is illustrated in the following picture (the blow-up of $\mathbb{A}^2$ is the same as in example 4.3.4):



It can in fact be shown that every singularity can be "resolved" in a similar way by successively blowing up the singular locus.

The good thing about smoothness is that is very easy to check:

Proposition 4.4.8.

(i) ***(Affine Jacobi criterion)*** *Let $X \subset \mathbb{A}^n$ be an affine variety with ideal $I(X) = (f_1,\dots,f_r)$, and let $P \in X$ be a point on X. Then X is smooth at P if and only if the rank of the $r \times n$ "Jacobi matrix" $\left(\frac{\partial f_i}{\partial x_j}(P)\right)$ is (at least) $n - \dim X$.*

(ii) ***(Projective Jacobi criterion)*** *Let $X \subset \mathbb{P}^n$ be a projective variety with ideal $I(X) = (f_1,\dots,f_r)$, and let $P \in X$ be a point on X. Then X is smooth at P if and only if the rank of the $r \times n$ Jacobi matrix $\left(\frac{\partial f_i}{\partial x_j}(P)\right)$ is (at least) $n - \dim X$.*

In particular, if the rank is r (the number of functions) then X is smooth of dimension $n - r$.

Proof. (i): By remark 4.4.2, the linearization of the functions f_i around the point $P = (a_1,\dots,a_n)$ is given by $\sum_j \frac{\partial f_i}{\partial x_j}(P) \cdot (x_i - a_i)$. By definition, X is smooth at P if these functions define a linear subspace of $\mathbb{A}^n$ of dimension (at most) $\dim X$, i. e. if and only if the linear subspace of $k[x_1,\dots,x_n]^{(1)}$ spanned by the above linearizations has dimension (at least) $n - \dim X$. But the dimension of this linear space is exactly the rank of the matrix whose entries are the coefficients of the various linear function.

(ii): This follows easily by covering the projective space $\mathbb{P}^n$ by the $n+1$ affine spaces $\{x_i \neq 0\} \cong \mathbb{A}^n$, and applying the criterion of (i) to these $n+1$ patches. □

Remark 4.4.9. Note that a matrix has rank less than k if and only if all $k \times k$ minors are zero. These minors are all polynomials in the entries of the matrix. In particular, the locus of singular points, i. e. where the Jacobi matrix has rank less than $n - \dim X$ as in the proposition, is closed.

It follows that the set

$$\{P \in X \ ; \ X \text{ is singular at } P\} \subset X$$

is closed. In other words, the set of smooth points of a variety is always open. One can show that the set of smooth points is also non-empty for every variety (see e. g. [H] theorem I.5.3). Hence the set of smooth points is always dense.

Example 4.4.10.

(i) For given n and d, let X be the so-called *Fermat hypersurface*
$$X = \{(x_0 : \cdots : x_n) \; ; \; x_0^d + \cdots + x_n^d = 0\}.$$
Then the Jacobi matrix has only one row, and the entries of this row are $d\, x_i^{d-1}$ for $i = 0, \dots, n$. Assuming that the characteristic of the ground field is zero (or at least not a divisor of d), it follows that at least one of the entries of this matrix is non-zero at every point. In other words, the rank of the Jacobi matrix is always 1. Therefore X is smooth by proposition 4.4.8.

(ii) Let X be the "twisted cubic curve" of exercise 3.5.2
$$X = \{(s^3 : s^2t : st^2 : t^3) \; ; \; (s:t) \in \mathbb{P}^1\}.$$
We have seen earlier that X can be given by the equations
$$X = \{(x_0 : x_1 : x_2 : x_3) \; ; \; x_1^2 - x_0x_2 = x_2^2 - x_1x_3 = x_0x_3 - x_1x_2 = 0\}.$$
So the Jacobi matrix is given by
$$\begin{pmatrix} -x_2 & 2x_1 & -x_0 & 0 \\ 0 & -x_3 & 2x_2 & -x_1 \\ x_3 & -x_2 & -x_1 & x_0 \end{pmatrix}.$$
By proposition 4.4.8, X is smooth if and only if the rank of this matrix is 2. (We know already that the rank cannot be bigger than 2, which is also easily checked directly).

The 2×2 minor given by the last two rows and the first two columns is x_3^2. The 2×2 minor given by last two rows and the first and last column is $x_1x_3 = x_2^2$. Similarly we find 2×2 minors that are x_1^2 and x_0^2. These cannot all be simultaneously zero; hence X is smooth. (Of course we have known this before, since X is just the degree-3 Veronese embedding of $\mathbb{P}^1$ (see example 3.4.11. In particular, X is isomorphic to $\mathbb{P}^1$ and therefore smooth.)

Remark 4.4.11. The Jacobi criterion of proposition 4.4.8 gives us a direct connection to complex analysis. Assume that we are given r holomorphic functions on $\mathbb{C}^n$ (e. g. polynomials), and that the matrix of the derivatives of the f_i has rank $n - \dim X$ at a point P, where X is the zero locus of the f_i. Assume for simplicity that the square matrix $\left(\frac{\partial f_i}{\partial x_j}(P)\right)_{1 \le i \le n - \dim X, \dim X < j \le n}$ of size $n - \dim X$ is invertible. Then the inverse function theorem states that the coordinates $x_{\dim X+1}, \dots, x_n$ are locally around P determined by the other coordinates $x_1, \dots, x_{\dim X}$. Thus there is a neighborhood U of P in $\mathbb{C}^n$ (in the classical topology!) and holomorphic functions $g_{\dim X+1}, \dots, g_n$ of $x_1, \dots, x_{\dim X}$ such that for every $P = (x_1, \dots, x_{\dim X}) \in U$ the functions f_i vanish at P if and only if $x_i = g_i(x_1, \dots, x_{\dim X})$ for $i = \dim X + 1, \dots, n$.

So the zero locus of the f_i is "locally the graph of a holomorphic map" given by the g_i. In other words, smoothness in algebraic geometry means in a sense the same thing as differentiability in analysis: the geometric object has "no edges".

Note however that the inverse function theorem is not true in the Zariski topology, because the open sets are too big. For example, consider the curve $X = \{(x,y) \; ; \; f(x,y) = y - x^2 = 0\} \subset \mathbb{C}^2$. Then $\frac{\partial f}{\partial x} \neq 0$ say at the point $P = (1,1) \in X$. Consequently, in complex analysis x can be expressed locally in terms of y around P: it is just the square root of y. But any non-empty Zariski open subset of X will contain pairs of points (x, x^2) and $(-x, x^2)$ for some x, so the inverse function theorem cannot hold here in algebraic geometry.

4.5. **The 27 lines on a smooth cubic surface.** As an application of the theory that we have developed so far, we now want to study lines on cubic surfaces in $\mathbb{P}^3$. We have already mentioned in example 0.1.7 that every smooth cubic surface has exactly 27 lines on it. We now want to show this. We also want to study the configuration of these lines, and show that every smooth cubic surface is birational to $\mathbb{P}^2$.

The results of this section will not be needed later on. Therefore we will not give all the proofs in every detail here. The goal of this section is rather to give an idea of what can be done with our current methods.

First let us recall some notation from exercise 3.5.4. Let $G = G(1,3)$ be the Grassmannian variety of lines in $\mathbb{P}^3$. This is a 4-dimensional projective variety. In this section we will use local affine coordinates on G: if $L_0 \in G$ is the line in $\mathbb{P}^3$ (with coordinates $x_0,\dots,x_3$) given by the equations $x_2 = x_3 = 0$ (of course every line is of this form after a linear change of coordinates), then there is an open neighborhood $\mathbb{A}^4 \subset G$ of L_0 in G given by sending a point $(a,b) := (a_2,b_2,a_3,b_3) \in \mathbb{A}^4$ to the line through the points $(1,0,a_2,a_3)$ and $(0,1,b_2,b_3)$.

The cubic surfaces in $\mathbb{P}^3$ are parametrized by homogeneous polynomials of degree 3 in x_0,x_1,x_2,x_3 up to scalars, which is a 19-dimensional projective space $\mathbb{P}^{19}$. A cubic surface given by the equation $f_c := \sum_\alpha c_\alpha x^\alpha = 0$ (in multi-index notation, so α runs over all quadruples of indices $(\alpha_0,\alpha_1,\alpha_2,\alpha_3)$ with $\alpha_i \geq 0$ and $\sum_i \alpha_i = 3$) corresponds to the point in $\mathbb{P}^{19}$ with homogeneous coordinates $c = (c_\alpha)$. We denote the corresponding cubic surface by $X_c = \{f_c = 0\}$.

To study lines in cubic surfaces, we consider the so-called *incidence correspondence*

$$M := \{(L,X)\ ;\ L \subset X\} \subset G \times \mathbb{P}^{19}$$

consisting of all pairs of a line and a cubic such that the line lies in the cubic. Let us start by proving some facts about this incidence correspondence.

Lemma 4.5.1. *With the above notation, the incidence correspondence M has an open cover by affine spaces $\mathbb{A}^{19}$. In particular, M is a smooth 19-dimensional variety.*

Proof. In the coordinates $(a,b,c) = (a_2,a_3,b_2,b_3,c_\alpha)$ as above, the incidence correspondence M is given by the equations

$$\begin{aligned}
(a,b,c) \in M &\iff s\,(1,0,a_2,a_3) + t\,(0,1,b_2,b_3) \in X_c \text{ for all } s,t \\
&\iff \sum_\alpha c_\alpha s^{\alpha_0} t^{\alpha_1} (s\,a_2 + t\,b_2)^{\alpha_2} (s\,a_3 + t\,b_3)^{\alpha_3} = 0 \text{ for all } s,t \\
&\iff: \sum_i s^i t^{3-i} F_i(a,b,c) = 0 \text{ for all } s,t \\
&\iff F_i(a,b,c) = 0 \text{ for } 0 \leq i \leq 3.
\end{aligned}$$

Note that the F_i are linear in the c_α. Moreover, $c_{i,3-i,0,0}$ occurs only in F_i for $i = 0,\dots,3$, and it occurs there with coefficient 1. So these equations can be written as $c_{i,3-i,0,0} = G_i(a,b,c)$ for $i = 0,\dots,3$, where the G_i depend only on those c_α where $\alpha_2 > 0$ or $\alpha_3 > 0$. Therefore the variety $\mathbb{A}^4 \times \mathbb{P}^{15}$ (with coordinates a_2,a_3,b_2,b_3, and all c_α with $\alpha_2 > 0$ or $\alpha_3 > 0$) is isomorphic to an open subvariety of M, with the isomorphism given by the equations $c_{i,3-i,0,0} = G(a,b,c)$. It follows that M has an open cover by affine spaces $\mathbb{A}^4 \times \mathbb{A}^{15} = \mathbb{A}^{19}$. $\square$

Lemma 4.5.2. *Again with notations as above, let $(a,b,c) \in M$ be a point such that the corresponding cubic surface X_c is smooth. Then the 4×4 matrix $\frac{\partial(F_0,F_1,F_2,F_3)}{\partial(a_2,a_3,b_2,b_3)}$ is invertible.*

Proof. After a change of coordinates we can assume for simplicity that $a = b = 0$. Then

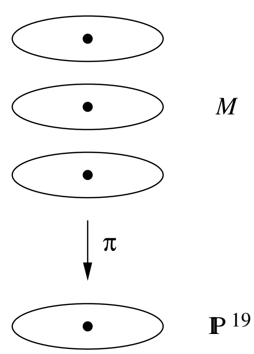
$$\frac{\partial}{\partial a_2}(\sum_i s^i t^{3-i} F_i)|_{(0,0,c)} = \frac{\partial}{\partial a_2} f_c(s,t,sa_2+tb_2,sa_3+tb_3)|_{(0,0,c)}$$
$$= s\,\frac{\partial f_c}{\partial x_2}(s,t,0,0).$$

The (s,t)-coefficients of this polynomial are the first row in the matrix $\frac{\partial F_i}{\partial(a,b)}(0,0,c)$. The other rows are obviously $s\,\frac{\partial f_c}{\partial x_3}(s,t,0,0)$, $t\,\frac{\partial f_c}{\partial x_2}(s,t,0,0)$, and $t\,\frac{\partial f_c}{\partial x_3}(s,t,0,0)$. So if the matrix $\frac{\partial F_i}{\partial(a,b)}(0,0,c)$ were not invertible, there would be a relation

$$(\lambda_2 s + \mu_2 t)\frac{\partial f_c}{\partial x_2}(s,t,0,0) + (\lambda_3 s + \mu_3 t)\frac{\partial f_c}{\partial x_3}(s,t,0,0) = 0$$

identically in s,t, with $(\lambda_2,\mu_2,\lambda_3,\mu_3) \neq (0,0,0,0)$. But this means that $\frac{\partial f_c}{\partial x_2}(s,t,0,0)$ and $\frac{\partial f_c}{\partial x_3}(s,t,0,0)$ have a common linear factor, i. e. there is a point $P = (x_0,x_1,0,0) \in \mathbb{P}^3$ such that $\frac{\partial f_c}{\partial x_2}(P) = \frac{\partial f_c}{\partial x_3}(P) = 0$. But as the line L_0 lies in the cubic f_c, we must have $f_c = x_2 \cdot g_2(x_0,x_1,x_2,x_3) + x_3 \cdot g_3(x_0,x_1,x_2,x_3)$ for some g_2, g_3. Hence $\frac{\partial f_c}{\partial x_0}(P) = \frac{\partial f_c}{\partial x_1}(P) = 0$ also, which means that P is a singular point of the cubic X_c. This is a contradiction to our assumptions. □

Remark 4.5.3. By remark 4.4.11, lemma 4.5.2 means that locally (in the classical topology) around any point $(a,b,c) \in M$ such that X_c is smooth, the coordinates a_2, a_3, b_2, b_3 are determined uniquely in M by the c_α. In other words, the projection map $\pi : M \to \mathbb{P}^{19}$ is a *local isomorphism* (again in the classical topology!) around such a point $(a,b,c) \in M$. So the local picture looks as follows:



As the number of lines in a given cubic X_c is just the number of inverse image points of $c \in \mathbb{P}^{19}$ under this projection map, it follows that *the number of lines on a smooth cubic surface is independent of the particular cubic chosen*.

Theorem 4.5.4. *Every smooth cubic surface $X \subset \mathbb{P}^3$ contains exactly 27 lines.*

Proof. We have just argued that the number of lines on a smooth cubic surface does not depend on the surface, so we can pick a special one. We take the surface X given by the equation $f = x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$ (which is smooth in characteristic not equal to 3). Up to a permutation of coordinates, every line in $\mathbb{P}^3$ can be written $x_0 = a_2x_2 + a_3x_3$,

$x_1 = b_2x_2 + b_3x_3$. Substituting this in the equation f yields the conditions

$$a_2^3 + b_2^3 = -1, \tag{1}$$

$$a_3^3 + b_3^3 = -1, \tag{2}$$

$$a_2^2 a_3 = -b_2^2 b_3, \tag{3}$$

$$a_2 a_3^2 = -b_2 b_3^2. \tag{4}$$

Assume that a_2, a_3, b_2, b_3 are all non-zero. Then $(3)^2/(4)$ gives $a_2^3 = -b_2^3$, while $(4)^2/(3)$ yields $a_3^3 = -b_3^3$. This is obviously a contradiction to (1) and (2). Hence at least one of the a_2, a_3, b_2, b_3 must be zero. Assume without loss of generality that $a_2 = 0$. Then $b_3 = 0$ and $a_3^3 = b_2^3 = -1$. This gives 9 lines by setting $a_3 = -\omega^i$ and $b_2 = -\omega^j$ for $0 \le i, j \le 2$ and ω a third root of unity. So by allowing permutations of the coordinates we find that there are exactly the following 27 lines on X:

$$\begin{aligned} x_0 + x_1\omega^i &= x_2 + x_3\omega^j = 0, \quad 0 \le i, j \le 2, \\ x_0 + x_2\omega^i &= x_1 + x_3\omega^j = 0, \quad 0 \le i, j \le 2, \\ x_0 + x_3\omega^i &= x_1 + x_2\omega^j = 0, \quad 0 \le i, j \le 2. \end{aligned}$$

□

Remark 4.5.5. We will now study to a certain extent the *configuration* of the 27 lines on a cubic surface, i. e. determine which of the lines intersect. Consider the special cubic X of the proof of theorem 4.5.4, and let L be the line

$$L = \{x_0 + x_1 = x_2 + x_3 = 0\}$$

in X. Then we can easily check that L meets exactly 10 of the other lines in X, namely

$$\begin{gathered} x_0 + x_1\omega^i = x_2 + x_3\omega^j = 0, \quad (i, j) \neq (0,0) \\ x_0 + x_2 = x_1 + x_3 = 0, \\ x_0 + x_3 = x_1 + x_2 = 0. \end{gathered}$$

The same is true for every other line in X. In fact, the statement is also true for every smooth cubic surface, and not just for the special one that we have just considered. The proof of this is very similar to the proof above that the number of lines on a smooth cubic surface does not depend on the particular cubic chosen.

Now let L_1 and L_2 be two disjoint lines on a smooth cubic surface X. We claim that there are exactly 5 lines on X that intersect both L_1 and L_2. To show this, one can proceed in the same way as above: check the statement directly on a special cubic surface, and then show that it must then be true for all other smooth cubic surfaces as well.

Proposition 4.5.6. *Any smooth cubic surface in $\mathbb{P}^3$ is birational to $\mathbb{P}^2$.*

Proof. By remark 4.5.5 there are two disjoint lines $L_1, L_2 \subset X$. The following mutually inverse rational maps $X \dashrightarrow L_1 \times L_2$ and $L_1 \times L_2 \dashrightarrow X$ show that X is birational to $\mathbb{P}^1 \times \mathbb{P}^1$ and hence to $\mathbb{P}^2$:

"$X \dashrightarrow L_1 \times L_2$": By exercise 3.5.1, for every point P not on L_1 or L_2 there is a unique line $L(P)$ in $\mathbb{P}^3$ through L_1, L_2 and P. Take the rational map $P \mapsto (L_1 \cap L(P), L_2 \cap L(P))$ that is obviously well-defined away from $L_1 \cup L_2$.

"$L_1 \times L_2 \dashrightarrow X$": Map any pair of points $(P, Q) \in L_1 \times L_2$ to the third intersection point of X with the line $\overline{PQ}$. This is well-defined whenever $\overline{PQ}$ is not contained in X. □

Proposition 4.5.7. *Any smooth cubic surface in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in 5 (suitably chosen) points, or equivalently, to $\mathbb{P}^2$ blown up in 6 (suitably chosen) points.*

Proof. We will only sketch the proof. Let X be a smooth cubic surface, and let $f : X \dashrightarrow L_1 \times L_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ be the rational map as in the proof of proposition 4.5.6.

First of all we claim that f is actually a morphism. To see this, note that there is a different description for f: if $P \in X \backslash L_1$, let H be the unique plane in $\mathbb{P}^3$ that contains L_1 and P, and let $f_2(P) = H \cap L_2$. If one defines $f_1(P)$ similarly, then $f(P) = (f_1(P), f_2(P))$. Now if the point P lies on L_1, let H be the tangent plane to X at P, and again let $f_2(P) = H \cap L_2$. Extending f_1 similarly, one can show that this extends $f = (f_1, f_2)$ to a well-defined morphism $X \to \mathbb{P}^1 \times \mathbb{P}^1$ on all of X.

Now let us investigate where the inverse map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow X$ is not well-defined. As already mentioned in the proof of proposition 4.5.6, this is the case if the point $(P,Q) \in L_1 \times L_2$ is such that $\overline{PQ} \subset X$. In this case, the whole line $\overline{PQ} \cong \mathbb{P}^1$ will be mapped to (P,Q) by f, and it can be checked that f is actually locally the blow-up of this point. By remark 4.5.5 there are exactly 5 such lines $\overline{PQ}$ on X. Hence f is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at 5 points.

By lemma 4.3.12 it then follows that f is also the blow-up of $\mathbb{P}^2$ in 6 suitably chosen points. □

Remark 4.5.8. It is interesting to see the 27 lines on a cubic surface X in the picture where one thinks of X as a blow-up of $\mathbb{P}^2$ in 6 points. It turns out that the 27 lines correspond to the following curves that we all already know (and that are all isomorphic to $\mathbb{P}^1$):

- the 6 exceptional hypersurfaces,
- the strict transforms of the $\binom{6}{2} = 15$ lines through two of the blown-up points,
- the strict transforms of the $\binom{6}{5} = 6$ conics through five of the blown-up points (see exercise 3.5.8).

In fact, it is easy to see by the above explicit description of the isomorphism of X with the blow-up of $\mathbb{P}^2$ that these curves on the blow-up actually correspond to lines on the cubic surface.

It is also interesting to see again in this picture that every such “line” meets 10 of the other “lines”, as mentioned in remark 4.5.5:

- Every exceptional hypersurface intersects the 5 lines and the 5 conics that pass through this blown-up point.
- Every line through two of the blown-up points meets
  - the 2 exceptional hypersurfaces of the blown-up points,
  - the $\binom{4}{2} = 6$ lines through two of the four remaining points,
  - the 2 conics through the four remaining points and one of the blown-up points.
- Every conic through five of the blown-up points meets the 5 exceptional hypersurfaces at these points, as well as the 5 lines through one of these five points and the remaining point.

4.6. **Exercises.**

Exercise 4.6.1. Let $X,Y \subset \mathbb{P}^n$ be projective varieties. Show that $X \cap Y$ is not empty if $\dim X + \dim Y \geq n$.

On the other hand, give an example of a projective variety Z and closed subsets $X,Y \subset Z$ with $\dim X + \dim Y \geq \dim Z$ and $X \cap Y = \emptyset$.

(Hint: Let H_1, H_2 be two disjoint linear subspaces of dimension n in $\mathbb{P}^{2n+1}$, and consider $X \subset \mathbb{P}^n \cong H_1 \subset \mathbb{P}^{2n+1}$ and $Y \subset \mathbb{P}^n \cong H_2 \subset \mathbb{P}^{2n+1}$ as subvarieties of $\mathbb{P}^{2n+1}$. Show that the join $J(X,Y) \subset \mathbb{P}^{2n+1}$ of exercise 3.5.7 has dimension $\dim X + \dim Y + 1$. Then construct $X \cap Y$ as a suitable intersection of $J(X,Y)$ with $n+1$ hyperplanes.)

Exercise 4.6.2. (This is a generalization of corollary 4.2.7). Let $f : X \to Y$ be a morphism of varieties. Show that there is a non-empty open subset U of Y such that every component of the fiber $f^{-1}(P)$ has dimension $\dim X - \dim Y$ for all $P \in U$.

(Hint: You can assume $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ to be affine. By considering the graph $(P, f(P)) \in \mathbb{A}^{n+m}$, reduce to the case where $f : \mathbb{A}^{n+1} \to \mathbb{A}^n$ is the projection map.)

Exercise 4.6.3. Let $f : X \to Y$ be a morphism of varieties, and let $Z \subset X$ be a closed subset. Assume that $f^{-1}(P) \cap Z$ is irreducible and of the same dimension for all $P \in Y$. Use exercise 4.6.2 to prove that then Z is irreducible too. (This is a quite useful criterion to check the irreducibility of closed subsets.)

Show by example that the conclusion is in general false if the $f^{-1}(P) \cap Z$ are irreducible but not all of the same dimension.

Exercise 4.6.4. Let X be a variety, and let $Y \subset X$ a closed subset. For every element in an open affine cover $\{U_i\}$ of X, let $V_i = U_i \cap Y$, and let $\tilde{U}_i$ be the blow-up of U_i at V_i. Show that the spaces $\tilde{U}_i$ can be glued together to give a variety $\tilde{X}$. (This variety is then called the blow-up of X at Y.)

Exercise 4.6.5. A quadric in $\mathbb{P}^n$ is a projective variety in $\mathbb{P}^n$ that can be given as the zero locus of a quadratic polynomial. Show that every quadric in $\mathbb{P}^n$ is birational to $\mathbb{P}^{n-1}$.

Exercise 4.6.6. Show that for four *general* lines $L_1, \dots, L_4 \subset \mathbb{P}^3$, there are exactly two lines in $\mathbb{P}^3$ intersecting all the L_i. (This means: the subset of $G(1,3)^4$ of all $(L_1, \dots, L_4)$ such that there are exactly two lines in $\mathbb{P}^3$ intersecting $L_1, \dots, L_4$ is dense. You may want to use the result of exercise 3.5.4 (iii) that $G(1,3)$ is a quadric in $\mathbb{P}^5$.)

Exercise 4.6.7. Let $P_1 = (1:0:0), P_2 = (0:1:0), P_3 = (0:0:1) \in \mathbb{P}^2$, and let $U = \mathbb{P}^2 \backslash \{P_1, P_2, P_3\}$. Consider the morphism

$$f : U \mapsto \mathbb{P}^2, (a_0 : a_1 : a_2) \mapsto (a_1 a_2 : a_0 a_2 : a_0 a_1).$$

(i) Show that there is no morphism $F : \mathbb{P}^2 \to \mathbb{P}^2$ extending f.
(ii) Let $\tilde{\mathbb{P}}^2$ be the blow-up of $\mathbb{P}^2$ in the three points P_1, P_2, P_3. Show that there is an *isomorphism* $\tilde{f} : \tilde{\mathbb{P}}^2 \to \tilde{\mathbb{P}}^2$ extending f. This is called the **Cremona transformation**.

Exercise 4.6.8. Let $X \subset \mathbb{A}^n$ be an affine variety. For every $f \in k[x_0, \dots, x_n]$ denote by f^{in} the *initial terms* of f, i. e. the terms of f of the lowest occurring degree (e. g. if $f = x_2^2 + 3x_1x_3 - x_2x_3^2$ then the lowest occurring degree in f is 2, so the initial terms are the terms of degree 2, namely $f^{in} = x_2^2 + 3x_1x_3$). Let $I(X)^{in} = \{f^{in} \,;\, f \in I(X)\}$ be the ideal of the initial terms in $I(X)$.

Now let $\pi : \tilde{X} \to X$ be the blow-up of X in the origin $\{0\} = Z(x_1, \dots, x_n)$. Show that the exceptional hypersurface $\pi^{-1}(0) \subset \mathbb{P}^n$ is precisely the projective zero locus of the homogeneous ideal $I(X)^{in}$.

Exercise 4.6.9. Let $X \subset \mathbb{A}^n$ be an affine variety, and let $P \in X$ be a point. Show that the coordinate ring $A(C_{X,P})$ of the tangent cone to X at P is equal to $\oplus_{k \geq 0} I(P)^k / I(P)^{k+1}$, where $I(P)$ is the ideal of P in $A(X)$.

Exercise 4.6.10. Let $X \subset \mathbb{A}^n$ be an affine variety, and let $Y_1, Y_2 \subsetneq X$ be irreducible, closed subsets, no-one contained in the other. Let $\tilde{X}$ be the blow-up of X at the (possibly non-radical, see exercise 1.4.1) ideal $I(Y_1) + I(Y_2)$. Then the strict transforms of Y_1 and Y_2 on $\tilde{X}$ are disjoint.

Exercise 4.6.11. Let $C \subset \mathbb{P}^2$ be a smooth curve, given as the zero locus of a homogeneous polynomial $f \in k[x_0, x_1, x_2]$. Consider the morphism

$$\varphi_C : C \to \mathbb{P}^2,\ P \mapsto \Big(\frac{\partial f}{\partial x_0}(P) : \frac{\partial f}{\partial x_1}(P) : \frac{\partial f}{\partial x_2}(P) \Big).$$

The image $\varphi_C(C) \subset \mathbb{P}^2$ is called the **dual curve** to C.

(i) Find a geometric description of φ. What does it mean geometrically if $\varphi(P) = \varphi(Q)$ for two distinct points $P, Q \in C$?
(ii) If C is a conic, prove that its dual $\varphi(C)$ is also a conic.
(iii) For any five lines in $\mathbb{P}^2$ in general position (what does this mean?) show that there is a unique conic in $\mathbb{P}^2$ that is tangent to these five lines. (Hint: Use exercise 3.5.8.)

Exercise 4.6.12. Resolve the singularities of the following curves by subsequent blow-ups of the singular points. This means: starting with the given curve C, blow up all singular points of C, and replace C by its strict transform. Continue this process until the resulting curve is smooth.

Also, describe the singularities that occur in the intermediate steps of the resolution process.

(i) $C = \{(x,y) \,;\, x^2 - x^4 - y^4 = 0\} \subset \mathbb{A}^2$,
(ii) $C = \{(x,y) \,;\, y^3 - x^5 = 0\} \subset \mathbb{A}^2$,
(iii) $C = \{(x,y) \,;\, y^2 - x^k = 0\} \subset \mathbb{A}^2$, $k \in \mathbb{N}$.

Exercise 4.6.13. Show that "a general hypersurface in $\mathbb{P}^n$ is smooth". In other words, for any given d we can consider $\mathbb{P}^{\binom{n+d}{d}-1}$ as the "space of all hypersurfaces of degree d in $\mathbb{P}^n$", by associating to any hypersurface $\{f(x_0,\dots,x_n) = 0\} \subset \mathbb{P}^n$ with f homogeneous of degree d the projective vector of all $\binom{n+d}{d}$ coefficients of f. Then show that the subset of $\mathbb{P}^{\binom{n+d}{d}-1}$ corresponding to *smooth* hypersurfaces is non-empty and open.

Exercise 4.6.14. (This is a generalization of exercises 3.5.8 and 4.6.11 (iii).) For $i = 0,\dots,5$, determine how many conics there are in $\mathbb{P}^2$ that are tangent to i given lines and in addition pass through $5 - i$ given points.

5. SCHEMES

To any commutative ring R with identity we associate a locally ringed space called $\operatorname{Spec} R$*, the spectrum of R. Its underlying set is the set of prime ideals of R, so if R is the coordinate ring of an affine variety X over an algebraically closed field, then* $\operatorname{Spec} R$ *as a set is the set of non-empty closed irreducible subvarieties of X. Moreover, in this case the open subsets of* $\operatorname{Spec} R$ *are in one-to-one correspondence with the open subsets of X, and the structure sheaves of* $\operatorname{Spec} R$ *and X coincide via this correspondence.*

A morphism of locally ringed spaces is a morphism of ringed spaces that respects the maximal ideals of the local rings. Locally ringed spaces of the form $\operatorname{Spec} R$ *are called affine schemes; locally ringed spaces that are locally of the form* $\operatorname{Spec} R$ *are called schemes. Schemes are the fundamental objects of study in algebraic geometry. Prevarieties correspond exactly to those schemes that are reduced, irreducible, and of finite type over an algebraically closed field.*

For any two morphisms of schemes $X \to S$ and $Y \to S$ there is a fiber product $X \times_S Y$; this is a scheme such that giving morphisms $Z \to X$ and $Z \to Y$ that commute with the given morphisms to S is "the same" as giving a morphism $Z \to X \times_S Y$. If X and Y are prevarieties over k and we take $S = \operatorname{Spec} k$, we get back our old notion of the product $X \times Y$ of prevarieties.

For any graded ring R there is a scheme $\operatorname{Proj} R$ *whose points are the homogeneous prime ideals of R that do not contain the irrelevant ideal. This construction generalizes our earlier construction of projective varieties; if R is the homogeneous coordinate ring of a projective variety X over an algebraically closed field then* $\operatorname{Proj} R$ *"is" just the projective variety X.*

5.1. **Affine schemes.** We now come to the definition of schemes, which are the main objects of study in algebraic geometry. The notion of schemes extends that of prevarieties in a number of ways. We have already met several instances where an extension of the category of prevarieties could be useful:

- We defined a prevariety to be irreducible. Obviously, it makes sense to also consider reducible spaces. In the case of affine and projective varieties we called them algebraic sets, but we did not give them any further structure or defined regular functions and morphisms of them. Now we want to make reducible spaces into full-featured objects of our category.
- At present we have no geometric objects corresponding to non-radical ideals in $k[x_1,\dots,x_n]$, or in other words to coordinate rings with nilpotent elements. These non-radical ideals pop up naturally however: e. g. we have seen in exercise 1.4.1 that intersections of affine varieties correspond to sums of their ideals, modulo taking the radical. It would seem more natural to define the intersection $X_1 \cap X_2$ of two affine varieties $X_1, X_2 \subset \mathbb{A}^n$ to be a geometric object associated to the ideal $I(X_1) + I(X_2) \subset k[x_1,\dots,x_n]$. This was especially obvious when we discussed blow-ups: blowing up $X_1 \cap X_2$ in $\mathbb{A}^n$ "separates" X_1 and X_2 (if none of these two sets is contained in the other), i. e. their strict transforms $\tilde{X}_1$ and $\tilde{X}_2$ are disjoint in $\tilde{\mathbb{A}}^n$, but this is only true if we blow-up at the ideal $I(X_1) + I(X_2)$ and not at its radical (see exercise 4.6.10).
- Recall that by lemma 2.3.7 and remark 2.3.14 we have a one-to-one correspondence between affine varieties over k and finitely generated k-algebras that are domains, both modulo isomorphism. We have just seen that we should drop the condition on the k-algebra to be a domain. We can go even further and also drop the condition that it is finitely generated — then we would expect to arrive at "infinite-dimensional" objects. Moreover, it turns out that we do not even need a k-algebra to do geometry; it is sufficient to start with any commutative ring with

identity, i. e. we do not have to have a ground field. This can be motivated by noting that most constructions we made with the coordinate ring of a variety — defining the structure sheaf, setting up correspondences between points and maximal ideals, and so on — actually only used the ring structure of the coordinate ring, and not the k-algebra structure.

All these generalizations are included in the definition of a scheme. Note that they apply already to *affine* varieties; so we will start by defining an *affine* scheme to be "an affine variety generalized as above". Later we will then say that a scheme is an object that looks locally like an affine scheme, just as we did it in the case of prevarieties.

We are now ready to construct from any ring R (which will always mean a commutative ring with identity) an affine scheme, which will be a ringed space and which will be denoted $\operatorname{Spec} R$, the spectrum of R.

Definition 5.1.1. Let R be a ring (commutative with identity, as always). We define $\operatorname{Spec} R$ to be the set of all prime ideals of R. (As usual, R itself does not count as a prime ideal, but (0) does if R is a domain.) We call $\operatorname{Spec} R$ the **spectrum** of R, or the **affine scheme** associated to R. For every $\mathfrak{p} \in \operatorname{Spec} R$, i. e. $\mathfrak{p} \subset R$ a prime ideal, let $k(\mathfrak{p})$ be the quotient field of the domain $R/\mathfrak{p}$.

Remark 5.1.2. Let $X = \operatorname{Spec} R$ be an affine scheme. We should think of X as the analogue of an affine variety, and of R as the analogue of its coordinate ring.

Remark 5.1.3. Any element $f \in R$ can be considered to be a "function" on $\operatorname{Spec} R$ in the following sense: for $\mathfrak{p} \in \operatorname{Spec} R$, denote by $f(\mathfrak{p})$ the image of f under the composite map $R \to R/\mathfrak{p} \to k(\mathfrak{p})$. We call $f(\mathfrak{p})$ the **value** of f at the point $\mathfrak{p}$. Note that these values will in general lie in different fields. If $R = k[x_1,\dots,x_n]/I(X)$ is the coordinate ring of an affine variety X and $\mathfrak{p}$ is a maximal ideal (i. e. a point in X), then $k(\mathfrak{p}) = k$ and the value of an element $f \in R$ as defined above is equal to the value of f at the point corresponding to $\mathfrak{p}$ in the classical sense. If $\mathfrak{p} \subset R$ is not maximal and corresponds to some subvariety $Y \subset X$, the value $f(\mathfrak{p})$ lies in the function field $K(Y)$ and can be thought of as the restriction of the function f to Y.

Example 5.1.4.

(i) If k is a field, then $\operatorname{Spec} k$ consists of a single point (0).

(ii) The space $\operatorname{Spec} \mathbb{C}[x]$ (that will correspond to the affine variety $\mathbb{A}^1$ over $\mathbb{C}$) contains a point $(x-a)$ for every $a \in \mathbb{A}^1$, together with a point (0) corresponding to the subvariety $\mathbb{A}^1$.

(iii) More generally, if $R = A(X)$ is the coordinate ring of an affine variety X over an algebraically closed field, then the set $\operatorname{Spec} R$ contains a point for every closed subvariety of X (as subvarieties correspond exactly to prime ideals). This affine scheme $\operatorname{Spec} R$ will be the analogue of the affine variety X. So an affine scheme has "more points" than the corresponding affine variety: we have enlarged the set by throwing in an additional point for every closed subvariety Y of X. This point is usually called the **generic point** (or **general point**) of Y. In other words, in the scheme corresponding to an affine variety with coordinate ring R we will have a point for every prime ideal in R, and not just for every maximal ideal. These additional points are sometimes important, but quite often one can ignore this fact. Many textbooks will even adopt the convention that a point of a scheme is always meant to be a point in the old geometric sense (i. e. a maximal ideal).

(iv) In contrast to (ii), the affine scheme $\operatorname{Spec} \mathbb{R}[x]$ contains points that are not of the form $(x-a)$ or (0), e. g. $(x^2+1) \in \operatorname{Spec} \mathbb{R}[x]$.

(v) The affine scheme $\operatorname{Spec} \mathbb{Z}$ contains an element for every prime number, and in addition the generic point (0).

So far we have defined $\operatorname{Spec} R$ as a set. This is not particularly interesting, so let us move on and make $\operatorname{Spec} R$ into a topological space. This is done in the same way as for affine varieties.

Definition 5.1.5. Let R be a ring. For every subset $S \subset R$, we define the **zero locus** of S to be the set

$$Z(S) := \{\mathfrak{p} \in \operatorname{Spec} R \,;\, f(\mathfrak{p}) = 0 \text{ for all } f \in S\} \subset \operatorname{Spec} R,$$

where $f(\mathfrak{p})$ is the value of f at $\mathfrak{p}$ as in remark 5.1.3. (Obviously, S and (S) define the same zero locus, so we will usually only consider zero loci of ideals.)

Remark 5.1.6. By the definition of the value of an element $f \in R$ at a point $\mathfrak{p} \in \operatorname{Spec} R$, we can also write the definition of the zero locus as

$$\begin{aligned} Z(S) &= \{\mathfrak{p} \in \operatorname{Spec} R \,;\, f \in \mathfrak{p} \text{ for all } f \in S\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} R \,;\, \mathfrak{p} \supset S\}. \end{aligned}$$

Lemma 5.1.7. *Let R be a ring.*

(i) *If $\{I_i\}$ is a family of ideals of R then $\bigcap_i Z(I_i) = Z(\sum_i I_i) \subset \operatorname{Spec} R$.*
(ii) *If $I_1, I_2 \subset R$ then $Z(I_1) \cup Z(I_2) = Z(I_1 I_2) \subset \operatorname{Spec} R$.*
(iii) *If $I_1, I_2 \subset R$ then $Z(I_1) \subset Z(I_2)$ if and only if $\sqrt{I_2} \subset \sqrt{I_1}$.*

Proof. The proof is literally the same as in the case of affine algebraic sets. $\square$

Hence we can define a topology on $\operatorname{Spec} R$ by taking the subsets of the form $Z(S)$ as the closed subsets. In particular, this defines the notions of irreducibility and dimension for $\operatorname{Spec} R$, as they are purely topological concepts.

Remark 5.1.8. Note that points $\mathfrak{p}$ in $\operatorname{Spec} R$ are not necessarily closed: in fact,

$$\overline{\{\mathfrak{p}\}} = Z(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} R \,;\, \mathfrak{q} \supset \mathfrak{p}\}.$$

This is equal to $\{\mathfrak{p}\}$ only if $\mathfrak{p}$ is maximal. Hence the closed points of $\operatorname{Spec} R$ correspond to the points of an affine variety in the classical sense. The other points are just generic points of irreducible closed subsets of $\operatorname{Spec} R$, as already mentioned in example 5.1.4.

Example 5.1.9. The motivation for the name "generic point" can be seen from the following example. Let k be an algebraically closed field, and let $R = \operatorname{Spec} k[x_1, x_2]$ be the affine scheme corresponding to $\mathbb{A}^2$. Consider $Z(x_2) \subset \operatorname{Spec} R$, which "is" just the x_1-axis; so its complement $\operatorname{Spec} R \backslash Z(x_2)$ should be the set of points that do not lie on the x_1-axis. But note that the element $\mathfrak{p} = (x_1)$ is contained in $\operatorname{Spec} R \backslash Z(x_2)$, although the zero locus of x_1, namely the x_2-axis, does intersect the x_1-axis. So the geometric way to express the fact that $(x_1) \in \operatorname{Spec} R \backslash Z(x_2)$ is to say that the *generic point* of the x_2-axis does not lie on the x_1-axis.

Remark 5.1.10. Let R be a ring, let $X = \operatorname{Spec} R$, and let $f \in R$. As in the case of affine varieties, we call $X_f := X \backslash Z(f)$ the **distinguished open subset** associated to f. Note that any open subset of X is a (not necessarily finite) union of distinguished open subsets. This is often expressed by saying that the distinguished open subsets form a *base* of the topology of X.

Now we come to the definition of the structure sheaf of $\operatorname{Spec} R$. Recall that in the case of an affine variety X, we first defined the local ring $\mathcal{O}_{X,P}$ of the functions regular at a point $P \in X$ to be the localization of $A(X)$ at the maximal ideal corresponding to P, and then said that an element in $\mathcal{O}_X(U)$ for an open subset $U \subset X$ is a function that is regular at every point $P \in U$. We could accomplish that in the case of varieties just by intersecting the local rings $\mathcal{O}_{X,P}$, as they were all contained in the function field $K(X)$. But in the case of a general affine scheme $\operatorname{Spec} R$ the various local rings $R_\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec} R$ do not lie inside some big space, so we cannot just take their intersection. The way around this problem is to say that an element in $\mathcal{O}_X(U)$ (for $X = \operatorname{Spec} R$ and $U \subset X$ open) is given by a collection of elements in the various local rings $R_\mathfrak{p}$ for all $\mathfrak{p} \in U$, and require that these elements can locally be written as quotients of elements of R (recall that we had a similar condition for affine varieties in lemma 2.1.8):

Definition 5.1.11. Let R be a ring, and let $X = \operatorname{Spec} R$. For every open subset $U \subset X$ we define $\mathcal{O}_X(U)$ to be

$$\begin{aligned}\mathcal{O}_X(U) := \{\varphi = (\varphi_\mathfrak{p})_{\mathfrak{p}\in U} &\text{ with } \varphi_\mathfrak{p} \in R_\mathfrak{p} \text{ for all } \mathfrak{p} \in U \\ &\text{such that ``}\varphi \text{ is locally of the form } \tfrac{f}{g} \text{ for } f,g \in R\text{''}\} \\ = \{\varphi = (\varphi_\mathfrak{p})_{\mathfrak{p}\in U} &\text{ with } \varphi_\mathfrak{p} \in R_\mathfrak{p} \text{ for all } \mathfrak{p} \in U \\ &\text{such that for every } \mathfrak{p} \in U \text{ there is a neighborhood } V \text{ in } U \text{ and } f,g \in R \\ &\text{with } g \notin \mathfrak{q} \text{ and } \varphi_\mathfrak{q} = \tfrac{f}{g} \in R_\mathfrak{q} \text{ for all } \mathfrak{q} \in V.\}\end{aligned}$$

As the conditions imposed on the elements of $\mathcal{O}_X(U)$ are local, it is easy to verify that this defines a sheaf $\mathcal{O}_X$ on $X = \operatorname{Spec} R$. The first thing to do is to check that this sheaf has the properties that we expect from the case of affine varieties (see definition 2.1.5, remark 2.1.6, and proposition 2.1.10).

Proposition 5.1.12. *Let R be a ring and $X = \operatorname{Spec} R$.*

(i) *For any $\mathfrak{p} \in X$ the stalk $\mathcal{O}_{X,\mathfrak{p}}$ of the sheaf $\mathcal{O}_X$ is isomorphic to the local ring $R_\mathfrak{p}$.*
(ii) *For any $f \in R$, the ring $\mathcal{O}_X(X_f)$ is isomorphic to the localized ring R_f. In particular, $\mathcal{O}_X(X) = R$.*

Proof. (i): There is a well-defined ring homomorphism

$$\psi : \mathcal{O}_{X,\mathfrak{p}} \to R_\mathfrak{p},\ (U,\varphi) \mapsto \varphi_\mathfrak{p}.$$

We have to show that ψ is a bijection.

ψ is surjective: Any element of $R_\mathfrak{p}$ has the form $\frac{f}{g}$ with $f,g \in R$ and $g \notin \mathfrak{p}$. The function $\frac{f}{g}$ is well-defined on X_g, so $(X_g, \frac{f}{g})$ defines an element in $\mathcal{O}_{X,\mathfrak{p}}$ that is mapped by ψ to the given element.

ψ is injective: Let $\varphi_1, \varphi_2 \in \mathcal{O}_X(U)$ for some neighborhood U of $\mathfrak{p}$, and assume that $(\varphi_1)_\mathfrak{p} = (\varphi_2)_\mathfrak{p}$. We have to show that φ_1 and φ_2 coincide in a neighborhood of $\mathfrak{p}$, so that they define the same element in $\mathcal{O}_{X,\mathfrak{p}}$. By shrinking U if necessary we may assume that $\varphi_i = \frac{f_i}{g_i}$ on U for $i = 1,2$, where $f_i, g_i \in R$ and $g_i \notin \mathfrak{p}$. As φ_1 and φ_2 have the same image in $R_\mathfrak{p}$, it follows that $h(f_1g_2 - f_2g_1) = 0$ in R for some $h \notin \mathfrak{p}$. Therefore we also have $\frac{f_1}{g_1} = \frac{f_2}{g_2}$ in every local ring $R_\mathfrak{q}$ such that $g_1, g_2, h \notin \mathfrak{q}$. But the set of such $\mathfrak{q}$ is the open set $X_{g_1} \cap X_{g_2} \cap X_h$, which contains $\mathfrak{p}$. Hence $\varphi_1 = \varphi_2$ on some neighborhood of $\mathfrak{p}$, as required.

(ii): There is a well-defined ring homomorphism

$$\psi : R_f \to \mathcal{O}_X(X_f),\ \frac{g}{f^r} \mapsto \frac{g}{f^r}$$

(i. e. we map $\frac{g}{f^r}$ to the element of $\mathcal{O}_X(X_f)$ that assigns to any $\mathfrak{p}$ the image of $\frac{g}{f^r}$ in $R_\mathfrak{p}$).

ψ is injective: Assume that $\psi(\frac{g_1}{f^{r_1}}) = \psi(\frac{g_2}{f^{r_2}})$, i. e. for every $\mathfrak{p} \in X_f$ there is an element $h \notin \mathfrak{p}$ such that $h(g_1 f^{r_2} - g_2 f^{r_1}) = 0$. Let $I \subset R$ be the annihilator of $g_1 f^{r_2} - g_2 f^{r_1}$, then we have just shown that $I \not\subset \mathfrak{p}$, as $h \in I$ but $h \notin \mathfrak{p}$. This holds for every $\mathfrak{p} \in X_f$, so $Z(I) \cap X_f = \emptyset$, or in other words $Z(I) \subset Z(f)$. By lemma 5.1.7 (iii) this means that $f^r \in I$ for some r, so $f^r(g_1 f^{r_2} - g_2 f^{r_1}) = 0$, hence $\frac{g_1}{f^{r_1}} = \frac{g_2}{f^{r_2}}$ in R_f.

ψ is surjective: Let $\varphi \in \mathcal{O}_X(X_f)$. By definition, we can cover X_f by open sets U_i on which φ is represented by a quotient $\frac{g_i}{f_i}$, with $f_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in U_i$, i. e. $U_i \subset X_{f_i}$. As the open subsets of the form X_{h_i} form a base for the topology of X, we may assume that $U_i = X_{h_i}$ for some h_i.

We want to show that we can assume $f_i = h_i$. In fact, as $X_{h_i} \subset X_{f_i}$, i. e. by taking complements we get $Z(f_i) \subset Z(h_i)$, and therefore $h_i \in \sqrt{f_i}$ by lemma 5.1.7 (iii). Hence $h_i^r = cf_i$, so $\frac{g_i}{f_i} = \frac{cg_i}{h_i^r}$. Replacing h_i by h_i^r (as $X_{h_i} = X_{h_i^r}$) and g_i by cg_i we can assume that X_f is covered by open subsets of the form X_{h_i}, and that φ is represented by $\frac{g_i}{h_i}$ on X_{h_i}.

Next we prove that X_f can actually be covered by finitely many such X_{h_i}. Indeed, $X_f \subset \bigcup_i X_{h_i}$ if and only if $Z(f) \supset \bigcap_i Z(h_i) = Z(\sum(h_i))$. By lemma 5.1.7 (iii) this is equivalent to saying that $f^r \in \sum(h_i)$ for some r. But this means that f^r can be written as a finite sum $f^r = \sum b_i h_i$. Hence we can assume that we have only finitely many h_i.

On $X_{h_i} \cap X_{h_j} = X_{h_i h_j}$, we have two elements $\frac{g_i}{h_i}$ and $\frac{g_j}{h_j}$ representing φ, so by the injectivity proven above it follows that $\frac{g_i}{h_i} = \frac{g_j}{h_j}$ in $R_{h_i h_j}$, hence $(h_i h_j)^n (g_i h_j - g_j h_i) = 0$ for some n. As we have only finitely many h_i, we may pick one n that works for all i, j. Now replace g_i by $g_i h_i^n$ and h_i by h_i^{n+1} for all i, then we still have φ represented by $\frac{g_i}{h_i}$ on X_{h_i}, and moreover $g_i h_j - g_j h_i = 0$ for all i, j.

Now write $f^r = \sum b_i h_i$ as above, which is possible since the X_{h_i} cover X_f. Let $g = \sum b_i g_i$. Then for every j we have

$$gh_j = \sum_i b_i g_i h_j = \sum_i b_i h_i g_j = f^r g_j,$$

so $\frac{f}{g} = \frac{h_j}{g_j}$ on X_{h_j}. Hence φ is represented on X_f by $\frac{g}{f^r} \in R_f$, i. e. ψ is surjective. $\square$

Remark 5.1.13. Note that a regular function is in general no longer determined by its values on points. For example, let $R = k[x]/(x^2)$ and $X = \operatorname{Spec} R$. Then X has just one point (x). On this point, the function $x \in R = \mathscr{O}_X(X)$ takes the value $0 = x \in (k[x]/(x^2))/(x) = k$. In particular, the functions 0 and x have the same values at all points of X, but they are not the same regular function.

5.2. **Morphisms and locally ringed spaces.** As in the case of varieties, the next step after defining regular functions on an affine scheme is to define morphisms between them. Of course one is tempted to define a morphism $f : X \to Y$ between affine schemes to be a morphism of ringed spaces as in definition 2.3.1, but recall that for this definition to work we needed a notion of pull-back f^* of regular functions. In the case of varieties we got this by requiring that the structure sheaves be sheaves of k-valued functions, so that a set-theoretic pull-back exists. But this is not possible for schemes, as we do not have a ground field, and the values $\varphi(\mathfrak{p})$ of a regular function φ lie in unrelated rings. Even worse, we have seen already in example 5.1.13 that a regular function is not determined by its values on points.

The way out of this dilemma is to make the pull-back maps $f^* : \mathscr{O}_Y(U) \to \mathscr{O}_X(f^{-1}(U))$ part of the data required to define a morphism. Hence we say that a morphism $f : X \to Y$ between affine schemes is given by a continuous map $f : X \to Y$ between the underlying topological spaces, together with pull-back maps $f^* = f_U^* : \mathscr{O}_Y(U) \to \mathscr{O}_X(f^{-1}(U))$ for every open subset $U \subset Y$. Of course we need some compatibility conditions among the f_U^*. The most obvious one is compatibility with the restriction maps, i. e. $f_V^* \circ \rho_{U,V} = \rho_{f^{-1}(U), f^{-1}(V)} \circ f_U^*$. But we also need some sort of compatibility between the f_U^* and the continuous map f. To explain this condition, note that the maps f_U^* give rise to a map between the stalks

$$f_P^* : \mathscr{O}_{Y,f(P)} \to \mathscr{O}_{X,P}, \ (U, \varphi) \mapsto (f^{-1}(U), f^* \varphi)$$

for every point $P \in X$ (this is easily seen to be well-defined). These stalks are local rings, call their maximal ideals $\mathfrak{m}_{Y,f(P)}$ and $\mathfrak{m}_{X,P}$, respectively. Now the fact that f maps P

to $f(P)$ should be reflected on the level of the pull-back maps f^* by the condition that $(f_P^*)^{-1}(\mathfrak{m}_{X,P}) = \mathfrak{m}_{Y,f(P)}$. This leads to the following definition.

Definition 5.2.1. A **locally ringed space** is a ringed space $(X, \mathcal{O}_X)$ such that at each point $P \in X$ the stalk $\mathcal{O}_{X,P}$ is a local ring. The maximal ideal of $\mathcal{O}_{X,P}$ will be denoted by $\mathfrak{m}_{X,P}$, and the **residue field** $\mathcal{O}_{X,P}/\mathfrak{m}_{X,P}$ will be denoted $k(P)$.

A **morphism** of locally ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is given by the following data:

- a continuous map $f : X \to Y$,
- for every open subset $U \subset Y$ a ring homomorphism $f_U^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$,

such that $f_V^* \circ \rho_{U,V} = \rho_{f^{-1}(U),f^{-1}(V)} \circ f_U^*$ for all $V \subset U \subset Y$ (i. e. the f^* are compatible with the restriction maps) and $(f_P^*)^{-1}(\mathfrak{m}_{X,P}) = \mathfrak{m}_{Y,f(P)}$, where the $f_P^* : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ are the maps induced on the stalks, as explained above. We will often omit the index of the various pull-back maps f^* if it is clear from the context on which spaces they act.

A morphism of affine schemes is a morphism as locally ringed spaces.

The following proposition is the analogue of lemma 2.3.7. It shows that definition 5.2.1 was "the correct one", because it gives us finally what we want.

Proposition 5.2.2. *Let R, S be rings, and let $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$ the corresponding affine schemes. There is a one-to-one correspondence between morphisms $X \to Y$ and ring homomorphisms $S \to R$.*

Proof. If $\psi : S \to R$ is a ring homomorphism, we define a map $f : X \to Y$ by $f(\mathfrak{p}) = \psi^{-1}(\mathfrak{p})$. For every ideal $I \subset S$ it follows that $f^{-1}(Z(I)) = Z(\psi(I))$, so f is continuous. For each $\mathfrak{p} \in \operatorname{Spec} R$, we can localize ψ to get a homomorphism of local rings $\psi_{\mathfrak{p}} : \mathcal{O}_{Y,f(\mathfrak{p})} = S_{\psi^{-1}(\mathfrak{p})} \to R_{\mathfrak{p}} = \mathcal{O}_{X,\mathfrak{p}}$ satisfying the condition $\psi_{\mathfrak{p}}^{-1}(\mathfrak{m}_{X,\mathfrak{p}}) = \mathfrak{m}_{Y,f(\mathfrak{p})}$. By definition of the structure sheaf, this gives homomorphisms of rings $f^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$, and by construction $f_{\mathfrak{p}}^* = \psi_{\mathfrak{p}}$, so we get a morphism of locally ringed spaces.

If $f : X \to Y$ is a morphism, we get a ring homomorphism $f^* : S = \mathcal{O}_Y(Y) \to \mathcal{O}_X(X) = R$ by proposition 5.1.12 (ii). By the above this again determines a morphism $g : X \to Y$. We leave it as an exercise to check that the various compatibility conditions imply that $f = g$. □

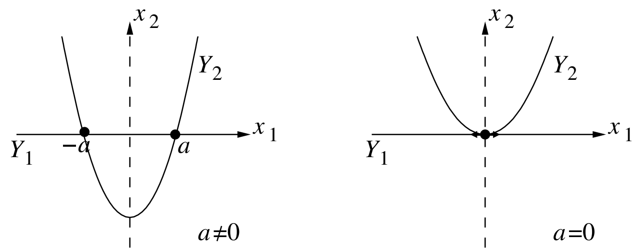
Example 5.2.3. Let $X = \operatorname{Spec} R$ be an affine scheme. If $I \subset R$ is an ideal, then we can form the affine scheme $Y = \operatorname{Spec}(R/I)$, and the ring homomorphism $R \to R/I$ gives us a morphism $Y \to X$. Note that the prime ideals of R/I are exactly the ideals $\mathfrak{p} \subset R$ with $\mathfrak{p} \supset I$, so the map $Y \to X$ is an inclusion with image $Z(I)$. So we can view Y as an affine "closed subscheme" of X. For a precise definition of this concept see example 7.2.10.

Now let $Y_1 = \operatorname{Spec}(R/I_1)$ and $Y_2 = \operatorname{Spec}(R/I_2)$ be closed subschemes of X. We define the intersection scheme $Y_1 \cap Y_2$ in X to be $Y_1 \cap Y_2 = \operatorname{Spec} R/(I_1 + I_2)$.

For example, let $X = \operatorname{Spec} \mathbb{C}[x_1, x_2]$, $Y_1 = \operatorname{Spec} \mathbb{C}[x_1, x_2]/(x_2)$, $Y_2 = \operatorname{Spec} \mathbb{C}[x_1, x_2]/(x_2 - x_1^2 + a^2)$ for some $a \in \mathbb{C}$. Then the intersection scheme $Y_1 \cap Y_2$ is $\operatorname{Spec} \mathbb{C}[x_1]/((x_1 - a)(x_1 + a))$. For $a \neq 0$ we have $\mathbb{C}[x_1]/((x_1 - a)(x_1 + a)) \cong \mathbb{C}[x_1]/(x_1 - a) \times \mathbb{C}[x_1]/(x_1 + a) \cong \mathbb{C} \times \mathbb{C}$, so $Y_1 \cap Y_2$ is just the disjoint union of the two points $(a, 0)$ and $(-a, 0)$ in $\mathbb{C}^2$. For $a = 0$ however we have $Y_1 \cap Y_2 = \operatorname{Spec} \mathbb{C}[x_1]/(x_1^2)$, which has only one point $(0, 0)$. But in all cases the ring $\mathbb{C}[x_1]/((x_1 - a)(x_1 + a))$ has dimension 2 as a vector space over $\mathbb{C}$. We say that $Y_1 \cap Y_2$ is a "scheme of length 2", which consists either of two distinct points of length 1 each, or of one point of length (i. e. multiplicity) 2.

Note also that there is always a unique line in $\mathbb{A}^2$ through $Y_1 \cap Y_2$, even in the case $a = 0$ where the scheme has only one geometric point. This is because the scheme $Y_1 \cap Y_2 = \operatorname{Spec} \mathbb{C}[x_1, x_2]/(x_2, (x_1 - a)(x_1 + a))$ is a subscheme of the line $L = \operatorname{Spec} \mathbb{C}[x_1, x_2]/(c_1 x_1 +$

$c_2x_2)$ if and only if $(c_1x_1+c_2x_2) \subset (x_2,(x_1-a)(x_1+a))$, which is the case only if $c_1 = 0$. In particular, the x_1-axis is the only line in $\mathbb{A}^2$ that contains $\operatorname{Spec}\mathbb{C}[x_1,x_2]/(x_2,x_1^2)$. One can therefore think of this scheme as "the origin together with a tangent direction along the x_1-axis".



Example 5.2.4. Again let $Y_1 = \operatorname{Spec}(R/I_1)$ and $Y_2 = \operatorname{Spec}(R/I_2)$ be closed subschemes of of the affine scheme $X = \operatorname{Spec} R$. Note that for affine varieties the ideal of the union of two closed subsets equals the intersection of their ideals (see exercise 1.4.1 (i)). So scheme-theoretically we just *define* the union $Y_1 \cup Y_2$ to be $\operatorname{Spec} R/(I_1 \cap I_2)$.

The following lemma is the scheme-theoretic analogue of lemma 2.3.16.

Lemma 5.2.5. *Let $X = \operatorname{Spec} R$ be an affine scheme, and let $f \in R$. Then the distinguished open subset X_f is the affine scheme $\operatorname{Spec} R_f$.*

Proof. Note that both X_f and $\operatorname{Spec} R_f$ have the description $\{\mathfrak{p} \in X \,;\, f \notin \mathfrak{p}\}$. So it only remains to be checked that the structure sheaves on X_f and $\operatorname{Spec} R_f$ agree. Now let $g \in R$ and consider the distinguished open subset $X_{fg} = (\operatorname{Spec} R_f)_g$. By proposition 5.1.12 (ii) we have

$$\begin{aligned} &\mathcal{O}_{X_f}(X_{fg}) = \mathcal{O}_X(X_{fg}) = R_{fg} \\ \text{and} \quad &\mathcal{O}_{\operatorname{Spec} R_f}((\operatorname{Spec} R_f)_g) = (R_f)_g = R_{fg}. \end{aligned}$$

So the rings of regular functions are the same for X_f and $\operatorname{Spec} R_f$ on every distinguished open subset. But every open subset is the intersection of such distinguished opens, so the rings of regular functions must be the same on every open subset. □

5.3. **Schemes and prevarieties.** Having defined affine schemes and their morphisms, we can now define schemes as objects that look locally like affine schemes — this is in parallel to the definition 2.4.1 of prevarieties.

Definition 5.3.1. A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ that can be covered by open subsets $U_i \subset X$ such that $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to an affine scheme $\operatorname{Spec} R_i$ for all i. A morphism of schemes is a morphism as locally ringed spaces.

Remark 5.3.2. From the point of view of prevarieties, it would seem more natural to call the objects defined above preschemes, and then say that a scheme is a prescheme having the "Hausdorff" property, i. e. a prescheme with closed diagonal (see definition 2.5.1 and lemma 2.5.3). This is in fact the terminology of [M1], but nowadays everyone seems to adopt the definition that we gave above, and then say that a scheme having the "Hausdorff property" is a *separated* scheme.

From our definitions we see that prevarieties are in a sense special cases of schemes — if we have an affine variety $X = Z(I) \subset \mathbb{A}^n$ with $I \subset k[x_1,\dots,x_n]$ an ideal, the scheme $\operatorname{Spec} A(X)$ corresponds to X (where $A(X) = k[x_1,\dots,x_n]$ is the coordinate ring of X); and any glueing along isomorphic open subsets that can be done in the category of prevarieties can be done equally well for the corresponding schemes. Hence we would like to say that every prevariety is a scheme. In the strict sense of the word this is not quite true however, because the topological space of a scheme contains a point for every irreducible closed subset, whereas the topological space of a prevariety consists only of the geometric points in the classical sense (i. e. the closed points). But of course there is a natural way to consider every prevariety as a scheme, by throwing in additional generic points for every irreducible closed subset. We give the precise statement and leave its proof as an exercise:

Proposition 5.3.3. *Let k be an algebraically closed field, and let X be a prevariety over k. Let X_{sch} be the space of all non-empty closed irreducible subsets of X. Then X_{sch} is a scheme in a natural way. The open subsets of X correspond bijectively to the open subsets of X_{sch}, and for every open subset U of X (which can then also be considered as an open subset of X_{sch}) we have $O_{X_{sch}}(U) = O_X(U)$. Every morphism $X \to Y$ of prevarieties over k extends to a morphism $X_{sch} \to Y_{sch}$ of schemes in a natural way.*

Let us now investigate the properties of schemes that arise from prevarieties in this way. As we have mentioned already, the glueing of schemes from affine schemes is exactly the same as that of prevarieties from varieties. Hence the special properties of schemes that come from prevarieties can already be seen on the level of affine schemes. We have also seen above that in an affine scheme $\operatorname{Spec} R$ the ring R corresponds to what is the coordinate ring $A(X)$ of an affine variety. Moreover we know by remark 2.3.14 that the coordinate ring of an affine variety is a finitely generated k-algebra that is a domain. So we have to write down conditions on a scheme that reflect the property that its local patches $\operatorname{Spec} R$ are not made from arbitrary rings, but rather from finitely generated k-algebras that are domains.

Definition 5.3.4. Let Y be a scheme. A **scheme over** Y is a scheme X together with a morphism $X \to Y$. A morphism of schemes X_1, X_2 over Y is a morphism of schemes $X_1 \to X_2$ such that

$$\begin{tikzcd} X_1 \arrow[rr] \arrow[dr] & & X_2 \arrow[dl] \\ & Y & \end{tikzcd}$$

commutes. If R is a ring, a scheme over R is a scheme over $\operatorname{Spec} R$.

A scheme X over Y is said to be **of finite type** over Y if there is a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$ such that $f^{-1}(V_i)$ can be covered by finitely many open affines $U_{i,j} = \operatorname{Spec} A_{i,j}$, where each $A_{i,j}$ is a finitely generated B_i-algebra. In particular, a scheme X over a field k is of finite type over k if it can be covered by finitely many open affines $U_i = \operatorname{Spec} A_i$, where each A_i is a finitely generated k-algebra.

A scheme X is called **reduced** if the rings $O_X(U)$ have no nilpotent elements for all open subsets $U \subset X$.

Now it is obvious what these conditions mean for an affine scheme $\operatorname{Spec} R$:

- $\operatorname{Spec} R$ is a scheme over k if and only if we are given a morphism $k \to R$, i. e. if R is a k-algebra. Moreover, a morphism $\operatorname{Spec} R \to \operatorname{Spec} S$ is a morphism of schemes over k if and only if the corresponding ring homomorphism $S \to R$ is a morphism of k-algebras.
- $\operatorname{Spec} R$ is of finite type over k if and only if R is a finitely generated k-algebra.
- $\operatorname{Spec} R$ is reduced and irreducible if and only if $f \cdot g = 0$ in R implies $f = 0$ or $g = 0$, i. e. if and only if R is a domain. To see this, assume that $f \cdot g = 0$, but $f \neq 0$ and $g \neq 0$. If f and g are the same up to a power, then R is not nilpotent-free, so $\operatorname{Spec} R$ is not reduced. Otherwise, we get a decomposition of $\operatorname{Spec} R$ into two proper closed subsets $Z(f)$ and $Z(g)$, so $\operatorname{Spec} R$ is not irreducible.

As glueing affine patches is allowed for varieties in the same way as for schemes, we get the following result:

Proposition 5.3.5. *Let k be an algebraically closed field. There is a one-to-one correspondence between prevarieties over k (and their morphisms) and reduced, irreducible schemes of finite type over k (and their morphisms).*

Hence, from now on a prevariety over k will mean a reduced and irreducible scheme of finite type over k.

Remark 5.3.6. As in the case of prevarieties, schemes and morphisms of schemes can (almost by definition) be glued together. As for glueing schemes lemma 2.4.7 holds in the same way (except that one may now also glue infinitely many patches X_i, and the isomorphic open subsets $U_{i,j} \subset X_i$ and $U_{j,i} \subset X_j$ can be empty, which might give rise to disconnected schemes). A morphism from the glued scheme X to some scheme Y can then be given by giving morphisms $X_i \to Y$ that are compatible on the overlaps in the obvious sense.

The following generalization of proposition 5.2.2 is an application of these glueing techniques.

Proposition 5.3.7. *Let X be any scheme, and let $Y = \operatorname{Spec} R$ be an affine scheme. Then there is a one-to-one correspondence between morphisms $X \to Y$ and ring homomorphisms $R = \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$.*

Proof. Let $\{U_i\}$ be an open affine cover of X, and let $\{U_{i,j,k}\}$ be an open affine cover of $U_i \cap U_j$. Then by remark 5.3.6 giving a morphism $f : X \to Y$ is the same as giving morphisms $f_i : U_i \to Y$ such that f_i and f_j agree on $U_i \cap U_j$, i. e. such that $f_i|_{U_{i,j,k}} = f_j|_{U_{i,j,k}}$ for all i, j, k. But as the U_i and $U_{i,j,k}$ are affine, by proposition 5.2.2 the morphisms f_i and $f_i|_{U_{i,j,k}}$ correspond exactly to ring homomorphisms $\mathcal{O}_Y(Y) \to \mathcal{O}_{U_i}(U_i) = \mathcal{O}_X(U_i)$ and $\mathcal{O}_Y(Y) \to \mathcal{O}_{U_{i,j,k}}(U_{i,j,k}) = \mathcal{O}_X(U_{i,j,k})$, respectively. Hence a morphism $f : X \to Y$ is the same as a collection of ring homomorphisms $f_i^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(U_i)$ such that the compositions $\rho_{U_i,U_{i,j,k}} \circ f_i^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(U_{i,j,k})$ and $\rho_{U_j,U_{i,j,k}} \circ f_j^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(U_{i,j,k})$ agree for all i, j, k. But by the sheaf axiom for $\mathcal{O}_X$, this is exactly the data of a ring homomorphism $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. □

Remark 5.3.8. By the above proposition, every scheme X admits a unique morphism to $\operatorname{Spec} \mathbb{Z}$, determined by the natural map $\mathbb{Z} \to \mathcal{O}_X(X)$. More explicitly, on points this map is given by associating to every point $P \in X$ the characteristic of its residue field $k(P)$. In particular, if X is a scheme over $\mathbb{C}$ (or any ground field of characteristic 0 for that matter) then the morphism $X \to \operatorname{Spec} \mathbb{Z}$ maps every point to the zero ideal (0).

5.4. **Fiber products.** In example 2.3.9 and exercise 2.6.13 we defined the product $X \times Y$ for two given prevarieties X and Y by giving the product set $X \times Y$ a suitable structure of a ringed space. The idea of this construction was that the coordinate ring $A(X \times Y)$ should be $A(X) \otimes A(Y)$ if X and Y are affine (see remark 2.3.13), and then to globalize this construction by glueing techniques. The characteristic property of the product $X \times Y$ was that giving a morphism to it is equivalent to giving a morphism to X and a morphism to Y (see lemma 2.3.11 and exercise 2.6.13).

Now we want to do the same thing for schemes. More generally, if X and Y are two schemes *over a third scheme* S (i. e. if morphisms $f : X \to S$ and $g : Y \to S$ are given) we want to construct the so-called fiber product $X \times_S Y$, that should naïvely correspond to the points $(x, y) \in X \times Y$ such that $f(x) = g(y)$. As in the case of prevarieties this will be done by first constructing this product in the affine case, and then glueing these products together to obtain the fiber product of general schemes. We start by defining fiber products using the characteristic property mentioned above.

Definition 5.4.1. Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes. We define the **fiber product** $X \times_S Y$ to be a scheme together with "projection" morphisms $\pi_X : X \times_S Y \to X$ and $\pi_Y : X \times_S Y \to Y$ such that the square in the following diagram commutes, and such that for any scheme Z and morphisms $Z \to X$ and $Z \to Y$ making a commutative diagram with f and g there is a unique morphism $Z \to X \times_S Y$ making the whole diagram commutative:

$$\begin{tikzcd}
Z \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_S Y \arrow[r, "\pi_Y"'] \arrow[d, "\pi_X"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & S
\end{tikzcd}$$

Let us first show that the fiber product is uniquely determined by this property:

Lemma 5.4.2. *The fiber product $X \times_S Y$ is unique if it exists. (In other words, if F_1 and F_2 are two fiber products satisfying the above characteristic property, then F_1 and F_2 are canonically isomorphic.)*

Proof. Let F_1 and F_2 be two fiber products satisfying the characteristic property of the definition. In particular, F_2 comes together with morphisms to X and Y. As F_1 is a fiber product, we get a morphism $\varphi : F_2 \to F_1$

$$\begin{tikzcd}
F_2 \arrow[dr, dotted, "\varphi"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& F_1 \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & S
\end{tikzcd}$$

so that this diagram commutes. By symmetry, we get a morphism $\psi : F_1 \to F_2$ as well. The diagram

$$\begin{tikzcd}
F_1 \arrow[dr, dotted, "\varphi\circ\psi"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& F_1 \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & S
\end{tikzcd}$$

is then commutative by construction. But the same diagram is commutative too if we replace $\varphi \circ \psi$ by the identity morphism. So by the uniqueness part of the definition of a fiber product it follows that $\varphi \circ \psi$ is the identity. Of course $\psi \circ \varphi$ is then also the identity by symmetry. So F_1 and F_2 are canonically isomorphic. □

Remark 5.4.3. The following two properties of fiber products are easily seen from the definition:

(i) If $S \subset U$ is an open subset, then $X \times_S Y = X \times_U Y$ (morphisms from any Z to X and Y commuting with f and g are then the same regardless of whether the base scheme is S or U).

(ii) If $U \subset X$ and $V \subset Y$ are open subsets, then the fiber product
$$U \times_S V = \pi_X^{-1}(U) \cap \pi_Y^{-1}(V) \subset X \times_S Y$$
is an open subset of the total fiber product $X \times_S Y$.

Now we want to show that fiber products always exist. We have already mentioned that in the affine case, fiber products should correspond to tensor products in commutative algebra. So let us define the corresponding tensor products first.

Definition 5.4.4. Let R be a ring, and let M and N be R-modules. For every $m \in M$ and $n \in N$ let $m \otimes n$ be a formal symbol. We let F be the "free R-module generated by the symbols $m \otimes n$", i. e. F is the R-module of formal *finite* linear combinations
$$F = \big\{ \sum_i r_i(m_i \otimes n_i) \; ; \; r_i \in R, m_i \in M, n_i \in N \big\}.$$
Now we define the **tensor product** $M \otimes_R N$ of M and N over R to be the R-module F modulo the relations
$$\begin{aligned}(m_1 + m_2) \otimes n &= m_1 \otimes n + m_2 \otimes n,\\ m \otimes (n_1 + n_2) &= m \otimes n_1 + m \otimes n_2,\\ r(m \otimes n) &= (rm) \otimes n = m \otimes (rn)\end{aligned}$$
for all $m, m_i \in M$, $n, n_i \in N$, and $r \in R$. Obviously, $M \otimes_R N$ is an R-module as well.

Example 5.4.5.

(i) Let k be a field. Then $k[x] \otimes_k k[y] = k[x,y]$, where the isomorphism is given by
$$k[x] \otimes_k k[y] \to k[x,y], \; f(x) \otimes g(y) \mapsto f(x) \cdot g(y)$$
and
$$k[x,y] \to k[x] \otimes_k k[y], \; \sum_{i,j} a_{i,j} x^i y^j \mapsto \sum_{i,j} a_{i,j} (x^i \otimes y^j).$$

(ii) Let R be a ring, and let I_1 and I_2 be ideals. Then R/I_1 and R/I_2 are R-modules, and we have $R/I_1 \otimes_R R/I_2 = R/(I_1 + I_2)$. In fact, the isomorphism is given by
$$R/I_1 \otimes_R R/I_2 \to R/(I_1 + I_2), \; r_1 \otimes r_2 \mapsto r_1 \cdot r_2$$
and
$$R/(I_1 + I_2) \to R/I_1 \otimes_R R/I_2, \; r \mapsto r(1 \otimes 1) = (r \otimes 1) = (1 \otimes r).$$

(iii) If M is any R-module, then $M \otimes_R R = R \otimes_R M = M$.

Remark 5.4.6. It is easy to see that the tensor product of modules satisfies the following characteristic property (which is exactly the same as that of definition 5.4.1, just with all the arrows reversed):

Let R, M, and N be rings, and assume that we are given ring homomorphisms $f : R \to M$ and $g : R \to N$ (that make M and N into R-modules). Then for every ring A and homomorphisms $M \to A$ and $N \to A$ making a commutative diagram with f and g there is a unique ring homomorphism $M \otimes_R N \to A$ making the whole diagram commutative:

$$\begin{tikzcd}
A & & \\
& M \otimes_R N \arrow[ul, dotted] & N \arrow[l] \arrow[ull, bend right] \\
& M \arrow[u] \arrow[uul, bend left] & R \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

where $M \to M \otimes_R N$ and $N \to M \otimes_R N$ are the obvious maps $m \mapsto m \otimes 1$ and $n \mapsto 1 \otimes n$. In fact, if $a : M \to A$ and $b : N \to A$ are the two ring homomorphisms, then $M \otimes_R N \to A$ is given by $m \otimes n \mapsto a(m) \cdot b(n)$.

Using the tensor product of modules, we can now construct the fiber product of schemes.

Lemma 5.4.7. *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes. Then there is a fiber product $X \times_S Y$.*

Proof. First assume that X, Y, and S are affine schemes, so $X = \operatorname{Spec} M$, $Y = \operatorname{Spec} N$, and $S = \operatorname{Spec} R$. The morphisms $X \to S$ and $Y \to S$ make M and N into R-modules by proposition 5.2.2. We claim that $\operatorname{Spec}(M \otimes_R N)$ is the fiber product $X \times_S Y$. Indeed, giving a morphism $Z \to \operatorname{Spec}(M \otimes_R N)$ is the same as giving a homomorphism $M \otimes_R N \to \mathscr{O}_Z(Z)$ by proposition 5.3.7. By remark 5.4.6, this is the same as giving homomorphisms $M \to \mathscr{O}_Z(Z)$ and $N \to \mathscr{O}_Z(Z)$ that induce the same homomorphism on R, which again by proposition 5.3.7 is the same as giving morphisms $Z \to X$ and $Z \to Y$ that give rise to the same morphism from $Z \to S$. Hence $\operatorname{Spec}(M \otimes_R N)$ is the desired product.

Now let X, Y and S be general schemes. Cover S by open affines S_i, then cover $f^{-1}(S_i)$ and $g^{-1}(S_i)$ by open affines $X_{i,j}$ and $Y_{i,k}$, respectively. Consider the fiber products $X_{i,j} \times_{S_i} Y_{i,k}$ that exist by the above tensor product construction. Note that by remark 5.4.3 (i) these will then be fiber products over S as well. Now if we have another such product $X_{i',j'} \times_S Y_{i',k'}$, both of them will contain the (unique) fiber product $(X_{i,j} \cap X_{i',j'}) \times_S (Y_{i,k} \cap Y_{i',k'})$ as an open subset by remark 5.4.3 (ii), hence they can be glued along these isomorphic open subsets. It is obvious that the final scheme $X \times_S Y$ obtained by glueing the patches satisfies the defining property of a fiber product. □

Example 5.4.8. Let X and Y be prevarieties over a field k. Then the scheme-theoretic fiber product $X \times_{\operatorname{Spec} k} Y$ is just the product prevariety $X \times Y$ considered earlier. In fact, this follows from remark 2.3.13 in the affine case, and the glueing is done in the same way for prevarieties and schemes.

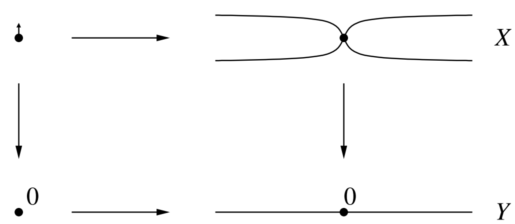
Consequently, we will still use the notation $X \times Y$ to denote the fiber product $X \times_{\operatorname{Spec} k} Y$ over $\operatorname{Spec} k$. Note however that for general schemes X and Y one also often defines $X \times Y$ to be $X \times_{\operatorname{Spec} \mathbb{Z}} Y$ (see remark 5.3.8). For schemes over k, $X \times_{\operatorname{Spec} k} Y$ and $X \times_{\operatorname{Spec} \mathbb{Z}} Y$ will in general be different (see exercise 5.6.10), so one has to make clear what is meant by the notation $X \times Y$.

Example 5.4.9. Let $Y_1 \to X$ and $Y_2 \to X$ be morphisms of schemes that are "inclusion morphisms", i. e. the Y_i might be open subsets of X, or closed subschemes as in example 5.2.3. Then Then $Y_1 \times_X Y_2$ is defined to be the intersection scheme of Y_1 and Y_2 in X and is usually written $Y_1 \cap Y_2$. For example, if $X = \operatorname{Spec} R$, $Y_1 = \operatorname{Spec} R/I_1$, and $Y_2 = \operatorname{Spec} R/I_2$ as in example 5.2.3, then $Y_1 \cap Y_2$ is $\operatorname{Spec} R/(I_1 + I_2)$, which is consistent with example 5.4.5 (ii).

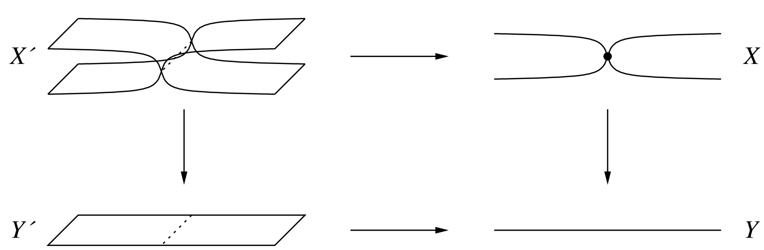
Example 5.4.10. Let Y be a scheme, and let $P \in Y$ be a point. Let $k = k(P)$ be the residue field of P. Then there is a natural morphism $\operatorname{Spec} k \to Y$ that maps the unique point of $\operatorname{Spec} k$ to P and pulls back a section $\varphi \in \mathcal{O}_Y(U)$ (with $P \in U$) to the element in $k(P)$ determined by the composition of maps $\mathcal{O}_Y(U) \to \mathcal{O}_{Y,P} \to k(P)$.

Now let $X \to Y$ be a morphism. Then the fiber product $X \times_Y \operatorname{Spec} k$ (with the morphism $\operatorname{Spec} k \to Y$ constructed above) is called the inverse image or **fiber** of $X \to Y$ over the point $P \in Y$ (hence the name "fiber product").

As an example, consider the morphism $X = \mathbb{A}^1_{\mathbb{C}} \to Y = \mathbb{A}^1_{\mathbb{C}}$ given by $x \mapsto y = x^2$. Over the point $0 \in Y$ the fiber is then $\operatorname{Spec}(\mathbb{C}[x] \otimes_{\mathbb{C}[y]} \mathbb{C})$, where the maps are given by $y \in \mathbb{C}[y] \mapsto x^2 \in \mathbb{C}[x]$ and $y \in \mathbb{C}[y] \mapsto 0 \in \mathbb{C}$. This tensor product is equal to $\mathbb{C}[x]/(x^2)$, so the fiber over 0 is the double point $\operatorname{Spec} \mathbb{C}[x]/(x^2)$; it is a non-reduced scheme and therefore different from the set-theoretic inverse image of 0 as defined earlier for prevarieties.



Example 5.4.11. Continuing the above example, one might want to think of a morphism $X \to Y$ as some sort of fibered object, giving a scheme $X \times_Y \operatorname{Spec} k(P)$ for every point $P \in Y$. (This is analogous to fibered objects in topology.) Now let $f : Y' \to Y$ be any morphism. Then the fiber product $X' = X \times_Y Y'$ has a natural projection morphism to Y', and its fiber over a point $P \in Y'$ is equal to the fiber of $X \to Y$ over the point $P \in Y$. This is usually called a **base extension** of the morphism $X \to Y$. (It corresponds to e. g. the pull-back of a vector bundle in topology.)



5.5. **Projective schemes.** We know that projective varieties are a special important class of varieties that are not affine, but still can be described globally without using glueing techniques. They arise from looking at *homogeneous* ideals, i. e. graded coordinate rings. A completely analogous construction exists in the category of schemes, starting with a graded ring and looking at homogeneous ideals in it.

Definition 5.5.1. Let R be a **graded ring** (think of the homogeneous coordinate ring $S(X)$ of a projective variety X), i. e. a ring together with a decomposition $R = \bigoplus_{d \geq 0} R^{(d)}$ into abelian groups such that $R^{(d)} \cdot R^{(e)} \subset R^{(d+e)}$. An element of $R^{(d)}$ is called homogeneous of degree d. An ideal $I \subset R$ is called homogeneous if it can be generated by homogeneous elements. Let R_+ be the ideal $\bigoplus_{d>0} R^{(d)}$.

We define the set $\operatorname{Proj} R$ to be the set of all homogeneous prime ideals $\mathfrak{p} \subset R$ with $R_+ \not\subset \mathfrak{p}$ (compare this to theorem 3.2.6; R_+ corresponds to the "irrelevant ideal" $(x_0, \dots, x_n) \subset k[x_0, \dots, x_n]$). If $I \subset R$ is a homogeneous ideal, we define $Z(I) = \{\mathfrak{p} \in \operatorname{Proj} R \,;\, \mathfrak{p} \supset I\}$ to be the **zero locus** of I.

The proof of the following lemma is the same as in the case of affine or projective varieties:

Lemma 5.5.2. *Let R be a graded ring.*

(i) *If $\{I_i\}$ is a family of homogeneous ideals of R then $\bigcap_i Z(I_i) = Z(\sum_i I_i) \subset \operatorname{Proj} R$.*
(ii) *If $I_1, I_2 \subset R$ are homogeneous ideals then $Z(I_1) \cup Z(I_2) = Z(I_1 I_2) \subset \operatorname{Proj} R$.*

In particular, we can define a topology on $\operatorname{Proj} R$ by taking the subsets of the form $Z(I)$ for some I to be the closed sets. Of course, the next thing to do is to define a structure of (locally) ringed space on $\operatorname{Proj} R$. This is in complete analogy to the affine case.

Next we have to define the rings of regular functions on $\operatorname{Proj} R$. This is a mixture of the case of affine schemes and projective varieties. We will more or less copy definition 5.1.11 for affine schemes, keeping in mind that in the projective (i. e. homogeneous) case our functions should locally be quotients of *homogeneous* elements of R *of the same degree*.

Definition 5.5.3. Let R be a graded ring, and let $X = \operatorname{Proj} R$. For every $\mathfrak{p} \in \operatorname{Proj} R$, let

$$R_{(\mathfrak{p})} = \left\{ \frac{f}{g} \; ; \; g \notin \mathfrak{p} \text{ and } f, g \in R^{(d)} \text{ for some } d \right\}$$

be the ring of degree zero elements of the localization of R with respect to the multiplicative system of all homogeneous elements of R that are not in $\mathfrak{p}$. (Of course, this will correspond to the local ring at the point $\mathfrak{p}$, see proposition 5.5.4 below.)

Now for every open subset $U \subset X$ we define $\mathcal{O}_X(U)$ to be

$$\begin{aligned} \mathcal{O}_X(U) := \{ & \varphi = (\varphi_\mathfrak{p})_{\mathfrak{p} \in U} \text{ with } \varphi_\mathfrak{p} \in R_{(\mathfrak{p})} \text{ for all } \mathfrak{p} \in U \\ & \text{such that "}\varphi \text{ is locally of the form } \tfrac{f}{g} \text{ for } f, g \in R^{(d)} \text{ for some } d\text{"} \} \\ = \{ & \varphi = (\varphi_\mathfrak{p})_{\mathfrak{p} \in U} \text{ with } \varphi_\mathfrak{p} \in R_{(\mathfrak{p})} \text{ for all } \mathfrak{p} \in U \\ & \text{such that for every } \mathfrak{p} \in U \text{ there is a neighborhood } V \text{ in } U \text{ and } f, g \in R^{(d)} \\ & \text{for some } d \text{ with } g \notin \mathfrak{q} \text{ and } \varphi_\mathfrak{q} = \tfrac{f}{g} \in R_{(\mathfrak{q})} \text{ for all } \mathfrak{q} \in V. \} \end{aligned}$$

It is clear from the local nature of the definition of $\mathcal{O}_X(U)$ that $\mathcal{O}_X$ is a sheaf.

Proposition 5.5.4. *Let R be a graded ring.*

(i) *For every $\mathfrak{p} \in \operatorname{Proj} R$ the stalk $\mathcal{O}_{X,\mathfrak{p}}$ is isomorphic to the local ring $R_{(\mathfrak{p})}$.*
(ii) *For every homogeneous $f \in R_+$, let $X_f \subset X$ be the distinguished open subset*

$$X_f := X \backslash Z(f) = \{ \mathfrak{p} \in \operatorname{Proj} R \; ; \; f \notin \mathfrak{p} \}.$$

These open sets cover X, and for each such open set we have an isomorphism of locally ringed spaces $(X_f, \mathcal{O}_X|_{X_f}) \cong \operatorname{Spec} R_{(f)}$, where

$$R_{(f)} = \left\{ \frac{g}{f^r} \; ; \; g \in R^{(r \cdot \deg f)} \right\}$$

is the ring of elements of degree zero in the localized ring R_f.

In particular, $\operatorname{Proj} R$ *is a scheme.*

Proof. (i): There is a well-defined homomorphism

$$\mathcal{O}_{X,\mathfrak{p}} \to R_{(\mathfrak{p})}, \; (U, \varphi) \mapsto \varphi(\mathfrak{p}).$$

The proof that this is an isomorphism is the same as in the affine case (see proposition 5.1.12 (i).

(ii): Let $\mathfrak{p} \in X$ be a point. By definition, $R_+ \not\subset \mathfrak{p}$, so there is a $f \in R_+$ with $f \notin \mathfrak{p}$. But then $\mathfrak{p} \in X_f$; hence the open subsets of the form X_f cover X.

Now fix $f \in R_+$; we will define an isomorphism $\psi : X_f \to \operatorname{Spec} R_{(f)}$. For any homogeneous ideal $I \subset R$, set $\psi(I) := (IR_f) \cap R_{(f)}$. In particular, restricting this to prime ideals gives a map of sets $X_f \to \operatorname{Spec} R_{(f)}$, which is easily seen to be a bijection. Moreover, if $I \subset R$ is any ideal then $\psi(\mathfrak{p}) \supset \psi(I)$ if and only if $\mathfrak{p} \supset I$, so $\psi : X_f \to \operatorname{Spec} R_{(f)}$ is a homeomorphism. Note also that for $\mathfrak{p} \in X_f$ the local rings

$$\mathscr{O}_{\operatorname{Proj} R,\mathfrak{p}} = R_{(\mathfrak{p})} = \left\{ \frac{g}{h} \,;\, g \text{ and } h \text{ homogeneous of the same degree, } h \notin \mathfrak{p} \right\}$$

and

$$\begin{aligned} \mathscr{O}_{\operatorname{Spec} R_{(f)},\psi(\mathfrak{p})} &= (R_{(f)})_{\psi(\mathfrak{p})} \\ &= \left\{ \frac{g/f^r}{h/f^s} \,;\, g \text{ and } h \text{ homogeneous of degrees } r \cdot \deg f \text{ and } s \cdot \deg f,\, h \notin \mathfrak{p} \right\} \end{aligned}$$

are isomorphic for $f \notin \mathfrak{p}$. This gives rise to isomorphisms between the rings of regular functions $\mathscr{O}_{X_f}(U)$ and $\mathscr{O}_{\operatorname{Spec} R_{(f)}}(U)$ (as they are by definition made up of the local rings). □

Example 5.5.5. If k is an algebraically closed field, then by construction $\operatorname{Proj} k[x_0,\dots,x_n]$ is the scheme that corresponds to projective n-space $\mathbb{P}^n_k$ over k. More generally, the scheme associated to a projective variety X is just $\operatorname{Proj} S(X)$, where $S(X) = k[x_0,\dots,x_n]/I(X)$ is the homogeneous coordinate ring of X.

Of course, scheme-theoretically we can now also consider schemes that are of the form $\operatorname{Proj} k[x_0,\dots,x_n]/I$ where I is any homogeneous ideal of the polynomial ring. This allows projective "subschemes of $\mathbb{P}^n$" that are not necessarily irreducible or reduced. Let us turn this into a definition.

Definition 5.5.6. Let k be an algebraically closed field. A **projective subscheme** of $\mathbb{P}^n_k$ is a scheme of the form $\operatorname{Proj} k[x_0,\dots,x_n]/I$ for some homogeneous ideal I.

As mentioned above, every projective variety is a projective subscheme of $\mathbb{P}^n$. However, the category of projective subschemes of $\mathbb{P}^n$ is bigger because it contains schemes that are reducible (e. g. the union of the coordinate axes in the plane $\operatorname{Proj} k[x_0,x_1,x_2]/(x_1x_2)$) or non-reduced (e. g. the double point $\operatorname{Proj} k[x_0,x_1]/(x_1^2)$).

As in the case of projective varieties, we now want to make precise the relation between projective subschemes of $\mathbb{P}^n$ and homogeneous ideals in $k[x_0,\dots,x_n]$. Note that the existence of the irrelevant ideal $(x_0,\dots,x_n)$ implies that this correspondence is not one-to-one: the example $\operatorname{Proj} k[x_0,\dots,x_n]/(f) = \operatorname{Proj} k[x_0,\dots,x_n]/(fx_0,\dots,fx_n)$ of remark 3.1.11 works for schemes as well.

Definition 5.5.7. Let $I \subset S = k[x_0,\dots,x_n]$ be a homogeneous ideal. The **saturation** $\bar{I}$ of I is defined to be

$$\bar{I} = \{ s \in S \,;\, x_i^m \cdot s \in I \text{ for some } m \text{ and all } i \}.$$

Example 5.5.8. If $I = (fx_0,\dots,fx_n)$ then $\bar{I} = (f)$. So in this case the saturation removes the ambiguity of the ideal associated to a projective subscheme of $\mathbb{P}^n$. We will now show that this is true in general:

Lemma 5.5.9. *Let $I,J \subset S = k[x_0,\dots,x_n]$ be homogeneous ideals. Then*

(i) *$\bar{I}$ is a homogeneous ideal.*
(ii) $\operatorname{Proj} S/I = \operatorname{Proj} S/\bar{I}$.
(iii) $\operatorname{Proj} S/\bar{I} = \operatorname{Proj} S/\bar{J}$ *if and only if* $\bar{I} = \bar{J}$.
(iv) $I^{(d)} = \bar{I}^{(d)}$ *for $d \gg 0$. Here and in the following we say that a statement holds for $d \gg 0$ if and only if it holds for large enough d, i. e. if and only if there is a number $D \geq 0$ such that the statement holds for all $d \geq D$.*

Proof. (i): Let $s \in \bar{I}$ any (possibly non-homogeneous) element. Then by definition $x_i^m \cdot s \in I$ for some m and all i. As I is homogeneous, it follows that the graded pieces $x_i^m \cdot s^{(d)}$ are in I as well for all d. Therefore, by definition, it follows that $s^{(d)} \in \bar{I}$ for all i. Hence $\bar{I}$ is homogeneous.

(ii): As the open affines $U_i := \{x_i \neq 0\} \subset \mathbb{P}^n$ cover $\mathbb{P}^n$, it suffices to show that $U_i \cap \operatorname{Proj} S/I = U_i \cap \operatorname{Proj} S/\bar{I}$. But this is obvious as $I|_{x_i=1} = \bar{I}_{x_i=1}$.

(iii): The direction "$\Rightarrow$" is trivial. For "$\Leftarrow$" it suffices to show that the saturated ideal $\bar{I}$ can be recovered from the projective scheme $X = \operatorname{Proj} S/\bar{I}$ alone. Thinking of projective varieties, $\bar{I}$ should just be "the ideal $I(X)$ of X", i. e. the ideal of functions vanishing on X. Now the elements of S do not define functions on X, but after setting one x_i equal to 1 they do define functions on $X \cap U_i$. Hence we can recover $\bar{I}$ from X as

$$\bar{I} = \{s \in S \,;\, s|_{x_i=1} = 0 \text{ on } X \cap U_i \text{ for all } i\}$$

(note that the right hand side depends only on the scheme X and not on its representation as $\operatorname{Proj} S/I$ for a certain I.

(iv): The inclusion $I^{(d)} \subset \bar{I}^{(d)}$ is obvious (for all d) as $I \subset \bar{I}$. So we only have to show that $\bar{I}^{(d)} \subset I^{(d)}$ for $d \gg 0$.

First of all note that $\bar{I}$ is finitely generated; let $f_1, \dots, f_m$ be (homogeneous) generators. Let D_1 be the maximum degree of the f_i. Next, by definition of $\bar{I}$ there is a number D_2 such that $x_j^d \cdot f_i \in I$ for all $0 \leq j \leq n$, $1 \leq i \leq m$, and $d \geq D_2$. Set $D = D_1 + (n+1)D_2$.

Now let $f \in \bar{I}^{(d)}$ be any homogeneous element in the saturation of degree $d \geq D$. We can write f as $\sum_i a_i f_i$, with the a_i homogeneous of degree at least $(n+1)D_2$. This degree bound implies that every monomial of a_i contains at least one x_j with a power of at least D_2. But then this power multiplied with f_i lies in I by construction. So it follows that $a_i f_i \in I$ for all i, and therefore $f \in I^{(d)}$. □

Definition 5.5.10. If X is a projective subscheme of $\mathbb{P}^n$, we let $I(X)$ be the saturation of any ideal $I \subset k[x_0, \dots, x_n]$ such that $X = \operatorname{Proj} k[x_0, \dots, x_n]/I$. (This is well-defined by lemma 5.5.9 (iii) and generalizes the notion of the ideal of a projective variety to projective subschemes of $\mathbb{P}^n$.) We define $S(X)$ to be $k[x_0, \dots, x_n]/I(X)$. As usual, we call $I(X)$ the ideal of X and $S(X)$ the homogeneous coordinate ring of X.

Corollary 5.5.11. *There is a one-to-one correspondence between projective subschemes of $\mathbb{P}_k^n$ and saturated homogeneous ideals in $k[x_0, \dots, x_n]$, given by $X \mapsto I(X)$ and $I \mapsto$ $\operatorname{Proj} k[x_0, \dots, x_n]/I$.*

5.6. **Exercises.**

Exercise 5.6.1. Find all closed points of the *real* affine plane $\mathbb{A}^2_{\mathbb{R}}$. What are their residue fields?

Exercise 5.6.2. Let $f(x,y) = y^2 - x^2 - x^3$. Describe the affine scheme $X = \operatorname{Spec} R/(f)$ set-theoretically for the following rings R:

(i) $R = \mathbb{C}[x,y]$ (the standard polynomial ring),
(ii) $R = \mathbb{C}[x,y]_{(x,y)}$ (the localization of the polynomial ring at the origin),
(iii) $R = \mathbb{C}[[x,y]]$ (the ring of formal power series).

Interpret the results geometrically. In which of the three cases is X irreducible?

Exercise 5.6.3. For each of these cases below give an example of an affine scheme X with that property, or prove that such an X does not exist:

(i) X has infinitely many points, and $\dim X = 0$.
(ii) X has exactly one point, and $\dim X = 1$.
(iii) X has exactly two points, and $\dim X = 1$.

(iv) $X = \operatorname{Spec} R$ with $R \subset \mathbb{C}[x]$, and $\dim X = 2$.

Exercise 5.6.4. Let X be a scheme, and let Y be an irreducible closed subset of X. If η_Y is the generic point of Y, we write $\mathscr{O}_{X,Y}$ for the stalk $\mathscr{O}_{X,\eta_Y}$. Show that $\mathscr{O}_{X,Y}$ is "the ring of rational functions on X that are regular at a general point of Y", i. e. it is isomorphic to the ring of equivalence classes of pairs (U,φ), where $U \subset X$ is open with $U \cap Y \neq \emptyset$ and $\varphi \in \mathscr{O}_X(U)$, and where two such pairs (U,φ) and (U',φ') are called equivalent if there is an open subset $V \subset U \cap U'$ with $V \cap Y \neq \emptyset$ such that $\varphi|_V = \varphi|_{V'}$.

(In particular, if X is a scheme that is a variety, then $\mathscr{O}_{X,\eta_X}$ is the function field of X as defined earlier. Hence the stalks of the structure sheaf of a scheme generalize both the concepts of the local rings and the function field of a variety.)

Exercise 5.6.5. Let X be a scheme of finite type over an algebraically closed field k. Show that the closed points of X are dense in every closed subset of X. Conversely, give an example of a scheme X such that the closed points of X are not dense in X.

Exercise 5.6.6. Let $X = \{(x,y,z) \in \mathbb{C}^3 \,;\, xy = xz = yz = 0\}$ be the union of the three coordinate lines in $\mathbb{C}^3$. Let $Y = \{(x,y) \in \mathbb{C}^2 \,;\, xy(x-y) = 0\}$ be the union of three concurrent lines in $\mathbb{C}^2$.

Are X and Y isomorphic as schemes? (Hint: Define and compute the tangent spaces of X and Y at the origin.)

Exercise 5.6.7. Let $X \subset \mathbb{P}^3$ the complex cubic surface

$$X = \{(x_0 : x_1 : x_2 : x_3) \,;\, x_0^3 = x_1 x_2 x_3\}.$$

(i) Show that X is singular.
(ii) Let $M \subset G(1,3)$ be the subset of the Grassmannian of lines in $\mathbb{P}^3$ that corresponds to all lines in $\mathbb{P}^3$ that lie in X. By writing down explicit equations for M, show that M has the structure of a scheme in a natural way.
(iii) Show that the scheme M contains exactly 3 points, but that it has length 27 over $\mathbb{C}$, i. e. it is of the form $M = \operatorname{Spec} R$ with R a 27-dimensional $\mathbb{C}$-algebra. Hence in a certain sense we can say that even the singular cubic surface X contains exactly 27 lines, if we count the lines with their correct multiplicities.

Exercise 5.6.8. Let k be an algebraically closed field. An n-fold point (over k) is a scheme of the form $X = \operatorname{Spec} R$ such that X has only one point and R is a k-algebra of vector space dimension n over k (i. e. X has length n). Show that every double point is isomorphic to $\operatorname{Spec} k[x]/(x^2)$. On the other hand, find two non-isomorphic triple points over k, and describe them geometrically.

Exercise 5.6.9. Show that for a scheme X the following are equivalent:

(i) X is reduced, i. e. for every open subset $U \subset X$ the ring $\mathscr{O}_X(U)$ has no nilpotent elements.
(ii) For any open subset U_i of an open affine cover $\{U_i\}$ of X, the ring $\mathscr{O}_X(U_i)$ has no nilpotent elements.
(iii) For every point $P \in X$ the local ring $\mathscr{O}_{X,P}$ has no nilpotent elements.

Exercise 5.6.10. Show that $\mathbb{A}^2_{\mathbb{C}} \not\cong \mathbb{A}^1_{\mathbb{C}} \times_{\operatorname{Spec}\mathbb{Z}} \mathbb{A}^1_{\mathbb{C}}$.

Exercise 5.6.11. Let $X = Z(x_1^2 x_2 + x_1 x_2^2 x_3) \subset \mathbb{A}^3_{\mathbb{C}}$, and denote by π_i the projection to the i-th coordinate. Compute the scheme-theoretic fibers $X_{x_i=a} = \pi_i^{-1}(a)$ for all $a \in \mathbb{C}$, and determine the set of isomorphism classes of these schemes.

Exercise 5.6.12. Let X be a prevariety over an algebraically closed field k, and let $P \in X$ be a (closed) point of X. Let $D = \operatorname{Spec} k[x]/(x^2)$ be the "double point". Show that the tangent space $T_{X,P}$ to X at P can be canonically identified with the set of morphisms $D \to X$ that map the unique point of D to P.

(In particular, this gives the *set* of morphisms $D \to X$ with fixed image point $P \in X$ the structure of a vector space over k. Can you see directly how to add two such morphisms, and how to multiply them with a scalar in k ?)

Exercise 5.6.13. Let X be an affine variety, let Y be a closed subscheme of X defined by the ideal $I \subset A(X)$, and let $\tilde{X}$ be the blow-up of X at I. Show that:

(i) $\tilde{X} = \operatorname{Proj}(\bigoplus_{d \geq 0} I^d)$, where we set $I^0 := A(X)$.

(ii) The projection map $\tilde{X} \to X$ is the morphism induced by the ring homomorphism $I^0 \to \bigoplus_{d \geq 0} I^d$.

(iii) The exceptional divisor of the blow-up, i. e. the fiber $Y \times_X \tilde{X}$ of the blow-up $\tilde{X} \to X$ over Y, is isomorphic to $\operatorname{Proj}(\bigoplus_{d \geq 0} I^d / I^{d+1})$.

Exercise 5.6.14. Let $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$ be affine schemes. Show that the disjoint union $X \sqcup Y$ is an affine scheme with

$$X \sqcup Y = \operatorname{Spec}(R \times S),$$

where as usual $R \times S = \{(r,s) \; ; \; r \in R, s \in S\}$ (with addition and multiplication defined componentwise).

6. First applications of scheme theory

To every projective subscheme of $\mathbb{P}^n_k$ we associate the Hilbert function $h_X : \mathbb{Z} \to \mathbb{Z}$, $d \mapsto \dim_k S(X)^{(d)}$. For large d the Hilbert function is a polynomial in d of degree $\dim X$, the so-called Hilbert polynomial χ_X.

We define $(\dim X)!$ times the leading coefficient of χ_X to be the degree of X; this is always a positive integer. For zero-dimensional schemes the degree is just the number of points in X counted with their scheme-theoretic multiplicities. The degree is additive for unions of equidimensional schemes and multiplicative for intersections with hypersurfaces (Bézout's theorem).

We give some elementary applications of Bézout's theorem for plane curves. Among others, we give upper bounds for the numbers of singularities of a plane curve and the numbers of loops of a real *plane curve.*

A divisor on a curve C is just a formal linear combination of points on C with integer coefficients. To every polynomial or rational function on C we can associate a divisor, namely the divisor of "zeros minus poles" of the polynomial or function. The group of all divisors modulo the subgroup of divisors of rational functions is called the Picard group $\operatorname{Pic} C$ of C.

We show that the degree-0 part of $\operatorname{Pic} C$ is trivial for $C = \mathbb{P}^1$, whereas it is bijective to C itself if C is a smooth plane cubic curve. This defines a group structure on such cubic curves that can also be interpreted geometrically. In complex analysis, plane cubic curves appear as complex tori of the form $\mathbb{C}/\Lambda$, where Λ is a rank-2 lattice in $\mathbb{C}$.

Finally, we give a short outlook to the important parts of algebraic geometry that have not been covered yet in this class.

6.1. **Hilbert polynomials.** In this section we will restrict our attention to projective subschemes of $\mathbb{P}^n$ over some fixed algebraically closed field. Let us start by defining some numerical invariants associated to a projective subscheme of $\mathbb{P}^n$.

Definition 6.1.1. Let X be a projective subscheme of $\mathbb{P}^n_k$. Note that the homogeneous coordinate ring $S(X)$ is a graded ring, and that each graded part $S(X)^{(d)}$ is a finite-dimensional vector space over k. We define the **Hilbert function** of X to be the function

$$\begin{aligned} h_X : \mathbb{Z} &\to \mathbb{Z} \\ d &\mapsto h_X(d) := \dim_k S(X)^{(d)}. \end{aligned}$$

(Note that we trivially have $h_X(d) = 0$ for $d < 0$ and $h_X(d) \geq 0$ for $d \geq 0$, so we will often consider h_X as a function $h_X : \mathbb{N} \to \mathbb{N}$.)

Example 6.1.2. Let $X = \mathbb{P}^n$ be projective space itself. Then $S(X) = k[x_0, \dots, x_n]$, so the Hilbert function $h_X(d) = \binom{d+n}{n}$ is just the number of degree-d monomials in $n+1$ variables $x_0, \dots, x_n$. In particular, note that $h_X(d) = \frac{(d+n)(d+n-1)\cdots(d+1)}{n!}$ is a polynomial in d of degree n with leading coefficient $\frac{1}{n!}$ (compare this to proposition 6.1.5).

Example 6.1.3. Let us now consider some examples of zero-dimensional schemes.

(i) Let $X = \{(1:0),(0:1)\} \subset \mathbb{P}^1$ be two points in $\mathbb{P}^1$. Then $I(X) = (x_0 x_1)$. So a basis of $S(X)^{(d)}$ is given by $\{1\}$ for $d = 0$, and $\{x_0^d, x_1^d\}$ for $d > 0$. We conclude that
$$h_X(d) = \begin{cases} 1 & \text{for } d = 0, \\ 2 & \text{for } d > 0. \end{cases}$$

(ii) Let $X = \{(1:0:0),(0:1:0),(0:0:1)\} \subset \mathbb{P}^2$ be three points in $\mathbb{P}^2$ that are not on a line. Then $I(X) = (x_0x_1, x_0x_2, x_1x_2)$. So in the same way as in (i), a basis of

$S(X)^{(d)}$ is given by $\{1\}$ for $d=0$ and $\{x_0^d, x_1^d, x_2^d\}$ for $d>0$. Therefore

$$h_X(d) = \begin{cases} 1 & \text{for } d=0, \\ 3 & \text{for } d>0. \end{cases}$$

(iii) Let $X = \{(1:0),(0:1),(1:1)\} \subset \mathbb{P}^1$ be three *collinear* points. Then $I(X) = (x_0x_1(x_0-x_1))$. The relation $x_0^2x_1 = x_0x_1^2$ allows us to reduce the number of x_0 in a monomial $x_0^ix_1^j$ provided that $i \geq 2$ and $j \geq 1$. So a basis of $S(X)^{(d)}$ is given by $\{1\}$ for $d=0$, $\{x_0,x_1\}$ for $d=1$, and $\{x_0^d, x_0x_1^{d-1}, x_1^d\}$ for $d>1$. Hence

$$h_X(d) = \begin{cases} 1 & \text{for } d=0, \\ 2 & \text{for } d=1, \\ 3 & \text{for } d>1. \end{cases}$$

It is easy to see that we get the same result for three collinear points in $\mathbb{P}^2$. So comparing this with (ii) we conclude that the Hilbert function does not only depend on the scheme X up to isomorphism, but also on the way the scheme is embedded into projective space.

(iv) Let $X \subset \mathbb{P}^1$ be the "double point" given by the ideal $I(X) = (x_0^2)$. A basis of $S(X)^{(d)}$ is given by $\{1\}$ for $d=0$ and $\{x_0x_1^{d-1}, x_1^d\}$ for $d>0$, so it follows that

$$h_X(d) = \begin{cases} 1 & \text{for } d=0, \\ 2 & \text{for } d>0. \end{cases}$$

just as in (i). So the double point "behaves like two separate points" for the Hilbert function.

So we see that in these examples the Hilbert function becomes constant for d large enough, whereas its initial values for small d may be different. We will now show that this is what happens in general *for zero-dimensional schemes*:

Lemma 6.1.4. *Let X be a zero-dimensional projective subscheme of $\mathbb{P}^n$. Then*

(i) *X is affine, so equal to* $\operatorname{Spec} R$ *for some k-algebra R.*
(ii) *This k-algebra R is a finite-dimensional vector space over k. Its dimension is called the* ***length*** *of X and can be interpreted as the number of points in X (counted with their scheme-theoretic multiplicities).*
(iii) $h_X(d) = \dim_k R$ *for $d \gg 0$. In particular, $h_X(d)$ is constant for large values of d.*

Proof. (i): As X is zero-dimensional, we can find a hyperplane that does not intersect X. Then $X = X \backslash H$ is affine by proposition 5.5.4 (ii).

(ii): First we may assume that X is irreducible, i. e. consists of only one point (but may have a non-trivial scheme structure), since in the reducible case $X = X_1 \sqcup \cdots \sqcup X_m$ with $X_i = \operatorname{Spec} R_i$ for $i = 1,\dots,m$ we have $R = R_1 \times \cdots \times R_m$ by exercise 5.6.14. Moreover, by a change of coordinates we can assume that this point is the origin in $\mathbb{A}^n$. If $X = \operatorname{Spec} k[x_1,\dots,x_n]/I$ we then must have $(x_1,\dots,x_n) = \sqrt{I}$ by the Nullstellensatz. It follows that $x_i^d \in I$ for some d and all i. Consequently, every monomial of degree at least $D := d \cdot n$ lies in I (as it must contain at least one x_i with a power of at least d). In other words, $k[x_1,\dots,x_n]/I$ has a basis (as a vector space over k) of polynomials of degree less than D. But the space of such polynomials is finite-dimensional.

(iii): Note that $I(X)$ is simply the homogenization of I. Conversely, I is equal to $I(X)|_{x_0=1}$. So for $d \geq D$ an isomorphism $S^{(d)} \to R$ as vector spaces over k is given by

$$(k[x_0,\dots,x_n]/I(X))^{(d)} \to k[x_1,\dots,x_n]/I, \quad f \mapsto f|_{x_0=1}$$

and the inverse

$$k[x_1,\dots,x_n]/I \mapsto (k[x_0,\dots,x_n]/I(X))^{(d)}, \quad f \mapsto f^h \cdot x_0^{d-\deg f}$$

where f^h denotes the homogenization of a polynomial as in exercise 3.5.3 (note that the second map is well-defined as $k[x_1,\dots,x_n]/I$ has a basis of polynomials of degree less than D). □

We will now discuss the Hilbert function of arbitrary projective subschemes of $\mathbb{P}^n$ (that are not necessarily zero-dimensional).

Proposition 6.1.5. *Let X be a (non-empty) m-dimensional projective subscheme of $\mathbb{P}^n$. Then there is a (unique) polynomial $\chi_X \in \mathbb{Z}[d]$ such that $h_X(d) = \chi_X(d)$ for $d \gg 0$. Moreover,*

(i) *The degree of χ_X is m.*
(ii) *The leading coefficient of χ_X is $\frac{1}{m!}$ times a positive integer.*

Remark 6.1.6. As the Hilbert polynomial is defined in terms of the Hilbert function *for large d*, it suffices to look at the graded parts of $I(X)$ (or $S(X)$) for $d \gg 0$. So by lemma 5.5.9 (iv) we do not necessarily need to take the *saturated* ideal of X for the computation of the Hilbert polynomial. We have as well that

$$\chi_X(d) = \dim_k (k[x_0,\dots,x_n]/I)^{(d)} \quad \text{for } d \gg 0$$

for *any* homogeneous ideal I such that $X = \operatorname{Proj} k[x_0,\dots,x_n]/I$.

Proof. We will prove the proposition by induction on the dimension m of X. The case $m = 0$ follows from lemma 6.1.4, so let us assume that $m > 0$. By a linear change of coordinates we can assume that no component of X lies in the hyperplane $H = \{x_0 = 0\}$. Then there is an exact sequence of graded vector spaces over k

$$0 \longrightarrow k[x_0,\dots,x_n]/I(X) \xrightarrow{\cdot x_0} k[x_0,\dots,x_n]/I(X) \longrightarrow k[x_0,\dots,x_n]/(I(X)+(x_0)) \longrightarrow 0.$$

(if the first map was not injective, there would be a homogeneous polynomial f such that $f \notin I(X)$ but $fx_0 \in I(X)$. We would then have $X = (X \cap Z(f)) \cup (X \cap H)$. But as no irreducible component lies in H by assumption, we must have $X = X \cap Z(f)$, in contradiction to $f \notin I(X)$). Taking the d-th graded part of this sequence (and using remark 6.1.6 for the ideal $I(X)+(x_0)$), we get

$$h_{X\cap H}(d) = h_X(d) - h_X(d-1).$$

for large d. By the induction assumption, $h_{X\cap H}(d)$ is a polynomial of degree $m-1$ for large d whose leading coefficient is $\frac{1}{(m-1)!}$ times a positive integer. We can therefore write

$$h_{X\cap H}(d) = \sum_{i=0}^{m-1} c_i \binom{d}{i} \qquad \text{for } d \gg 0$$

for some constants c_i, where c_{m-1} is a positive integer (note that $\binom{d}{i}$ is a polynomial of degree i in d with leading coefficient $\frac{1}{i!}$). We claim that

$$h_X(d) = c + \sum_{i=0}^{m-1} c_i \binom{d+1}{i+1} \qquad \text{for } d \gg 0$$

for some $c \in \mathbb{Z}$. In fact, this follows by induction on d, as

$$\begin{aligned} h_X(d) &= h_{X\cap H}(d) + h_X(d-1) \\ &= \sum_{i=0}^{m-1} c_i \binom{d}{i} + c + \sum_{i=0}^{m-1} c_i \binom{d}{i+1} \\ &= c + \sum_{i=0}^{m-1} c_i \binom{d+1}{i+1}. \end{aligned}$$

□

The statement of proposition 6.1.5 motivates the following definition:

Definition 6.1.7. Let X be a projective subscheme of $\mathbb{P}^n$. The **degree** $\deg X$ of X is defined to be $(\dim X)!$ times the leading coefficient of the Hilbert polynomial χ_X. (By proposition 6.1.5, this is a positive integer.)

Example 6.1.8.

(i) If X is a zero-dimensional scheme then $\deg X$ is equal to the length of X, i. e. to "the number of points in X counted with their scheme-theoretic multiplicities".
(ii) $\deg \mathbb{P}^n = 1$ by example 6.1.2.
(iii) Let $X = \operatorname{Proj} k[x_0,\dots,x_n]/(f)$ be the zero locus of a homogeneous polynomial. We claim that $\deg X = \deg f$. In fact, taking the d-th graded part of $S(X) = k[x_0,\dots,x_n]/f\cdot k[x_0,\dots,x_n]$ we get

$$\begin{aligned} h_X(d) &= \dim_k k[x_0,\dots,x_n]^{(d)} - \dim_k k[x_0,\dots,x_n]^{(d-\deg f)} \\ &= \binom{d+n}{n} - \binom{d-\deg f+n}{n} \\ &= \frac{1}{n!}\left((d+n)\cdots(d+1) - (d-\deg f+n)\cdots(d-\deg f+1)\right) \\ &= \frac{\deg f}{(n-1)!}\, d^{n-1} + \text{lower order terms.} \end{aligned}$$

Proposition 6.1.9. *Let X_1 and X_2 be m-dimensional projective subschemes of $\mathbb{P}^n$, and assume that* $\dim(X_1 \cap X_2) < m$. *Then* $\deg(X_1 \cup X_2) = \deg X_1 + \deg X_2$.

Proof. For simplicity of notation let us set $S = k[x_0,\dots,x_n]$. Note that

$$X_1 \cap X_2 = \operatorname{Proj} S/(I(X_1)+I(X_2)) \quad \text{and} \quad X_1 \cup X_2 = \operatorname{Proj} S/(I(X_1)\cap I(X_2)).$$

So from the exact sequence

$$\begin{array}{ccccccccc} 0 & \to & S/(I(X_1)\cap I(X_2)) & \to & S/I(X_1)\oplus S/I(X_2) & \to & S/(I(X_1)+I(X_2)) & \to & 0 \\ & & f & \mapsto & (f,f) & & & & \\ & & & & (f,g) & \mapsto & f-g & & \end{array}$$

we conclude that

$$h_{X_1}(d) + h_{X_2}(d) = h_{X_1\cup X_2}(d) + h_{X_1\cap X_2}(d)$$

for large d. In particular, the same equation follows for the Hilbert polynomials. Comparing only the leading (i. e. d^m) coefficient we then get the desired result, since the degree of $\chi_{X_1\cap X_2}$ is less than m by assumption. □

Example 6.1.10. Let X be a projective subscheme of $\mathbb{P}^n$. We call

$$g(X) := (-1)^{\dim X}\cdot(\chi_X(0)-1)$$

the **(arithmetic) genus** of X. The importance of this number comes from the following two facts (that we unfortunately cannot prove yet with our current techniques):

(i) The genus of X is independent of the projective embedding, i. e. if X and Y are isomorphic projective subschemes then $g(X) = g(Y)$. See section 6.6.3 and exercise 10.6.8 for more details.
(ii) If X is a smooth curve over $\mathbb{C}$, then $g(X)$ is precisely the "topological genus" introduced in example 0.1.1. (Compare for example the degree-genus formula of example 0.1.3 with exercise 6.7.3 (ii).)

Remark 6.1.11. In general, the explicit computation of the Hilbert polynomial h_X of a projective subscheme $X = \operatorname{Proj} k[x_0,\dots,x_n]/I$ from the ideal I is quite complicated and requires methods of computer algebra.

6.2. **Bézout's theorem.** We will now prove the main property of the degree of a projective variety: that it is "multiplicative when taking intersections". We will prove this here only for intersections with hypersurfaces, but there is a more general version about intersections in arbitrary codimension (see e. g. cite Ha theorem 18.4).

Theorem 6.2.1. *(Bézout's theorem) Let X be a projective subscheme of $\mathbb{P}^n$ of positive dimension, and let $f \in k[x_0,\dots,x_n]$ be a homogeneous polynomial such that no component of X is contained in $Z(f)$. Then*

$$\deg(X \cap Z(f)) = \deg X \cdot \deg f.$$

Proof. The proof is very similar to that of the existence of the Hilbert polynomial in proposition 6.1.5. Again we get an exact sequence

$$0 \longrightarrow k[x_0,\dots,x_n]/I(X) \xrightarrow{\cdot f} k[x_0,\dots,x_n]/I(X) \longrightarrow k[x_0,\dots,x_n]/(I(X)+(f)) \longrightarrow 0$$

from which it follows that

$$\chi_{X\cap Z(f)} = \chi_X(d) - \chi_X(d-\deg f).$$

But we know that

$$\chi_X(d) = \frac{\deg X}{m!}\, d^m + c_{m-1}d^{m-1} + \text{terms of order at most } d^{m-2},$$

where $m = \dim X$. Therefore it follows that

$$\begin{aligned}\chi_{X\cap Z(f)} &= \frac{\deg X}{m!}\,(d^m - (d-\deg f)^m) + c_{m-1}\,(d^{m-1} - (d-\deg f)^{m-1}) \\ &\qquad + \text{terms of order at most } d^{m-2} \\ &= \frac{\deg X}{m!}\cdot m \deg f \cdot d^{m-1} + \text{terms of order at most } d^{m-2}.\end{aligned}$$

We conclude that $\deg(X \cap Z(f)) = \deg X \cdot \deg f$. □

Example 6.2.2. Let C_1 and C_2 be two curves in $\mathbb{P}^2$ without common irreducible components. These curves are then given as the zero locus of homogeneous polynomials of degrees d_1 and d_2, respectively. We conclude that $\deg(C_1 \cap C_2) = d_1 \cdot d_2$ by Bézout's theorem. By example 6.1.8 (i) this means that C_1 and C_2 intersect in exactly $d_1 \cdot d_2$ points, if we count these points with their scheme-theoretic multiplicities in the intersection scheme $C_1 \cap C_2$. In particular, as these multiplicities are always positive integers, it follows that C_1 and C_2 intersect set-theoretically in *at most* $d_1 \cdot d_2$ points, and in *at least* one point. This special case of theorem 6.2.1 is also often called Bézout's theorem in textbooks.

Example 6.2.3. In the previous example, the scheme-theoretic multiplicity of a point in the intersection scheme $C_1 \cap C_2$ is often easy to read off from geometry: let $P \in C_1 \cap C_2$ be a point. Then:

(i) If C_1 and C_2 are smooth at P and have different tangent lines at P then P counts with multiplicity 1 (we say: the intersection multiplicity of C_1 and C_2 at P is 1).

(ii) If C_1 and C_2 are smooth at P and are tangent to each other at P then the intersection multiplicity at P is at least 2.
(iii) If C_1 is singular and C_2 is smooth at P then the intersection multiplicity at P is at least 2.
(iv) If C_1 and C_2 are singular at P then the intersection multiplicity at P is at least 3.

The key to proving these statements is the following. As the computation is local around P we can assume that the curves are affine in $\mathbb{A}^2$, that $P = (0,0)$ is the origin, and that the two curves are given as the zero locus of one equation

$$\begin{aligned} C_1 &= \{f_1 = 0\} && \text{where } f_1 = a_1x + b_1y + \text{higher order terms}, \\ C_2 &= \{f_2 = 0\} && \text{where } f_2 = a_2x + b_2y + \text{higher order terms}. \end{aligned}$$

If both curves are singular at the origin, their tangent space at P must be two-dimensional, i. e. all of $\mathbb{A}^2$. This means that $a_1 = b_1 = a_2 = b_2 = 0$. It follows that 1, x, and y are three linearly independent elements in $k[x,y]/(f_1, f_2)$ (whose spectrum is by definition the intersection scheme). So the intersection multiplicity is at least 3. In the same way, we get at least 2 linearly independent elements (the constant 1 and one linear function) if only one of the curves is singular, or both curves have the same tangent line (i. e. the linear parts of their equations are linearly dependent).

Example 6.2.4. Consider again the twisted cubic curve in $\mathbb{P}^3$

$$\begin{aligned} C &= \{(s^3 : s^2t : st^2 : t^3) \,;\, (s:t) \in \mathbb{P}^1\} \\ &= \{(x_0 : x_1 : x_2 : x_3) \,;\, x_1^2 - x_0x_2 = x_2^2 - x_1x_3 = x_0x_3 - x_1x_2 = 0\}. \end{aligned}$$

We have met this variety as the easiest example of a curve in $\mathbb{P}^3$ that cannot be written as the zero locus of two polynomials. We are now able to prove this statement very easily using Bézout's theorem: assume that $I(C) = (f, g)$ for some homogeneous polynomials f and g. As the degree of C is 3 by exercise 6.7.2, it follows that $\deg f \cdot \deg g = 3$. This is only possible if $\deg f = 3$ and $\deg g = 1$ (or vice versa), i. e. one of the polynomials has to be linear. But C is not contained in a linear space (its ideal does not contain linear functions).

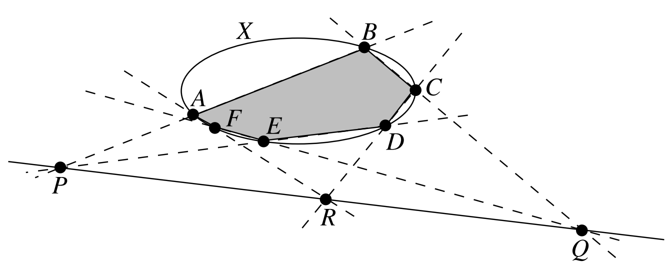
In particular we see that C cannot be the intersection of two of the quadratic polynomials given above, as this intersection must have degree 4. In fact,

$$Z(x_1^2 - x_0x_2, x_2^2 - x_1x_3) = C \cup \{x_1 = x_2 = 0\}$$

in accordance with Bézout's theorem and proposition 6.1.9 (note that $\{x_1 = x_2 = 0\}$ is a line and thus has degree 1).

Let us now prove some corollaries of Bézout's theorem.

Corollary 6.2.5. *(**Pascal's theorem**) Let $X \subset \mathbb{P}^2$ be a conic (i. e. the zero locus of a homogeneous polynomial f of degree 2). Pick six points A,B,C,D,E,F on X that form the vertices of a hexagon inscribed in X. Then the three intersection points of the opposite edges of the hexagon (i. e. $P = \overline{AB} \cap \overline{DE}$, $Q = \overline{BC} \cap \overline{EF}$, and $R = \overline{CD} \cap \overline{FA}$) lie on a line.*



Proof. Consider the two reducible cubics $X_1 = \overline{AB} \cup \overline{CD} \cup \overline{EF}$ and $X_2 = \overline{BC} \cup \overline{DE} \cup \overline{FA}$, and let $f_1 = 0$ and $f_2 = 0$ be the equations of X_1 and X_2, respectively. In accordance with Bézout's theorem, X_1 and X_2 meet in the 9 points A,B,C,D,E,F,P,Q,R.

Now pick any point $S \in X$ not equal to the previously chosen ones. Of course there are $\lambda, \mu \in k$ such that $\lambda f_1 + \mu f_2$ vanishes at S. Set $X' = Z(\lambda f_1 + \mu f_2)$; this is a cubic curve too.

Note that X' meets X in the 7 points A,B,C,D,E,F,S, although $\deg X' \cdot \deg X = 6$. We conclude by Bézout's theorem that X' and X have a common component. For degree reasons the only possibility for this is that the cubic X' is reducible and contains the conic X as a factor. Therefore $X' = X \cup L$, where L is a line.

Finally note that P,Q,R lie on X' as they lie on X_1 and X_2. Therefore $P,Q,R \in X \cup L$. But these points are not on X, so they must be on the line L. □

Corollary 6.2.6. *Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree d. Then C has at most $\binom{d-1}{2}$ singular points.*

Remark 6.2.7. For $d = 1$ C must be a line, so there is no singular point. A conic is either irreducible (and smooth) or a union of two lines, so for $d = 2$ the statement is obvious too. For $d = 3$ the corollary states that there is at most one singular point on an irreducible curve. In fact, the projectivization of the singular cubic affine curve $y^2 = x^2 + x^3$ is such an example with one singular point (namely the origin).

Proof. Assume the contrary and let $P_1, \dots, P_{\binom{d-1}{2}+1}$ be distinct singular points of C. Moreover, pick arbitrary further distinct points $Q_1, \dots, Q_{d-3}$ on C (we can assume $d \geq 3$ by remark 6.2.7). We thus have a total of $\binom{d-1}{2} + 1 + d - 3 = \frac{d^2}{2} - \frac{d}{2} - 1$ points P_i and Q_j.

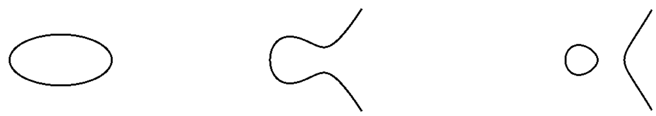
We claim that there is a curve C' of degree $d-2$ that passes through all P_i and Q_j. In fact, the space of all homogeneous degree-$(d-2)$ polynomials in three variables is a $\binom{d}{2}$-dimensional vector space over k, so the space of hypersurfaces of degree $d-2$ is a projective space $\mathbb{P}^N$ of dimension $N = \binom{d}{2} - 1$, with the coefficients of the equation as the homogeneous coordinates. Now the condition that such a hypersurface passes through a given point is obviously a linear condition in this $\mathbb{P}^N$. As N hyperplanes in $\mathbb{P}^N$ always have a non-empty intersection, it follows that there is a hypersurface passing through any N given points. But $N = \binom{d}{2} - 1 = \frac{d^2}{2} - \frac{d}{2} - 1$ is precisely the number of points we have. (Compare this argument to exercise 3.5.8 and the parametrization of cubic surfaces at the beginning of section 4.5.)

Now compute the degree of the intersection scheme $C \cap C'$. By Bézout's theorem, it must be $\deg C \cdot \deg C' = d(d-2)$. Counting the intersection points, we see that we have the $d-3$ points Q_i, and the $\binom{d-1}{2} + 1$ points P_j *that count with multiplicity at least 2 as they are singular points of* C (see example 6.2.3). So we get

$$\deg(C \cap C') \geq (d-3) + 2\left(\binom{d-1}{2} + 1\right) = d^2 - 2d + 1 > \deg C \cdot \deg C'.$$

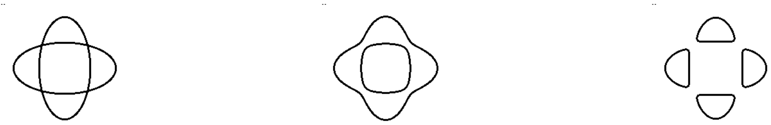
By Bézout's theorem it follows that C and C' must have a common component. But C is irreducible of degree $\deg C > \deg C'$, so this is impossible. We thus arrive at a contradiction and conclude that the assumption of the existence of $\binom{d-1}{2} + 1$ singular points was false. □

The following statement about *real* plane curves looks quite different from corollary 6.2.6, yet the proof is largely identical. Note that every smooth real plane curve consists of a certain number of connected components (in the classical topology); here are examples with one real component (the left two curves) and with two real components (the right curve):



$$\frac{x^2}{4}+y^2-4=0 \qquad y^2-x^2-\frac{x^3}{4}-1=0 \qquad y^2-x^2-\frac{x^3}{4}+1=0$$

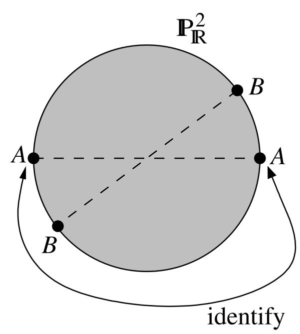
We want to know the maximum number of such components that a real smooth curve of degree d can have. One way of constructing curves with many components is to start with a singular curve, and then to deform the equation a little bit to obtain a smooth curve. The following example starts with a reducible quartic curve and deforms it into a smooth curve with two and four components, respectively.



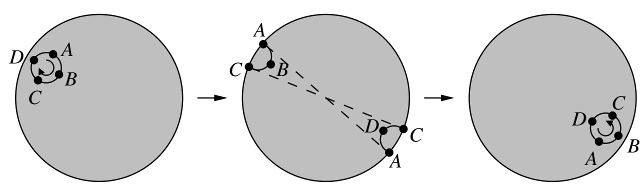
$$\left(\frac{x^2}{4}+y^2-4\right)\left(x^2+\frac{y^2}{4}-4\right)=0 \qquad \left(\frac{x^2}{4}+y^2-4\right)\left(x^2+\frac{y^2}{4}-4\right)=1 \qquad \left(\frac{x^2}{4}+y^2-4\right)\left(x^2+\frac{y^2}{4}-4\right)=-1$$

As in the complex case, it is more convenient to pass to the *projective* plane $\mathbb{P}^2_{\mathbb{R}}$ instead of $\mathbb{A}^2_{\mathbb{R}}$. This will add points at infinity of the curves so that every component becomes a loop (i. e. it has no ends). For example, in the two cubic curves above one point each is added to the curves, so that the components extending to infinity become a loop. We are therefore asking for the maximum number of loops that a *projective* smooth real plane curve of degree d can have.

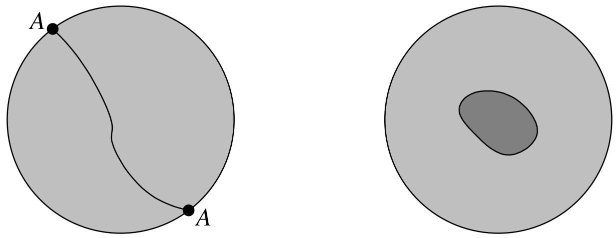
There is an extra topological twist in $\mathbb{P}^2_{\mathbb{R}}$ that we have not encountered before. As usual, we construct $\mathbb{P}^2_{\mathbb{R}}$ by taking $\mathbb{A}^2_{\mathbb{R}}$ (which we will draw topologically as an open disc here) and adding a point at infinity for every direction in $\mathbb{A}^2_{\mathbb{R}}$. This has the effect of adding a boundary to the disc (with the boundary point corresponding to the point at infinity). But note that opposite points of the boundary of the disc belong to the same direction in $\mathbb{A}^2_{\mathbb{R}}$ and hence are the same point in $\mathbb{P}^2_{\mathbb{R}}$. In other words, $\mathbb{P}^2_{\mathbb{R}}$ is topologically equivalent to a closed disc with opposite boundary points identified:



It is easy to see that this is a *non-orientable* surface: if we start with a small circle and move it across the boundary of the disc (i. e. across the infinity locus of $\mathbb{P}^2_{\mathbb{R}}$ then it comes out with opposite orientation:



Consequently, we have two different types of loops. A “type 1 loop” is a loop such that its complement has only one component (which is topologically a disc). A “type 2 loop” is a loop such that its complement has two components (an “interior” and “exterior” of the loop). It is interesting to note that of these two components one is a disc, and the other is a Möbius strip.



Type 1 loop Type 2 loop

(Those of you who know some algebraic topology will note that the homology group $H_1(\mathbb{P}^2_{\mathbb{R}})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$; so the two types of curves correspond to the two elements of $\mathbb{Z}/2\mathbb{Z}$.)

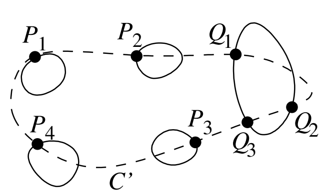
With these prerequisites at hand, we can now prove the following statement (modulo some topology statements that should be intuitively clear):

Corollary 6.2.8. ***(Harnack’s theorem)*** *A real smooth curve in $\mathbb{P}^2_{\mathbb{R}}$ of degree d has at most $\binom{d-1}{2}+1$ loops.*

Remark 6.2.9. A line ($d=1$) has always exactly one loop. A non-empty conic ($d=2$) is a hyperbola, parabola, or ellipse, so in every case the number of loops is 1. For $d=3$ the corollary gives a maximum number of 2 loops, and for $d=4$ we get at most 4 loops. We have just seen examples of these numbers of loops above. One can show that the bound given in Harnack’s theorem is indeed sharp, i. e. for every d one can find smooth real curves of degree d with exactly $\binom{d-1}{2}+1$ loops.

Proof. Assume that the statement is false, so that there are $\binom{d-1}{2}+2$ loops in a smooth real plane curve C. Note that any two type 1 loops must intersect (which is impossible for a *smooth* curve), so there can be at most one type 1 loop. Hence assume that the first $\binom{d-1}{2}+1$ loops are of type 2, and pick one point $P_1,\dots,P_{\binom{d-1}{2}+1}$ on each of them. By remark 6.2.9 we can assume that $d\geq 3$, so pick $d-3$ further distinct points $Q_1,\dots,Q_{d-3}$ on the last loop (which can be of any type). We thus have a total of $\binom{d-1}{2}+1+d-3=\frac{d^2}{2}-\frac{d}{2}-1$ points P_i and Q_j.

As in the proof of corollary 6.2.6 there is a curve C' of degree $d-2$ that passes through all P_i and Q_j. Compute the degree of the intersection scheme $C\cap C'$. By Bézout’s theorem, it must be $\deg C\cdot\deg C'=d(d-2)$. Counting the intersection points, we see that we have the $d-3$ points Q_i, and the $\binom{d-1}{2}+1$ points P_j *that count with multiplicity at least 2 as every type 2 loop divides the real projective plane in an interior and exterior region; so if C' enters the interior of a type 2 loop it must exit it again somewhere*. (It may also be that C' is tangent to the loop or singular at the intersection point, but in this case the intersection multiplicity must be at least 2 too.)



So we get

$$\deg(C \cap C') \geq (d-3) + 2\left(\binom{d-1}{2} + 1\right) = d^2 - 2d + 1 > \deg C \cdot \deg C'.$$

By Bézout's theorem it follows that C and C' must have a common component. But C is irreducible of degree $\deg C > \deg C'$, so this is impossible. We thus arrive at a contradiction and conclude that the assumption of the existence of $\binom{d-1}{2} + 2$ loops was false. □

Corollary 6.2.10. *Every isomorphism $f : \mathbb{P}^n \to \mathbb{P}^n$ is linear, i. e. it is of the form $f(x) = A \cdot x$, where $x = (x_0, \dots, x_n)$ and A is an invertible $(n+1) \times (n+1)$ matrix with elements in the ground field.*

Proof. Let $H \subset \mathbb{P}^n$ be a hyperplane, and let $L \subset \mathbb{P}^n$ be a line not contained in H. Of course, $H \cap L$ is scheme-theoretically just one reduced point. As f is an isomorphism, $f(H) \cap f(L)$ must also be scheme-theoretically one reduced point, i. e. $\deg(f(H) \cap f(L)) = 1$. As degrees are always positive integers, it follows by Bézout's theorem that $\deg f(H) = \deg f(L) = 1$. In particular, f maps hyperplanes to hyperplanes. Applying this to all hyperplanes $\{x_i = 0\}$ in turn, we conclude that f maps all coordinate functions x_i to linear functions, so $f(x) = A \cdot x$ for some scalar matrix A. Of course A must be invertible if f has an inverse. □

6.3. **Divisors on curves.** Bézout's theorem counts the number of intersection points of a projective curve with a hypersurface. For example, if $C \subset \mathbb{P}^2$ is a plane cubic then the intersection of C with any line consists of 3 points (counted with their scheme-theoretic multiplicities). But of course not every collection of three points on C can arise this way, as three points will in general not lie on a line. So by reducing the intersections of curves to just the number of intersection points we are losing information about the possible configurations of intersection schemes. In contrast, we will now present a theory that is able to keep track of the configurations of (intersection) points on curves.

Definition 6.3.1. Let $C \subset \mathbb{P}^n$ be a smooth irreducible projective curve. A **divisor** on C is a formal finite linear combination $D = a_1 P_1 + \cdots + a_m P_m$ of points $P_i \in C$ with integer coefficients a_i. Obviously, divisors can be added and subtracted. The group of divisors on C is denoted $\operatorname{Div} C$.

Equivalently, $\operatorname{Div} C$ is the free abelian group generated by the points of C.

The **degree** $\deg D$ of a divisor $D = a_1 P_1 + \cdots + a_m P_m$ is defined to be the integer $a_1 + \cdots + a_m$. Obviously, the degree function is a group homomorphism $\deg : \operatorname{Div} C \to \mathbb{Z}$.

Example 6.3.2. Divisors on a curve C can be associated to several objects:

(i) Let $Z \subset \mathbb{P}^n$ be a zero-dimensional projective subscheme of $\mathbb{P}^n$, and let $P_1, \dots, P_m$ be the points in Z. Each of these points comes with a scheme-theoretic multiplicity a_i (the length of the component of Z at P_i) which is a positive integer. If the points P_i are on C, then $a_1 P_1 + \cdots a_m P_m$ is a divisor on C which we denote by (Z). It is called the divisor associated to Z.

(ii) Let $f \in k[x_0, \dots, x_n]$ be a homogeneous polynomial such that C is not contained in $Z(f)$. Then $C \cap Z(f)$ is a zero-dimensional subscheme of $\mathbb{P}^n$ whose points lie in C, so by (i) there is an associated divisor $(C \cap Z(f))$ on C. It is called the divisor of f and denoted (f); we can think of it as the zeros of f on C counted with their respective multiplicities. By Bézout's theorem, the number of such zeros is $\deg(f) = \deg C \cdot \deg f$.

(iii) Note that the intersection scheme $C \cap Z(f)$ and therefore the divisor (f) do not change if we add to f an element of the ideal $I(C)$. Hence there is a well-defined divisor (f) for every non-zero $f \in S(C)^{(d)}$.

(iv) Assume that $C \subset \mathbb{P}^2$, and that $C' = Z(f') \subset \mathbb{P}^2$ is another (not necessarily irreducible) curve that does not contain C as a component. Then the divisor (f') is also called the **intersection product** of C and C' and denoted $C \cdot C' \in \operatorname{Div} C$.

Lemma 6.3.3. *Let $C \subset \mathbb{P}^n$ be a smooth irreducible curve, and let $f,g \in S(C)$ be non-zero homogeneous elements in the coordinate ring of C. Then $(fg) = (f) + (g)$.*

Proof. Let $(fg) = a_1P_1 + \cdots + a_mP_m$. It is obvious that set-theoretically the zeros of fg are the union of the zeros of f and g, so f and g vanish at most at the points P_i. Let $(f) = b_1P_1 + \cdots + b_mP_m$ and $(g) = c_1P_1 + \cdots c_mP_m$. We have to show that $a_i = b_i + c_i$ for all $i = 1,\dots,m$.

Fix a certain i and choose an affine open subset $U = \operatorname{Spec} R \subset C$ that contains P_i, but no other zero of fg. Then by definition we have $a_i = \dim_k R/(fg)$, $b_i = \dim_k R/(f)$, and $c_i = \dim_k R/(g)$. The statement now follows from the exact sequence

$$0 \longrightarrow R/(f) \xrightarrow{\cdot g} R/(fg) \xrightarrow{\cdot 1} R/(g) \longrightarrow 0.$$

□

Definition 6.3.4. Let $C \subset \mathbb{P}^n$ be a smooth irreducible curve, and let $\varphi \in K(C)$ be a non-zero rational function. By definition we can write $\varphi = \frac{f}{g}$ for some non-zero $f,g \in S(C)^{(d)}$. We define the divisor of φ to be $(\varphi) = (f) - (g)$ (this is well-defined by lemma 6.3.3). It can be thought of as the zeros minus the poles of the rational function.

Remark 6.3.5. Note that the divisor of a rational function always has degree zero: if $\varphi = \frac{f}{g}$ with $f,g \in S(C)^{(d)}$, then

$$\deg(\varphi) = \deg(f) - \deg(g) = d \deg C - d \deg C = 0$$

by Bézout's theorem.

Example 6.3.6. Let $C = \mathbb{P}^1$, and consider the two homogeneous polynomials $f(x_0,x_1) = x_0x_1$ and $g(x_0,x_1) = (x_0 - x_1)^2$. Then $(f) = P_1 + P_2$ with $P_1 = (1:0)$ and $P_2 = (0:1)$, and $(g) = 2P_3$ with $P_3 = (1:1)$. The quotient $\frac{f}{g}$ defines a rational function φ on $\mathbb{P}^1$ with $(\varphi) = P_1 + P_2 - 2P_3$. We have $\deg(f) = \deg(g) = 2$ and $\deg(\varphi) = 0$ (in accordance with remark 6.3.5).

Remark 6.3.7. By lemma 6.3.3, the map $K(C)\backslash\{0\} \to \operatorname{Div} C$ that sends every rational function φ to its divisor (φ) is a group homomorphism, if we regard $K(C)\backslash\{0\}$ as an abelian group under multiplication. In particular, the subset of $\operatorname{Div} C$ of all divisors of the form (φ) is a subgroup of $\operatorname{Div} C$.

Definition 6.3.8. The **Picard group** (or **divisor class group**) $\operatorname{Pic} C$ of C is defined to be the group $\operatorname{Div} C$ modulo the subgroup of all divisors of the form (φ) for $\varphi \in K(C)\backslash\{0\}$. If $f \in S(C)^{(d)}$, we will usually still write (f) for the divisor class in $\operatorname{Pic} C$ associated to f. Two divisors D_1 and D_2 are said to be **linearly equivalent** if $D_1 - D_2 = 0 \in \operatorname{Pic} C$, i. e. if they define the same divisor class.

Remark 6.3.9. By remark 6.3.5, the degree function $\deg : \operatorname{Div} C \to \mathbb{Z}$ passes to a group homomorphism $\deg : \operatorname{Pic} C \to \mathbb{Z}$. So it makes sense to talk about the degree of a divisor class. We define $\operatorname{Pic}^0 C \subset \operatorname{Pic} C$ to be the group of divisor classes of degree 0.

Remark 6.3.10. The divisor group $\operatorname{Div} C$ is a free (and very "big") abelian group and therefore not very interesting. In contrast, the divisor *class* group $\operatorname{Pic} C$ has quite a rich structure that we want to study now in some easy examples.

Lemma 6.3.11. $\operatorname{Pic} \mathbb{P}^1 \cong \mathbb{Z}$ *(with an isomorphism being the degree homomorphism). In other words, on $\mathbb{P}^1$ all divisors of the same degree are linearly equivalent.*

Proof. Let $D = a_1P_1 + \cdots a_mP_m$ be a divisor of degree zero, i. e. $a_1 + \cdots + a_m = 0$. We have to show that D is the divisor of a rational function. In fact, assume the P_i have homogeneous coordinates $(x_i : y_i)$; then

$$\varphi = \prod_{i=1}^{m} (xy_i - yx_i)^{a_i}$$

is a rational function such that $(\varphi) = D$. □

Let us now move on to more complicated curves. We know already that smooth conics in $\mathbb{P}^2$ are isomorphic to $\mathbb{P}^1$, so their Picard group is isomorphic to the integers too. Let us therefore consider cubic curves in $\mathbb{P}^2$. We will compute $\operatorname{Pic} C$ and show that it is *not* isomorphic to $\mathbb{Z}$ (thereby showing that cubic curves are not isomorphic to $\mathbb{P}^1$). Let us prove a lemma first.

Lemma 6.3.12. *Let $C = Z(f) \subset \mathbb{P}^2$ be a smooth cubic curve, and let $C' = Z(g)$ with $g \in k[x_0,x_1,x_2]^{(d)}$ be another curve that does not have C as a component. Assume that "three points of $C \cap C'$ lie on a line", i. e. that $C \cdot C'$ contains three points P_1, P_2, P_3 (that need not be distinct) such that there is a line $L = Z(l)$ with $C \cdot L = P_1 + P_2 + P_3$. Then there is a polynomial $g' \in k[x_0,x_1,x_2]^{(d-1)}$ such that $g = l \cdot g'$ in $S(C)$.*

Proof. By Bézout's theorem we have $C' \cdot L = P_1 + \cdots + P_d$ for some points P_i (that need not be distinct, but they must contain the first three given points P_1, P_2, P_3). Let $a \in k[x_0,x_1,x_2]^{(d-3)}$ be a homogeneous polynomial such that $Z(a) \cdot L = P_4 + \cdots + P_d$ (it is obvious that this can always be found). Then $Z(af) \cdot L = P_1 + \cdots P_d$ too.

Now pick any point $Q \in L$ distinct from the P_i. As g and af do not vanish at Q, we can find a $\lambda \in k$ such that $g + \lambda af$ vanishes at Q. It follows that $g + \lambda af$ vanishes on L at least at the $d+1$ points $P_1, \dots, P_d, Q$. So it follows by Bézout's theorem that $Z(g + \lambda af)$ contains the line L, or in other words that $g + \lambda af = lg'$ for some g'. Passing to the coordinate ring $S(C) = k[x_0,x_1,x_2]/I(C)$ we get the desired result. □

Proposition 6.3.13. *Let $C \subset \mathbb{P}^2$ be a smooth cubic curve, and let P, Q be distinct points on C. Then $P - Q \neq 0$ in $\operatorname{Pic} C$. In other words, there is no rational function $\varphi \in K(C) \backslash \{0\}$ with $(\varphi) = P - Q$, i. e. no rational function that has exactly one zero which is at P, and exactly one pole which is at Q.*

Remark 6.3.14. It follows from this proposition already that a smooth plane cubic curve is not isomorphic to $\mathbb{P}^1$ (as the statement of the proposition is false for $\mathbb{P}^1$ by lemma 6.3.11).

Proof. Assume the contrary. Then there is a positive integer d and homogeneous polynomials $f, g \in S(C)^{(d)}$ such that

(i) There are points $P_1, \dots, P_{3d-1}$ and $P \neq Q$ such that

$$(f) = P_1 + \cdots + P_{3d-1} + P \quad \text{and} \quad (g) = P_1 + \cdots + P_{3d-1} + Q$$

(hence $(\varphi) = P - Q$ for $\varphi = \frac{f}{g}$).

(ii) Among the $P_1, \dots, P_{3d-1}$ there are at least $2d-1$ distinct points. (If this is not the case in the first place, we can replace f by $f \cdot l$ and g by $g \cdot l$ some linear function l that vanishes on C at three distinct points that are not among the P_i. This raises the degree of the polynomials by 1 and the number of distinct points by 3, so by doing this often enough we can get at least $2d-1$ distinct points.)

Pick d minimal with these properties.

If $d = 1$ then $(f) = P_1 + P_2 + P$ and $(g) = P_1 + P_2 + Q$, so both f and g define the unique line through P_1 and P_2 (or the tangent to C at P_1 if $P_1 = P_2$). In particular, it follows that $P = Q$ as well, which is a contradiction. So we can assume that $d > 1$. We can rearrange the P_i such that $P_2 \neq P_3$, and such that $P_1 = P_2$ if there are any equal points among the P_i.

Now consider curves given by linear combinations $\lambda f + \mu g$. These curves will intersect C at least in the points $P_1, \dots, P_{3d-1}$ (as $Z(f)$ and $Z(g)$ do). Note that for any point $R \in C$ we can adjust λ and μ so that $(\lambda f + \mu g)(R) = 0$. Such a curve will then have intersection divisor $P_1 + \cdots + P_{3d-1} + R$ with C. In other words, by passing to linear combinations of f and g we can assume that the last points P and Q in the divisors of f and g are any two points we like. We choose P to be the third intersection point of $\overline{P_1 P_2}$ with C, and Q to be the third intersection point of $\overline{P_1 P_3}$ with C.

By lemma 6.3.12, it now follows that $f = l \cdot f'$ and $g = l' \cdot g'$ in $S(C)$ for some linear functions l and l' that have intersection divisors $P_1 + P_2 + P$ and $P_1 + P_3 + Q$ with C. Hence

$$(f') = P_4 + \cdots + P_{3d-1} + P_3 \quad \text{and} \quad (g') = P_4 + \cdots + P_{3d-1} + P_2$$

Note that these f' and g' satisfy (i) for d replaced by $d-1$, as $P_2 \neq P_3$ by assumption. Moreover, f' and g' satisfy (ii) because if there are any equal points among the P_i at all, then by our relabeling of the P_i there are only two distinct points among P_1, P_2, P_3, so there must still be at least $2(d-1)-1$ distinct points among $P_4, \dots, P_{3d-1}$.

This contradicts the minimality of d and therefore proves the proposition. □

Corollary 6.3.15. *Let C be a smooth cubic curve, and let $P_0 \in C$ be a point. Then the map*

$$C \to \operatorname{Pic}^0 C, \quad P \mapsto P - P_0$$

is a bijection.

Proof. The map is well-defined and injective by proposition 6.3.13. We will show that it is surjective. Let $D = P_1 + \cdots + P_m - Q_1 - \cdots - Q_m$ be any divisor of degree 0.

If $m > 1$ let P be the third intersection point of $\overline{P_1 P_2}$ with C, and let Q be the third intersection point of $\overline{Q_1 Q_2}$ with C. Then $P_1 + P_2 + P$ and $Q_1 + Q_2 + Q$ are both the divisors of linear forms on C. The quotient of these linear forms is a rational function whose divisor $P_1 + P_2 + P - Q_1 - Q_2 - Q$ is therefore 0 in $\operatorname{Pic} C$. It follows that $D = P_3 + \cdots + P_m + Q - Q_3 - \cdots - Q_m - P$. We have thus reduced the number m of (positive and negative) points in D by 1. Continuing this process, we can assume that $m = 1$, i. e. $D = P - Q$ for some $P, Q \in C$.

Now let P' be the third intersection point of $\overline{PP_0}$ with C, and let Q' be the third intersection point of $\overline{P'Q}$ with C. Then $P' + P + P_0 = P' + Q + Q'$ in $\operatorname{Pic} C$ as above, so $D = P - Q = Q' - P_0$, as desired. □

6.4. **The group structure on a plane cubic curve.** Let $C \subset \mathbb{P}^2$ be a smooth cubic curve. Corollary 6.3.15 gives a canonical bijection between the variety C and the abelian group $\operatorname{Pic}^0 C$, so between two totally different mathematical objects. Using this bijection, we can give C a group structure (after choosing a base point P_0 as in the corollary) and $\operatorname{Pic}^0 C$ the structure of a smooth projective variety.

We should mention that $\operatorname{Pic}^0 C$ can be made into a variety (the so-called Picard variety) for every smooth projective curve C; it is in general not isomorphic to C however. (If C is not $\mathbb{P}^1$ one can show that the map $P \mapsto P - P_0$ of corollary 6.3.15 is at least injective, so we can think of C as a subvariety of the Picard variety.)

In contrast, the statement that C can be made into an abelian group is very special to cubic plane curves (or to be precise, to curves of genus 1). Curves of other types do not admit such a group structure.

Example 6.4.1. Let us investigate the group structure on C geometrically. If P and Q are two points on C (not necessarily distinct), we denote by $\varphi(P,Q)$ the third point of intersection of the line $\overline{PQ}$ with C, i. e. the unique point of C such that $P + Q + \varphi(P,Q)$ is linearly equivalent to the divisor of a linear function. We will denote the group structure on C by $\oplus$, to distinguish it from the addition of points in $\operatorname{Div} C$ or $\operatorname{Pic} C$. Consequently, we write $\ominus P$ for the inverse of P, and $n \odot P$ for $P \oplus \cdots \oplus P$ (n times).

Of course, the zero element of the group structure on C is just P_0.

By construction, $P \oplus Q$ is the unique point of C such that $(P - P_0) + (Q - P_0) = (P \oplus Q) - P_0$ in $\operatorname{Pic} C$, i. e. $P + Q = (P \oplus Q) + P_0$. Now let $R = \varphi(P, Q)$. Then $P + Q + R = (P \oplus Q) + P_0 + R \in \operatorname{Pic} C$, so

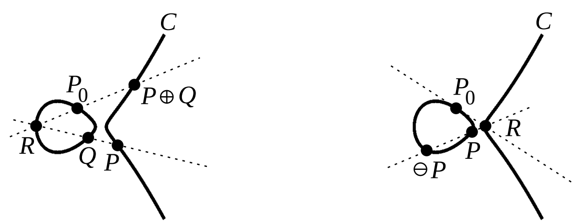
$$P \oplus Q = \varphi(R, P_0) = \varphi(\varphi(P, Q), P_0).$$

In other words, to construct the point $P \oplus Q$ we draw a line through P and Q. Then we draw another line through the third intersection point R of this line with C and the point P_0. The third intersection point of this second line with C is $P \oplus Q$ (see the picture below on the left).

Similarly, to construct $\ominus P$ we are looking for a point such that $(P - P_0) + ((\ominus P) - P_0) = 0$, so $P + (\ominus P) = 2P_0$. In the same way as above we conclude

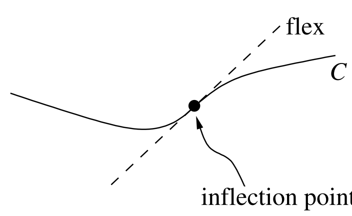
$$\ominus P = \varphi(\varphi(P_0, P_0), P).$$

In other words, to construct the inverse $\ominus P$ we draw the tangent to C through P_0. Then we draw another line through the (scheme-theoretic) third intersection point R of this tangent with C and the point P. The third intersection point of this second line with C is $\ominus P$:



Of special geometric importance are the (tangent) lines that meet C in a point with multiplicity (at least) 3. In analogy with the real analysis case such points will be called inflection points:

Definition 6.4.2. Let $C \subset \mathbb{P}^2$ be a smooth curve. A point $P \in C$ is called an **inflection point** of C if the tangent line to C at P intersects C in P with multiplicity at least 3. Such a tangent line is then called a **flex**.



For cubic curves C, any line intersects C in three points, so $P \in C$ is a flex if and only if $3P$ is the divisor of a linear function. Let us first prove that there are some inflection points on every smooth cubic curve.

Lemma 6.4.3. *Let $C = Z(f) \subset \mathbb{P}^2$ be a smooth curve of degree d. Then*

$$h = \det \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{0 \le i,j \le 2}$$

*is a homogeneous polynomial of degree $3(d-2)$. (It is called the **Hessian polynomial** of C. The corresponding curve $H = Z(h) \subset \mathbb{P}^2$ is called the **Hessian curve** of C.)*

Then $P \in C$ is an inflection point of C if and only if $P \in H$.

Proof. By a linear change of coordinates we can assume that $P = (1:0:0)$ and that the tangent line to C at P is $L = \{x_2 = 0\}$. Let $f = \sum_{i+j+k=d} a_{i,j,k}\, x_0^i x_1^j x_2^k$. In inhomogeneous coordinates ($x_0 = 1$) the restriction of f to L is

$$f(1,x_1,0) = \sum_{i=0}^{d} a_{d-i,i,0} x_1^i.$$

As f passes through P and is tangent to L there, $f|_L(x_1)$ must have a zero of order at least 2 at P, so $a_{d,0,0} = a_{d-1,1,0} = 0$. Now note that

$$\frac{\partial^2 f}{\partial x_0^2}(P) = d(d-1)a_{d,0,0}, \qquad \frac{\partial^2 f}{\partial x_0 \partial x_1}(P) = (d-1)a_{d-1,1,0},$$
$$\frac{\partial^2 f}{\partial x_0 \partial x_2}(P) = (d-1)a_{d-1,0,1}, \qquad \frac{\partial^2 f}{\partial x_1^2}(P) = 2a_{d-2,2,0}.$$

So the Hessian polynomial at P has the form

$$h(P) = \det \begin{pmatrix} 0 & 0 & (d-1)a_{d-1,0,1} \\ 0 & 2a_{d-2,2,0} & * \\ (d-1)a_{d-1,0,1} & * & * \end{pmatrix}.$$

In the same way, note that

$$\left(\frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right)(P) = (da_{d,0,0}, a_{d-1,1,0}, a_{d-1,0,1}) = (0,0,a_{d-1,0,1}),$$

which must be a non-zero vector by the Jacobian criterion of proposition 4.4.8 (ii) as C is smooth at P. So $a_{d-1,0,1} \neq 0$, and therefore $h(P) = 0$ if and only if $a_{d-2,2,0} = 0$. This is the case if and only if $f|_L(x_1)$ vanishes to order at least 3 at P, i. e. if and only if P is an inflection point. □

Corollary 6.4.4. *Every smooth cubic curve in $\mathbb{P}^2$ has exactly 9 inflection points.*

Proof. By lemma 6.4.3 the inflection points of C are precisely the points of $C \cap H \subset \mathbb{P}^2$, where H is the Hessian curve of C. But by Bézout's theorem, $\deg(C \cap H) = d \cdot 3(d-2) = 9$ for $d = 3$. So we only have to check that every point in $C \cap H$ occurs with intersection multiplicity 1.

Let us continue with the notation of the proof of lemma 6.4.3, and assume that P is an inflection point, so that $a_{3,0,0} = a_{2,1,0} = a_{1,2,0} = 0$. We will show that the Hessian curve H is smooth at P and has a tangent line different from that of C (i. e. its tangent line is not $L = \{x_2 = 0\}$. Both statements follow if we can prove that $h(1,x_1,x_2)$ contains the monomial x_1 with a non-zero coefficient, i. e. that h contains the monomial $x_0^2 x_1$ with a non-zero coefficient. But note that

$$h(x_2 = 0) = \det \begin{pmatrix} 0 & 0 & 2a_{2,0,1}x_0 + a_{1,1,1}x_1 \\ 0 & 6a_{0,3,0}x_1 & * \\ 2a_{2,0,1}x_0 + a_{1,1,1}x_1 & * & * \end{pmatrix},$$

so the $x_0^2 x_1$-coefficient of h is $-24a_{2,0,1}^2 a_{0,3,0}$. The corollary now follows from the following two observations:

(i) the Jacobian matrix of f at P is $(3a_{3,0,0}, a_{2,1,0}, a_{2,0,1})$. As C is smooth this matrix must have rank 1 by proposition 4.4.8 (ii). But $a_{3,0,0}$ and $a_{2,1,0}$ are zero already, so $a_{2,0,1} \neq 0$.

(ii) We know already that $f|_L = a_{0,3,0}x_1^3$. As L cannot be a component of C, it follows that $a_{0,3,0} \neq 0$.

□

Remark 6.4.5. If C is a smooth curve of degree d in $\mathbb{P}^2$, we would still expect from Bézout's theorem that C has $3d(d-2)$ inflection points. This is indeed the "general" number, but for $d > 3$ it may occur that C and its Hessian H do not intersect at all points with multiplicity 1, so that there are fewer than $3d(d-2)$ inflection points.

Lemma 6.4.6. *Let $C \subset \mathbb{P}^2$ be a smooth cubic curve, and choose an inflection point P_0 as the zero element of the group structure on C. Then a point $P \in C$ is an inflection point if and only if $3 \odot P = P_0$. In particular, there are exactly 9 3-torsion points in* $\operatorname{Pic} C$*, i. e. 9 points $P \in C$ such that $3 \odot P = P_0$.*

Proof. Assume that P_0 is an inflection point, i. e. $3P_0$ is the divisor of a linear function on C. Then P is an inflection point if and only if $3P$ is the divisor of a linear function too, which is the case if and only if $3P - 3P_0 = 3(P - P_0)$ is the divisor of a rational function (a quotient of two linear functions). This in turn is by definition the case if and only if $3 \odot P = P_0$. It then follows by corollary 6.4.4 that there are exactly 9 3-torsion points in $\operatorname{Pic} C$. □

Corollary 6.4.7. *Let $C \subset \mathbb{P}^2$ be a smooth cubic curve. Then any line through two inflection points of C passes through a third inflection point of C.*

Proof. Choose an inflection point $P_0 \in C$ as the zero element for the group structure on C. Now let P and Q be two inflection points, and let $R = \varphi(P,Q)$ be the third intersection point of $\overline{PQ}$ with C. Then $P + Q + R$ is the divisor of a linear function and hence equal to $3P_0$ in $\operatorname{Pic} C$. It follows that

$$3(R - P_0) = 3(2P_0 - P - Q) = 3(P_0 - P) + 3(P_0 - Q) = 0 \in \operatorname{Pic} C.$$

So $3 \odot R = P_0$, i. e. R is an inflection point by lemma 6.4.6. □

Example 6.4.8. There is an interesting application of the group structure on a cubic curve to cryptography. The key observation is that "multiplication is easy, but division is hard". More precisely, assume that we are given a specific cubic curve C and a zero point $P_0 \in C$ for the group structure. (For practical computations one will usually do this over a finite field to avoid rounding errors. The group structure exists in these cases too by exercise 6.7.10.) Then:

(i) Given any point P and a positive integer n, the point $n \odot P$ can be computed quickly, even for very large n (think of numbers with hundreds of digits):
  (a) By repeatedly applying the operation $P \mapsto P \oplus P$, we can compute all points $2^k \odot P$ for all k such that $2^k < n$.
  (b) Now we just have to add these points $2^k \odot P$ for all k such that the k-th digit in the binary representation of n is 1.

  This computes the point $n \odot P$ in a time proportional to $\log n$ (i. e. in a very short time).

(ii) On the other hand, given a point P and a positive integer n, it is essentially impossible to compute a point Q such that $n \odot Q = P$ (assuming that such a point exists). This is not a mathematically precise statement; there is just no algorithm known to exist that can perform the "inverse" of the multiplication $P \mapsto n \odot P$ in shorter time than a simple trial-and-error approach. Of course, if the ground field is large and C contains enough points, this is practically impossible. In the same way, given two points P and Q on C, there is no way to find the (smallest) number n such that $n \odot Q = P$ except trying out all integers in turn. Again, if n has hundreds of digits this is of course practically impossible.

Using this idea, assume that Alice wants to send a secret message to Bob. We can think of this message as just a number N (every message can be converted into a sequence of numbers, of course). There is an easy way to achieve this if they both know a secret key number N_0: Alice just sends Bob the number $N+N_0$ in public, and then Bob can reconstruct the secret N by subtracting the key N_0 from the transmitted number $N+N_0$. Any person who observed the number $N+N_0$ in transit but does not know the secret key N_0 is not able to reconstruct the message N.

The problem is of course that Alice and Bob must first have agreed on a secret key N_0, which seems impossible as they do not have a method for secure communication yet.

This is where our cubic curve can help. Let us describe a (simplified) version of what they might do. Alice and Bob first (publicly) agree on a ground field, a specific cubic curve C, a zero point $P_0 \in C$, and another point $P \in C$. Now Alice picks a secret (very large) integer a, and Bob picks a secret integer b. They are not telling each other what their secret numbers are. Instead, Alice computes $a \odot P$ and sends (the coordinates of) this point to Bob. In the same way, Bob computes $b \odot P$ and sends this point to Alice. Now the point $ab \odot P$ can be used as a secret key number N_0:

(i) Alice got the information about $b \odot P$ from Bob and knows her own secret number a, so she can compute $ab \odot P = a \odot (b \odot P)$.
(ii) In the same way, Bob knows $ab \odot P = b \odot (a \odot P)$.
(iii) The only information that Alice and Bob exchanged was the data of the cubic curve chosen, P, $a \odot P$, and $b \odot P$. But we have just noted that there is no practical way to reconstruct a and b from this information, so anybody else will not be able to determine the secret key $ab \odot P$ from this data.

6.5. **Plane cubic curves as complex tori.** We will now restrict our attention to the ground field $k = \mathbb{C}$ and see how smooth plane cubic curves arise in complex analysis in a totally different way. We will only sketch most arguments; more details can be found e. g. in [K] section 5.1 (and many other books on complex analysis).

Let $U \subset \mathbb{C}$ be an open set *in the classical topology*. Recall that a (set-theoretic) function $f : U \to \mathbb{C}$ is called **holomorphic** at $z_0 \in U$ if it is complex differentiable at z_0, i. e. if the limit

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. A function $f : U \backslash \{z_0\} \to \mathbb{C}$ is called **meromorphic** if there is a number $n \in \mathbb{Z}$ and a holomorphic function $\tilde{f} : V \to \mathbb{C}$ in a neighborhood V of z_0 in U such that

$$f(z) = (z - z_0)^n \cdot \tilde{f}(z) \quad \text{and} \quad \tilde{f}(z_0) \neq 0$$

on V. Note that the number n is then uniquely determined; it is called the **order** of f at z_0 and denoted $\operatorname{ord}_{z_0} f$. If $n > 0$ we say that $f(z)$ has a zero of order n at z_0. If $n < 0$ we say that $f(z)$ has a pole of order $-n$ at z_0. A function that is meromorphic at z_0 is holomorphic at z_0 if and only if its order is non-negative.

Example 6.5.1. Any regular function on $\mathbb{A}^1_{\mathbb{C}}$ (i. e. any polynomial in z) is a holomorphic function on $\mathbb{C}$. Similarly, any rational function φ on $\mathbb{A}^1_{\mathbb{C}}$ is a meromorphic function on $\mathbb{C}$. The notion of zeros and poles of φ as a meromorphic function agrees with our old one of definition 6.3.4, so the multiplicity of a point $z \in \mathbb{C}$ in the divisor of φ is precisely the order of φ at z.

Conversely, there are holomorphic (resp. meromorphic) functions on $\mathbb{C}$ that are not regular (resp. rational), e. g. $f(z) = e^z$.

Remark 6.5.2. Although the definition of holomorphic, i. e. *complex* differentiable functions is formally exactly the same as that of *real* differentiable functions, the behavior of the complex and real cases is totally different. The most notable differences that we will need are:

(i) Every holomorphic function is automatically infinitely differentiable: all higher derivatives $f^{(k)}$ exist for $k > 0$ and are again holomorphic functions.

(ii) Every holomorphic function f is analytic, i. e. it can be represented locally around every point z_0 by its Taylor series. The radius of convergence is "as large as it can be", i. e. if f is holomorphic in an open ball B around z_0, then the Taylor series of f at z_0 converges and represents f at least on B. Consequently, a meromorphic function f of order n at z_0 can be expanded in a **Laurent series** as $f(z) = \sum_{k \geq n} c_k (z - z_0)^k$. The coefficient c_{-1} of this series is called the **residue** of f at z_0 and denoted $\operatorname{res}_{z_0} f$.

(iii) (**Liouville's theorem**) Every function f that is holomorphic and bounded on the whole complex plane $\mathbb{C}$ is constant.

(iv) (**Identity theorem**) Let f and g be holomorphic functions on a *connected* open subset $U \subset \mathbb{C}$. If f and g agree on any open subset $V \subset U$ then they agree on U. By (ii) this is e. g. the case if their Taylor series agree at some point in U. One should compare this to the algebro-geometric version of remark 2.1.9.

(v) (**Residue theorem**) If γ is a closed (positively oriented) contour in $\mathbb{C}$ and f is a meromorphic function in a neighborhood of γ and its interior that has no poles on γ itself, then

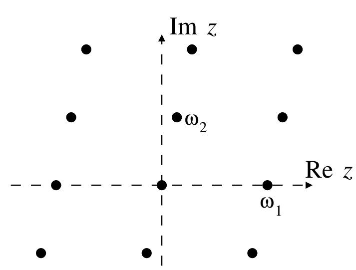
$$\int_\gamma f(z)\, dz = 2\pi i \sum_{z_0} \operatorname{res}_{z_0} f(z),$$

with the sum taken over all z_0 in the interior of γ (at which f has poles). In particular, if f is holomorphic then this integral vanishes.

In this section we will study a particular meromorphic function on $\mathbb{C}$ associated to a *lattice*. Let us describe the construction. Fix once and for all two complex numbers $\omega_1, \omega_2 \in \mathbb{C}$ that are linearly independent *over* $\mathbb{R}$, i. e. that do not lie on the same real line in $\mathbb{C}$ through the origin. Then the subset

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{m\omega_1 + n\omega_2 \,;\, m, n \in \mathbb{Z}\} \subset \mathbb{C}$$

is called a **lattice** in $\mathbb{C}$. Obviously, the same lattice in $\mathbb{C}$ can be obtained by different choices of ω_1 and ω_2. The constructions that we will make in this section will only depend on the lattice Λ and not on the particular choice of basis ω_1, ω_2.



Proposition and Definition 6.5.3. *Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$. There is a meromorphic function $\wp(z)$ on $\mathbb{C}$ defined by*

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

It is called the ***Weierstraß $\wp$-function****. It has poles of order 2 exactly at the lattice points.*

Proof. It is a standard fact that an (infinite) sum of holomorphic functions is holomorphic at z_0 provided that the sum converges uniformly in a neighborhood of z_0. We will only sketch the proof of this convergence: let $z_0 \in \mathbb{C} \setminus \Lambda$ be a fixed point that is not in the lattice. Then every summand is a holomorphic function in a neighborhood of z_0. The expansions of these summands for large ω are

$$\frac{1}{(z_0-\omega)^2}-\frac{1}{\omega^2}=\frac{1}{\omega^2}\left(\frac{1}{(1-\frac{z_0}{\omega})^2}-1\right)=\frac{z_0}{\omega^3}+\text{terms of order at least }\frac{1}{\omega^4}$$

so the summands grow like ω^3. Let us add up these values according to the absolute value of ω. As the number of lattice points with a given absolute value (approximately) equal to N grows linearly with N, the final sum behaves like $\sum_N N\cdot\frac{1}{N^3}=\sum_N\frac{1}{N^2}$, which is convergent.

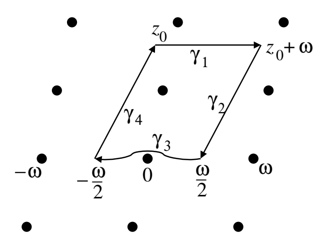
Note that the sum would not have been convergent without subtraction of the constant $\frac{1}{\omega^2}$ in each summand, as then the individual terms would grow like $\frac{1}{\omega^2}$ and therefore the final sum would be of the type $\sum_N\frac{1}{N}$, which is divergent. □

Remark 6.5.4. It is a standard fact that in an absolutely convergent series as above all manipulations (reordering of the summands, term-wise differentiation) can be performed as expected. In particular, the following properties of the $\wp$-function are obvious:

(i) The $\wp$-function is an *even* function, i. e. $\wp(z)=\wp(-z)$ for all $z\in\mathbb{C}$. In particular, its Laurent series at 0 contains only even exponents.

(ii) Its derivative is $\wp'(z)=\sum_{\omega\in\Lambda}\frac{-2}{(z-\omega)^3}$. It is an *odd* function, i. e. $\wp'(z)=-\wp'(-z)$. In particular, its Laurent series at 0 contains only odd exponents. It has poles of order 3 exactly at the lattice points.

(iii) The $\wp$-function is doubly periodic with respect to Λ, i. e. $\wp(z_0)=\wp(z_0+\omega)$ for all $z_0\in\mathbb{C}$ and $\omega\in\Lambda$. To show this note first that it is obvious from (ii) that $\wp'(z_0)=\wp'(z_0+\omega)$. Now integrate $\wp'(z)$ along the closed contour $\gamma=\gamma_1+\gamma_2+\gamma_3+\gamma_4$ shown in this picture:



Of course, the result is 0, since $\wp$ is an integral of $\wp'$. But also the integral along γ_2 cancels the integral along γ_4 as $\wp'(z)$ is periodic. The integral along γ_3 is equal to $\wp(-\frac{\omega}{2})-\wp(\frac{\omega}{2})$ and hence vanishes too as $\wp(z)$ is an even function. So we conclude that

$$0=\int_{\gamma_1}\wp'(z)\,dz=\wp(z_0+\omega)-\wp(z_0),$$

i. e. $\wp(z)$ is periodic with respect to Λ too.

Lemma 6.5.5. *The $\wp$-function associated to a lattice Λ satisfies a differential equation*

$$\wp'(z)^2=c_3\wp(z)^3+c_2\wp(z)^2+c_1\wp(z)+c_0$$

for some constants $c_i\in\mathbb{C}$ that depend on Λ.

Proof. By remark 6.5.4 (ii) $\wp'(z)^2$ is an even function with a pole of order 6 at 0. Hence its Laurent series around 0 is

$$\wp'(z)^2=\frac{a_{-6}}{z^6}+\frac{a_{-4}}{z^4}+\frac{a_{-2}}{z^2}+a_0+\text{terms of order }z^{>0}$$

for some constants $a_{-6},a_{-4},a_{-2}\in\mathbb{C}$. The functions $\wp(z)^3$, $\wp(z)^2$, $\wp(z)$, and 1 are also even, and they have poles of order 6,4,2, and 0, respectively. Hence there are constants $c_{-6},c_{-4},c_{-2},c_0\in\mathbb{C}$ such that the series of the linear combination

$$f(z):=\wp'(z)^2-c_3\wp(z)^3-c_2\wp(z)^2-c_1\wp(z)-c_0$$

has only positive powers of z. We conclude that $f(z)$ is holomorphic around 0 and vanishes at 0. By the identity theorem of remark 6.5.2 (iv) it then follows that $f = 0$ everywhere. $\square$

Remark 6.5.6. An explicit computation shows that the coefficients c_i in lemma 6.5.5 are given by

$$c_3 = 4, \quad c_2 = 0, \quad c_1 = -60 \sum_{\omega \in \Lambda \backslash \{0\}} \frac{1}{\omega^4}, \quad c_0 = -140 \sum_{\omega \in \Lambda \backslash \{0\}} \frac{1}{\omega^6}.$$

Proposition 6.5.7. *Let $\Lambda \subset \mathbb{C}$ be a given lattice, and let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be the cubic curve*

$$C = \{(x_0 : x_1 : x_2)\ ;\ x_2^2 x_0 = c_3 x_1^3 + c_2 x_1^2 x_0 + c_1 x_1 x_0^2 + c_0 x_0^3\}$$

for the constants $c_i \in \mathbb{C}$ of lemma 6.5.5. Then there is a bijection

$$\Phi : \mathbb{C}/\Lambda \to C, \quad z \mapsto (1 : \wp(z) : \wp'(z)).$$

Proof. As $\wp(z)$ and $\wp'(z)$ are periodic with respect to Λ and satisfy the differential equation of lemma 6.5.5, it is clear that Φ is well-defined. (Strictly speaking, for $z = 0$ we have to note that $\wp(z)$ has a pole of order 2 and $\wp'(z)$ has a pole of order 3, so $\wp(z) = \frac{f(z)}{z^2}$ and $\wp'(z) = \frac{g(z)}{z^3}$ locally around 0 for some holomorphic functions f, g around 0 that do not vanish at 0. Then

$$(1 : \wp(0) : \wp'(0)) = (z^3 : z f(z) : g(z))|_{z=0} = (0 : 0 : 1),$$

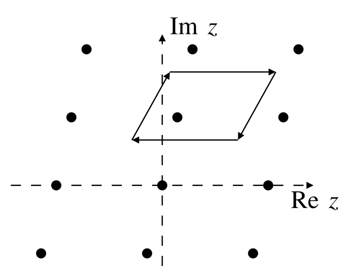
so Φ is well-defined at 0 too.)

Now let $(x_0 : x_1 : x_2) \in C$ be a given point; we will show that it has exactly one inverse image point under Φ. By what we have just said this is obvious for the "point at infinity" $(0 : 0 : 1)$, so let us assume that we are not at this point and hence pass to inhomogeneous coordinates where $x_0 = 1$.

We will first look for a number $z \in \mathbb{C}$ such that $\wp(z) = x_1$. To do so, consider the integral

$$\int_\gamma \frac{\wp'(z)}{\wp(z) - x_1}\, dz$$

over the boundary of any "parallelogram of periodicity" as in the following picture:



The integrals along opposite sides of the parallelogram vanish because of the periodicity of $\wp$ and $\wp'$, so the integral must be 0. So by the residue theorem of remark 6.5.2 (v) we get

$$0 = \sum_{z_0 \in \mathbb{C}/\Lambda} \operatorname{res}_{z_0} \frac{\wp'(z)}{\wp(z) - x_1}. \qquad (*)$$

Now note that if $F(z)$ is any meromorphic function of order n around 0 then

$$\operatorname{res}_0 \frac{F'(z)}{F(z)} = \operatorname{res}_0 \frac{n a_n z^{n-1} + \cdots}{a_n z^n + \cdots} = n,$$

so we conclude from $(*)$ that $\sum_{z_0 \in \mathbb{C}/\Lambda} \operatorname{ord}_{z_0}(\wp(z) - x_1) = 0$: the function $\wp(z) - x_1$ has as many zeros as it has poles in $\mathbb{C}/\Lambda$, counted with multiplicities. (This is a statement in complex analysis corresponding to remark 6.3.5.) As $\wp(z)$ has a pole of order 2 in the lattice points, it thus follows that there are exactly two points $z_1, z_2 \in \mathbb{C}/\Lambda$ such that $\wp(z) = x_1$. Since the $\wp$-function is an even function, these two points are obviously negatives of each other. Now as $\wp'$ is an odd function, it follows that $\wp'(z_1) = -\wp'(z_2)$. So if we specify $\wp(z)$ and $\wp'(z)$ there is exactly one point $z \in \mathbb{C}/\Lambda$ leading to the given image point in C. □

Remark 6.5.8. We are again in a similar situation as in corollary 6.3.15: we have a bijection between a group $\mathbb{C}/\Lambda$ and a variety C. In fact, one can show that the group structure of $\mathbb{C}/\Lambda$ is precisely the same as that of $\operatorname{Pic}^0 C$, so we have just rediscovered our old group structure on a plane cubic curve. But the group structure is a lot more obvious in this new picture: e. g. the n-torsion points of C are easily read off to be

$$\left\{ \frac{1}{n}\,(i\omega_1 + j\omega_2)\;;\; 0 \le i, j < n \right\}.$$

In particular, there are exactly n^2 points $P \in C$ such that $n \odot P = 0$, in accordance with exercise 6.7.11 and lemma 6.4.6.

It should be said however that the bijection of proposition 6.5.7 differs from that of corollary 6.3.15 in that both $\mathbb{C}/\Lambda$ and C can independently be made into a *complex manifold* (which you should roughly think of as a variety whose structure sheaf consists of *holomorphic* functions instead of just *polynomial* functions). The map Φ of the above proposition is then an isomorphism between these two complex manifolds.

Remark 6.5.9. The topology of a plane cubic curve becomes very clear from proposition 6.5.7: it is just a parallelogram with opposite sides identified, i. e. a torus. This agrees with our earlier statements that a smooth plane cubic curve has genus 1, and that the genus should be thought of as the number of "holes" in the (real) surface.

6.6. **Where to go from here.** After having discussed some basic algebraic geometry we now want to sketch which important parts of the general theory are still missing in our framework.

Example 6.6.1. *Intersection theory.* Let $X \subset \mathbb{P}^n$ be a projective variety of dimension r, and let $X_1, \dots, X_r \subset \mathbb{P}^n$ be r hypersurfaces. If the hypersurfaces are in sufficiently general position, the intersection $X_1 \cap \cdots \cap X_r \cap X$ will be zero-dimensional. Bézout's theorem then tells us that the intersection consists of exactly $\deg X_1 \cdot \cdots \cdot \deg X_r \cdot \deg X$ points, counted with multiplicities.

There is obvious room for generalizations here. Assume that we do not have r hypersurfaces $X_1, \dots, X_r$, but rather closed subvarieties $X_1, \dots, X_s$ of X whose codimensions in X add up to r. If these subvarieties are in sufficiently general position then we still expect the intersection $X_1 \cap \cdots \cap X_s \cap X$ to be zero-dimensional. So we can still ask for the number of points in the intersection and expect a finite answer.

If $X = \mathbb{P}^r$ is projective space itself, then the answer is still just $\deg X_1 \cdot \cdots \cdot \deg X_s$: in $\mathbb{P}^r$ the degree is multiplicative when taking intersections. For general X the situation is a lot more subtle though — there is no single number that can be associated to any subvariety of X and that is just multiplicative with respect to intersections. This is easy to see: if e. g. $X = \mathbb{P}^1 \times \mathbb{P}^1$ and we consider the three 1-dimensional subvarieties of X

$$X_1 = \mathbb{P}^1 \times \{0\}, \quad X_2 = \mathbb{P}^1 \times \{1\}, \quad X_3 = \{0\} \times \mathbb{P}^1,$$

then $X_1 \cap X_2$ is empty, so if there were numbers associated to X_1 and X_2 whose product gives the number of intersection points (namely zero), then one of these two numbers (say for X_1) must obviously itself be zero. But then the product of the numbers for X_1 and X_3 would also be zero, although X_1 and X_3 intersect in precisely one point.

It turns out however that there is a *finite collection* of numbers that can be associated to any subvariety of X such that the number of points in $X_1 \cap \cdots \cap X_s$ is given by an explicit multilinear form in these collections of numbers. For example, in the $\mathbb{P}^1 \times \mathbb{P}^1$ case above, curves (like X_1, X_2, X_3 given above) are characterized by their *bidegree* (i. e. the bidegree of the defining equation). In our example, the bidegrees of X_1, X_2, and X_3 are $(1,0)$, $(1,0)$, and $(0,1)$, respectively. Two curves of bidegrees (d_1,e_1) and (d_2,e_2) then intersect in exactly $d_1e_2 + d_2e_1$ points.

Setting up a corresponding theory for *any* variety X is the object of *intersection theory*. It is essentially a well-established theory that can be set up both in algebraic geometry and (for the ground field $\mathbb{C}$) topology. In the latter case it is a part of *algebraic topology*. In both cases the theory allows you to answer most questions concerning numbers of intersection points quite effectively (and without the need for computer algebra techniques). Intersection theory is used in one form or the other in virtually every geometric field of mathematics.

Example 6.6.2. *Sheaves and vector bundles.* Let us illustrate the idea behind vector bundles by an example. In section 4.5 we have shown that every smooth cubic surface in $\mathbb{P}^3$ has exactly 27 lines on it. We did this by first proving that the number of lines does not depend on the particular cubic chosen, and then calculating the number for a specific cubic for which the answer happened to be directly computable.

Now let us consider a slightly more difficult setting. Let $X \subset \mathbb{P}^4$ be a (3-dimensional) smooth hypersurface of degree 5. We will see momentarily that we again expect there to be a finite number of lines in X. So again we ask for the number of such lines. Compared to the cubic surface case it is still true that the answer does not depend on the particular quintic hypersurface chosen. There is no specific quintic any more however for which we can read off the answer by simply writing down all the lines explicitly. So we need to apply a different technique to obtain the answer.

As before, we first consider again the Grassmannian variety $G(1,4)$ of lines in $\mathbb{P}^4$ (see exercise 3.5.4). The dimension of $G(1,4)$ is 6. Now define the set

$$E := \{(L,f)\ ;\ L \in G(1,4),\ f \text{ is a homogeneous polynomial of degree 5 on } L \cong \mathbb{P}^1\},$$

so elements of E are pairs of a line in $\mathbb{P}^4$ and a quintic equation on this line. There is an obvious projection map $\pi : E \to G(1,4)$ given by forgetting f.

We claim that E is a variety in a natural way. In fact, as in exercise 3.5.4 consider the open subset $U \subset G(1,4)$ isomorphic to $\mathbb{A}^6$ (with coordinates a_2,b_2,a_3,b_3,a_4,b_4) where the line $L \in U$ can be represented by the matrix

$$\begin{pmatrix} 1 & 0 & a_2 & a_3 & a_4 \\ 0 & 1 & b_2 & b_3 & b_4 \end{pmatrix}. \tag{1}$$

For every such line we can obviously take x_0 and x_1 as homogeneous coordinates on $L \cong \mathbb{P}^1$, so every quintic equation on L is of the form $\sum_i c_i x_0^i x_1^{5-i}$ for some $c_0,\dots,c_5$. Then $\pi^{-1}(U)$ can obviously be thought of as a 12-dimensional affine space with coordinates $a_2,a_3,a_4,b_2,b_3,b_4,c_0,\dots,c_5$. As E can be covered by these spaces, it is a 12-dimensional variety.

Note that the fibers $\pi^{-1}(L)$ for $L \in G(1,4)$ are all 6-dimensional vector spaces, namely the spaces of degree-5 homogeneous polynomials on L. They are not just 6-dimensional affine spaces but rather *linear* affine spaces in the sense that it is meaningful to add two polynomials on L, and to multiply them with a scalar. So two points in E that map to the same base point in $G(1,4)$ can be "added", just by summing up their coordinates c_i. In contrast, it does *not* make much sense to add the coefficients a_i and b_i in two matrices as in (1), as the resulting line is not related to the two original lines in any obvious way. So although the coordinates $a_2, a_3, a_4, b_2, b_3, b_4$ in U live in an affine space $\mathbb{A}^6$, it does not make sense to think of this $\mathbb{A}^6$ as a *vector* space.

Note also that E is not just the direct product of $G(1,4)$ with a constant 6-dimensional vector space $k[x_0,x_1]^{(5)}$, as the coordinates that we can use on the line L vary with the line. Only the *fibers* of π are all 6-dimensional vector spaces. We say that E is a *vector bundle* of rank 6 on $G(1,4)$.

Now let us return to our original question: to count the lines on X. Let $f \in k[x_0,\dots,x_4]^{(5)}$ be the polynomial whose zero locus is X. There is an obvious morphism

$$\sigma : G(1,4) \to E, \quad L \mapsto (L, f|_L) \tag{2}$$

such that $\pi \circ \sigma = \mathrm{id}_{G(1,4)}$. Such a morphism is called a *section* of E: it assigns to every point L in the base $G(1,4)$ an element in the vector space $\pi^{-1}(L)$ "sitting over" L. Note that this can indeed be thought of as a section in the sheaf-theoretic sense: suppose that we have an open cover $\{U_i\}$ of $G(1,4)$ and morphisms $\sigma_i : U_i \to \pi^{-1}(U_i)$ such that $\pi \circ \sigma_i = \mathrm{id}_{U_i}$ (i. e. on every U_i we associate to any point $L \in U_i$ an element in the vector space $\pi^{-1}(L)$). If $\sigma_i = \sigma_j$ on $U_i \cap U_j$ for all i, j, then there is obviously a global section $\sigma : U \to E$ that restricts to the σ_i on the U_i. In other words, we can think of the vector bundle E as a *sheaf*, with $E(U)$ (in the sense of definition 2.2.1) being the space of all morphisms $\sigma : U \to \pi^{-1}(U)$ such that $\pi \circ \sigma = \mathrm{id}_U$.

Finally, return to our specific section σ in (2). As the fibers of π are vector spaces, there is also a well-defined *zero section*

$$\sigma_0 : G(1,4) \to E, \quad L \mapsto (L, 0).$$

Obviously, a line L lies in the quintic hypersurface X if and only if $f|_L = 0$, i. e. if and only if $\sigma(L) = \sigma_0(L)$. So the number of lines we are looking for is simply the number of intersection points of $\sigma(G(1,4))$ and $\sigma_0(G(1,4))$. As these are both 6-dimensional varieties in the 12-dimensional variety E, we expect a finite number of such intersection points, showing that we expect a finite number of lines in X. Their number is now given by intersection theory methods as explained in example 6.6.1. It can be computed explicitly and the result turns out to be 2875. (To mention the corresponding keywords: we need the 6th Chern class of the vector bundle E on $G(1,4)$, and the result can be obtained using Schubert calculus, i. e. the intersection theory on the Grassmannian $G(1,4)$.)

Another example of a vector bundle on a smooth r-dimensional variety X is the *tangent bundle*: it is just the rank-r vector bundle whose fiber over a point $P \in X$ is the tangent space $T_{X,P}$. The dual vector bundle (i. e. the rank-r bundle whose fiber over a point $P \in X$ is the dual vector space to $T_{X,P}$) is called the *cotangent bundle* and denoted $\Omega_{X,P}$. It can be thought of as the vector bundle of *differential forms* on X.

Any operations that can be done with vector spaces can be done with vector bundles as well, just by performing the corresponding operation in every fiber. So there are e. g. direct sums of vector bundles, tensor products, symmetric products, exterior products, and so on.

If X is a smooth r-dimensional variety, the r-th exterior power $\Lambda^r \Omega_X$ of the cotangent bundle is called the *canonical bundle* and denoted K_X. Obviously it is a vector bundle of rank 1: such bundles are called *line bundles*. Its importance (and name) stems from the fact that it is canonically given for any smooth variety X.

Vector bundles (and corresponding sheaves) occur in almost any branch of algebraic geometry, as well as in topology and differential geometry.

Example 6.6.3. *Sheaf cohomology.* Let X be a variety, and let E be a vector bundle on X. By the remark above, (global) sections $\sigma : X \to E$ can be added and multiplied with a scalar, so the space of global sections is in fact a vector space over the ground field k. It is denoted $H^0(X,E)$.

As an example, let $X \subset \mathbb{P}^2$ be a curve, and let n be an integer. For an open subset $U \subset X$ define

$$E(U) = \left\{ \frac{f}{g};\ f,g \in S(X) \text{ homogeneous with } \deg f - \deg g = n,\ g(P) \neq 0 \text{ for all } P \in U \right\}.$$

These data form a sheaf E that can be thought of as the sheaf of regular "functions" $\varphi(x_0,x_1,x_2)$ on X that satisfy $\varphi(\lambda x_0,\lambda x_1,\lambda x_2) = \lambda^n \varphi(x_0,x_1,x_2)$ under rescaling of the homogeneous coordinates. An element in the fiber of E over a point P is then just given by a number in k that rescales with λ^n. So E is a line bundle. We will usually denote it by $O(n)$. For $n = 0$ we obviously just get the ordinary structure sheaf O.

The spaces $H^0(X, O(n))$ of sections are easily written down:

$$H^0(X, O(n)) = \begin{cases} S(X)^{(n)} & \text{for } n \geq 0, \\ 0 & \text{for } n < 0. \end{cases}$$

In particular, their dimensions (usually denoted $h^0(X, O(n))$) are just the values $h_X(n)$ of the Hilbert function. So the Hilbert function can be thought of as the dimension of the space of global sections of a line bundle $O(n)$.

In our study of Hilbert polynomials we have seen that Hilbert functions and polynomials are usually computed using exact sequences (of graded vector spaces). In the same way, the spaces of sections $H^0(X,E)$ are usually computed using exact sequences of vector bundles. For example, if Y is a smooth subvariety of a smooth variety X, then there is an exact sequence of vector bundles on X

$$0 \to T_Y \to T_X|_Y \to N_{Y/X} \to 0,$$

where $N_{Y/X}$ is the *normal bundle* of Y in X — it is by definition simply the vector bundle whose fibers are the normal spaces $T_{X,P}/T_{Y,P}$. The sequence is then exact by definition (i. e. it is exact *locally* at every fiber). This does not mean however that the spaces of *global* sections necessarily form an exact sequence

$$0 \to H^0(Y,T_Y) \to H^0(Y,T_X|_Y) \to H^0(Y,N_{Y/X}) \to 0.$$

In fact one can show that one always gets an exact sequence

$$0 \to H^0(Y,T_Y) \to H^0(Y,T_X|_Y) \to H^0(Y,N_{Y/X}),$$

but exactness need not be preserved in the last term: a surjective map $E \to F$ of vector bundles need not give rise to a surjective map $H^0(X,E) \to H^0(X,F)$ of global sections. An example is easily found: consider the morphism of vector bundles

$$O \oplus O \to O(2), \quad (\varphi_1,\varphi_2) \mapsto x_0^2\varphi_1 + x_1^2\varphi_2$$

on $\mathbb{P}^1$. This is obviously surjective in every fiber — for every point $P = (x_0 : x_1) \in \mathbb{P}^1$ at least one of the coordinates is non-zero, so by picking suitable $\varphi_1(P)$ and $\varphi_2(P)$ we can get any number for $x_0^2\varphi_1(P) + x_1^2\varphi_2(P)$. But the corresponding morphism of global sections

$$H^0(\mathbb{P}^1, O \oplus O) \to H^0(\mathbb{P}^1, O(2))$$

cannot be surjective simply for dimensional reasons, as the dimensions of these vector spaces are 2 and 3, respectively.

It turns out however that there are canonically defined *cohomology groups* $H^i(X,E)$ for $i > 0$ and every vector bundle E (in fact even for more general sheaves) such that every exact sequence

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

of the bundles gives rise to an exact sequence of cohomology groups

$$0 \to H^0(X,E_1) \to H^0(X,E_2) \to H^0(X,E_3) \to H^1(X,E_1) \to H^1(X,E_2) \to H^1(X,E_3) \to H^2(X,E_1) \to \cdots.$$

So every such sequence of vector bundles gives rise to a relation between the (dimensions of the) cohomology groups: if we set

$$h^i(X,E) = \dim H^i(X,E) \quad \text{and} \quad \chi(X,E) = \sum_i (-1)^i h^i(X,E)$$

then

$$\chi(X,E_2) = \chi(X,E_1) + \chi(X,E_3).$$

It can be shown that the sums in the definition of $\chi(X,E)$ are always finite. In fact, the higher cohomology groups vanish in many cases anyway (there are a lot of so-called "vanishing theorems"), so that the above long sequence between the cohomology groups is usually by far not as complicated as it seems to be here.

The problem of computing these numbers $h^i(X,E)$ (or rather $\chi(X,E)$) is solved by the *Riemann-Roch theorem*: expressed in simple terms this theorem states that $\chi(X,E)$ can always be computed using the intersection-theoretic data of the vector bundle (namely the Chern classes mentioned above in example 6.6.2). It is an explicit multilinear function in these Chern classes that is usually easily computable. In particular, $\chi(X,\mathcal{O}(n))$ turns out to be a polynomial in n — it is just the Hilbert polynomial of X. There is a vanishing theorem that implies $h^i(X,\mathcal{O}(n)) = 0$ for $i > 0$ and $n \gg 0$, so we arrive at our old characterization of the Hilbert polynomial as the polynomial that agrees with the Hilbert function for large n.

In particular, we see that the arithmetic genus of a variety (see example 6.1.10) is just $(-1)^{\dim X}(\chi(X,\mathcal{O})-1)$, which obviously does *not* depend on the embedding of X in projective space.

The easiest case of the Riemann-Roch theorem is that of line bundles on smooth curves. If E is a line bundle on a curve X (e. g. a bundle of the form $\mathcal{O}(n)$ if X is projective), we can associate to it:

(i) *intersection-theoretic data*: given a (rational) section of E, how many zeros and poles does this section have? This number is called the *degree* of E. For example, the degree of $\mathcal{O}(n)$ on a plane curve of degree d is $d \cdot n$, as every global section of $\mathcal{O}(n)$ (i. e. a polynomial of degree n) vanishes on X at $d \cdot n$ points by Bézout's theorem.

(ii) *cohomological data*: how many sections of E are there? Ideally we would like to know $h^0(X,E)$, but the Riemann-Roch theorem will only give us $\chi(X,E) = h^0(X,E) - h^1(X,E)$.

The Riemann-Roch theorem then states that

$$\chi(X,E) = \deg E + 1 - g,$$

where g is the genus of the curve X. For example, for $X = \mathbb{P}^1$ we get $\chi(X,\mathcal{O}(n)) = n+1-0$, which is indeed the Hilbert polynomial of $\mathbb{P}^1$.

Example 6.6.4. *Moduli spaces.* We have now met several instances already where it proved useful to make the *set* of all geometric objects of a certain type into a *scheme* (or maybe a variety):

(i) The Grassmannian $G(1,n)$ is a variety that can be thought of as the set of all lines in $\mathbb{P}^n$.

(ii) The affine space $\mathbb{A}^N = k[x_0,\dots,x_n]^{(d)}$ (with $N = \binom{n+d}{d}$) can be thought of as the set of all degree-d hypersurfaces in $\mathbb{P}^n$.

(iii) The vector bundle E of example 6.6.2 can be thought of as the set of pairs (L,f), where L is a line in $\mathbb{P}^4$ and f is a quintic polynomial on L.

Schemes whose points describe geometric objects in this sense are called *moduli spaces*. So we say e. g. that $G(1,n)$ is the moduli space of lines in $\mathbb{P}^n$. There are many other moduli spaces one may want to consider. The most prominent ones are:

(i) moduli spaces of curves (with a fixed given genus),
(ii) moduli spaces of projective subschemes of $\mathbb{P}^n$ with a fixed given Hilbert polynomial (the so-called *Hilbert schemes*),
(iii) moduli spaces of vector bundles over a given variety,

but you can try to give more or less every set of geometric objects a scheme structure. Such a scheme structure may or may not exist, and it may or may not behave nicely.

Moduli spaces come into play when you want to consider *families* of geometric objects, e. g. families of varieties. For example, a family of lines in $\mathbb{P}^n$ over a base scheme B is simply a morphism $f : B \to G(1,n)$ to the moduli space of lines. This assigns to every point of B a line in $\mathbb{P}^n$ in a continuously varying way (as a morphism is given by continuous functions). For example, if the ground field is $\mathbb{C}$ and you have a sequence of points P_i in B converging to a point $P \in B$, then we get a corresponding sequence of lines $f(P_i)$ in $\mathbb{P}^n$ that converges to $f(P)$. We can thus talk about convergence, limits, or "small deformations" of the objects for which we have a moduli space. Deformations are often a powerful tool to make complicated objects into easier ones. For example, in example 0.1.3 we computed the genus of a plane curve by deforming it into a union of lines, for which the genus could be read off easily.

Example 6.6.5. *Classification theory.* Closely related to the study of moduli spaces is the desire to "classify all algebraic varieties" (or other objects occurring in algebraic geometry). For smooth curves the result is quite easy to state:

(i) Every smooth curve has a genus (see e. g. example 0.1.1 and 6.1.10) that is a non-negative integer.
(ii) The moduli space of all smooth curves of a given genus g is an irreducible projective variety (with only mild singularities). Its dimension is 0 for $g = 0$, 1 for $g = 1$, and $3g-3$ for $g > 1$.

So this result says that curves are characterized by one *discrete* invariant, namely its genus. Once the genus is fixed, every curve of this genus can be deformed continuously into any other curve of the same genus. In contrast, curves cannot be deformed into each other if their genera are different.

For higher-dimensional varieties the situation is a lot more complicated. As above, one first looks for discrete invariants, i. e. "integers that can be associated to the variety in a natural way" and that are invariant under deformation. In a second step, one can then ask for the dimension (and other properties) of the moduli space of varieties with the given fixed discrete invariants.

Examples of discrete invariants are:

(i) the dimension (of course),
(ii) cohomological or intersection-theoretic properties of the tangent bundle and related bundles, e. g. $h^i(X,T_X)$, $h^i(X,\Omega_X)$, the Chern classes of the tangent bundle, ...
(iii) the genus $(-1)^{\dim X}(\chi(X,\mathcal{O})-1)$,
(iv) various intersection-theoretic data, e. g. the collection of numbers and the multilinear functions describing intersection products as in example 6.6.1.

For surfaces, this classification problem is solved, but the result is quite complicated. For higher-dimensional varieties, the problem is still largely unsolved.

6.7. **Exercises.**

Exercise 6.7.1. Let X be a collection of four distinct points in some $\mathbb{P}^n$. What are the possible Hilbert functions h_X ?

Exercise 6.7.2. Compute the Hilbert function and the Hilbert polynomial of the "twisted cubic curve" $C = \{(s^3 : s^2t : st^2 : t^3) \,;\, (s:t) \in \mathbb{P}^1\} \subset \mathbb{P}^3$.

Exercise 6.7.3. Let $X \subset \mathbb{P}^n$ be a projective scheme with Hilbert polynomial χ. As in example 6.1.10 define the *arithmetic genus* of X to be $g(X) = (-1)^{\dim X} \cdot (\chi(0)-1)$.

(i) Show that $g(\mathbb{P}^n) = 0$.
(ii) If X is a hypersurface of degree d in $\mathbb{P}^n$, show that $g(X) = \binom{d-1}{n}$. In particular, if $C \subset \mathbb{P}^2$ is a plane curve of degree d, then $g(C) = \frac{1}{2}(d-1)(d-2)$ (compare this to example 0.1.3).
(iii) Compute the arithmetic genus of the union of the three coordinate axes

$$Z(x_1x_2, x_1x_3, x_2x_3) \subset \mathbb{P}^3.$$

Exercise 6.7.4. For $N = (n+1)(m+1)-1$ let $X \subset \mathbb{P}^N$ be the image of the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$. Show that the degree of X is $\binom{n+m}{n}$.

Exercise 6.7.5. Let X be an ellipse in the real plane $\mathbb{R}^2$, and let P be a given point on X. Using only a ruler with no markings, construct the tangent line to X at P.

(In other words: start with a piece of paper which has only the ellipse X and the marked point $P \in X$ on it. The only thing you are now allowed to do is to repeatedly draw straight lines through two points that have already been constructed (the point P, intersection points of previously drawn curves, or arbitrarily chosen points). No measuring of lengths or angles is permitted. Give an algorithm that finally allows you to draw the tangent line to X at P this way.)

Exercise 6.7.6. Let $C \subset \mathbb{P}^n$ be an irreducible curve of degree d. Show that C is contained in a linear subspace of $\mathbb{P}^n$ of dimension d.

Exercise 6.7.7. Let X and Y be subvarieties of $\mathbb{P}^n_k$ that lie in disjoint linear subspaces of $\mathbb{P}^n_k$. Recall from exercises 3.5.7 and 4.6.1 that the join $J(X,Y) \subset \mathbb{P}^n_k$ of X and Y is defined to be the union of all lines $\overline{PQ}$ with $P \in X$ and $Q \in Y$.

(i) Show that $S(J(X,Y))^{(d)} \cong \bigoplus_{i+j=d} S(X)^{(i)} \otimes_k S(X)^{(j)}$.
(ii) Show that $\deg J(X,Y) = \deg X \cdot \deg Y$.

Exercise 6.7.8. Let $C_1 = \{f_1 = 0\}$ and $C_2 = \{f_2 = 0\}$ be affine curves in $\mathbb{A}^2_k$, and let $P \in C_1 \cap C_2$ be a point. Show that the intersection multiplicity of C_1 and C_2 at P (i. e. the length of the component at P of the intersection scheme $C_1 \cap C_2$) is equal to the dimension of the vector space $\mathscr{O}_{\mathbb{A}^2,P}/(f_1,f_2)$ over k.

Exercise 6.7.9. Let $C_1, C_2 \subset \mathbb{P}^2$ be distinct smooth cubic curves, and assume that C_1 and C_2 intersect in 9 (distinct) points $P_1, \dots, P_9$. Prove that every cubic curve passing through $P_1, \dots, P_8$ also has to pass through P_9.

Can you find a stronger version of this statement that applies in the case that the intersection multiplicities in $C_1 \cap C_2$ are not all equal to 1 ?

Exercise 6.7.10. Let C be a smooth cubic curve of the form

$$C = \{(x:y:z) \,;\, y^2z = x^3 + axz^2 + bz^3\} \subset \mathbb{P}^2_k$$

for some given $a, b \in k$. (It can be shown that every cubic can be brought into this form by a change of coordinates.) Pick the point $P_0 = (0:1:0)$ as the zero element for the group structure on C. For given points $P_1 = (x_1 : y_1 : 1)$ and $P_2 = (x_2 : y_2 : 1)$ compute explicitly the coordinates of the inverse $\ominus P_1$ and of the sum $P_1 \oplus P_2$. Conclude that the group structure on C is well-defined even if k is not necessarily algebraically closed.

Exercise 6.7.11. Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a smooth cubic curve, and let $P \in C$ be an inflection point of C. Show that there are exactly 4 tangents of C that pass through P. Conclude that there are exactly 4 divisor classes D in $\operatorname{Pic} C$ such that $2D = 0$.

Exercise 6.7.12. Let $C \subset \mathbb{P}^2$ be a smooth cubic curve, and let $P, Q \in C$ be two points. Show that there is an isomorphism $f : C \to C$ with $f(P) = Q$. Is this isomorphism unique?

Exercise 6.7.13. Check that the cubic curve $C \subset \mathbb{P}^2_{\mathbb{C}}$ defined by a lattice $\Lambda \subset \mathbb{C}$ as in proposition 6.5.7 is smooth.

Exercise 6.7.14. Using the complex analysis methods of section 6.5, reprove the statement of proposition 6.3.13 that there is no rational function φ on a smooth plane complex cubic curve C with divisor $(\varphi) = P - Q$ if P and Q are two distinct points on C.

Exercise 6.7.15. Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a smooth cubic curve arising from a lattice $\Lambda \subset \mathbb{C}$. Show that the group structure of Pic^0_C is isomorphic to the natural group structure of $\mathbb{C}/\Lambda$.

Exercise 6.7.16. Let $\Lambda \subset \mathbb{C}$ be a lattice. Given a point $z \in \mathbb{C}/\Lambda$ and any $n \in \mathbb{Z}$, it is obviously very easy to find a point $w \in \mathbb{C}/\Lambda$ such that $n \cdot w = z$ (in the group structure of $\mathbb{C}/\Lambda$). Isn't this a contradiction to the idea of example 6.4.8?

7. More about sheaves

We present a detailed study of sheaves on a scheme X, in particular sheaves of $\mathcal{O}_X$-modules. For any presheaf $\mathcal{F}'$ on X there is an associated sheaf $\mathcal{F}$ that describes "the same objects as $\mathcal{F}'$ but with the conditions on the sections made local". This allows us to define sheaves by constructions that would otherwise only yield presheaves. We can thus construct e. g. direct sums of sheaves, tensor products, kernels and cokernels of morphisms of sheaves, as well as push-forwards and pull-backs along morphisms of schemes.

A sheaf of $\mathcal{O}_X$-modules is called quasi-coherent if it is induced by an R-module on every affine open subset $U = \operatorname{Spec} R$ of X. Almost all sheaves that we will consider are of this form. This reduces local computations regarding these sheaves to computations in commutative algebra.

A quasi-coherent sheaf on X is called locally free of rank r if it is locally isomorphic to $\mathcal{O}_X^{\oplus r}$. Locally free sheaves are the most well-behaved sheaves; they correspond to vector bundles in topology. Any construction and theorem valid for vector spaces can be carried over to the category of locally free sheaves. Locally free sheaves of rank 1 are called line bundles.

For any morphism $f: X \to Y$ we define the sheaf of relative differential forms $\Omega_{X/Y}$ on X relative Y. The most important case is when Y is a point, in which case we arrive at the sheaf Ω_X of differential forms on X. It is locally free of rank $\dim X$ if and only if X is smooth. In this case, its top alternating power $\Lambda^{\dim X}\Omega_X$ is a line bundle ω_X called the canonical bundle. On a smooth projective curve it has degree $2g-2$, where g is the genus of the curve.

On every smooth curve X the line bundles form a group which is isomorphic to the Picard group $\operatorname{Pic} X$ of divisor classes. A line bundle together with a collection of sections that do not vanish simultaneously at any point determines a morphism to projective space.

If $f: X \to Y$ is a morphism of smooth projective curves, the Riemann-Hurwitz formula states that the canonical bundles of X and Y are related by $\omega_X = f^*\omega_Y \otimes \mathcal{O}_X(R)$, where R is the ramification divisor. For any smooth projective curve X of genus g and any divisor D the Riemann-Roch theorem states that $h^0(D) - h^0(K_X - D) = \deg D + 1 - g$, where $h^0(D)$ denotes the dimension of the space of global sections of the line bundle $\mathcal{O}(D)$ associated to D.

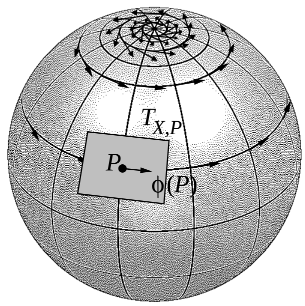
7.1. **Sheaves and sheafification.** The first thing we have to do to discuss the more advanced topics mentioned in section 6.6 is to get a more detailed understanding of sheaves. Recall from section 2.2 that we defined a sheaf to be a structure on a topological space X that describes "function-like" objects that can be patched together from local data. Let us first consider an informal example of a sheaf that is not just the sheaf of regular functions on a scheme.

Example 7.1.1. Let X be a smooth complex curve. For any open subset $U \subset X$, we have seen that the ring of regular functions $\mathcal{O}_X(U)$ on U can be thought of as the ring of complex-valued functions $\varphi : U \to \mathbb{C}, P \mapsto \varphi(P)$ "varying nicely" (i. e. as a rational function) with P.

Now consider the "tangent sheaf" T_X, i. e. the sheaf "defined" by

$$T_X(U) = \{\varphi = (\varphi(P))_{P \in U} \; ; \; \varphi(P) \in T_{X,P} \text{"varying nicely with } P\text{"}\}$$

(of course we will have to make precise what "varying nicely" means). In other words, a section $\varphi \in T_X(U)$ is just given by specifying a tangent vector at every point in U. As an example, here is a picture of a section of $T_{\mathbb{P}^1}(\mathbb{P}^1)$:



As the tangent spaces $T_{X,P}$ are all one-dimensional complex vector spaces, $\varphi(P)$ can again be thought of as being specified by a single complex number, just as for the structure sheaf $\mathcal{O}_X$. The important difference (that is already visible from the definition above) is that these one-dimensional vector spaces vary with P and thus *have no canonical identification with the complex numbers*. For example, it does not make sense to talk about "the tangent vector 1" at a point P. Consequently, there is no analogue of "constant functions" for sections of the tangent sheaf. In fact, we will see in lemma 7.4.15 that every global section of $T_{\mathbb{P}^1}$ has two zeros, so there is really no analogue of constant functions. (In the picture above, the north pole of the sphere is a point where the section of $T_{\mathbb{P}^1}$ would be ill-defined if we do not choose a section in which the lengths of the tangent vectors approach zero towards the north pole.) Hence we have seen that the tangent sheaf of $\mathbb{P}^1$ is a sheaf that is not isomorphic to the structure sheaf $\mathcal{O}_{\mathbb{P}^1}$ although its sections are given locally by "one complex number varying nicely".

(We should mention that the above property of $\mathbb{P}^1$ is purely topological: there is not even a continuous nowhere-zero tangent field on the unit ball in $\mathbb{R}^3$. This is usually called the "hairy ball theorem" and stated as saying that "you cannot comb a hedgehog (i. e. a ball) without a bald spot".)

Let us now get more rigorous. Recall that a presheaf of rings $\mathcal{F}$ on a topological space X was defined to be given by the data:

- for every open set $U \subset X$ a ring $\mathcal{F}(U)$,
- for every inclusion $U \subset V$ of open sets in X a ring homomorphism $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ called the restriction map,

such that

- $\mathcal{F}(\emptyset) = 0$,
- $\rho_{U,U}$ is the identity map for all U,
- for any inclusion $U \subset V \subset W$ of open sets in X we have $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$.

The elements of $\mathcal{F}(U)$ are then called the sections of $\mathcal{F}$ over U, and the restriction maps $\rho_{V,U}$ are written as $f \mapsto f|_U$. The space of global sections $\mathcal{F}(X)$ is often denoted $\Gamma(\mathcal{F})$.

A presheaf $\mathcal{F}$ of rings is called a sheaf of rings if it satisfies the following glueing property: if $U \subset X$ is an open set, $\{U_i\}$ an open cover of U and $f_i \in \mathcal{F}(U_i)$ sections for all i such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j, then there is a unique $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$ for all i. In other words, sections of a sheaf can be patched from compatible local data.

The same definition applies equally to categories other than rings, e. g. we can define sheaves of Abelian groups, k-algebras, and so on. For a ringed space $(X, \mathcal{O}_X)$, e. g. a scheme, we can also define **sheaves of $\mathcal{O}_X$-modules** in the obvious way: every $\mathcal{F}(U)$ is required to be an $\mathcal{O}_X(U)$-module, and these module structures have to be compatible with the restriction maps in the obvious sense. For example, the tangent sheaf of example 7.1.1 on a curve X is a sheaf of $\mathcal{O}_X$-modules: "sections of the tangent sheaf can be multiplied with regular functions".

Example 7.1.2. Let $X \subset \mathbb{P}^N$ be a projective variety over an algebraically closed field k, and let $S(X) = S = \bigoplus_{d \geq 0} S^{(d)}$ be its homogeneous coordinate ring. For any integer n, let $K(n)$ be the n-th graded piece of the localization of S at the non-zero homogeneous elements, i. e.

$$K(n) = \left\{ \frac{f}{g} \; ; \; f \in S^{(d+n)}, g \in S^{(d)} \text{ for some } d \geq 0 \text{ and } g \neq 0 \right\}.$$

Now for any $P \in X$ and open set $U \subset X$ we set

$$\mathcal{O}_X(n)_P = \left\{ \frac{f}{g} \in K(n) \; ; \; g(P) \neq 0 \right\} \quad \text{and} \quad \mathcal{O}_X(n)(U) = \bigcap_{P \in U} \mathcal{O}_X(n)_P.$$

For $n = 0$ this is precisely the definition of the structure sheaf, so $\mathcal{O}_X(0) = \mathcal{O}_X$. In general, $\mathcal{O}_X(n)$ is a sheaf of $\mathcal{O}_X$-modules whose sections can be thought of as "functions" of degree n in the homogeneous coordinates of X. For example:

(i) Every homogeneous polynomial of degree n defines a global section of $\mathcal{O}_X(n)$.
(ii) There are no global sections of $\mathcal{O}_X(n)$ for $n < 0$.
(iii) In $\mathbb{P}^1$ with homogeneous coordinates x_0, x_1, we have

$$\frac{1}{x_0} \in \mathcal{O}_{\mathbb{P}^1}(-1)(U)$$

for $U = \{(x_0 : x_1) \; ; \; x_0 \neq 0\}$.

Note that on the distinguished open subset X_{x_i} (where x_i are the coordinates of $\mathbb{P}^N$) the sheaf $\mathcal{O}_X(n)$ is isomorphic to the structure sheaf $\mathcal{O}_X$: for every open subset $U \subset X_{x_i}$ the maps

$$\mathcal{O}_X(U) \to \mathcal{O}_X(n)(U),\ \varphi \mapsto \varphi \cdot x_i^n \quad \text{and} \quad \mathcal{O}_X(n)(U) \to \mathcal{O}_X(U),\ \varphi \mapsto \frac{\varphi}{x_i^n}$$

give an isomorphism, hence $\mathcal{O}_X(n)|_{X_{x_i}} \cong \mathcal{O}_X|_{X_{x_i}}$. So $\mathcal{O}_X(n)$ is *locally* isomorphic to the structure sheaf, but not *globally*. (This is the same situation as for the tangent sheaf of a smooth curve in example 7.1.1.)

The sheaves $\mathcal{O}(n)$ on a projective variety (or more generally on a projective scheme) are called the **twisting sheaves**. They are probably the most important sheaves after the structure sheaf.

If we want to deal with more general sheaves, we certainly need to set up a suitable category, i. e. we have to define morphisms of sheaves, kernels, cokernels, and so on. Let us start with some simple definitions.

Definition 7.1.3. Let X be a topological space. A **morphism** $f : \mathcal{F}_1 \to \mathcal{F}_2$ of presheaves of abelian groups (or rings, sheaves of $\mathcal{O}_X$-modules etc.) on X is a collection of group homomorphisms (resp. ring homomorphisms, $\mathcal{O}_X(U)$-module homomorphisms etc.) $f_U : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$ for every open subset $U \subset X$ that commute with the restriction maps, i. e. the diagram

$$\begin{array}{ccc} \mathcal{F}_1(U) & \xrightarrow{\rho_{U,V}} & \mathcal{F}_1(V) \\ \downarrow{\scriptstyle f_U} & & \downarrow{\scriptstyle f_V} \\ \mathcal{F}_2(U) & \xrightarrow{\rho_{U,V}} & \mathcal{F}_2(V) \end{array}$$

is required to be commutative.

Example 7.1.4. If $X \subset \mathbb{P}^N$ is a projective variety and $f \in k[x_0,\dots,x_N]$ is a homogeneous polynomial of degree d, we get morphisms of sheaves of $\mathcal{O}_X$-modules

$$\mathcal{O}_X(n) \to \mathcal{O}_X(n+d), \quad \varphi \mapsto f \cdot \varphi$$

for all n.

Definition 7.1.5. If $f : X \to Y$ is a morphism of topological spaces and $\mathcal{F}$ is a sheaf on X, then we define the **push-forward** $f_*\mathcal{F}$ of $\mathcal{F}$ to be the sheaf on Y given by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ for all open subsets $U \subset Y$.

Example 7.1.6. By definition, a morphism $f : X \to Y$ of ringed spaces comes equipped with a morphism of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$. This is exactly given by the data of the pull-back morphisms $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ for all open subsets $U \subset Y$ (see definition 5.2.1).

Definition 7.1.7. Let $f : \mathcal{F}_1 \to \mathcal{F}_2$ be a morphism of sheaves of e. g. Abelian groups on a topological space X. We define the **kernel** $\ker f$ of f by setting

$$(\ker f)(U) = \ker(f_U : \mathcal{F}_1(U) \to \mathcal{F}_2(U)).$$

We claim that $\ker f$ is a sheaf on X. In fact, it is easy to see that $\ker f$ with the obvious restriction maps is a presheaf. Now let $\{U_i\}$ be an open cover of an open subset $U \subset X$, and assume we are given $\varphi_i \in \ker(\mathcal{F}_1(U_i) \to \mathcal{F}_2(U_i))$ that agree on the overlaps $U_i \cap U_j$. In particular, the φ_i are then in $\mathcal{F}_1(U_i)$, so we get a unique $\varphi \in \mathcal{F}_1(U)$ with $\varphi|_{U_i} = \varphi_i$ as $\mathcal{F}_1$ is a sheaf. Moreover, $f(\varphi_i) = 0$, so $(f(\varphi))|_{U_i} = 0$ by definition 7.1.3. As $\mathcal{F}_2$ is a sheaf, it follows that $f(\varphi) = 0$, so $\varphi \in \ker f$.

What the above argument boils down to is simply that the property of being in the kernel, i. e. of being mapped to zero under a morphism, is a *local* property — a function is zero if it is zero on every subset of an open cover. So the kernel is again a sheaf.

Remark 7.1.8. Now consider the dual case to definition 7.1.7, namely cokernels. Again let $f : \mathcal{F}_1 \to \mathcal{F}_2$ be a morphism of sheaves of e. g. Abelian groups on a topological space X. As above we define a presheaf $\operatorname{coker}' f$ by setting

$$(\operatorname{coker}' f)(U) = \operatorname{coker}(f_U : \mathcal{F}_1(U) \to \mathcal{F}_2(U)) = \mathcal{F}_2(U)/\operatorname{im} f_U.$$

Note however that $\operatorname{coker}' f$ is *not* a sheaf. To see this, consider the following example. Let $X = \mathbb{A}^1\backslash\{0\}$, $Y = \mathbb{A}^2\backslash\{0\}$, and let $i : X \to Y$ be the inclusion morphism $(x_1) \mapsto (x_1, 0)$. Let $i^\# : \mathcal{O}_Y \to i_*\mathcal{O}_X$ be the induced morphisms of sheaves on Y of example 7.1.6, and consider the presheaf $\operatorname{coker}' i^\#$ on Y. Look at the cover of Y by the affine open subsets $U_1 = \{x_1 \neq 0\} \subset Y$ and $U_2 = \{x_2 \neq 0\} \subset Y$. Then the maps

$$k\left[x_1, \frac{1}{x_1}, x_2\right] = \mathcal{O}_Y(U_1) \to \mathcal{O}_X(U_1 \cap X) = k\left[x_1, \frac{1}{x_1}\right]$$
$$\text{and} \quad k\left[x_1, x_2, \frac{1}{x_2}\right] = \mathcal{O}_Y(U_2) \to \mathcal{O}_X(U_2 \cap X) = 0$$

are surjective, hence $(\operatorname{coker}' i^\#)(U_1) = (\operatorname{coker}' i^\#)(U_2) = 0$. But on global sections the map

$$k[x_1, x_2] = \mathcal{O}_Y(Y) \to \mathcal{O}_X(X) = k\left[x_1, \frac{1}{x_1}\right]$$

is not surjective, hence $(\operatorname{coker}' i^\#)(Y) \neq 0$. This shows that $\operatorname{coker}' i^\#$ cannot be a sheaf — the zero section on the open cover $\{U_1, U_2\}$ has no *unique* extension to a global section on Y.

What the above argument boils down to is simply that being in the cokernel of a morphism, i. e. of being a quotient in $\mathcal{F}_2(U)/\operatorname{im} f_U$, is *not* a local property — it is a question about finding a global section of $\mathcal{F}_2$ on U that cannot be answered locally.

Example 7.1.9. Here is another example showing that quite natural constructions involving sheaves often lead to only presheaves because the constructions are not local. Let $X \subset \mathbb{P}^N$ be a projective variety. Consider the tensor product presheaf of the sheaves $\mathcal{O}_X(1)$ and $\mathcal{O}_X(-1)$, defined by

$$(\mathcal{O}_X(1) \otimes' \mathcal{O}_X(-1))(U) = \mathcal{O}_X(1)(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(-1)(U).$$

As $\mathcal{O}_X(1)$ describes "functions" of degree 1 and $\mathcal{O}_X(-1)$ "functions" of degree -1, we expect products of them to be true functions of pure degree 0 in the homogeneous coordinates of X. In other words, the tensor product of $\mathcal{O}_X(1)$ with $\mathcal{O}_X(-1)$ should just be the structure sheaf $\mathcal{O}_X$. However, $\mathcal{O}_X(1) \otimes' \mathcal{O}_X(-1)$ is not even a sheaf: consider the case $X = \mathbb{P}^1$ and the open subsets $U_0 = \{x_0 \neq 0\}$ and $U_1 = \{x_1 \neq 0\}$. On these open subsets we have the sections

$$x_0 \otimes \frac{1}{x_0} \in (\mathcal{O}_X(1) \otimes' \mathcal{O}_X(-1))(U_0)$$

$$\text{and} \quad x_1 \otimes \frac{1}{x_1} \in (\mathcal{O}_X(1) \otimes' \mathcal{O}_X(-1))(U_1).$$

Obviously, both these local sections are the constant function 1, so in particular they agree on the overlap $U_0 \cap U_1$. But there is no global section in $\mathcal{O}_X(1)(X) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_X(-1)(X)$ that corresponds to the constant function 1, as $\mathcal{O}_X(-1)$ has no non-zero global sections at all.

The way out of this trouble is called sheafification. This means that for any presheaf $\mathcal{F}'$ there is an associated sheaf $\mathcal{F}$ that is "very close" to $\mathcal{F}'$ and that should usually be the object that one wants. Intuitively speaking, if the sections of a presheaf are thought of as function-like objects satisfying some conditions, then the associated sheaf describes the same objects *with the conditions on them made local*. In particular, if we look at $\mathcal{F}'$ locally, i. e. at the stalks, then we should not change anything; it is just the global structure that changes. We have done this construction quite often already without explicitly saying so, e. g. in the construction of the structure sheaf of schemes in definition 5.1.11. Here is the general construction:

Definition 7.1.10. Let $\mathcal{F}'$ be a presheaf on a topological space X. The **sheafification** of $\mathcal{F}'$, or the sheaf **associated to** the presheaf $\mathcal{F}'$, is defined to be the sheaf $\mathcal{F}$ such that

$$\begin{aligned} \mathcal{F}(U) := \{\varphi = (\varphi_P)_{P \in U} & \text{ with } \varphi_P \in \mathcal{F}'_P \text{ for all } P \in U \\ & \text{such that for every } P \in U \text{ there is a neighborhood } V \text{ in } U \\ & \text{and a section } \varphi' \in \mathcal{F}'(V) \text{ with } \varphi_Q = \varphi'_Q \in \mathcal{F}'_Q \text{ for all } Q \in V.\} \end{aligned}$$

(For the notion of the stalk $\mathcal{F}'_P$ of a presheaf $\mathcal{F}'$ at a point $P \in X$ see definition 2.2.7.) It is obvious that this defines a sheaf.

Example 7.1.11. Let $X \subset \mathbb{A}^N$ be an affine variety. Let $\mathcal{O}'_X$ be the presheaf given by

$$\mathcal{O}'_X(U) = \Big\{ \varphi : U \to k \,;\, \text{there are } f, g \in k[x_1, \dots, x_N] \text{ with } g(P) \neq 0 \\ \text{and } \varphi(P) = \tfrac{f(P)}{g(P)} \text{ for all } P \in U \Big\}$$

for all open subsets $U \subset X$, i. e. the "presheaf of functions that are (globally) quotients of polynomials". Then the structure sheaf $\mathcal{O}_X$ is the sheafification of $\mathcal{O}'_X$, i. e. the sheaf of functions that are locally quotients of polynomials. We have seen in example 2.1.7 that in general $\mathcal{O}'_X$ differs from $\mathcal{O}_X$, i. e. it is in general not a sheaf.

Example 7.1.12. If X is a topological space and $\mathcal{F}$ the presheaf of constant real-valued functions on X, then the sheafification of $\mathcal{F}$ is the sheaf of *locally* constant functions on X (see also remark 2.2.5).

The sheafification has the following nice and expected properties:

Lemma 7.1.13. *Let $\mathcal{F}'$ be a presheaf on a topological space X, and let $\mathcal{F}$ be its sheafification.*

(i) *The stalks $\mathcal{F}_P$ and $\mathcal{F}'_P$ agree at every point $P \in X$.*
(ii) *If $\mathcal{F}'$ is a sheaf, then $\mathcal{F} = \mathcal{F}'$.*

Proof. (i): The isomorphism between the stalks is given by the following construction:

- An element of $\mathcal{F}_P$ is by definition represented by (U, φ), where U is an open neighborhood of P and $\varphi = (\varphi_Q)_{Q \in U}$ is a section of $\mathcal{F}$ over U. To this we can associate the element $\varphi_P \in \mathcal{F}'_P$.
- An element of $\mathcal{F}'_P$ is by definition represented by (U, φ), where $\varphi \in \mathcal{F}'(U)$. To this we can associate the element $(\varphi_Q)_{Q \in U}$ in $\mathcal{F}(U)$, which in turn defines an element of $\mathcal{F}_P$.

(ii): Note that there is always a morphism of presheaves $\mathcal{F}' \to \mathcal{F}$ given by $\mathcal{F}'(U) \to \mathcal{F}(U), \varphi \mapsto (\varphi_P)_{P \in U}$.

Now assume that $\mathcal{F}'$ is a sheaf; we will construct an inverse morphism $\mathcal{F} \to \mathcal{F}'$. Let $U \subset X$ be an open subset and $\varphi = (\varphi_P)_{P \in U} \in \mathcal{F}(U)$ a section of F. For every $P \in U$ the germ $\varphi_P \in \mathcal{F}'_P$ is represented by some (V, φ) with $\varphi \in \mathcal{F}'(V)$. As P varies over U, we are thus getting sections of $\mathcal{F}'$ on an open cover of U that agree on the overlaps. As $\mathcal{F}'$ is a sheaf, we can glue these sections together to give a global section in $\mathcal{F}'(U)$. $\square$

Using sheafification, we can now define all the “natural” constructions that we would expect to be possible:

Definition 7.1.14. Let $f : \mathcal{F}_1 \to \mathcal{F}_2$ be a morphism of sheaves of e. g. Abelian groups on a topological space X.

(i) The **cokernel** $\operatorname{coker} f$ of f is defined to be the sheaf associated to the presheaf $\operatorname{coker}' f$.
(ii) The morphism f is called **injective** if $\ker f = 0$. It is called **surjective** if $\operatorname{coker} f = 0$.
(iii) If the morphism f is injective, its cokernel is also denoted $\mathcal{F}_2/\mathcal{F}_1$ and called the **quotient** of $\mathcal{F}_2$ by $\mathcal{F}_1$.
(iv) As usual, a sequence of sheaves and morphisms

$$\cdots \to \mathcal{F}_{i-1} \to \mathcal{F}_i \to \mathcal{F}_{i+1} \to \cdots$$

is called **exact** if $\ker(\mathcal{F}_i \to \mathcal{F}_{i+1}) = \operatorname{im}(\mathcal{F}_{i-1} \to \mathcal{F}_i)$ for all i.

Remark 7.1.15. Let us rephrase again the results of definition 7.1.7 and remark 7.1.8 in this new language:

(i) A morphism $f : \mathcal{F}_1 \to \mathcal{F}_2$ of sheaves is injective if and only if the maps $f_U : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$ are injective for all U.
(ii) If a morphism $f : \mathcal{F}_1 \to \mathcal{F}_2$ of sheaves is surjective, this does *not* imply that all maps $f_U : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$ are surjective. (The converse of this is obviously true however: if all maps $f_U : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$ are surjective, then $\operatorname{coker}' f = 0$, so $\operatorname{coker} f = 0$.)

This very important fact is the basis of the theory of cohomology, see chapter 8.

Example 7.1.16. Let $X = \mathbb{P}^1_k$ with homogeneous coordinates x_0, x_1. Consider the morphism of sheaves $f : \mathcal{O}_X(-1) \to \mathcal{O}_X$ given by the linear polynomial x_0 (see example 7.1.4).

We claim that f is injective. In fact, every section of $\mathcal{O}_X(-1)$ over an open subset of X has the form $\frac{g(x_0,x_1)}{h(x_0,x_1)}$ for some homogeneous polynomials g, h with $\deg g - \deg h = -1$. But $f(\frac{g}{h}) = \frac{g x_0}{h}$ is zero *on an open subset of* X if and only if $g = 0$ (note that we are not asking for zeros of $\frac{gx_0}{h}$, but we are asking whether this function vanishes on a whole open subset). As this means that $\frac{g}{h}$ itself is zero, we see that the kernel of f is trivial, i. e. f is injective.

We have seen already in example 7.1.2 that f is in fact an isomorphism when restricted to $U = X\backslash\{P\}$ where $P := (0:1)$. In particular, f is surjective when restricted to U. However, f is not surjective on X (otherwise it would be an isomorphism, which is not true as we already know). Let us determine its cokernel.

First we have to compute the cokernel presheaf $\operatorname{coker}' f$. Consider an open subset $U \subset X$. By the above argument, $(\operatorname{coker}' f)(U) = 0$ if $P \notin U$. So assume that $P \in U$. Then we have an exact sequence of $\mathcal{O}_X(U)$-modules

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_X(-1)(U) & \to & \mathcal{O}_X(U) & \to & k & \to & 0 \\ & & \frac{g}{h} & \mapsto & \frac{gx_0}{h} & & & & \\ & & & & \varphi = \frac{g}{h} & \mapsto & \varphi(P) & & \end{array}$$

as the functions in the image of $\mathcal{O}_X(-1)(U) \to \mathcal{O}_X$ are precisely those that vanish on P. So we have found that

$$(\operatorname{coker}' f)(U) = \begin{cases} k & \text{if } P \in U, \\ 0 & \text{if } P \notin U. \end{cases}$$

It is easily verified that $\operatorname{coker}' f$ is in fact a sheaf. It can be thought of as the ground field k "concentrated at the point P". For this reason it is often called a **skyscraper sheaf** and denoted k_P.

Summarizing, we have found the exact sequence of sheaves of $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_X(-1) \stackrel{\cdot x_0}{\to} \mathcal{O}_X \to k_P \to 0.$$

Example 7.1.17. Let $\mathcal{F}_1, \mathcal{F}_2$ be two sheaves of $\mathcal{O}_X$-modules on a ringed space X. Then we can define the direct sum, the tensor product, and the dual sheaf in the obvious way:

(i) The **direct sum** $\mathcal{F}_1 \oplus \mathcal{F}_2$ is the sheaf of $\mathcal{O}_X$-modules defined by $(\mathcal{F}_1 \oplus \mathcal{F}_2)(U) = \mathcal{F}_1(U) \oplus \mathcal{F}_2(U)$. (It is easy to see that this is a sheaf already, so that we do not need sheafification.)
(ii) The **tensor product** $\mathcal{F}_1 \otimes \mathcal{F}_2$ is the sheaf of $\mathcal{O}_X$-modules associated to the presheaf $U \mapsto \mathcal{F}_1(U) \otimes_{\mathcal{O}_X(U)} \mathcal{F}_2(U)$.
(iii) The **dual** $\mathcal{F}_1^\vee$ of $\mathcal{F}_1$ is the sheaf of $\mathcal{O}_X$-modules associated to the presheaf $U \mapsto \mathcal{F}_1(U)^\vee = \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}_1(U), \mathcal{O}_X(U))$.

Example 7.1.18. Similarly to example 7.1.16 consider the morphism $f : \mathcal{O}_X(-2) \to \mathcal{O}_X$ of sheaves on $X = \mathbb{P}^1_k$ given by multiplication with x_0x_1 (instead of with x_0). The only difference to the above example is that the function x_0x_1 vanishes at two points $P_0 = (0:1)$, $P_1 = (1:0)$. So this time we get an exact sequence of sheaves

$$0 \to \mathcal{O}_X(-2) \stackrel{\cdot x_0x_1}{\to} \mathcal{O}_X \to k_{P_0} \oplus k_{P_1} \to 0,$$

where the last morphism is evaluation at the points P_0 and P_1.

The important difference is that this time the cokernel presheaf is *not* equal to the cokernel sheaf: if we consider our exact sequence on global sections, we get

$$0 \to \Gamma(\mathcal{O}_X(-2)) \to \Gamma(\mathcal{O}_X) \to k \oplus k,$$

where $\Gamma(\mathcal{O}_X(-2)) = 0$, and $\Gamma(\mathcal{O}_X)$ are just the constant functions. But the last morphism is evaluation at P and Q, and constant functions must have the same value at P and Q. So the last map $\Gamma(\mathcal{O}_X) \to k \oplus k$ is not surjective, indicating that some sheafification is going on. (In example 7.1.16 we only had to evaluate at one point, and the corresponding map was surjective.)

Example 7.1.19. On $X=\mathbb{P}^N$, we have $\mathcal{O}_X(n)\otimes\mathcal{O}_X(m)=\mathcal{O}_X(n+m)$, with the isomorphism given on sections by
$$\frac{f_1}{g_1}\otimes\frac{f_2}{g_2}\mapsto\frac{f_1f_2}{g_1g_2}.$$
Similarly, we have $\mathcal{O}_X(n)^\vee=\mathcal{O}_X(-n)$, as the $\mathcal{O}_X(U)$-linear homomorphisms from $\mathcal{O}_X(n)$ to $\mathcal{O}_X$ are precisely given by multiplication with sections of $\mathcal{O}_X(-n)$.

7.2. **Quasi-coherent sheaves.** It turns out that sheaves of modules are still too general objects for many applications — usually one wants to restrict to a smaller class of sheaves. Recall that any ring R determines an affine scheme $X=\operatorname{Spec}R$ together with its structure sheaf $\mathcal{O}_X$. Hence one would expect that any R-module M determines a sheaf $\tilde{M}$ of $\mathcal{O}_X$-modules on X. This is indeed the case, and almost any sheaf of $\mathcal{O}_X$-modules appearing in practice is of this form. For computations, this means that statements about this sheaf $\tilde{M}$ on X are finally reduced to statements about the R-module M. But it does not follow from the definitions that a sheaf of $\mathcal{O}_X$-modules has to be induced by some R-module in this way (see example 7.2.3), so we will say that it is quasi-coherent if it does, and in most cases restrict our attention to these quasi-coherent sheaves. If X is a general scheme, we accordingly require that it has an open cover by affine schemes $\operatorname{Spec}R_i$ over which the sheaf is induced by an R_i-module for all i.

Let us start by showing how an R-module M determines a sheaf of modules $\tilde{M}$ on $X=\operatorname{Spec}R$. This is essentially the same construction as for the structure sheaf in definition 5.1.11.

Definition 7.2.1. Let R be a ring, $X=\operatorname{Spec}R$, and let M be an R-module. We define a sheaf of $\mathcal{O}_X$-modules $\tilde{M}$ on X by setting
$$\begin{aligned}\tilde{M}(U):=\{&\varphi=(\varphi_\mathfrak{p})_{\mathfrak{p}\in U}\text{ with }\varphi_\mathfrak{p}\in M_\mathfrak{p}\text{ for all }\mathfrak{p}\in U\\ &\text{such that “}\varphi\text{ is locally of the form }\tfrac{m}{r}\text{ for }m\in M,r\in R\text{”}\}\\ =\{&\varphi=(\varphi_\mathfrak{p})_{\mathfrak{p}\in U}\text{ with }\varphi_\mathfrak{p}\in M_\mathfrak{p}\text{ for all }\mathfrak{p}\in U\\ &\text{such that for every }\mathfrak{p}\in U\text{ there is a neighborhood }V\text{ in }U\text{ and }m\in M,r\in R\\ &\text{with }r\notin\mathfrak{q}\text{ and }\varphi_\mathfrak{q}=\tfrac{m}{r}\in M_\mathfrak{q}\text{ for all }\mathfrak{q}\in V\}.\end{aligned}$$
It is clear from the local nature of the definition that $\tilde{M}$ is a sheaf.

The following proposition corresponds exactly to the statement of proposition 5.1.12 for structure sheaves. Its proof can be copied literally, replacing R by M at appropriate places.

Proposition 7.2.2. *Let R be a ring, $X=\operatorname{Spec}R$, and let M be an R-module.*

(i) *For every $\mathfrak{p}\in X$ the stalk of $\tilde{M}$ at $\mathfrak{p}$ is $M_\mathfrak{p}$.*
(ii) *For every $f\in R$ we have $\tilde{M}(X_f)=M_f$. In particular, $\tilde{M}(X)=M$.*

Example 7.2.3. The following example shows that not all sheaves of $\mathcal{O}_X$-modules on $X=\operatorname{Spec}R$ have to be of the form $\tilde{M}$ for some R-module M.

Let $X=\mathbb{A}^1_k$, and let $\mathcal{F}$ be the sheaf associated to the presheaf
$$U\mapsto\begin{cases}\mathcal{O}_X(U) & \text{if }0\notin U,\\ 0 & \text{if }0\in U.\end{cases}$$
with the obvious restriction maps. Then $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules. The stalk $\mathcal{F}_0$ is zero, whereas $\mathcal{F}_P=\mathcal{O}_{X,P}$ for all other points $P\in X$.

Note that $\mathcal{F}$ has no non-trivial global sections: if $\varphi\in\mathcal{F}(X)$ then we obviously must have $\varphi_0=0\in\mathcal{F}_0$, which by definition of sheafification means that φ is zero in some neighborhood of 0. But as X is irreducible, φ must then be the zero function. Hence it follows that $\mathcal{F}(X) = 0$. So if $\mathcal{F}$ was of the form $\tilde{M}$ for some R-module M, it would follow from proposition 7.2.2 (ii) that $M = 0$, hence $\mathcal{F}$ would have to be the zero sheaf, which it obviously is not.

Definition 7.2.4. Let X be a scheme, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is **quasi-coherent** if for every affine open subset $U = \operatorname{Spec} R \subset X$ the restricted sheaf $\mathcal{F}|_U$ is of the form $\tilde{M}$ for some R-module M.

Remark 7.2.5. It can be shown that it is sufficient to require the condition of the definition only for every open subset in an affine open cover of X (see e. g. [H] proposition II.5.4). In other words, quasi-coherence is a local property.

Example 7.2.6. On any scheme the structure sheaf is quasi-coherent. The sheaves $\mathcal{O}_X(n)$ are quasi-coherent on any projective subscheme of $\mathbb{P}^N$ as they are locally isomorphic to the structure sheaf. In the rest of this section we will show that essentially all operations that you can do with quasi-coherent sheaves yield again quasi-coherent sheaves. Therefore almost all sheaves that occur in practice are in fact quasi-coherent.

Lemma 7.2.7. *Let R be a ring and $X = \operatorname{Spec} R$.*

(i) *For any R-modules M, N there is a one-to-one correspondence*

$$\{\text{morphisms of sheaves } \tilde{M} \to \tilde{N}\} \leftrightarrow \{R\text{-module homomorphisms } M \to N\}.$$

(ii) *A sequence of R-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact if and only if the sequence of sheaves $0 \to \tilde{M}_1 \to \tilde{M}_2 \to \tilde{M}_3 \to 0$ is exact on X.*

(iii) *For any R-modules M, N we have $\tilde{M} \oplus \tilde{N} = (M \oplus N)\tilde{}$.*

(iv) *For any R-modules M, N we have $\tilde{M} \otimes \tilde{N} = (M \otimes N)\tilde{}$.*

(v) *For any R-module M we have $(\tilde{M})^\vee = (M^\vee)\tilde{}$.*

In particular, kernels, cokernels, direct sums, tensor products, and duals of quasi-coherent sheaves are again quasi-coherent on any scheme X.

Proof. (i): Given a morphism $\tilde{M} \to \tilde{N}$, taking global sections gives an R-module homomorphism $M \to N$ by proposition 7.2.2 (ii). Conversely, an R-module homomorphism $M \to N$ gives rise to morphisms between the stalks $M_\mathfrak{p} \to N_\mathfrak{p}$ for all $\mathfrak{p}$, and therefore by definition determines a morphism $\tilde{M} \to \tilde{N}$ of sheaves. It is obvious that these two operations are inverse to each other.

(ii): By exercise 7.8.2, exactness of a sequence of sheaves can be seen on the stalks. Hence by proposition 7.2.2 (i) the statement follows from the algebraic fact that the sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact if and only if $0 \to (M_1)_\mathfrak{p} \to (M_2)_\mathfrak{p} \to (M_3)_\mathfrak{p} \to 0$ is for all prime ideals $\mathfrak{p} \in R$.

(iii), (iv), and (v) follow in the same way as (ii): the statement can be checked on the stalks, hence it follows from the corresponding algebraic fact about localizations of modules. □

Example 7.2.8. Let $X = \mathbb{P}^1$ and $P = (0:1) \in X$. The skyscraper sheaf k_P of example 7.1.16 is quasi-coherent by lemma 7.2.7 as it is the cokernel of a morphism of quasi-coherent sheaves. One can also check explicitly that k_P is quasi-coherent: if $U_0 = \{x_0 \neq 0\} = \mathbb{P}^1 \backslash \{P\}$ and $U_1 = \{x_1 \neq 0\} = \operatorname{Spec} k[x_0] \cong \mathbb{A}^1$ then $k_P|_{U_0} = 0$ (so it is the sheaf associated to the zero module) and $k_P|_{U_1} \cong \tilde{M}$ where $M = k$ is the $k[x_0]$-module with the module structure

$$\begin{aligned} k[x_0] \times k &\to k \\ (f, \lambda) &\mapsto f(0) \cdot \lambda. \end{aligned}$$

Proposition 7.2.9. *Let $f : X \to Y$ be a morphism of schemes, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Assume moreover that every open subset of X can be covered by* finitely many *affine open subsets (this should be thought of as a technical condition that is essentially always satisfied — it is e. g. certainly true for all subschemes of projective spaces). Then $f_*\mathcal{F}$ is quasi-coherent on Y.*

Proof. Let us first assume that X and Y are affine, so $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$, and $\mathcal{F} = \tilde{M}$ for some R-module M. Then it follows immediately from the definitions that $f_*\mathcal{F} = (M \text{ as an } S\text{-module})\tilde{}$, hence push-forwards of quasi-coherent sheaves are quasi-coherent if X and Y are affine.

In the general case, note that the statement is local on Y, so we can assume that Y is affine. But it is not local on X, so we cannot assume that X is affine. Instead, cover X by affine opens U_i, and cover $U_i \cap U_j$ by affine opens $U_{i,j,k}$. By our assumption, we can take these covers to be finite.

Now the sheaf property for $\mathcal{F}$ says that for every open set $V \subset Y$ the sequence

$$0 \to \mathcal{F}(f^{-1}(V)) \to \bigoplus_i \mathcal{F}(f^{-1}(V) \cap U_i) \to \bigoplus_{i,j,k} \mathcal{F}(f^{-1}(V) \cap U_{i,j,k})$$

is exact, where the last map is given by $(\dots, s_i, \dots) \mapsto (\dots, s_i|_{U_{i,j,k}} - s_j|_{U_{i,j,k}}, \dots)$. This means that the sequence of sheaves on Y

$$0 \to f_*\mathcal{F} \to \bigoplus_i f_*(\mathcal{F}|_{U_i}) \to \bigoplus_{i,j,k} f_*(\mathcal{F}|_{U_{i,j,k}})$$

is exact. But as we have shown the statement already for morphisms between affine schemes and as finite direct sums of quasi-coherent sheaves are quasi-coherent, the last two terms in this sequence are quasi-coherent. Hence the kernel $f_*\mathcal{F}$ is also quasi-coherent by lemma 7.2.7. □

Example 7.2.10. With this result we can now define (and motivate) what a closed embedding of schemes is. Note that for historical reasons closed embeddings are usually called closed immersions in algebraic geometry (in contrast to differential geometry, where an immersion is defined to be a *local* embedding).

We say that a morphism $f : X \to Y$ of schemes is a **closed immersion** if

(i) f is a homeomorphism onto a closed subset of Y, and
(ii) the induced morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.

The kernel of the morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is then called the **ideal sheaf** $\mathcal{I}_{X/Y}$ of the immersion.

Let us motivate this definition. We certainly want condition (i) to hold on the level of topological spaces. But this is not enough — we have seen that even isomorphisms cannot be detected on the level of topological spaces (see example 2.3.8), so we need some conditions on the structure sheaves as well. We have seen in example 5.2.3 that a closed immersion should be a morphism that is locally of the form $\operatorname{Spec} R/I \to \operatorname{Spec} R$ for some ideal $I \subset R$. In fact, this is exactly what condition (ii) means: assume that we are in the affine case, i. e. $X = \operatorname{Spec} S$ and $Y = \operatorname{Spec} R$. As $\mathcal{O}_Y$ and $f_*\mathcal{O}_X$ are quasi-coherent (the former by definition and the latter by proposition 7.2.9), so is the kernel of $\mathcal{O}_Y \to f_*\mathcal{O}_X$ by lemma 7.2.7. So the exact sequence

$$0 \to \mathcal{I}_{X/Y} \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to 0$$

comes from an exact sequence of R-modules

$$0 \to I \to R \to S \to 0$$

by lemma 7.2.7 (ii). In other words, $I \subset R$ is an ideal of R, and $S = R/I$. So indeed the morphism f is of the form $\operatorname{Spec} R/I \to \operatorname{Spec} R$ and therefore corresponds to an inclusion morphism of a closed subset.

Example 7.2.11. Having studied push-forwards of sheaves, we now want to consider pull-backs, i. e. the "dual" situation: given a morphism $f : X \to Y$ and a sheaf $\mathcal{F}$ on Y, we want to construct a "pull-back" sheaf $f^*\mathcal{F}$ on X. Note that this should be "more natural" than the push-forward, as sheaves describe "function-like" objects, and for functions pull-back is more natural than push-forward: given a "function" $\varphi : Y \to k$, there is set-theoretically a well-defined pull-back function $\varphi \circ f : X \to k$. In contrast, a function $\varphi : X \to k$ does not give rise to a function $Y \to k$ in a natural way.

Let us first consider the affine case: assume that $X = \operatorname{Spec} R$, and $Y = \operatorname{Spec} S$, so that the morphism f corresponds to a ring homomorphism $S \to R$. Assume moreover that the sheaf $\mathcal{F}$ on Y is quasi-coherent, so that it corresponds to an S-module M. Then $M \otimes_S R$ is a well-defined R-module, and the corresponding sheaf on X should be the pull-back $f^*\mathcal{F}$. Indeed, if e. g. $M = S$, i. e. $\mathcal{F} = \mathcal{O}_Y$, then $M \otimes_S R = S \otimes_S R = R$, so $f^*\mathcal{F} = \mathcal{O}_X$: pull-backs of regular functions are just regular functions.

This is our "local model" for the pull-back of sheaves. To show that this extends to the global case (and to sheaves that are not necessarily quasi-coherent), we need a different description though. So assume now that X, Y, and $\mathcal{F}$ are arbitrary. The first thing to do is to define a sheaf of abelian groups on X from $\mathcal{F}$. This is more complicated than for the push-forward constructed in definition 7.1.5, because $f(U)$ need not be open if U is.

We let $f^{-1}\mathcal{F}$ be the sheaf on X associated to the presheaf $U \mapsto \lim_{V \supset f(U)} \mathcal{F}(V)$, where the limit is taken over all open subsets V with $f(U) \subset V \subset Y$. This notion of limit means that an element in $\lim_{V \supset f(U)} \mathcal{F}(V)$ is given by a pair (V, φ) with $V \supset f(U)$ and $\varphi \in \mathcal{F}(V)$, and that two such pairs (V, φ) and (V', φ') define the same element if and only if there is an open subset W with $f(U) \subset W \subset V \cap V'$ such that $\varphi|_W = \varphi'|_W$. This is the best we can do to adapt definition 7.1.5 to the pull-back case. It is easily checked that this construction does what we want on the stalks: we have $(f^{-1}\mathcal{F})_P = \mathcal{F}_{f(P)}$ for all $P \in X$.

Note that $f^{-1}\mathcal{F}$ is obviously a sheaf of $(f^{-1}\mathcal{O}_Y)$-modules, but not a sheaf of $\mathcal{O}_X$-modules. (This corresponds to the statement that in the affine case considered above, M is an S-module, but not an R-module.) We have seen in our affine case what we have to do: we have to take the tensor product over $f^{-1}\mathcal{O}_Y$ with $\mathcal{O}_X$ (i. e. over S with R). In other words, we define the **pull-back** $f^*\mathcal{F}$ of $\mathcal{F}$ to be

$$f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X,$$

which is then obviously a sheaf of $\mathcal{O}_X$-modules. As this construction restricts to the one given above if X and Y are affine and $\mathcal{F}$ quasi-coherent, it also follows that pull-backs of quasi-coherent sheaves are again quasi-coherent.

It should be stressed that this complicated limit construction is only needed to prove the existence of $f^*\mathcal{F}$ in the general case. To compute the pull-back in practice, one will almost always restrict to affine open subsets and then use the tensor product construction given above.

Example 7.2.12. Here is a concrete example in which we can see again why the tensor product construction is necessary in the construction of the pull-back. Consider the morphism $f : X = \mathbb{P}^1 \to Y = \mathbb{P}^1$ given by $(s : t) \mapsto (x : y) = (s^2 : t^2)$. We want to compute the pull-back sheaf $f^*\mathcal{O}_Y(1)$ on X.

As we already know, local sections of $\mathcal{O}_Y(1)$ are of the form $\frac{g(x,y)}{h(x,y)}$, with g and h homogeneous such that $\deg g - \deg h = 1$. Pulling this back just means inserting the equations

$x = s^2$ and $y = t^2$ of f into this expression; so the sheaf $f^{-1}\mathcal{O}_Y(1)$ has local sections $\frac{g(s^2,t^2)}{h(s^2,t^2)}$, where now $\deg(g(s^2,t^2)) - \deg(h(s^2,t^2)) = 2$.

But note that these sections do not even describe a sheaf of $\mathcal{O}_X$-modules: if we try to multiply the section s^2 with the function $\frac{t}{s}$ (i. e. a section of $\mathcal{O}_X$) on the open subset where $s \neq 0$, we get st, which is not of the form $\frac{g(s^2,t^2)}{h(s^2,t^2)}$. We have just seen the solution to this problem: consider the tensor product with $\mathcal{O}_X$. So sections of $f^*\mathcal{O}_Y(1)$ are of the form

$$\frac{g(s^2,t^2)}{h(s^2,t^2)} \otimes \frac{g'(s,t)}{h'(s,t)}$$

with $\deg(g(s^2,t^2)) - \deg(h(s^2,t^2)) = 2$ and $\deg g' - \deg h' = 0$. It is easy to see that this describes precisely all expressions of the form $\frac{g''(s,t)}{h''(s,t)}$ with $\deg g'' - \deg h'' = 2$, so the result we get is $f^*\mathcal{O}_Y(1) = \mathcal{O}_X(2)$.

In the same way one shows that $f^*\mathcal{O}_Y(n) = \mathcal{O}_X(dn)$ for all $n \in \mathbb{Z}$ and any morphism $f : X \to Y$ between projective schemes that is given by a collection of homogeneous polynomials of degree d.

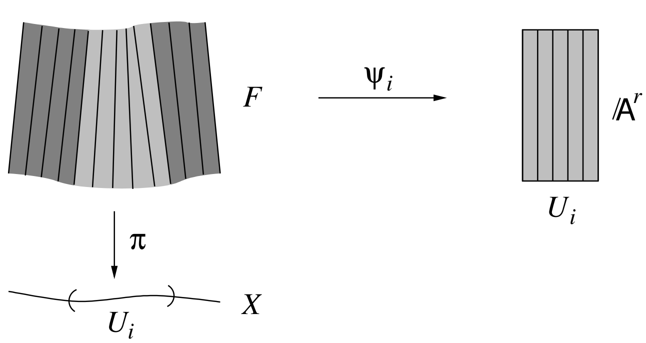
We have seen now that most sheaves occurring in practice are in fact quasi-coherent. So when we talk about sheaves from now on, we will usually think of quasi-coherent sheaves. This has the advantage that, on affine open subsets, sheaves (that form a somewhat complicated object) are essentially replaced by modules, which are usually much easier to handle.

7.3. **Locally free sheaves.** We now come to the discussion of locally free sheaves, i. e. sheaves that are locally just a finite direct sum of copies of the structure sheaf. These are the most important and best-behaved sheaves one can imagine.

Definition 7.3.1. Let X be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called **locally free** of **rank** r if there is an open cover $\{U_i\}$ of X such that $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r}$ for all i. Obviously, every locally free sheaf is also quasi-coherent.

Remark 7.3.2. The geometric interpretation of locally free sheaves is that they correspond to "vector bundles" as known from topology — objects that associate to every point P of a space X a vector bundle. For example, the "tangent sheaf" of $\mathbb{P}^1$ in example 7.1.1 is such a vector bundle (of rank 1). Let us make this correspondence precise.

A **vector bundle** of rank r on a scheme X over a field k is a k-scheme F and a k-morphism $\pi : F \to X$, together with the additional data consisting of an open covering $\{U_i\}$ of X and isomorphisms $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{A}^r_k$ over U_i, such that the automorphism $\psi_i \circ \psi_j^{-1}$ of $(U_i \cap U_j) \times \mathbb{A}^r$ is linear in the coordinates of $\mathbb{A}^r$ for all i, j. In other words, the morphism $\pi : F \to X$ looks locally like the projection morphism $U \times \mathbb{A}^r_k \to U$ for sufficiently small open subsets $U \subset X$.



We claim that there is a one-to-one correspondence

$$\{\text{vector bundles } \pi : F \to X \text{ of rank } r\} \leftrightarrow \{\text{locally free sheaves } \mathcal{F} \text{ of rank } r \text{ on } X\}$$

given by the following constructions:

(i) Let $\pi : F \to X$ be a vector bundle of rank r. Define a sheaf $\mathcal{F}$ on X by

$$\mathcal{F}(U) = \{k\text{-morphisms } s : U \to F \text{ such that } \pi \circ s = \mathrm{id}_U\}.$$

(This is called the sheaf of sections of F.) Note that this has a natural structure of a sheaf of $\mathcal{O}_X$-modules (over every point in X we can multiply a vector with a scalar — doing this on an open subset means that we can multiply a section in $\mathcal{F}(U)$ with a regular function in $\mathcal{O}_X(U)$).

Locally, on an open subset U on which π is of the form $U \times \mathbb{A}^r_k \to U$, we obviously have

$$\mathcal{F}(U) = \{k\text{-morphisms } s : U \to \mathbb{A}^r_k\},$$

so sections are just given by r independent functions. In other words, $\mathcal{F}|_U$ is isomorphic to $\mathcal{O}_U^{\oplus r}$. So $\mathcal{F}$ is locally free by definition.

(ii) Conversely, let $\mathcal{F}$ be a locally free sheaf. Take an open cover $\{U_i\}$ of X such that there are isomorphisms $\psi_i : \mathcal{F}|_{U_i} \to \mathcal{O}_{U_i}^{\oplus r}$. Now consider the schemes $U_i \times \mathbb{A}^r_k$ and glue them together as follows: for all i, j we glue $U_i \times \mathbb{A}^r_k$ and $U_j \times \mathbb{A}^r_k$ on the common open subset $(U_i \cap U_j) \times \mathbb{A}^r_k$ along the isomorphism

$$(U_i \cap U_j) \times \mathbb{A}^r_k \to (U_i \cap U_j) \times \mathbb{A}^r_k, \qquad (P, s) \mapsto (P, \psi_i \circ \psi_j^{-1}).$$

Note that $\psi_i \circ \psi_j^{-1}$ is an isomorphism of sheaves of $\mathcal{O}_X$-modules and therefore linear in the coordinates of $\mathbb{A}^r_k$.

It is obvious that this gives exactly the inverse construction to (i).

Remark 7.3.3. Let $\pi : F \to X$ be a vector bundle of rank r, and let $P \in X$ be a point. We call $\pi^{-1}(P)$ the **fiber** of F over P; it is an r-dimensional vector space. If $\mathcal{F}$ is the corresponding locally free sheaf, the fiber can be realized as $i^*\mathcal{F}$ where $i : P \to X$ denotes the inclusion morphism (note that $i^*\mathcal{F}$ is a sheaf on a one-point space, so its data consists only of one k-vector space $(i^*\mathcal{F})(P)$, which is precisely the fiber F_P).

Lemma 7.3.4. *Let X be a scheme. If $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves of $\mathcal{O}_X$-modules of rank r and s, respectively, then the following sheaves are also locally free: $\mathcal{F} \oplus \mathcal{G}$ (of rank $r+s$), $\mathcal{F} \otimes \mathcal{G}$ (of rank $r \cdot s$), and $\mathcal{F}^\vee$ (of rank r). If $f : X \to Y$ is a morphism of schemes and $\mathcal{F}$ is a locally free sheaf on Y, then $f^*\mathcal{F}$ is a locally free sheaf on X of the same rank. (The push-forward of a locally free sheaf is in general not locally free.)*

Proof. The proofs all follow from the corresponding statements about vector spaces (or free modules over a ring): for example, if M and N are free R-modules of dimension r and s respectively, then $M \oplus N$ is a free R-module of dimension $r+s$. Applying this to an open affine subset $U = \operatorname{Spec} R$ in X on which $\mathcal{F}$ and $\mathcal{G}$ are isomorphic to $\mathcal{O}_U^{\oplus r} = \tilde{M}$ and $\mathcal{O}_U^{\oplus s} = \tilde{N}$ gives the desired result. The statement about tensor products and duals follows in the same way. As for pull-backs, we have already seen that $f^*\mathcal{O}_Y = \mathcal{O}_X$, so $f^*\mathcal{F}$ will be of the form $\mathcal{O}_{f^{-1}(U)}^{\oplus r}$ on the inverse image $f^{-1}(U) \subset X$ of an open subset $U \subset Y$ on which $\mathcal{F}$ is of the form $\mathcal{O}_U^{\oplus r}$. □

Remark 7.3.5. Lemma 7.3.4 is an example of the general principle that any "canonical" construction or statement that works for vector spaces (or free modules) also works for vector bundles. Here is another example: recall that for any vector space V over k (or any free module) one can define the n-th **symmetric product** $S^n V$ and the n-th **alternating**

product $\Lambda^n V$ to be the vector space of formal totally symmetric (resp. antisymmetric) products

$$v_1 \cdot \;\cdots\; \cdot v_n \in S^n V \qquad \text{and} \qquad v_1 \wedge \cdots \wedge v_n \in \Lambda^n V.$$

If V has dimension r, then $S^n V$ and $\Lambda^n V$ have dimension $\binom{n+r-1}{n}$ and $\binom{r}{n}$, respectively. More precisely, if $\{v_1, \dots, v_r\}$ is a basis of V, then

$$\{v_{i_1} \cdot \;\cdots\; \cdot v_{i_n}\,;\, i_1 \le \cdots \le i_n\} \quad \text{is a basis of } S^n V\text{, and}$$
$$\{v_{i_1} \wedge \;\cdots\; \wedge v_{i_n}\,;\, i_1 < \cdots < i_n\} \quad \text{is a basis of } \Lambda^n V.$$

Using the same construction, we can get symmetric and alternating products $S^n \mathcal{F}$ and $\Lambda^n \mathcal{F}$ on X for every locally free sheaf $\mathcal{F}$ on X of rank r. They are locally free sheaves of ranks $\binom{n+r-1}{n}$ and $\binom{r}{n}$, respectively.

Here is an example of a linear algebra lemma that translates directly into the language of locally free sheaves:

Lemma 7.3.6. *Let $0 \to U \to V \to W \to 0$ be an exact sequence of vector spaces of dimensions a, $a+b$, and b, respectively. Then $\Lambda^{a+b} V = \Lambda^a U \otimes \Lambda^b W$.*

Proof. Denote the two homomorphisms by $i: U \to V$ and $p: V \to W$. Then there is a canonical isomorphism

$$\Lambda^a U \otimes \Lambda^b W \to \Lambda^{a+b} V$$
$$(u_1 \wedge \cdots \wedge u_a) \otimes (w_1 \wedge \cdots \wedge w_b) \mapsto i(u_1) \wedge \cdots \wedge i(u_a) \wedge p^{-1}(w_1) \wedge \cdots \wedge p^{-1}(w_b).$$

The key remark here is that the $p^{-1}(w_i)$ are well-defined up to an element of U by the exact sequence. But if the above expression is non-zero at all, the $u_1, \dots, u_a$ must form a basis of U, so if we plug in any element of U in the last b entries of the alternating product we will get zero. Therefore the ambiguity in the $p^{-1}(w_i)$ does not matter and the above homomorphism is well-defined. It is obviously not the zero map, and it is then an isomorphism for dimensional reasons (both sides are one-dimensional vector spaces). $\square$

Corollary 7.3.7. *Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be an exact sequence of locally free sheaves of ranks a_1, a_2, a_3 on a scheme X. Then $\Lambda^{a_2} \mathcal{F}_2 = \Lambda^{a_1} \mathcal{F}_1 \otimes \Lambda^{a_3} \mathcal{F}_3$.*

Proof. Immediately from lemma 7.3.6 using the above principle. $\square$

7.4. **Differentials.** We have seen in proposition 4.4.8 that (formal) differentiation of functions is useful to compute the tangent spaces at the (closed) points of a scheme X. We now want to introduce this language of differentials. The idea is that the various tangent spaces T_P for $P \in X$ should not just be independent vector spaces at every point, but rather come from a global object on X. For example, if X is smooth over $\mathbb{C}$, so that it is a complex manifold, we know from complex geometry that X has a cotangent bundle whose fiber at a point P is just the cotangent space, i. e. the dual of the tangent space, at P. We want to give an algebro-geometric analogue of this construction. So let us first define the process of formal differentiation. We start with the affine case.

Definition 7.4.1. Let $f: X = \operatorname{Spec} R \to Y = \operatorname{Spec} S$ be a morphism of affine schemes, corresponding to a ring homomorphism $S \to R$. We define the R-module $\Omega_{R/S}$, the **module of relative differentials**, to be the free R-module generated by formal symbols $\{dr\,;\, r \in R\}$, modulo the relations:

- $d(r_1 + r_2) = dr_1 + dr_2$ for $r_1, r_2 \in R$,
- $d(r_1 r_2) = r_1 dr_2 + r_2 dr_1$ for $r_1, r_2 \in R$,
- $ds = 0$ for $s \in S$.

Example 7.4.2. Let $S = k$ be a field and $R = k[x_1, \dots, x_n]$, so that we consider the morphism $f : \mathbb{A}^n_k \to \text{pt}$. Then by the relations in $\Omega_{R/k}$, which are exactly the rules of differentiation with the elements of k being the "constant" functions, it follows that $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$ for all $f \in k[x_1, \dots, x_n]$. So $\Omega_{R/k}$ is just the free R-module generated by the symbols $dx_1, \dots, dx_n$.

Again let $S = k$, but now let $R = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$ be the coordinate ring of an affine variety. By the same calculation as above, $\Omega_{R/S}$ is still generated as an R-module by $dx_1, \dots, dx_n$, but the relations f_i give rise to relations $df_i = 0$ in $\Omega_{R/S}$. It is easy to see that these are all relations in $\Omega_{R/S}$, so we have

$$\Omega_{R/S} = (R dx_1 + \dots + R dx_n)/(\sum_i \frac{\partial f_j}{\partial x_i} dx_i, j = 1, \dots, m).$$

In particular, if $X = \operatorname{Spec} R$, k is algebraically closed, and $P \in X$ is a closed point of X corresponding to a morphism $R \to k$, then by definition 4.4.1 we see that

$$\Omega_{R/S} \otimes_R k = \langle dx_1, \dots, dx_n \rangle/(\sum_i \frac{\partial f_j}{\partial x_i}(P) dx_i, j = 1, \dots, m)$$

is just the dual $T^\vee_{X,P}$ of the tangent space to X at P.

Example 7.4.3. If Y is not a point, then the difference in the module of differentials is just that all elements of S (i. e. all differentials that come from Y) are treated as "constants". So then $\Omega_{R/S}$ can be thought of as "the differentials on X modulo pull-backs of differentials on Y". We will probably not need this very often.

Of course, if $f : X \to Y$ is a morphism of general (not necessarily affine) schemes, we want to consider the relative differentials of every restriction of f to affine opens of X and Y, and glue them together to get a quasi-coherent sheaf $\Omega_{X/Y}$. To do this, we have to give a different description of the relative differentials, as the construction given above does not glue very well.

Lemma 7.4.4. *Let $S \to R$ be a homomorphism of rings. Consider the map $\delta : R \otimes_S R \to R$ given by $\delta(r_1 \otimes r_2) = r_1 r_2$ and let $I \subset R \otimes_S R$ be its kernel. Then I/I^2 is an R-module that is isomorphic to $\Omega_{R/S}$.*

Proof. The R-module structure of I/I^2 is given by $r \cdot (r_1 \otimes r_2) := r r_1 \otimes r_2 = r_1 \otimes r r_2$, where the second equality follows from

$$r r_1 \otimes r_2 - r_1 \otimes r r_2 = (r_1 \otimes r_2) \cdot (r \otimes 1 - 1 \otimes r) \in I \cdot I$$

if $r_1 \otimes r_2 \in I$. Define a map of R-modules $\Omega_{R/S} \to I/I^2$ by $dr \mapsto 1 \otimes r - r \otimes 1$. Now we construct its inverse. The R-module $E := R \oplus \Omega_{R/S}$ is a ring by setting $(r_1 \oplus dr'_1) \cdot (r_2 \oplus dr'_2) := r_1 r_2 \oplus (r_1 dr'_2 + r_2 dr'_1)$. It is easy to check that the map $R \times R \to E$ given by $(r_1, r_2) \mapsto (r_1 r_2, r_1 dr_2)$ is an S-bilinear ring homomorphism, hence gives rise to a map $g : R \otimes_S R \to E$. As $g(I) \subset \Omega_{R/S}$ by definition and $g(I^2) = 0$, this induces a map $I/I^2 \to \Omega_{R/S}$. It is easy to see that this is in fact the inverse of the map $\Omega_{R/S} \to I/I^2$ given above. □

Remark 7.4.5. It is easy to translate this lemma into the language of schemes: let $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$, so that the ring homomorphism $S \to R$ corresponds to a map $X \to Y$. Then $\operatorname{Spec} R \otimes_S R = X \times_Y X$, and $\delta : R \otimes_S R \to R$ corresponds to the diagonal morphism $X \to X \times_Y X$. Hence $I \subset R \otimes_S R$ is the ideal of the diagonal $\Delta(X) \subset X \times_Y X$. This motivates the following construction.

Definition 7.4.6. Let $f : X \to Y$ be a morphism of schemes. Let $\Delta : X \to X \times_Y X$ be the diagonal morphism, and let $\mathcal{I} = \mathcal{I}_{\Delta(X)/X \times_Y X}$ be its ideal sheaf. Then the **sheaf of relative differentials** $\Omega_{X/Y}$ is defined to be the sheaf $\Delta^*(\mathcal{I}/\mathcal{I}^2)$ on X. If X is a scheme over a field k and $Y = \operatorname{Spec} k$ is a point, then we will usually write $\Omega_{X/Y}$ as Ω_X.

Remark 7.4.7. Here we assume that the diagonal morphism Δ is a closed immersion, which is the case if the schemes in question are separated (this is the analogue of lemma 2.5.3 for schemes). We will always assume this here to avoid further complications.

Remark 7.4.8. It should be stressed that definition 7.4.6 is essentially useless for practical computations. Its only use is to show that a global object $\Omega_{X/Y}$ exists that restricts to the old definition 7.4.1 on affine open subsets. For applications, we will always use definition 7.4.1 and example 7.4.2 on open subsets.

Remark 7.4.9. The sheaf $\Omega_{X/Y}$ is always quasi-coherent: on affine open subsets it restricts to the sheaf associated to the module $\Omega_{R/S}$ constructed above.

Remark 7.4.10. Any morphism $f : X \to Y$ of schemes over a field induces a morphism of sheaves $f^*\Omega_Y \to \Omega_X$ on X that is just given by $d\varphi \mapsto d(f^*\varphi) = d(\varphi \circ f)$ for any function φ on Y.

Proposition 7.4.11. *An n-dimensional scheme X (of finite type over an algebraically closed field, e. g. a variety) is smooth if and only if Ω_X is locally free of rank n. (Actually, this is a local statement: $P \in X$ is a smooth point of X if and only if Ω_X is (locally) free in a neighborhood of P.)*

Proof. One direction is obvious: if Ω_X is locally free of rank n then its fibers at any point P, i. e. the cotangent spaces $T^\vee_{X,P}$, have dimension n. By definition this means that P is a smooth point of X.

Now let us assume that X is smooth (at P). As the proposition is of local nature we can assume that $X = \operatorname{Spec} R$ with $R = k[x_1,\dots,x_r]/(f_1,\dots,f_m)$. By example 7.4.2 we then have

$$T^\vee_{X,P} = \langle dx_1,\dots,dx_r\rangle/(\sum_i \frac{\partial f_j}{\partial x_i}(P)dx_i, j = 1,\dots,m).$$

As this vector space has dimension n, we know that the matrix of differentials $D(P) = (\frac{\partial f_l}{\partial x_i}(P))$ at the point P has rank $r-n$. Without loss of generality we can assume that the submatrix of D given by the first $r-n$ columns and rows has non-zero determinant. This means that $dx_{r-n+1},\dots,dx_r$ form a basis of $T^\vee_{X,P}$.

But the condition for a determinant to be non-zero is an "open condition", i. e. the set on which it is satisfied is open. In other words, there is a neighborhood U of P in X such that the submatrix of $D(Q)$ given by the first $r-n$ columns and rows has non-zero determinant for all $Q \in U$. Consequently, the differentials $dx_{r-n+1},\dots,dx_r$ generate $T^\vee_{X,Q}$ for all $Q \in U$. In particular, the dimension of $T^\vee_{X,Q}$ is at most n. But the opposite inequality $\dim T^\vee_{X,Q} \geq n$ is always true; so we conclude that the differentials $dx_{r-n+1},\dots,dx_r$ actually form a basis of the cotangent space at all points $Q \in U$. So

$$\Omega_X|_U = \mathcal{O}_U dx_{r-n+1} \oplus \cdots \oplus \mathcal{O}_U dx_r,$$

i. e. Ω_X is locally free. □

Remark 7.4.12. There is a similar statement for any quasi-coherent sheaf $\mathcal{F}$. It says that:

(i) The dimension of the fibers is an *upper semi-continuous function*. This means that if the dimension of the fiber of $\mathcal{F}$ at a point P is n, then it is at most n in some neighborhood of P.
(ii) If the dimension of the fibers is constant on some open subset U, then $\mathcal{F}|_U$ is locally free.

The idea of the proof of this statement is very similar to that of proposition 7.4.11.

Definition 7.4.13. Let X be a smooth n-dimensional scheme over an algebraically closed field. The dual bundle $\Omega_X^\vee$ of the cotangent bundle is called the **tangent bundle** and is denoted T_X. It is a locally free sheaf of rank n. The top exterior power $\Lambda^n \Omega_X$ of the cotangent bundle is a locally free sheaf of rank 1; it is called the **canonical bundle** ω_X of X.

Remark 7.4.14. The importance of the cotangent / canonical bundles stems from the fact that these bundles are *canonically defined* (hence the name) for any smooth scheme n. This gives e. g. a new method to show that two varieties are not isomorphic: if we have two varieties whose canonical bundles have different properties (say their spaces of global sections have different dimensions), then the varieties cannot be isomorphic.

As an example, let us now compute the cotangent / tangent / canonical bundles of some easy varieties.

Lemma 7.4.15. *The cotangent bundle of $\mathbb{P}^n$ is determined by the exact sequence*

$$0 \to \Omega_{\mathbb{P}^n} \to O(-1)^{\oplus(n+1)} \to O \to 0.$$

*(This sequence is usually called the **Euler sequence**.) Consequently, the tangent bundle fits into the dual exact sequence*

$$0 \to O \to O(1)^{\oplus(n+1)} \to T_{\mathbb{P}^n} \to 0,$$

and the canonical bundle is $\omega_{\mathbb{P}^n} = O(-n-1)$.

Proof. We know already from example 7.4.2 that the cotangent bundle $\Omega_{\mathbb{P}^n}$ is generated on the standard open subsets $U_i = \{x_i \neq 0\} \cong \mathbb{A}^n$ by the differentials $d(\frac{x_0}{x_i}), \dots, d(\frac{x_n}{x_i})$ of the affine coordinates. Therefore the differentials $d(\frac{x_i}{x_j})$, where defined, generate all of $\Omega_{\mathbb{P}^n}$. By the rules of differentiation we have to require formally that

$$d\left(\frac{x_i}{x_j}\right) = \frac{x_j dx_i - x_i dx_j}{x_j^2}.$$

Note that the dx_i are not well-defined objects, as the x_i are not functions. But if we formally let the symbols $dx_0, \dots, dx_n$ be the names of the generators of $O(-1)^{\oplus(n+1)}$, the morphism of sheaves

$$\Omega_{\mathbb{P}^n} \to O(-1)^{\oplus(n+1)}, \quad d\left(\frac{x_i}{x_j}\right) \mapsto \frac{1}{x_j} \cdot dx_i - \frac{x_i}{x_j^2} \cdot dx_j$$

is obviously well-defined and injective. It is now easily checked that the sequence of the lemma is exact, with the last morphism given by

$$O(-1)^{\oplus(n+1)} \mapsto O, \quad dx_i \mapsto x_i.$$

The sequence for the tangent bundle is obtained by dualizing. The statement about the canonical bundle then follows from corollary 7.3.7. $\square$

Lemma 7.4.16. *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d, and let $i : X \to \mathbb{P}^n$ be the inclusion morphism. Then the cotangent bundle Ω_X is determined by the exact sequence*

$$0 \to O_X(-d) \to i^* \Omega_{\mathbb{P}^n} \to \Omega_X \to 0.$$

Consequently, the tangent bundle is determined by the exact sequence

$$0 \to T_X \to i^* T_{\mathbb{P}^n} \to O_X(d) \to 0,$$

and the canonical bundle is $\omega_X = O_X(d-n-1)$.

Proof. We claim that the exact sequence is given by

$$\begin{array}{ccccccccc} 0 & \to & O_X(-d) & \to & i^*\Omega_{\mathbb{P}^n} & \to & \Omega_X & \to & 0 \\ & & \varphi & \mapsto & d(f\cdot\varphi), & & & & \\ & & & & d\varphi & \mapsto & d(\varphi|_X), & & \end{array}$$

where f is the equation defining X. In fact, the second map is just the usual pull-back of differential forms as in remark 7.4.10 (which is just a restriction in this case). It is surjective because functions on X are *locally* of the form $\frac{g}{h}$ for some homogeneous polynomials g and h of the same degree, so they are locally obtained by restricting a function on $\mathbb{P}^n$ to X. It is not an isomorphism though, because we have the identity $f = 0$ on X. Consequently, differentials $d\varphi$ are zero when restricted to X if and only if φ contains f as a factor. This explains the first map of the above sequence.

As in the previous lemma, the statements about the tangent and canonical bundles are obtained by dualizing and applying corollary 7.3.7, respectively. □

Remark 7.4.17. In general, if $i : X \to Y$ is a closed immersion of smooth schemes over a field, there is an injective morphism $T_X \to i^*T_Y$ of sheaves on X. In other words, at points in X the tangent spaces of X are just subspaces of the tangent spaces of Y. The quotient $T_{Y,P}/T_{X,P}$ is called the normal space, and consequently the quotient bundle $N_{X/Y} = i^*T_Y/T_X$ is called the **normal bundle**. This is the same construction as in differential geometry. Thus lemma 7.4.16 just tells us that the normal bundle of a degree-d hypersurface in $\mathbb{P}^n$ is $N_{X/\mathbb{P}^n} = O_X(d)$.

Example 7.4.18. Let us evaluate lemma 7.4.16 in the simplest cases, namely for curves $X \subset \mathbb{P}^2$ of low degrees d.

(i) $d = 1$: A linear curve in $\mathbb{P}^2$ is just isomorphic to $\mathbb{P}^1$. We get $\Omega_X = \omega_X = O(1-2-1) = O(-2)$ by lemma 7.4.16. This is consistent with lemma 7.4.15 for $n = 1$.

(ii) $d = 2$: We know from example 3.3.11 that a smooth plane conic is again just isomorphic to $\mathbb{P}^1$ by means of a quadratic map $f : \mathbb{P}^1 \to X \subset \mathbb{P}^2$. Our formula of lemma 7.4.16 gives $\omega_X = O_X(2-2-1) = O_X(-1)$. By pulling this back via f we obtain $\omega_X = O_{\mathbb{P}^1}(-2)$ by example 7.2.12. So by applying the isomorphism to case (i) we get the same canonical bundle back — which has to be the case, as the cotangent bundle is canonically defined and cannot change with the embedding in projective space.

(iii) $d = 3$: Here we get $\omega_X = O(3-2-1) = O$, i. e. the canonical bundle is simply isomorphic to the sheaf of regular functions. We can understand this from our representation in proposition 6.5.7 of cubic curves as complex tori of the form $\mathbb{C}/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}$. If z is the complex coordinate on $\mathbb{C}$, note that the differential form dz is invariant under shifts in Λ, as $d(z+a) = dz$ for all $a \in \mathbb{C}$. Therefore dz descends to a global differential form on $X = \mathbb{C}/\Lambda$ without zeros or poles. It follows that we have an isomorphism $O_X \to \omega_X$ given by $\varphi \mapsto \varphi \cdot dz$.

7.5. **Line bundles on curves.** We now want to specialize even further and consider vector bundles of rank 1 (also called "line bundles", because their fibers are just lines) on smooth curves. This section should be compared to section 6.3 where we considered divisors on such curves. We will show that divisor classes and line bundles are essentially the same thing.

Recall that the group $\operatorname{Pic} X$ of divisor classes on a smooth curve X has a group structure in a natural way. So let us first make the set of all line bundles on X into a group as well. In fact, this can be done for any scheme:

Definition 7.5.1. Let X be a scheme. A **line bundle** on X is a vector bundle (i. e. a locally free sheaf) of rank 1. We denote the set of all line bundles on X by $\operatorname{Pic}' X$. This set has a natural structure of Abelian group, with multiplication given by tensor products, inverses by taking duals, and the neutral element by the structure sheaf.

We will now restrict our attention to smooth curves. To set up a correspondence between line bundles and divisors, we will have to define the divisor of a (rational) section of a line bundle. This is totally analogous to the divisor of a rational function in definition 6.3.4.

Definition 7.5.2. Let $\mathcal{L}$ be a line bundle on a smooth curve X, and let $P \in X$ be a point. Assume that we are given a section $s \in \mathcal{L}(U)$ of $\mathcal{L}$ on some neighborhood U of P. As $\mathcal{L}$ is a line bundle, there is an isomorphism $\psi : \mathcal{L}|_U \to \mathcal{O}_U$ (possibly after shrinking U). The order of vanishing $\operatorname{ord}_P s$ of the section s at P is defined to be the order of vanishing of the regular function $\psi(s)$ at P.

Remark 7.5.3. Note that this definition does not depend on the choice of ψ: if $\psi' : \mathcal{L}|_U \to \mathcal{O}_U$ is another isomorphism, then the composition $\psi' \circ \psi^{-1} : \mathcal{O}_U \to \mathcal{O}_U$ is an isomorphism of the structure sheaf, which must be given by multiplication with a function φ that is nowhere zero (in particular not at P). So we have an equation of divisors

$$(\psi'(s)) = (\psi'\psi^{-1}\psi(s)) = (\varphi \cdot \psi(s)) = (\varphi) + (\psi(s)) = (\psi(s)),$$

which shows that $\operatorname{ord}_P s$ is well-defined.

Definition 7.5.4. Let $\mathcal{L}$ be a line bundle on a smooth curve X. A **rational section of** $\mathcal{L}$ over U is a section of the sheaf $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X$, where $\mathcal{K}_X$ denotes the "sheaf of rational functions" whose value at every open subset $U \subset X$ is just $K(X)$. In other words, a rational section of a line bundle is given by an ordinary section of the line bundle, possibly multiplied with a rational function.

Now let $P \in X$ be a point, and let s be a rational section of $\mathcal{L}$ in a neighborhood of P. With the same isomorphism ψ as in definition 7.5.2, the order $\operatorname{ord}_P s$ of s at P is defined to be the order of the rational function $\psi(s)$ at P. (This is well-defined for the reason stated in remark 7.5.3.)

If s is a global rational section of $\mathcal{L}$, we define the divisor (s) of s to be

$$(s) = \sum_{P \in X} \operatorname{ord}_P s \cdot P \in \operatorname{Div} X.$$

Example 7.5.5. Let $X = \mathbb{P}^1$ with homogeneous coordinates x_0, x_1.

(i) Consider the global section $s = x_0 x_1$ of $\mathcal{O}_X(2)$. It vanishes at the points $P = (0:1)$ and $Q = (1:0)$ with multiplicity 1 each, so $(s) = P + Q$.
(ii) The divisor of the global *rational* section $s = \frac{1}{x_0}$ of $\mathcal{O}_X(-1)$ is $(s) = -P$.

To show that $\operatorname{Pic}' X \cong \operatorname{Pic} X$ for smooth curves we need the following key lemma (which is the only point at which smoothness is needed).

Lemma 7.5.6. *Let X be a curve (over some algebraically closed field), and let $P \in X$ be a smooth point. Then there is a function φ_P in a neighborhood of P such that*

(i) *φ_P vanishes at P with multiplicity 1, i. e. its divisor contains the point P with multiplicity 1.*
(ii) *φ_P is non-zero at all points distinct from P.*

Proof. We can assume that $X = \operatorname{Spec} R$ is affine, with $R = k[x_1, \dots, x_r]/(f_1, \dots, f_m)$ being the coordinate ring of X. As P is a smooth point of X, its cotangent space

$$T_{X,P}^\vee = \langle dx_1, \dots, dx_r \rangle / (\sum_i \frac{\partial f_j}{\partial x_i}(P)\, dx_i \text{ for all } j)$$

is one-dimensional. Let φ_P be any linear function such that $d\varphi_P$ generates this vector space. Then φ_P vanishes at P with multiplicity 1 by construction. We can now pick a neighborhood of P such that φ_P does not vanish at any other point. $\square$

Remark 7.5.7. If the ground field is $\mathbb{C}$ and one thinks of X as a complex one-dimensional manifold, one can think of the function φ_P of lemma 7.5.6 as a "local coordinate" of X around P, i. e. a function that gives a local isomorphism of X with $\mathbb{C}$, with P mapping to $0 \in \mathbb{C}$. Note however that this is *not* true in the algebraic category, as the Zariski open subsets are too big.

We are now ready to prove the main proposition of this section.

Definition 7.5.8. A divisor $D = \sum_P a_P P$ on a smooth curve X is called **effective** (written $D \geq 0$) if $a_P \geq 0$ for all P.

Proposition 7.5.9. *Let X be a smooth curve. Then there is an isomorphism of Abelian groups*

$$\begin{array}{rcl} \operatorname{Pic}' X & \to & \operatorname{Pic} X \\ \mathcal{L} & \mapsto & (s) \quad \textit{for any rational section } s \textit{ of } \mathcal{L}. \end{array}$$

Its inverse is given by

$$\begin{array}{rcl} \operatorname{Pic} X & \to & \operatorname{Pic}' X \\ D & \mapsto & \mathcal{O}(D), \end{array}$$

where $\mathcal{O}(D)$ is the line bundle defined by

$$\mathcal{O}(D)(U) = \{\varphi \in K(X)\ ;\ (\varphi) + D \geq 0 \textit{ on } U\}.$$

Proof. We have to check a couple of things:

(i) If $\mathcal{L}$ is a line bundle, then there is a rational section s of $\mathcal{L}$: This is obvious, as $\mathcal{L}$ is isomorphic to $\mathcal{O}$ on an open subset of X. So we can find a section of $\mathcal{L}$ on this open subset (corresponding to the constant function 1). This will be a rational section of $\mathcal{L}$ on all of X.

(ii) The divisor *class* (s) of a rational section s of $\mathcal{L}$ does not depend on the choice of s: If we have another section s', then the quotient $\frac{s}{s'}$ will be a rational function, which has divisor class zero by definition of $\operatorname{Pic} X$. So $(s) = (\frac{s}{s'} \cdot s') = (\frac{s}{s'}) + (s') = (s')$ in $\operatorname{Pic} X$.

(iii) If D is a divisor then $\mathcal{O}(D)$ is actually a line bundle: let $P \in X$ be a point and choose a neighborhood U of P such that no point of $U \backslash P$ is contained in D. Let n be the coefficient of P in D. Then an isomorphism $\psi : \mathcal{O}(D) \to \mathcal{O}$ on U is given by multiplication with φ_P^n, where φ_P is the function of lemma 7.5.6. In fact, a rational function φ in $K(X)$ is by definition a section of $\mathcal{O}(D)$ if and only if $\operatorname{ord}_P \varphi + n \geq 0$, which is the case if and only if $\varphi \cdot \varphi_P^n$ is regular at P.

(iv) If the divisors D and D' define the same element in $\operatorname{Pic} X$ then $\mathcal{O}(D) = \mathcal{O}(D')$: By assumption we have $D - D' = (\varphi)$ in $\operatorname{Pic} X$ for some rational function φ. Obviously, this induces an isomorphism $\mathcal{O}(D) \to \mathcal{O}(D')$ through multiplication with φ.

We have now shown that the maps stated in the proposition are well-defined. Let us now check that the two maps are inverse to each other.

(v) $\operatorname{Pic}' X \to \operatorname{Pic} X \to \operatorname{Pic}' X$: Let s_0 be a rational section of a line bundle $\mathcal{L}$, and consider $\mathcal{O}((s_0)) = \{\varphi \in K(X)\ ;\ (\varphi) + (s_0) \geq 0\}$. We have an isomorphism

$$\mathcal{L} \to \mathcal{O}((s_0)), \quad s \mapsto \frac{s}{s_0}.$$

(vi) $\operatorname{Pic} X \to \operatorname{Pic}' X \to \operatorname{Pic} X$: The (constant) rational function 1 defines a rational section of $\mathcal{O}(D)$. To determine its order at a point P we have to apply the local isomorphism with $\mathcal{O}$ constructed in (iii): the order of this rational section at P is just the order of $1 \cdot \varphi_P^n$, which is n. This is exactly the multiplicity of P in D, so the divisor of our section is precisely D.

Finally, we have to check that the map is a homomorphism of groups. But this is clear: if s and s' are rational sections of $\mathcal{L}$ and $\mathcal{L}'$, respectively, then ss' is a rational section of $\mathcal{L} \otimes \mathcal{L}'$, and $(ss') = (s) + (s')$. Hence tensor products of line bundles correspond to addition of divisors under our correspondence. □

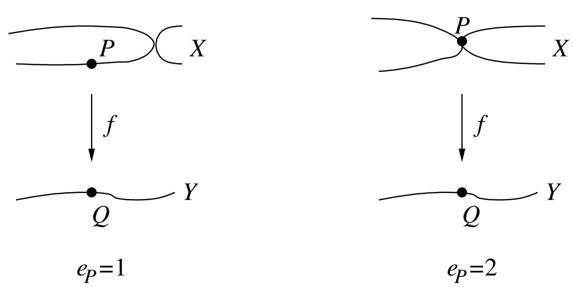
Definition 7.5.10. Let X be a smooth curve. From now on we will identify line bundles with divisor classes and call both groups $\operatorname{Pic} X$. In particular, this defines the degree of a line bundle (to be the degree of the associated divisor class). The divisor class associated to the canonical bundle ω_X is denoted K_X; it is called the **canonical divisor (class)**.

Example 7.5.11. We have seen in lemma 6.3.11 that $\operatorname{Pic} \mathbb{P}^1 = \mathbb{Z}$, i. e. there is exactly one divisor class in every degree. Consequently, there is exactly one line bundle for every degree n, which is of course just $O(n)$. On the other hand, if $X \subset \mathbb{P}^2$ is a smooth cubic curve we know from corollary 6.3.15 that $\operatorname{Pic} X$ consists of a copy of X in every degree. So on a cubic curve there are (many) more line bundles than just the bundles of the form $O(n)$.

Remark 7.5.12. The correspondence of proposition 7.5.9 allows us to define the pull-back f^*D of a divisor class D on Y for any (surjective) morphism of smooth curves $f : X \to Y$: it is just given by pulling back the corresponding line bundle.

In fact, we can even define a pull-back f^*D for any *divisor* $D \in \operatorname{Div} Y$ that induces this construction on the corresponding divisor class: let $P \in X$ be any point, and let $Q = f(P)$ be its image, considered as an element of $\operatorname{Div} Y$. Then the subscheme $f^{-1}(Q)$ of X has a component whose underlying point is P. We define the **ramification index** e_P of f at P to be the length of this component subscheme. In more down to earth terms, this means that we take a function φ_Q as in lemma 7.5.6 that vanishes at Q with multiplicity 1, and define e_P to be the order of vanishing of the pull-back function $f^*\varphi_Q = \varphi_Q \circ f$ at P.

The ramification index has a simple interpretation in complex analysis: in the ordinary topology the curves X and Y are locally isomorphic to the complex plane, so we can pick local coordinates z on X around P and w on Y around Q. Every holomorphic map is now locally of the form $z \mapsto w = uz^n$ for some $n \geq 1$ and an invertible function u (i. e. a function that is non-zero at P). The number n is just the ramification index defined above. It is 1 if and only if f is a local isomorphism at P in complex analysis. We say that f is ramified at P if $n = e_P > 1$, and unramified at P otherwise.



If we now consider a point Q as an element of $\operatorname{Div} Y$, we simply define

$$f^*Q = \sum_{P: f(P)=Q} e_P \cdot P$$

and extend this by linearity to obtain a homomorphism $f^* : \operatorname{Div} Y \to \operatorname{Div} X$. In other words, f^*D is just obtained by taking the inverse image points of the points in D with the appropriate multiplicities.

Using the correspondence of proposition 7.5.9 it is now easily checked that the induced map $f^* : \operatorname{Pic} Y \to \operatorname{Pic} X$ on the Picard groups agrees with the pull-back of line bundles.

Example 7.5.13. Let $f : X = \mathbb{P}^1 \to Y = \mathbb{P}^1$ be the morphism given by $(x_0 : x_1) \mapsto (x_0^2 : x_1^2)$. Then $f^*(1:0) = 2\cdot(1:0)$ and $f^*(1:1) = (1:1)+(1:-1)$ as divisors in X.

As an application of line bundles, we will now see how they can be used to describe morphisms to projective spaces. This works for all schemes (not just curves).

Lemma 7.5.14. *Let X be a scheme over an algebraically closed field. There is a one-to-one correspondence*

$$\{\textit{morphisms } f : X \to \mathbb{P}^r\} \longleftrightarrow \left\{ \begin{array}{l} \textit{line bundles } \mathcal{L} \textit{ on } X \textit{ together with global} \\ \textit{sections } s_0,\dots,s_r \in \Gamma(X,\mathcal{L}) \textit{ such that:} \\ \textit{for all } P \in X \textit{ there is some } s_i \textit{ with } s_i(P) \neq 0 \end{array} \right\}$$

Proof. "$\longleftarrow$": Given $r+1$ sections of a line bundle $\mathcal{L}$ on X that do not vanish simultaneously, we can define a morphism $f : X \to \mathbb{P}^r$ by setting $f(P) = (s_0(P) : \cdots : s_r(P))$. Note that the values $s_i(P)$ are not well-defined numbers, but their quotients $\frac{s_i}{s_j}(P)$ are (as they are sections of $\mathcal{L} \otimes \mathcal{L}^\vee = \mathcal{O}$, i. e. ordinary functions). Therefore $f(P)$ is a well-defined point in projective space.

"$\longrightarrow$": Given a morphism $f : X \to \mathbb{P}^r$, we set $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^r}(1)$ and $s_i = f^*x_i$, where we consider the x_i as sections of $\mathcal{O}(1)$ (and thus the s_i as sections of $f^*\mathcal{O}(1)$). □

Remark 7.5.15. One should regard this lemma as a generalization of lemma 3.3.9 where we have seen that a morphism to $\mathbb{P}^r$ can be given by specifying $r+1$ homogeneous polynomials of the same degree. Of course, this was just the special case in which the line bundle of lemma 7.5.14 is $\mathcal{O}(d)$. We had mentioned already in remark 3.3.10 that not all morphisms are of this form; this translates now into the statement that not all line bundles are of the form $\mathcal{O}(n)$.

7.6. **The Riemann-Hurwitz formula.** Let X and Y be smooth projective curves, and let $f : X \to Y$ be a surjective morphism. We want to compare the sheaves of differentials on X and Y. Note that every projective curve admits a surjective morphism to $\mathbb{P}^1$: by definition it sits in some $\mathbb{P}^n$ to start with, so we can find a morphism to $\mathbb{P}^1$ by repeated projections from points not in X. So if we know the canonical bundle of $\mathbb{P}^1$ (which we do by lemma 7.4.15: it is just $\mathcal{O}_{\mathbb{P}^1}(-2)$) and how canonical bundles transform under morphisms, we can at least in theory compute the canonical bundles of every curve.

Definition 7.6.1. Let $f : X \to Y$ be a surjective morphism of smooth projective curves. We define the **ramification divisor** of f to be $R = \sum_{P\in X}(e_P - 1)\cdot P \in \operatorname{Div} X$, where e_P is the ramification index of f at P defined in remark 7.5.12. So the divisor R contains all points at which f is ramified, with appropriate multiplicities.

Proposition 7.6.2. ***(Riemann-Hurwitz formula)*** *Let $f : X \to Y$ be a surjective morphism of smooth projective curves, and let R be the ramification divisor of f. Then $K_X = f^*K_Y + R$ (or equivalently $\omega_X = f^*\omega_Y \otimes \mathcal{O}_X(R)$) in* $\operatorname{Pic} X$.

Proof. Let $P \in X$ be any point, and let $Q = f(P)$ be its image point. Choose local functions φ_P and φ_Q around P (resp. Q) that vanish at P (resp. Q) with multiplicity 1 as in lemma 7.5.6. Then by the definition of the ramification index we have

$$f^*\varphi_Q = u \cdot \varphi_P^{e_P}$$

for some local function u on X with no zero or pole at P. Now pick a global rational section α of ω_Y. If its divisor (α) contains the point Q with multiplicity n, we can write locally

$$\alpha = v \cdot \varphi_Q^n d\varphi_Q,$$

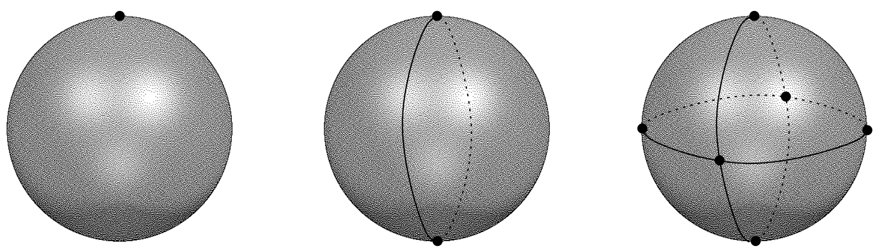
where v is a local function on Y with no zero or pole at Q. Inserting these equations into each other, we see that

$$f^*\alpha = f^*v \cdot (f^*\varphi_Q^n) d(f^*\varphi_Q) = u^n f^*v \cdot \varphi_P^{ne_P} \cdot (\varphi_P^{e_P} du + ue_P \varphi_P^{e_P-1} d\varphi_P).$$

This vanishes at P to order $ne_P + e_P - 1$. Summing this over all points $P \in X$ we see that the divisor of $f^*\alpha$ is $f^*(\alpha) + R$. As $K_X = (f^*\alpha)$ and $f^*K_Y = f^*(\alpha)$, the proposition follows. □

We will now study the same situation from a topological point of view (if the ground field is $\mathbb{C}$). Then X and Y are two-dimensional compact manifolds.

For such a space X, we say that a **cell decomposition** of X is given by writing X as a finite disjoint union of points, (open) lines, and discs. This decomposition should be "nice" in a certain topological sense, e. g. the boundary points of every line in the decomposition must be points of the decomposition. It takes some work to make this definition (and the following propositions) bullet-proof. We do not want to elaborate on this, but only remark that every "reasonable" decomposition that one could think of will be allowed. For example, here are three valid decompositions of the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$:



(i) (ii) (iii)

(In (i), we have only one point (the north pole), no line, and one "disc", namely $\mathbb{P}^1$ minus the north pole). We denote by $\sigma_0, \sigma_1, \sigma_2$ the number of points, lines and discs in the decomposition, respectively. So in the above examples we have $(\sigma_0, \sigma_1, \sigma_2) = (1,0,1)$, $(2,2,2)$, and $(6,8,4)$, respectively.

Of course there are many possible decompositions for a given curve X. But there is an important number that is invariant:

Lemma 7.6.3. *The number $\sigma_0 - \sigma_1 + \sigma_2$ depends only on X and not on the chosen decomposition. It is called the **(topological) Euler characteristic** $\chi(X)$ of X.*

Proof. Let us first consider the case when we move from one decomposition to a "finer" one, i. e. if we add points or lines to the decomposition. For example, in the above pictures (iii) is a refinement of (ii), which is itself a refinement of (i). Note that every refinement is obtained by applying the following steps a finite number of times:

(i) Adding another point on a line: In this case we raise σ_0 and σ_1 by 1, so the alternating sum $\sigma_0 - \sigma_1 + \sigma_2$ does not change (see the picture below).

add a point add a line

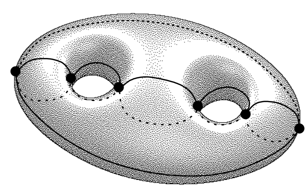
(ii) Adding another line in a disc: In this case we raise σ_1 and σ_2 by 1, so the alternating sum $\sigma_0 - \sigma_1 + \sigma_2$ again does not change (see the picture above).

So we conclude that the alternating sum $\sigma_0 - \sigma_1 + \sigma_2$ does not change under refinements. But it is easily seen that any two decompositions have a common refinement (which is essentially given by taking all the points and lines in both decompositions, and maybe add more points where two such lines intersect. For example, the common refinement of decomposition (ii) above and the same decomposition rotated clockwise by 90 degrees would be (iii)). It follows that the alternating sum is independent of the decomposition. □

We have already noted in example 0.1.1 that a smooth complex curve is topologically a (real) closed surface with a certain number g of "holes". The number g is called the genus of the curve. Let us compute the topological Euler characteristic of such a curve of genus g:

Lemma 7.6.4. *The Euler characteristic of a curve of genus g is equal to $2-2g$.*

Proof. Take e. g. the decomposition illustrated in the following picture:



It has $2g+2$ points, $4g+4$ lines, and 4 discs, so the result follows. □

Let us now compare the Euler characteristics of two curves X and Y if we have a morphism $f : X \to Y$:

Lemma 7.6.5. *Let $f : X \to Y$ be a morphism of complex smooth projective curves. Let n be the number of inverse image points of any point of Y under f. As in proposition 7.6.2 let R be the ramification divisor of f. Then $-\chi(X) = -n \cdot \chi(Y) + \deg R$.*

Proof. Choose "compatible" decompositions of X and Y, i. e. loosely speaking decompositions such that the inverse images of the points / lines / discs of the decomposition of Y are (finite) unions of points / lines / discs of the decomposition of X, and such that all points / lines / discs of the decomposition of X arise in this way. Moreover, we require that all ramification points of f are points of the decomposition of X. (It is easily seen that this can always be achieved.) Denote by σ_0^X, σ_1^X, σ_2^X the number of points / lines / discs of the decomposition of X, and similarly for Y.

As every point of Y that is not the image of a ramification point has n inverse images under f, it follows that $\sigma_1^X = n\sigma_1^Y$ and $\sigma_2^X = n\sigma_2^Y$. We do not have $\sigma_0^X = n\sigma_0^Y$ however: if P is a ramification point, i. e. $e_P > 1$, then f is locally e_P-to-one around P, i. e. P counts for e_P in $n\sigma_0^Y$, whereas it is actually only one point in the decomposition of X. Hence we have to subtract $e_P - 1$ for any ramification point P from $n\sigma_0^Y$ to get the correct value of σ_0^X. This means that $\sigma_0^X = n\sigma_0^Y - \deg R$ and hence $-\chi(X) = -n\chi(Y) + \deg R$. □

Corollary 7.6.6. *Let X be a (complex) smooth projective curve. Then $\deg K_X = 2g-2$.*

Proof. As we have already remarked, any such curve X admits a surjective morphism f to $\mathbb{P}^1$ by projection. Using that $\deg K_{\mathbb{P}^1} = -\chi(\mathbb{P}^1) = -2$ (by lemma 7.4.15 and lemma 7.6.4) and applying lemma 7.6.5 together with the Riemann-Hurwitz formula 7.6.2, we see that $\deg K_X = -\chi(X)$. The result therefore follows from lemma 7.6.4. □

7.7. **The Riemann-Roch theorem.** As in the last section let X be a smooth projective curve of genus g over an algebraically closed field. For any line bundle $\mathcal{L}$ we want to compute the dimensions of the vector spaces $\Gamma(\mathcal{L})$ of global sections of $\mathcal{L}$. We will denote this dimension by $h^0(\mathcal{L})$ (the reason for this notation will become obvious when we discuss cohomology in chapter 8). By abuse of notation we will also write $h^0(D)$ instead of $h^0(\mathcal{O}(D))$ for any divisor D.

We should remark that this is a classical question that was one of the first problems studied in algebraic geometry: given a smooth projective curve X (resp. a compact one-dimensional complex manifold), points $P_1, \dots, P_r \in X$, and numbers $a_1, \dots, a_r \geq 0$, what is the dimension of the space of rational (resp. meromorphic) functions on X that have poles of order at most a_i at the points P_i and are regular (resp. holomorphic) everywhere else? In our language, this just means that we are looking for the number $h^0(a_1 P_1 + \cdots + a_r P_r)$.

Example 7.7.1. Let D be a divisor on X with negative degree. Recall that sections of $\mathcal{O}(D)$ are just rational functions φ on X such that $(\varphi) + D$ is effective. Taking degrees, this certainly implies that $\deg(\varphi) + \deg D \geq 0$. But $\deg(\varphi) = 0$ by remark 6.3.5 and $\deg D < 0$ by assumption, which is a contradiction. Hence we conclude that $h^0(D) = 0$ if $\deg D < 0$: there are no global sections of $\mathcal{O}(D)$ in this case.

Example 7.7.2. Let $\mathcal{L}$ be the line bundle $\mathcal{O}_X(n)$ for some $n \in \mathbb{Z}$. Recall that sections of $\mathcal{L}$ are of the form $\frac{f}{g}$ with f and g homogeneous such that $\deg f - \deg g = n$. Now for *global* sections g must be a constant function (otherwise we would have a pole somewhere), so we conclude that $\Gamma(\mathcal{L})$ is simply the n-th graded piece of the homogeneous coordinate ring $S(X)$.In other words, $h^0(\mathcal{L})$ is by definition equal to the value $h_X(n)$ of the Hilbert function introduced in section 6.1. We have seen in proposition 6.1.5 that $h_X(n)$ is equal to a linear polynomial $\chi_X(n)$ in n for $n \gg 0$. Moreover, the linear coefficient of $\chi_X(n)$ is the degree of $\mathcal{O}_X(n)$, and the constant coefficient is $1-g$ by definition of g (see example 6.1.10). So we conclude that

$$h^0(D) = \deg D + 1 - g$$

if D is the divisor class associated to a line bundle $\mathcal{O}_X(n)$ for $n \gg 0$.

Theorem 7.7.3. ***(Riemann-Roch theorem for line bundles on curves)*** *Let X be a complex smooth projective curve of genus g. Then for any divisor D on X we have*

$$h^0(D) - h^0(K_X - D) = \deg D + 1 - g.$$

Proof. Step 1. Recall that for any point $P \in X$ and any divisor D we have the exact "skyscraper sequence" by exercise 7.8.4

$$0 \to \mathcal{O}(D) \to \mathcal{O}(D+P) \to k_P \to 0$$

where the last morphism is given by evaluation at the point P. From this we get an exact sequence of global sections

$$0 \to \Gamma(\mathcal{O}(D)) \to \Gamma(\mathcal{O}(D+P)) \to \mathbb{C}$$

(where the last map is in general not surjective, see example 7.1.18). Therefore $h^0(D + P) - h^0(D)$ is either 0 or 1. If we denote the left hand side of the Riemann-Roch theorem by $\chi(D) = h^0(D) - h^0(K_X - D)$, we conclude that

$$\chi(D+P) - \chi(D) = (h^0(D+P) - h^0(D)) + (h^0(K_X - D) - h^0(K_X - D - P))$$

is either 0, 1, or 2. (Of course, what we want to prove is that $\chi(D+P) - \chi(D)$ is always equal to 1.)

Step 2. We want to rule out the case that $\chi(D+P)-\chi(D)=2$. For this we actually have to borrow a theorem from complex analysis.

So assume that $h^0(D+P)-h^0(D)=1$ and $h^0(K_X-D)-h^0(K_X-D-P)=1$. The fact that $h^0(D+P)-h^0(D)=1$ means precisely that there is a global section φ of $\mathscr{O}_X(D+P)$ that is not a global section of $\mathscr{O}_X(D)$, i. e. that φ is a rational section of $\mathscr{O}_X(D)$ that has a simple pole at P and is regular at all other points. Similarly, there is a global section α of $\mathscr{O}_X(K_X-D)$ that is not a global section of $\mathscr{O}_X(K_X-D-P)$. In other words, α is a global section of $\omega_X\otimes\mathscr{L}^\vee$ that does not vanish at P. By multiplication we see that $\varphi\cdot\alpha$ is a rational section of $\mathscr{L}\otimes(\omega_X\otimes\mathscr{L}^\vee)=\omega_X$ that has a simple pole at P and is regular at all other points. In other words, $\varphi\cdot\alpha$ is a global rational differential form with just a single pole which is of order 1. But this is a contradiction to the residue theorem of complex analysis: the sum of the residues of any rational (or meromorphic) differential form on a compact Riemann surface is zero, but in our case we have $\sum_{Q\in X}\operatorname{res}_Q(\varphi\cdot\alpha)=\operatorname{res}_P(\varphi\cdot\alpha)\neq 0$.

Step 3. We claim that

$$\chi(D)\geq\deg D+1-g$$

for all divisors D. Note that we can choose points $P_1,\dots,P_r$ such that $D+P_1+\cdots+P_r$ is precisely the intersection divisor of X with a certain number n of hyperplanes: for every point in D we just choose a hyperplane through that point and add all other intersection points with X to the P_i. This then means that $\mathscr{O}(D+P_1+\cdots+P_r)=\mathscr{O}(n)$. By possibly adding more intersection points of X with hyperplanes we can make n arbitrarily large. So by example 7.7.2 we find that

$$h^0(D+P_1+\cdots+P_r)=\deg D+r+1-g.$$

Moreover, if n (and thus r) is large enough we see by example 7.7.1 that $h^0(K_X-D-P_1-\cdots-P_r)=0$ and therefore

$$\chi(D+P_1+\cdots+P_r)=\deg D+r+1-g.$$

But by step 2 we know that subtracting a point from the divisor will decrease $\chi(\cdot)$ by 0 or 1. If we apply this r times to the points $P_1,\dots,P_r$ we conclude that $\chi(D)\geq(\deg D+r+1-g)-r$, as we have claimed.

Step 4. Replacing D by K_X-D in the inequality of step 3 yields

$$\begin{aligned}-\chi(D)=h^0(K_X-D)-h^0(D)&\geq\deg K_X-\deg D+1-g\\&=-\deg D-1+g\end{aligned}$$

as $\deg K_X=2g-2$ by corollary 7.6.6. Combining the two inequalities of steps 3 and 4 proves the theorem. □

Remark 7.7.4. If D is the divisor associated to the line bundle $\mathscr{O}(n)$ (for any n), note that $\chi(D)$ is just the value $\chi_X(n)$ of the Hilbert polynomial. So for these line bundles we can reinterpret our main proposition 6.1.5 about Hilbert polynomials as follows: the difference between $h_X(n)$ and $\chi_X(n)$ is simply $h^0(\omega_X\otimes\mathscr{O}_X(-n))$. As this vanishes for large n by degree reasons, it follows that $h_X(n)=\chi_X(n)$ for large n.

Example 7.7.5. Setting $D=0$ in the Riemann-Roch theorem yields $h^0(K_X)=g$. This gives an alternate definition of the genus of a smooth projective curve: one could define the genus of such a curve as the dimension of the space of global differential forms. This definition has the advantage that it is immediately clear that it is well-defined and independent of the projective embedding (compare this to example 6.1.10).

Remark 7.7.6. In general one should think of the Riemann-Roch theorem as a formula to compute $h^0(D)$ for any D, modulo an "unwanted" correction term $h^0(K_X-D)$. In many applications one can make this correction term vanish, e. g. by making the degree of D large enough so that $\deg(K_X-D)$ becomes negative.

Remark 7.7.7. There are numerous generalizations of the Riemann-Roch theorem. In fact, there are whole books on Riemann-Roch type theorems. Let us mention some of the generalizations without proof:

(i) The requirement that the ground field be $\mathbb{C}$ is not essential. The very same statement holds over any algebraically closed ground field (the proof has to be changed though at step 2 where we invoked complex analysis).

(ii) The requirement that the curve be projective is not essential either, it only needs to be complete (i. e. "compact").

(iii) Instead of a line bundle one can take a vector bundle: if $\mathcal{F}$ is any vector bundle on X of rank r then

$$h^0(\mathcal{F}) - h^0(\omega_X \otimes \mathcal{F}^\vee) = \deg \Lambda^r \mathcal{F} + r(1-g)$$

(see example 10.4.7).

(iv) There are versions of the Riemann-Roch theorem for *singular* curves as well. (Note that in the singular case we do not have a canonical bundle, so one needs a new idea here.)

(v) There are also versions of the Riemann-Roch theorem for varieties of dimension bigger than 1 (see theorem 10.4.5).

(vi) Finally, the same theorem can be proven (with the same proof actually) in complex analysis, where $h^0(D)$ then denotes the dimension of the space of *meromorphic* functions with the specified zeros and poles. As the resulting dimension does change we conclude that on a projective smooth complex curve *every meromorphic function is in fact rational*. This is an example of a very general result that says that complex analysis essentially reduces to algebraic geometry in the projective case (in other words, we "do not gain much" by allowing holomorphic functions instead of rational ones in the first place).

As an application of the Riemann-Roch theorem let us consider again morphisms to projective spaces. Let X be a smooth projective curve, and let D be a divisor on X. Let $s_0, \dots, s_r$ be a basis of the space $\Gamma(O(D))$ of global sections of $O(D)$. Then we have seen in lemma 7.5.14 that we get a morphism

$$X \to \mathbb{P}^r, \quad P \mapsto (s_0(P) : \cdots : s_r(P))$$

provided that the sections s_i do not vanish simultaneously at any point. Using the Riemann-Roch theorem we can now give an easy criterion when this is the case. Note first however that picking a different basis of section would result in a morphism that differs from the old one simply by a linear automorphism of $\mathbb{P}^r$. Thus we usually say that the divisor D (or its associated line bundle) determines a morphism to $\mathbb{P}^r$ up to automorphisms of $\mathbb{P}^r$.

Proposition 7.7.8. *Let X be a smooth projective curve of genus g, and let D be a divisor on X.*

(i) *If $\deg D \geq 2g$ then the divisor D determines a morphism $X \to \mathbb{P}^r$ as above.*

(ii) *If $\deg D \geq 2g+1$ then moreover this morphism is an embedding (i. e. an isomorphism onto its image).*

Proof. (i): By what we have said above we simply have to show that for every point $P \in X$ there is a global section $s \in \Gamma(O(D))$ that does not vanish at P.

By the degree condition we know that $\deg(K_X - D) \leq 2g-2-2g < 0$ and $\deg(K_X - D+P) \leq 2g-2-2g+1 < 0$. So by example 7.7.1 we get from the Riemann-Roch theorem that

$$h^0(D) = \deg D + 1 - g \quad \text{and} \quad h^0(D-P) = (\deg D - 1) + 1 - g.$$

In particular we have $h^0(D) - h^0(D-P) = 1$, i. e. there is a section $s \in \Gamma(O(D))$ that is not a section of $O(D-P)$, i. e. that does not vanish at P.

(ii): The idea of the proof is the same as in (i). However, as we have not developed enough powerful techniques yet to prove that a morphism has an inverse, we will restrict ourselves to proving that the morphism determined by D is bijective. So let P and Q be distinct points of X. To prove that they are mapped to different points it suffices to show that there is a section $s \in \Gamma(\mathcal{O}(D))$ with $s(P) = 0, s(Q) \neq 0$: the morphism $R \mapsto (s(R) : s'(R) : \cdots)$ then maps P to a point with the first coordinate 0, while the first coordinate is non-zero for the image point of Q.

To find this section s, simply apply the argument of (i) to $D-P$ and the point Q: we get $h^0(D-P) - h^0(D-P-Q) = 1$, i. e. there is a section $s \in \Gamma(\mathcal{O}(D-P))$ that is not a section of $\mathcal{O}(D-P-Q)$, i. e. it is a section of $\mathcal{O}(D)$ that vanishes at P but not at Q. □

Example 7.7.9. If X is a smooth projective curve of genus $g \geq 2$ we get a *canonical* embedding $X \to \mathbb{P}^r$ into a projective space (up to automorphisms by $\mathbb{P}^r$) by taking the morphism associated to the divisor $3K_X$. This follows by part (ii) of proposition 7.7.8 as $3(2g-2) \geq 2g+1$ if $g \geq 2$. By remark 7.7.7 (ii) the same is true for any complete (i. e. "compact") curve that is not necessarily given initially as a subvariety of projective space.

7.8. **Exercises.**

Exercise 7.8.1. Let $\mathcal{F}'$ be a presheaf on a topological space X, and let $\mathcal{F}$ be its sheafification as in definition 7.1.10. Show that

(i) There is a natural morphism $\theta : \mathcal{F}' \to \mathcal{F}$.
(ii) Any morphism from $\mathcal{F}'$ to a sheaf factors uniquely through θ.

Exercise 7.8.2. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups on a topological space X. Show that f is injective / surjective / an isomorphism if and only if all induced maps $f_P : \mathcal{F}_P \to \mathcal{G}_P$ on the stalks are injective / surjective / isomorphisms.

Exercise 7.8.3. Let $f : \mathcal{F}_1 \to \mathcal{F}_2$ be a morphism of locally free sheaves on a scheme X over a field k. Let $P \in X$ be a point, and denote by $(\mathcal{F}_i)_P$ the fiber of the vector bundle $\mathcal{F}_i$ over P, which is a k-vector space. Are the following statements true or false:

(i) If $\mathcal{F}_1 \to \mathcal{F}_2$ is injective then the induced map $(\mathcal{F}_1)_P \to (\mathcal{F}_2)_P$ is injective for all $P \in X$.
(ii) If $\mathcal{F}_1 \to \mathcal{F}_2$ is surjective then the induced map $(\mathcal{F}_1)_P \to (\mathcal{F}_2)_P$ is surjective for all $P \in X$.

Exercise 7.8.4. Prove the following generalization of example 7.1.16: If X is a smooth curve over some field k, $\mathcal{L}$ a line bundle on X, and $P \in X$ a point, then there is an exact sequence

$$0 \to \mathcal{L}(-P) \to \mathcal{L} \to k_P \to 0,$$

where k_P denotes the "skyscraper sheaf"

$$k_P(U) = \begin{cases} k & \text{if } P \in U, \\ 0 & \text{if } P \notin U. \end{cases}$$

Exercise 7.8.5. If X is an *affine* variety over a field k and $\mathcal{F}$ a locally free sheaf of rank r on X, is then necessarily $\mathcal{F} \cong \mathcal{O}_X^{\oplus r}$?

Exercise 7.8.6. Let X be a scheme, and let $\mathcal{F}$ be a locally free sheaf on X. Show that $(\mathcal{F}^\vee)^\vee \cong \mathcal{F}$. Show by example that this statement is in general false if $\mathcal{F}$ is only quasi-coherent but not locally free.

Exercise 7.8.7. Figure out what exactly goes wrong with the correspondence between line bundles and divisor classes on a curve X if X is singular. Can we still associate a divisor to any section of a line bundle? Can we still construct a line bundle from any divisor?

Exercise 7.8.8. What is the line bundle on $\mathbb{P}^n \times \mathbb{P}^m$ leading to the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ by the correspondence of lemma 7.5.14? What is the line bundle leading to the degree-d Veronese embedding $\mathbb{P}^n \to \mathbb{P}^N$?

Exercise 7.8.9. Show that any smooth projective curve of genus 2...

(i) can be realized as a curve of degree 5 in $\mathbb{P}^3$,
(ii) admits a two-to-one morphism to $\mathbb{P}^1$. How many ramification points does such a morphism have?

Exercise 7.8.10. Let X be a smooth projective curve, and let $P \in X$ be a point. Show that there is a rational function on X that is regular everywhere except at P.

8. COHOMOLOGY OF SHEAVES

For any quasi-coherent sheaf $\mathcal{F}$ on a scheme X we construct the cohomology groups $H^i(X,\mathcal{F})$ for $i \geq 0$ using the Čech complex associated to an affine open cover of X. We show that the cohomology groups do not depend on the choice of affine open cover. The cohomology groups $H^i(X,\mathcal{F})$ vanish for $i > 0$ if X is affine, and in any case for $i > \dim X$.

For any short exact sequence of sheaves on X there is an associated long exact sequence of the corresponding cohomology groups.

If $\mathcal{L}$ is a line bundle of degree at least $2g-1$ on a smooth projective curve of genus g then the cohomology group $H^1(X,\mathcal{L})$ is zero. Using this "vanishing theorem" we reprove the Riemann-Roch theorem in a cohomological version. Comparing this to the old version yields the equality $\dim H^0(K_X - D) = \dim H^1(D)$ for any divisor D, which is a special case of the Serre duality theorem. As an application we can now define the genus of a possibly singular curve to be $\dim H^1(X,\mathcal{O}_X)$.

We compute the cohomology groups of all line bundles on projective spaces. As a consequence, we obtain the result that the cohomology groups of coherent sheaves on projective schemes are always finite-dimensional vector spaces, and that $H^i(X,\mathcal{F} \otimes \mathcal{O}_X(d)) = 0$ for all $i > 0$ and $d \gg 0$.

8.1. **Motivation and definitions.** There are numerous ways to motivate the theory of cohomology of sheaves. Almost all of them are based on the observation that "the functor of taking global sections of a sheaf is not exact", i. e. given an exact sequence of sheaves of Abelian groups

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

on a scheme (or topological space) X, by taking global sections we get an exact sequence

$$0 \to \Gamma(\mathcal{F}_1) \to \Gamma(\mathcal{F}_2) \to \Gamma(\mathcal{F}_3)$$

of Abelian groups in which the last map $\Gamma(\mathcal{F}_2) \to \Gamma(\mathcal{F}_3)$ is in general not surjective. We have seen one example of this in example 7.1.18. Here is one more example:

Example 8.1.1. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d with inclusion morphism $i : X \to \mathbb{P}^n$. We know from lemma 7.4.15 that the cotangent sheaf of $\mathbb{P}^n$ fits into an exact sequence of vector bundles

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O} \to 0.$$

Pulling this sequence back by i and taking global sections, we see that we have an exact sequence

$$0 \to \Gamma(i^*\Omega_{\mathbb{P}^n}) \to \Gamma(\mathcal{O}_X(-1)^{\oplus(n+1)}) \to \cdots.$$

But $\mathcal{O}_X(-1)$ has no global sections, so we conclude that $i^*\Omega_{\mathbb{P}^n}$ has no global sections either. Now consider the exact sequence of lemma 7.4.16

$$0 \to \mathcal{O}_X(-d) \to i^*\Omega_{\mathbb{P}^n} \to \Omega_X \to 0,$$

from which we deduce the exact sequence

$$0 \to \Gamma(\mathcal{O}_X(-d)) \to \Gamma(i^*\Omega_{\mathbb{P}^n}) \to \Gamma(\Omega_X).$$

We have just seen that the first two groups in this sequence are trivial. But $\Gamma(\Omega_X)$ is not trivial in general (e. g. for a cubic curve in $\mathbb{P}^2$ we have $\Omega_X = \mathcal{O}_X$ and thus $\Gamma(\Omega_X) = k$). Hence the last map in the above sequence of global sections cannot be surjective in general.

We have however already met a case in which the induced map on global sections *is* exact: if $X = \operatorname{Spec} R$ is an *affine* scheme and $\mathcal{F}_i = \tilde{M}_i$ for some R-modules M_i are *quasi-coherent* sheaves on X then by lemma 7.2.7 (ii) the sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

is exact if and only if the sequence

$$0 \to \Gamma(\mathcal{F}_1) \to \Gamma(\mathcal{F}_2) \to \Gamma(\mathcal{F}_3) \to 0$$

is exact (note that $\Gamma(\mathcal{F}_i) = M_i$ by proposition 7.2.2 (ii)). We have mentioned already that essentially all sheaves occurring in practice are quasi-coherent, so we will assume this from now on for the rest of this chapter.

The conclusion is that we know that taking global sections is an exact functor if the underlying scheme is affine. The goal of the theory of cohomology is to extend the global section sequence to the right for all schemes X in the following sense: for any (quasi-coherent) sheaf $\mathcal{F}$ on X we will define natural cohomology groups $H^i(X,\mathcal{F})$ for all $i > 0$ satisfying (among other things) the following property: given any exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ of sheaves on X, there is an induced long exact sequence of cohomology groups

$$0 \to \Gamma(\mathcal{F}_1) \to \Gamma(\mathcal{F}_2) \to \Gamma(\mathcal{F}_3) \to H^1(X,\mathcal{F}_1) \to H^1(X,\mathcal{F}_2) \to H^1(X,\mathcal{F}_3) \to H^2(X,\mathcal{F}_1) \to \cdots.$$

If X is an affine scheme then $H^i(X,\mathcal{F}) = 0$ for all $i > 0$, so that we arrive again at our old result that the sequence of global sections is exact in this case.

Let us now give the definition of these cohomology groups. There are various ways to define these groups. In these notes we will use the approach of so-called Čech cohomology. This is the most suitable approach for actual applications (but maybe not the best one from a purely theoretical point of view). The idea of Čech cohomology is simple: we have seen above that the global section functor is exact (i. e. does what we finally want) if X is an affine scheme. So if X is any scheme we will just choose an affine open cover $\{U_i\}$ of X and consider sections of our sheaves on these affine open subsets and their intersections.

Definition 8.1.2. Let X be a scheme, and let $\mathcal{F}$ be a (quasi-coherent) sheaf on X. Fix an affine open cover $\{U_i\}_{i\in I}$ of X, and assume for simplicity that I is an ordered set. For all $p \geq 0$ we define the Abelian group

$$C^p(\mathcal{F}) = \prod_{i_0<\cdots<i_p} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}).$$

In other words, an element $\alpha \in C^P(\mathcal{F})$ is a collection $\alpha = (\alpha_{i_0,\dots,i_p})$ of sections of $\mathcal{F}$ over all intersections of $p+1$ sets taken from the cover. These sections can be totally unrelated.

For every $p \geq 0$ we define a "boundary operator" $d^p : C^p(\mathcal{F}) \to C^{p+1}(\mathcal{F})$ by

$$(d^p\alpha)_{i_0,\dots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\dots,i_{k-1},i_{k+1},\dots,i_{p+1}}|_{U_{i_0}\cap\cdots\cap U_{i_{p+1}}}.$$

Note that this makes sense as the $\alpha_{i_0,\dots,i_{k-1},i_{k+1},i_{p+1}}$ are sections of $\mathcal{F}$ on $U_{i_0} \cap \cdots \cap U_{i_{k-1}} \cap U_{i_{k+1}} \cap \cdots \cap U_{i_{p+1}}$, which contains $U_{i_0} \cap \cdots \cap U_{i_{p+1}}$ as an open subset.

By abuse of notation we will denote all these operators simply by d if it is clear from the context on which $C^p(\mathcal{F})$ they act.

Lemma 8.1.3. *Let $\mathcal{F}$ be a sheaf on a scheme X. Then $d^{p+1} \circ d^p : C^p(\mathcal{F}) \to C^{p+2}(\mathcal{F})$ is the zero map for all $p \geq 0$.*

Proof. This statement is essentially due to the sign in the definition of $d\alpha$: for every $\alpha \in C^p(\mathcal{F})$ we have

$$\begin{aligned}
(d^{p+1}d^p\alpha)_{i_0,\dots,i_{p+2}} &= \sum_{k=0}^{p+2}(-1)^k(d\alpha)_{i_0,\dots,i_{k-1},i_{k+1},\dots,i_{p+2}} \\
&= \sum_{k=0}^{p+2}\sum_{m=0}^{k-1}(-1)^{k+m}\alpha_{i_0,\dots,i_{m-1},i_{m+1},\dots,i_{k-1},i_{k+1},\dots,i_{p+2}} \\
&\quad + \sum_{k=0}^{p+2}\sum_{m=k+1}^{p+2}(-1)^{k+m-1}\alpha_{i_0,\dots,i_{k-1},i_{k+1},\dots,i_{m-1},i_{m+1},\dots,i_{p+2}} \\
&= 0
\end{aligned}$$

(omitting the restriction maps). □

We have thus defined a sequence of Abelian groups and homomorphisms

$$C^0(\mathcal{F}) \xrightarrow{d^0} C^1(\mathcal{F}) \xrightarrow{d^1} C^2(\mathcal{F}) \xrightarrow{d^2} \cdots$$

such that $d^{p+1} \circ d^p = 0$ at every step. Such a sequence is usually called a **complex** of Abelian groups. The maps d^p are then called the **boundary operators**.

Definition 8.1.4. Let $\mathcal{F}$ be a sheaf on a scheme X. Pick an affine open cover $\{U_i\}$ of X and consider the associated groups $C^p(\mathcal{F})$ and homomorphisms $d^p : C^p(\mathcal{F}) \to C^{p+1}(\mathcal{F})$ for $p \geq 0$. We define the p-th **cohomology group** of $\mathcal{F}$ to be

$$H^p(X,\mathcal{F}) = \ker d^p / \operatorname{im} d^{p-1}$$

with the convention that $C^p(\mathcal{F})$ and d^p are zero for $p < 0$. Note that this is well-defined as $\operatorname{im} d^{p-1} \subset \ker d^p$ by lemma 8.1.3. If X is a scheme over a field k then the cohomology groups will be vector spaces over k. The dimension of the cohomology groups $H^i(X,\mathcal{F})$ as a k-vector space is then denoted $h^i(X,\mathcal{F})$.

Remark 8.1.5. The definition of the cohomology groups as it stands depends on the choice of the affine open cover of X. It is a very crucial (and non-trivial) fact that the $H^i(X,\mathcal{F})$ actually do *not* depend on this choice (as we have already indicated by the notation). It is the main disadvantage of our Čech approach to cohomology that this independence is not obvious from the definition. There are other constructions of the cohomology groups (for example the "derived functor approach" of [H] chapter III) that never use such affine open covers and therefore do not face this problem. On the other hand, these other approaches are essentially useless for actual computations. This is why we have given the Čech approach here. We will prove the independence of our cohomology groups of the open cover in section 8.5. For now we will just assume this independence and rather discuss the properties and applications of the cohomology groups.

Example 8.1.6. The following examples follow immediately from the definition and the assumption of remark 8.1.5:

(i) For any X and $\mathcal{F}$ we have $H^0(X,\mathcal{F}) = \Gamma(\mathcal{F})$. In fact, we have $H^0(X,\mathcal{F}) = \ker(d^0 : C^0(\mathcal{F}) \to C^1(\mathcal{F}))$ by definition. But an element $\alpha \in C^0(\mathcal{F})$ is just given by a section $\alpha_i \in \mathcal{F}(U_i)$ for every element of the open cover, and the map d^0 is given by $(\alpha_i - \alpha_j)|_{U_i \cap U_j}$. By the sheaf axiom this is zero for all i and j if and only if the α_i come from a global section of $\mathcal{F}$. Hence $H^0(X,\mathcal{F}) = \Gamma(\mathcal{F})$. (In particular, our definition of $h^0(\mathcal{L})$ in section 7.7 is consistent with our current definition of $h^0(X,\mathcal{L})$.)

(ii) If X is an affine scheme then $H^i(X,\mathcal{F}) = 0$ for $i > 0$. In fact, if X is affine we can pick the open cover consisting of the single element X, in which case the groups $C^i(\mathcal{F})$ and hence the $H^i(X,\mathcal{F})$ are trivially zero for $i > 0$.

(iii) If X is a projective scheme of dimension n then $H^i(X,\mathcal{F}) = 0$ whenever $i > n$. In fact, by proposition 4.1.9 we can pick homogeneous polynomials $f_0,\dots,f_n$ such that $X \cap Z(f_0,\dots,f_n) = \emptyset$. We thus get an open cover of X by the $n+1$ subsets $X \backslash Z(f_i)$ which are all affine by proposition 5.5.4. Using this open cover for the definition of the cohomology groups, we see that the $C^i(\mathcal{F})$ and hence the $H^i(X,\mathcal{F})$ are zero for $i > n$. Note that the same is true for any scheme that can be covered by $n+1$ affine open subsets.

Note that for (i) we did not need the independence of the cohomology groups of the open cover, but for (ii) and (iii) we did. In fact, the last two statements are both highly non-trivial theorems about cohomology groups. They only follow so easily in our setup because we assumed the independence of the cover.

Example 8.1.7. Let $X = \mathbb{P}^1$ and $\mathcal{F} = \mathcal{O}$. By example 8.1.6 (i) we know that $H^0(\mathbb{P}^1,\mathcal{O}) \cong k$ is simply the space of (constant) global regular functions, and by part (iii) we know that $H^i(\mathbb{P}^1,\mathcal{O}) = 0$ for $i > 1$. So let us determine $H^1(\mathbb{P}^1,\mathcal{O})$. To compute this cohomology group let us pick the obvious affine open cover $U_i = \{x_i \neq 0\}$ for $i = 0,1$. Then

$$\begin{aligned} C^1(\mathcal{O}) &= \mathcal{O}(U_0 \cap U_1) \\ &= \left\{ \frac{f}{x_0^a x_1^b} \;;\; f \text{ homogeneous of degree } a+b \right\} \\ &= \left\langle \frac{x_0^m x_1^n}{x_0^a x_1^b} \;;\; m+n = a+b \text{ and } m,n,a,b \geq 0 \right\rangle. \end{aligned}$$

Of course the condition $m+n = a+b$ implies that we always have $m \geq a$ or $n \geq b$. So every such generator is regular on at least one of the open subsets U_0 and U_1. It follows that every such generator is in the image of the boundary map

$$d^0 : C^0(\mathcal{O}) = \mathcal{O}(U_0) \times \mathcal{O}(U_1) \to \mathcal{O}(U_0 \cap U_1), \quad (\alpha_0,\alpha_1) \mapsto \alpha_1 - \alpha_0|_{U_0 \cap U_1}.$$

Consequently $H^1(\mathbb{P}^1,\mathcal{O}) = 0$ by definition of the cohomology groups.

Example 8.1.8. In the same way as in example 8.1.7 let us now compute the cohomology group $H^1(\mathbb{P}^1,\mathcal{O}(-2))$. With the same notations as above we have now

$$\begin{aligned} C^1(\mathcal{O}(-2)) &= \mathcal{O}(-2)(U_0 \cap U_1) \\ &= \left\{ \frac{f}{x_0^a x_1^b} \;;\; f \text{ homogeneous of degree } a+b-2 \right\} \\ &= \left\langle \frac{x_0^m x_1^n}{x_0^a x_1^b} \;;\; m+n = a+b-2 \right\rangle. \end{aligned}$$

The condition $m+n = a+b-2$ implies that $m \geq a-1$ or $n \geq b-1$. If one of these inequalities is strict, then the corresponding generator $\frac{x_0^m x_1^n}{x_0^a x_1^b}$ is regular on U_0 or U_1 and is therefore zero in the cohomology group $H^1(\mathbb{P}^1,\mathcal{O}(-2))$ as above. So we are only left with the function $\frac{1}{x_0 x_1}$ where neither inequality is strict. As $C^2(\mathcal{O}(-2)) = 0$ and so the boundary operator d^1 is trivial, we conclude that $H^1(\mathbb{P}^1,\mathcal{O}(-2))$ is one-dimensional, with the function $\frac{1}{x_0 x_1}$ as a generator.

8.2. **The long exact cohomology sequence.** The main property of the cohomology groups is that they solve the problem of finding an exact sequence of sections associated to a short exact sequence of sheaves:

Proposition 8.2.1. *Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be an exact sequence of sheaves on a (separated) scheme X. Then there is a canonical long exact sequence of cohomology groups*

$$\begin{aligned} 0 &\to H^0(X,\mathcal{F}_1) \to H^0(X,\mathcal{F}_2) \to H^0(X,\mathcal{F}_3) \\ &\to H^1(X,\mathcal{F}_1) \to H^1(X,\mathcal{F}_2) \to H^1(X,\mathcal{F}_3) \\ &\to H^2(X,\mathcal{F}_1) \to \cdots . \end{aligned}$$

Proof. Consider the diagram of Abelian groups and homomorphisms

$$\begin{tikzcd} & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\ 0 \arrow[r] & C^{p-1}(\mathcal{F}_1) \arrow[r, "f"] \arrow[d, "d"] & C^{p-1}(\mathcal{F}_2) \arrow[r, "g"] \arrow[d, "d"] & C^{p-1}(\mathcal{F}_3) \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & C^{p}(\mathcal{F}_1) \arrow[r, "f"] \arrow[d, "d"] & C^{p}(\mathcal{F}_2) \arrow[r, "g"] \arrow[d, "d"] & C^{p}(\mathcal{F}_3) \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & C^{p+1}(\mathcal{F}_1) \arrow[r, "f"] \arrow[d, "d"] & C^{p+1}(\mathcal{F}_2) \arrow[r, "g"] \arrow[d, "d"] & C^{p+1}(\mathcal{F}_3) \arrow[r] \arrow[d, "d"] & 0 \\ & \cdots & \cdots & \cdots & \end{tikzcd}$$

The columns of this diagram are complexes (i. e. $d \circ d = 0$ at all places) by lemma 8.1.3. We claim that the rows of this diagram are all exact: by lemma 7.2.7 (ii) and what we have said in section 8.1 we know that the sequences $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U) \to 0$ are exact on every *affine* open subset U of X. But the intersection of two (and hence finitely many) affine open subsets of X is again affine as $U \cap V = \Delta_X \cap (U \times V)$ is a closed subset of an affine scheme $U \times V$ (where $\Delta_X \subset X \times X$ denotes the diagonal of X). As the $C^p(\mathcal{F}_i)$ are made up from sections on such open subsets, the claim follows. Moreover, note that all squares in this diagram are commutative by construction.

The statement of the proposition now follows from a basic lemma of homological algebra: □

Lemma 8.2.2. *Any short exact sequence of complexes*

$$\begin{tikzcd} & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\ 0 \arrow[r] & C^{p-1} \arrow[r, "f"] \arrow[d, "d"] & D^{p-1} \arrow[r, "g"] \arrow[d, "d"] & E^{p-1} \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & C^{p} \arrow[r, "f"] \arrow[d, "d"] & D^{p} \arrow[r, "g"] \arrow[d, "d"] & E^{p} \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & C^{p+1} \arrow[r, "f"] \arrow[d, "d"] & D^{p+1} \arrow[r, "g"] \arrow[d, "d"] & E^{p+1} \arrow[r] \arrow[d, "d"] & 0 \\ & \cdots & \cdots & \cdots & \end{tikzcd}$$

(i. e. the C^p, D^p, E^p are Abelian groups, the diagram commutes, the rows are exact and the columns are complexes) gives rise to a long exact sequence in cohomology

$$\cdots \to H^{p-1}(E) \to H^p(C) \to H^p(D) \to H^p(E) \to H^{p+1}(C) \to \cdots$$

where $H^p(C) = \ker(C^p \to C^{p+1}) / \operatorname{im}(C^{p-1} \to C^p)$, and similarly for D and E.

Proof. The proof is done by pure "diagram chasing". We will give some examples.

(i) Existence of the morphisms $\psi : H^p(C) \to H^p(D)$: let $\alpha \in H^p(C)$ be represented by an element in C^p (which we denote by the same letter by abuse of notation). Then $d\alpha = 0 \in C^{p+1}$. Set $\psi(\alpha) = f(\alpha)$. Note that $d\psi(\alpha) = f(d\alpha) = 0$, so $\psi(\alpha)$ is a well-defined cohomology element. We still have to check that this definition does not depend on the representative chosen in C^p. So if $\alpha = d\alpha'$ for some $\alpha' \in C^{p-1}$ (so that $\alpha = 0$ in $H^p(C)$) then $\psi(\alpha) = f(d\alpha') = df(\alpha')$ (so that $\psi(\alpha) = 0$ in $H^p(D)$).

(ii) The existence of the morphisms $H^p(D) \to H^p(E)$ follows in the same way: they are simply induced by the morphisms g.

(iii) Existence of the morphisms $\phi : H^p(E) \to H^{p+1}(C)$: The existence of these "connecting morphisms" is probably the most unexpected part of this lemma. Let α be a (representative of a) cohomology element in E^p, so that $d\alpha = 0$. As $g : D^p \to E^p$ is surjective, we can pick a $\beta \in D^p$ such that $g(\beta) = \alpha$. Consider the element $d\beta \in D^{p+1}$. We have $g(d\beta) = dg(\beta) = d\alpha = 0$, so $d\beta$ is in fact of the form $f(\gamma)$ for a (unique) $\gamma \in C^{p+1}$. We set $\phi(\alpha) = \gamma$.

We have to check that this is well-defined:

(a) $d\gamma = 0$ (so that γ actually defines an element in cohomology): we have $f(d\gamma) = df(\gamma) = d(d\beta) = 0$ as the middle column is a complex, so $d\gamma = 0$ as the f are injective.

(b) The construction does not depend on the choice of β: if we pick another β' with $g(\beta') = \alpha$ then $g(\beta - \beta') = 0$, so $\beta - \beta' = f(\delta)$ for some $\delta \in C^p$ as the p-th row is exact. Now if γ and γ' are the elements such that $f(\gamma) = d\beta$ and $f(\gamma') = \beta'$ then $f(\gamma - \gamma') = d(\beta - \beta') = df(\delta) = f(d\delta)$. As f is injective we conclude that $\gamma - \gamma' = d\delta$, so γ and γ' define the same element in $H^{p+1}(C)$.

(c) If $\alpha = d\alpha'$ for some $\alpha' \in E^{p-1}$ (so that α defines the zero element in cohomology) then we can pick an inverse image β' with $g(\beta') = \alpha'$ as g is surjective. For β we can then take $d\beta'$. But then $d\beta = d(d\beta') = 0$ as the middle column is a complex, so the resulting element in $H^{p+1}(C)$ is zero.

Summarizing, we can say that the morphism $H^p(E) \to H^{p+1}(C)$ is obtained by going "left, down, left" in our diagram. We have just checked that this really gives rise to a well-defined map.

We have now seen that there is a canonical sequence of morphisms between the cohomology groups as stated in the lemma. It remains to be shown that the sequence is actually exact. We will check exactness at the $H^p(D)$ stage only (i. e. show that $\ker(H^p(D) \to H^p(E)) = \operatorname{im}(H^p(C) \to H^p(D))$ and leave the other two checks (at $H^p(C)$ and $H^p(E)$) that are completely analogous as an exercise.

$\operatorname{im}(H^p(C) \to H^p(D)) \subset \ker(H^p(D) \to H^p(E))$: Let $\alpha \in H^p(D)$ be of the form $\alpha = f(\beta)$ for some $\beta \in H^p(C)$). Then $g(\alpha) = g(f(\beta)) = 0$ as the p-th row is exact.

$\ker(H^p(D) \to H^p(E)) \subset \operatorname{im}(H^p(C) \to H^p(D))$: Let $\alpha \in H^p(D)$ be a cohomology element (i. e. $d\alpha = 0$) such that $g(\alpha) = 0$ in cohomology, i. e. $g(\alpha) = d\beta$ for some E^{p-1}. As g is surjective we can pick an inverse image $\gamma \in D^{p-1}$ of β. Then

$$g(\alpha - d\gamma) = g(\alpha) - g(d\gamma) = g(\alpha) - d\beta = 0,$$

so there is a $\delta \in C^p$ such that $f(\delta) = \alpha - d\gamma$ as the p-th row is exact. Note that δ defines an element in $H^p(C)$ as $f(d\delta) = d(\alpha - d\gamma) = 0 - 0 = 0$ and thus $d\delta = 0$ as f is injective. Moreover, $f(\delta) = \alpha$ in $H^p(D)$ by construction, so $\alpha \in \operatorname{im}(H^p(C) \to H^p(D))$. $\square$

Example 8.2.3. Consider the exact sequence of sheaves on $X = \mathbb{P}^1$

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{\cdot x_0 x_1} \mathcal{O} \longrightarrow k_P \oplus k_Q \longrightarrow 0$$

from example 7.1.18, where $P = (0:1)$ and $Q = (1:0)$, and the last map is given by evaluation at P and Q. From proposition 8.2.1 we deduce an associated long exact cohomology sequence

$$0 \to H^0(X, \mathcal{O}(-2)) \to H^0(X, \mathcal{O}) \to H^0(X, k_P \oplus k_Q) \to H^1(X, \mathcal{O}(-2)) \to H^1(X, \mathcal{O}) \to \cdots .$$

Now $H^0(X, \mathcal{O}(-2)) = 0$ by example 7.7.1 and $H^1(X, \mathcal{O}) = 0$ by example 8.1.7. Moreover, $H^0(X, \mathcal{O})$ is just the space of global (constant) functions, $H^0(X, k_P \oplus k_Q)$ is isomorphic to $k \times k$ (given by specifying values at the points P and Q), and $H^1(X, \mathcal{O}(-2)) = \langle \frac{1}{x_0 x_1} \rangle$ is 1-dimensional by example 8.1.8. So our exact sequence is just

$$0 \to k \to k \times k \to k \to 0.$$

We can actually also identify the morphisms. The first morphism in this sequence is $a \mapsto (a, a)$ as it is the evaluation of the constant function a at the points P and Q. The second morphism is given by the "left, down, left" procedure of part (iii) of the proof of lemma 8.2.2 in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & C^0(\mathcal{O}(-2)) \arrow[r] \arrow[d] & C^0(\mathcal{O}) \arrow[r] \arrow[d] & C^0(k_P \oplus k_Q) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^1(\mathcal{O}(-2)) \arrow[r] & C^1(\mathcal{O}) \arrow[r] & C^1(k_P \oplus k_Q) \arrow[r] & 0
\end{tikzcd}$$

Starting with any element $(a, b) \in C^0(k_P \oplus k_Q)$ we can find an inverse image in $C^0(\mathcal{O}) = \mathcal{O}(U_0) \times \mathcal{O}(U_1)$ (with $U_i = \{x_i \neq 0\}$, namely the pair of constant functions (b, a) (as $P \in U_1$ and $Q \in U_0$). Going down in the diagram yields the function $a - b \in \mathcal{O}(U_0 \cap U_1)$ by the definition of the boundary operator. Recalling that the morphism from $\mathcal{O}(-2)$ to $\mathcal{O}$ is given by multiplication with $x_0 x_1$, we find that $\frac{a-b}{x_0 x_1}$ is the element in $C^1(\mathcal{O}(-2))$ that we were looking for. In terms of the basis vector $\frac{1}{x_0 x_1}$ of $H^1(X, \mathcal{O}(-2))$ this function has the single coordinate $a - b$. So in this basis our exact cohomology sequence becomes

$$\begin{array}{ccccccccc}
0 & \to & k & \to & k \times k & \to & k & \to & 0 \\
 & & a & \mapsto & (a,a) & & & & \\
 & & & & (a,b) & \mapsto & a-b, & &
\end{array}$$

which is indeed exact.

8.3. **The Riemann-Roch theorem revisited.** Let us now study the cohomology groups of line bundles on smooth projective curves in some more detail. So let X be such a curve, and let $\mathcal{L}$ be a line bundle on X. Of course by example 8.1.6 (i) and (iii) the only interesting cohomology group is $H^1(X, \mathcal{L})$. We will show that this group is trivial if $\mathcal{L}$ is "positive enough":

Proposition 8.3.1. ***(Kodaira vanishing theorem for line bundles on curves)*** *Let X be a smooth projective curve of genus g, and let $\mathcal{L}$ be a line bundle on X such that* $\deg \mathcal{L} \geq 2g - 1$. *Then* $H^1(X, \mathcal{L}) = 0$.

Proof. We compute $H^1(X, \mathcal{L})$ using our definition of cohomology groups. So let $U_0 \subset X$ be an affine open subset of X. It must be of the form $X \backslash \{P_1, \dots, P_r\}$ for some points P_i on X. Now pick any other affine open subset $U_1 \subset X$ that contains the points P_i. Then U_1

is of the form $X \backslash \{Q_1, \dots, Q_s\}$ with $P_i \neq Q_j$ for all i, j. So we have an affine open cover $X = U_0 \cup U_1$.

By definition we have $H^1(X, \mathcal{L}) = \mathcal{L}(U_0 \cap U_1)/(\mathcal{L}(U_0) + \mathcal{L}(U_1))$. Note that $\mathcal{L}(U_0 \cap U_1)$ is precisely the space of rational sections of $\mathcal{L}$ that may have poles at the points P_i and Q_j, and similarly for $\mathcal{L}(U_0)$ and $\mathcal{L}(U_1)$. In other words, to prove the proposition we have to show that any rational section α of $\mathcal{L}$ with poles at the P_i and Q_j can be written as the sum of two rational sections α_0 and α_1, where α_0 has poles only at the P_i and α_1 only at the Q_j.

So let α be such a rational section. It is a global section of $\mathcal{L} \otimes \mathcal{O}_X(a_1 P_1 + \cdots + a_r P_r + b_1 Q_1 + \cdots + b_s Q_s)$ for some $a_i, b_j \geq 0$.

Let us assume that $a_1 \geq 1$. Note that then the degree of the line bundle $\omega_X \otimes \mathcal{L}^\vee \otimes \mathcal{O}_X(-a_1 P_1 - \cdots - a_r P_r)$ is at most -2 by assumption and corollary 7.6.6. Hence by the Riemann-Roch theorem 7.7.3 (and example 7.7.1) it follows that

$$h^0(\mathcal{L} \otimes \mathcal{O}_X(a_1 P_1 + \cdots + a_r P_r)) = \deg L + a_1 + \cdots + a_r + 1 - g.$$

In the same way we get

$$h^0(\mathcal{L} \otimes \mathcal{O}_X((a_1 - 1)P_1 + \cdots + a_r P_r)) = \deg L + a_1 - 1 + a_2 + \cdots + a_r + 1 - g.$$

We conclude that

$$h^0(\mathcal{L} \otimes \mathcal{O}_X(a_1 P_1 + \cdots + a_r P_r)) - h^0(\mathcal{L} \otimes \mathcal{O}_X((a_1 - 1)P_1 + \cdots + a_r P_r)) = 1.$$

So we can pick a rational section α_0' in $\Gamma(\mathcal{L} \otimes \mathcal{O}_X(a_1 P_1 + \cdots + a_r P_r))$ that is not in $\Gamma(h^0(\mathcal{L} \otimes \mathcal{O}_X((a_1 - 1)P_1 + \cdots + a_r P_r)))$, i. e. a section that has a pole of order exactly a_1 at P_1.

Now α and α_0' are both sections of the one-dimensional vector space

$$\Gamma(\mathcal{L} \otimes \mathcal{O}_X(a_1 P_1 + \cdots + a_r P_r))/\Gamma(\mathcal{L} \otimes \mathcal{O}_X((a_1 - 1)P_1 + \cdots + a_r P_r)),$$

and moreover α_0' is not zero in this quotient. So by possibly multiplying α_0' with a constant scalar we can assume that $\alpha - \alpha_0'$ is in $\Gamma(\mathcal{L} \otimes \mathcal{O}_X((a_1 - 1)P_1 + \cdots + a_r P_r))$.

Note now that α_0' has poles only at the P_i, whereas the total order of the poles of $\alpha - \alpha_0'$ at the P_i is at most $a_1 + \cdots + a_r - 1$. Repeating this process we arrive after $a_1 + \cdots + a_r$ steps at a rational section α_0 with poles only at the P_i such that $\alpha_1 := \alpha - \alpha_0$ has no poles any more at the P_i. This is precisely what we had to construct. □

Remark 8.3.2. As in the case of the Riemann-Roch theorem there are vast generalizations of the Kodaira vanishing theorem, e. g. to higher-dimensional spaces. One version is the following: if X is a smooth projective variety then $H^i(X, \omega_X \otimes \mathcal{O}_X(n)) = 0$ for all $i > 0$ and $n > 0$. Note that in the case of a smooth curve this follows from our version of proposition 8.3.1, as $\deg(\omega_X \otimes \mathcal{O}_X(n)) = 2g - 2 + 1 \geq 2g - 1$.

In general cohomology groups "tend to be zero quite often". There are many so-called vanishing theorems that assert that certain cohomology groups are zero under some conditions that can hopefully easily be checked. We will prove one more vanishing theorem in theorem 8.4.7 (ii). Of course, the advantage of vanishing cohomology groups is that they break up the long exact cohomology sequence of proposition 8.2.1 into smaller pieces.

Using our Kodaira vanishing theorem we can now reprove the Riemann-Roch theorem in a "cohomological version". In analogy to the notation of section 7.7 let us denote $H^1(X, \mathcal{O}_X(D))$ also by $H^1(D)$ for any divisor D, and similarly for $h^1(D)$.

Corollary 8.3.3. ***(Riemann-Roch theorem for line bundles on curves, second version)*** *Let X be a smooth projective curve of genus g. Then for any divisor D on X we have*

$$h^0(D) - h^1(D) = \deg D + 1 - g.$$

Proof. From the exact skyscraper sequence

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D+P) \to k_P \to 0$$

for any point $P \in X$ we get the long exact sequence in cohomology

$$0 \to H^0(D) \to H^0(D+P) \to k \to H^1(D) \to H^1(D+P) \to 0$$

by proposition 8.2.1. Taking dimensions, we conclude that $\chi(D+P) - \chi(D) = 1$, where $\chi(D) := h^0(D) - h^1(D)$. It follows by induction that we must have

$$h^0(D) - h^1(D) = \deg D + c$$

for some constant c (that does not depend on D). But by our first version of the Riemann-Roch theorem 7.7.3 we have

$$h^0(D) - h^0(K_X - D) = \deg D + 1 - g.$$

So to determine the constant c we can pick a divisor D of degree at least $2g-1$: then $h^1(D)$ vanishes by proposition 8.3.1 and $h^0(K_X - D)$ by example 7.7.1. So we conclude that $c = 1 - g$, as desired. □

Remark 8.3.4. Comparing our two versions of the Riemann-Roch theorem we see that we must have $h^0(\omega_X \otimes \mathcal{L}^\vee) = h^1(\mathcal{L})$ for all line bundles $\mathcal{L}$ on a smooth projective curve X. In fact, this is just a special case of the **Serre duality** theorem that asserts that for any smooth n-dimensional variety X and any locally free sheaf $\mathcal{F}$ there are canonical isomorphisms

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \omega_X \otimes \mathcal{F}^\vee)^\vee$$

for all $i = 0, \dots, n$. Unfortunately, these isomorphisms cannot easily be written down. There are even more general versions for singular varieties X and more general sheaves $\mathcal{F}$. We refer to [H] section III.7 for details.

Note that our new version of the Riemann-Roch theorem can be used to define the genus of singular curves:

Definition 8.3.5. Let X be a (possibly singular) curve. Then the **genus** of X is defined to be the non-negative integer $h^1(X, \mathcal{O}_X)$. (This definition is consistent with our old ones as we can see by setting $\mathcal{L} = \mathcal{O}_X$ in corollary 8.3.3.)

Let us investigate the geometric meaning of the genus of singular curves in two cases:

Example 8.3.6. Let $C_1, \dots, C_n$ be smooth irreducible curves of genera $g_1, \dots, g_n$, and denote by $\tilde{C} = C_1 \cup \cdots \cup C_n$ their disjoint union. Now pick r pairs of points $P_i, Q_i \in \tilde{C}$ that are all distinct, and denote by C the curve obtained from $\tilde{C}$ by identifying every P_i with the corresponding Q_i for $i = 1, \dots, r$. Curves obtained by this procedure are called **nodal curves**.

To compute the genus of the nodal curve C we consider the exact sequence

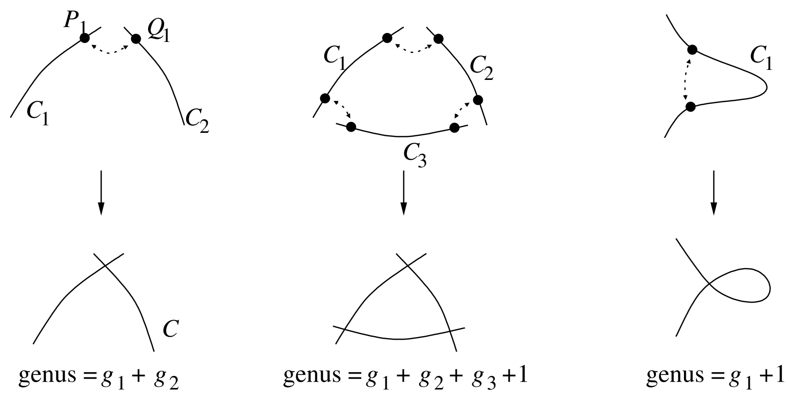
$$0 \to \mathcal{O}_C \to \oplus_{i=1}^n \mathcal{O}_{C_i} \to \oplus_{i=1}^r k_{P_i} \to 0$$

where the last maps $\oplus_{i=1}^n \mathcal{O}_{C_i} \to k_{P_i}$ are given by evaluation at P_i minus evaluation at Q_i. The sequence just describes the fact that regular functions on C are precisely functions on $\tilde{C}$ that have the same value at P_i and Q_i for all i.

By proposition 8.2.1 we obtain a long exact cohomology sequence

$$0 \to H^0(C, \mathcal{O}_C) \to \oplus_{i=1}^n H^0(C_i, \mathcal{O}_{C_i}) \to k^{\oplus r} \to H^1(C, \mathcal{O}_C) \to \oplus_{i=1}^n H^1(C_i, \mathcal{O}_{C_i}) \to 0.$$

Taking dimensions, we get $1 - n + r - h^1(C, \mathcal{O}_C) + \sum_i g_i = 0$, so we see that the genus of C is $\sum_i g_i + r + 1 - n$. If C is connected, note that $r + 1 - n$ is precisely the number of "loops" in the graph of C. So the genus of a nodal curve is the sum of the genera of its components plus the number of "loops". This fits well with our topological interpretation of the genus given in examples 0.1.2 and 0.1.3.



Proposition 8.3.7. *Let $X \subset \mathbb{P}^2$ be a (possibly singular) curve of degree d, given as the zero locus of a homogeneous polynomial f of degree d. Then the genus of X is equal to* $\frac{1}{2}(d-1)(d-2)$.

Proof. Let x_0, x_1, x_2 be the coordinates of $\mathbb{P}^2$. By a change of coordinates we can assume that the point $(0:0:1)$ is not on X. Then the affine open subsets $U_0 = \{x_0 \neq 0\}$ and $U_1 = \{x_1 \neq 0\}$ cover X. So in the same way as in the proof of proposition 8.3.1 we get

$$H^1(X, \mathcal{O}_X) = \mathcal{O}_X(U_0 \cap U_1)/(\mathcal{O}_X(U_0) + \mathcal{O}_X(U_1)).$$

Moreover, the equation of f must contain an x_2^d-term, so the relation $f = 0$ can be used to restrict the degrees in x_2 of functions on X to at most $d-1$. Hence we get

$$\mathcal{O}_X(U_0 \cap U_1) = \left\{ \frac{x_2^i}{x_0^j x_1^k} \; ; \, 0 \le i \le d-1 \text{ and } i = j+k \right\}$$

and

$$\mathcal{O}_X(U_0) = \left\{ \frac{x_2^i}{x_0^j x_1^k} \; ; \, 0 \le i \le d-1, \, k \le 0, \text{ and } i = j+k \right\}$$

(and similarly for $\mathcal{O}_X(U_1)$). We conclude that

$$H^1(X, \mathcal{O}_X) = \left\{ \frac{x_2^i}{x_0^j x_1^k} \; ; \, 0 \le i \le d-1, \, j > 0, \, k > 0, \text{ and } i = j+k \right\}.$$

To compute the dimension of this space note that for a given value of i (which can run from 0 to $d-1$) we get $i-1$ choices of j and k (namely $(1, i-1), (2, i-2), \dots, (i-1, 1)$). So the total dimension is $h^1(X, \mathcal{O}_X) = 1 + 2 + \cdots + (d-2) = \frac{1}{2}(d-1)(d-2)$. □

Remark 8.3.8. The important point of proposition 8.3.7 is that the genus of a curve is *constant in families*: if we degenerate a smooth curve into a singular one (by varying the coefficients in its equation) then the genus of the singular curve will be the same as the genus of the original smooth curve. This also fits well with our idea in examples 0.1.2 and 0.1.3 that we can compute the genus of a plane curve by degenerating it into a singular one, where the result is then easy to read off.

Remark 8.3.9. Our second (cohomological) version of the Riemann-Roch theorem is in fact the one that is needed for generalizations to higher-dimensional varieties. If X is an n-dimensional projective variety and $\mathcal{F}$ a sheaf on X then the generalized Riemann-Roch theorem mentioned in remark 7.7.7 (v) will compute the *Euler characteristic*

$$\chi(X, \mathcal{F}) := \sum_{i=0}^{n} (-1)^i h^i(X, \mathcal{F})$$

in terms of other data that are usually easier to determine than the cohomology groups themselves.

8.4. **The cohomology of line bundles on projective spaces.** Let us now turn to higher-dimensional varieties and compute the cohomology groups of the line bundles $\mathcal{O}_X(d)$ on the projective space $X = \mathbb{P}^n$.

Proposition 8.4.1. *Let $X = \mathbb{P}^n$, and denote by $S = k[x_0, \dots, x_n]$ the graded coordinate ring of X. Then the sheaf cohomology groups of the line bundles $\mathcal{O}_X(d)$ on X are given by:*

(i) $\bigoplus_{d\in\mathbb{Z}} H^0(X, \mathcal{O}_X(d)) = S$ *as graded k-algebras,*
(ii) $\bigoplus_{d\in\mathbb{Z}} H^n(X, \mathcal{O}_X(d)) = S'$ *as graded k-vector spaces, where $S' \cong S$ with the grading given by $S'_d = S_{-n-1-d}$.*
(iii) $H^i(X, \mathcal{O}_X(d)) = 0$ *whenever $i \neq 0$ and $i \neq n$.*

Remark 8.4.2. By splitting up the equations of (i) and (ii) into the graded pieces one obtains the individual cohomology groups $H^i(X, \mathcal{O}_X(d))$. So for example we have

$$h^n(X, \mathcal{O}_X(d)) = h^0(X, \mathcal{O}_X(-n-1-d)) = \begin{cases} \binom{-d-1}{n} & \text{if } d \le -n-1, \\ 0 & \text{if } d > -n-1. \end{cases}$$

(Note that the equality of these two dimensions is consistent with the Serre duality theorem of remark 8.3.4, since $\omega_X = \mathcal{O}_X(-n-1)$ by lemma 7.4.15.)

Proof. (i) is clear from example 8.1.6 (i).

(ii): Let $\{U_i\}$ for $0 \le i \le n$ be the standard affine open cover of X, i. e. $U_i = \{x_i \neq 0\}$. We will prove the proposition for all d together by computing the cohomology of the quasi-coherent *graded* sheaf $\mathcal{F}_X = \bigoplus_{d\in\mathbb{Z}} \mathcal{O}_X(d)$ while keeping track of the grading (note that cohomology commutes with direct sums). This is just a notational simplification.

Of course we have $U_{i_0,\dots,i_k} = \{x_{i_0} \cdot \cdots \cdot x_{i_k} \neq 0\}$. So $\mathcal{F}(U_{i_0,\dots,i_k})$ is just the localization $S_{x_{i_0}\cdots x_{i_k}}$. It follows that the sequence of groups $C^k(\mathcal{F}_X)$ reads

$$\prod_{i_0} S_{x_{i_0}} \to \prod_{i_0<i_1} S_{x_{i_0}x_{i_1}} \to \cdots \to \prod_j S_{x_0\cdots x_{j-1}x_{j+1}\cdots x_n} \to S_{x_0\cdots x_n}. \tag{$*$}$$

Looking at the last term in this sequence, we compute that

$$\begin{aligned} H^n(X, \mathcal{F}) &= \operatorname{coker}\Big(\prod_j S_{x_0\cdots x_{j-1}x_{j+1}\cdots x_n} \to S_{x_0\cdots x_n}\Big) \\ &= \langle x_0^{j_0}\cdots x_n^{j_n} \,;\, j_i \in \mathbb{Z}\rangle / \langle x_0^{j_0}\cdots x_n^{j_n} \,;\, j_i \ge 0 \text{ for some } i\rangle \\ &= \langle x_0^{j_0}\cdots x_n^{j_n} \,;\, j_i < 0 \text{ for all } i\rangle \\ &= \frac{1}{x_0\cdots x_n} k[x_0^{-1}, \dots, x_n^{-1}], \end{aligned}$$

so up to a shift of $\deg x_0 \cdots x_n = n+1$ these are just the polynomials in x_i with non-positive exponents. This shows (ii).

(iii): We prove this by induction on n. There is nothing to show for $n = 1$. Let $H = \{x_n = 0\} \cong \mathbb{P}^{n-1}$ be a hyperplane. Note that there is an exact sequence of sheaves on X

$$0 \to \mathcal{O}_X(d-1) \to \mathcal{O}_X(d) \to \mathcal{O}_H(d) \to 0$$

for all d, where the first map is given by multiplication with x_n, and the second one by setting x_n to 0. Taking these sequences together for all $d \in \mathbb{Z}$ we obtain the exact sequence

$$0 \to \mathcal{F}(-1) \xrightarrow{\cdot x_n} \mathcal{F} \to \mathcal{F}_H \to 0$$

where we set $\mathcal{F}(-1) = \mathcal{F} \otimes \mathcal{O}_X(-1)$. From the associated long exact cohomology sequence and the induction hypothesis we get the following exact sequences:

$$0 \to H^0(X,\mathcal{F}(-1)) \to H^0(X,\mathcal{F}) \to H^0(H,\mathcal{F}_H) \to H^1(X,\mathcal{F}(-1)) \to H^1(X,\mathcal{F}) \to 0,$$
$$0 \to H^i(X,\mathcal{F}(-1)) \to H^i(X,\mathcal{F}) \to 0 \quad \text{for } 1 < i < n-1,$$
$$0 \to H^{n-1}(X,\mathcal{F}(-1)) \to H^{n-1}(X,\mathcal{F}) \to H^{n-1}(H,\mathcal{F}_H) \to H^n(X,\mathcal{F}(-1)) \to H^n(X,\mathcal{F}) \to 0.$$

So first of all we see that $H^i(X,\mathcal{F}(-1)) \cong H^i(X,\mathcal{F})$ for all $1 < i < n-1$. We claim that this holds in fact for $1 \le i \le n-1$. To see this for $i = 1$ note that the first exact sequence above starts with

$$0 \to k[x_0,\dots,x_n] \xrightarrow{\cdot x_n} k[x_0,\dots,x_n] \to k[x_0,\dots,x_{n-1}] \to \cdots,$$

which is obviously exact on the right, so it follows that $H^1(X,\mathcal{F}(-1)) \cong H^1(X,\mathcal{F})$. A similar analysis of the third exact sequence above, using the explicit description of the proof of part (ii), shows that $H^{n-1}(X,\mathcal{F}(-1)) \cong H^{n-1}(X,\mathcal{F})$. So we see that the map $H^i(X,\mathcal{F}(-1)) \xrightarrow{\cdot x_n} H^i(X,\mathcal{F})$ is an isomorphism for all $1 \le i \le n-1$. (Splitting this up into the graded parts, this means that $H^i(X,\mathcal{O}_X(d-1)) \cong H^i(X,\mathcal{O}_X(d))$ for all d, i. e. the cohomology groups do not depend on d. We still have to show that they are in fact zero.)

Now localize the Čech complex $(*)$ with respect to x_n. Geometrically this just means that we arrive at the complex that computes the cohomology of $\mathcal{F}$ on $U_n = \{x_n \neq 0\}$. As U_n is an affine scheme and therefore does not have higher cohomology groups by example 8.1.6 (ii), we see that

$$H^i(X,\mathcal{F})_{x_n} = H^i(U_n,\mathcal{F}|_{U_n}) = 0.$$

So for any $\alpha \in H^i(X,\mathcal{F})$ we know that $x_n^k \cdot \alpha = 0$ for some k. But we have shown above that multiplication with x_n in $H^i(X,\mathcal{F})$ is an isomorphism, so $\alpha = 0$. This means that $H^i(X,\mathcal{F}) = 0$, as desired. □

Example 8.4.3. As a consequence of this computation we can now of course compute the cohomology groups of all sheaves on $\mathbb{P}^n$ that are made up of line bundles in some way. Let us calculate the cohomology groups $H^i(X,\Omega_X)$ as an example. By the Euler sequence of lemma 7.4.15

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}(-1)^{\oplus(n+1)} \to \mathcal{O} \to 0$$

we get the long exact cohomology sequence

$$\begin{aligned} 0 &\to H^0(\Omega_{\mathbb{P}^n}) \to H^0(\mathcal{O}(-1))^{\oplus(n+1)} \to H^0(\mathcal{O}) \\ &\to H^1(\Omega_{\mathbb{P}^n}) \to H^1(\mathcal{O}(-1))^{\oplus(n+1)} \to H^1(\mathcal{O}) \\ &\to H^2(\Omega_{\mathbb{P}^n}) \to \cdots. \end{aligned}$$

By proposition 8.4.1 the cohomology groups of $\mathcal{O}(-1)$ are all zero, while the cohomology groups $H^i(\mathcal{O})$ are zero unless $i = 0$, in which case we have $h^0(\mathcal{O}) = 1$. So we conclude that

$$h^i(\mathbb{P}^n,\Omega_{\mathbb{P}^n}) = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

As an application of our computation of the cohomology groups of line bundles on projective spaces, we now want to prove in the rest of this section that the cohomology groups of certain "finitely generated" quasi-coherent sheaves on projective schemes are always finite-dimensional. Let us first define what we mean by this notion of finite generation.

Definition 8.4.4. Let X be a scheme. A sheaf $\mathcal{F}$ on X is called **coherent** if for every affine open subset $U = \operatorname{Spec} R \subset X$ the restricted sheaf $\mathcal{F}|_U$ is the sheaf associated to a *finitely generated* R-module in the sense of definition 7.2.1.

Remark 8.4.5. Except for the finite generation condition this definition is precisely the same as for quasi-coherent sheaves. Consequently, our results that essentially all operations that one can do with quasi-coherent sheaves yield again quasi-coherent sheaves carry over to coherent sheaves without much change.

To show that the cohomology groups of coherent sheaves on projective schemes are finite-dimensional we need an auxiliary lemma first.

Lemma 8.4.6. *Let X be a projective scheme over a field, and let $\mathcal{F}$ be a coherent sheaf on X. Then there is a surjective morphism $\mathcal{O}_X(-d)^{\oplus n} \to \mathcal{F}$ for some d and n.*

Proof. Let $X \subset \mathbb{P}^r = \operatorname{Proj} k[x_0, \dots, x_r]$ and consider the standard affine open subsets $U_i = \operatorname{Spec} R_i \subset X$ given by $x_i \neq 0$. As $\mathcal{F}$ is coherent, $\mathcal{F}|_{U_i}$ is of the form $\tilde{M}_i$, where M_i is a finitely generated R_i-module. Let $s_{i,1}, \dots, s_{i,k_i}$ be generators. Then the $s_{i,j}$ define sections of $\mathcal{F}$ over U_i, and their germs generate the stalk of $\mathcal{F}$ at every point of U_i.

The $s_{i,j}$ do not need to extend to global sections of $\mathcal{F}$, but we will now show that after multiplying with x_i^d for some d we get global sections $s_{i,j} \cdot x_i^d \in \Gamma(\mathcal{F} \otimes \mathcal{O}_X(d))$. As $X \backslash U_i$ is covered by the affine open subsets U_k for $k \neq i$, it is sufficient to show that we can extend $s_{i,j}$ to all U_k in this way. But $\mathcal{F}(U_k) = M_k$ and $\mathcal{F}(U_i \cap U_k) = (M_k)_{x_i}$ by proposition 7.2.2 (ii), so $s_{i,j} \in \mathcal{F}(U_i \cap U_k) \in (M_k)_{x_i}$ obviously gives an element in $\mathcal{F}(U_k) = M_k$ after multiplying with a sufficiently high power of x_i.

Hence we have shown that for some d we get global sections $s_{i,j} \in \Gamma(\mathcal{F} \otimes \mathcal{O}_X(d))$ that generate the stalk of $\mathcal{F} \otimes \mathcal{O}_X(d)$ at all points of X. So these sections define a surjective morphism $\mathcal{O} \to \mathcal{F} \otimes \mathcal{O}_X(d)^{\oplus n}$ (where n is the total number of sections chosen) and hence a surjective morphism $\mathcal{O}_X(-d)^{\oplus n} \to \mathcal{F}$. □

Theorem 8.4.7. *Let X be a projective scheme over a field, and let $\mathcal{F}$ be a coherent sheaf on X.*

(i) *The cohomology groups $H^i(X, \mathcal{F})$ are finite-dimensional vector spaces for all i.*
(ii) *We have $H^i(X, \mathcal{F} \otimes \mathcal{O}_X(d)) = 0$ for all $i > 0$ and $d \gg 0$.*

Proof. Let $i : X \to \mathbb{P}^r$ be the inclusion morphism. As $i_*\mathcal{F}$ is coherent by proposition 7.2.9 (or rather its analogue for coherent sheaves) and the cohomology groups of $\mathcal{F}$ and $i_*\mathcal{F}$ agree by definition, we can assume that $X = \mathbb{P}^r$.

We will prove the proposition by *descending* induction on i. By example 8.1.6 (iii) there is nothing to show for $i > r$. By lemma 8.4.6 there is an exact sequence $0 \to \mathcal{R} \to \mathcal{O}_X(-d)^{\oplus n} \to \mathcal{F} \to 0$ for some d and n, where $\mathcal{R}$ is a coherent sheaf on X by lemma 7.2.7. Tensoring with $\mathcal{O}_X(e)$ for some $e \in \mathbb{Z}$ and taking the corresponding long exact cohomology sequence, we get

$$\cdots \to H^i(X, \mathcal{O}_X(e-d)^{\oplus n}) \to H^i(X, \mathcal{F} \otimes \mathcal{O}_X(e)) \to H^{i+1}(X, \mathcal{R} \otimes \mathcal{O}_X(e)) \to \cdots.$$

(i): Take $e = 0$. Then the vector space on the left is always finite-dimensional by the explicit computation of proposition 8.4.1, and the one on the right is finite-dimensional by the induction hypothesis. Hence $H^i(X, \mathcal{F})$ is finite-dimensional as well.

(ii): For $e \gg 0$ the group on the left is zero again by the explicit calculation of proposition 8.4.1, and the one on the right is zero by the induction hypothesis. Hence $H^i(X, \mathcal{F} \otimes \mathcal{O}_X(e)) = 0$ for $e \gg 0$. □

Remark 8.4.8. Of course the assumption of projectivity is essential in theorem 8.4.7, as for example $H^0(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}) = k[x]$ is not finite-dimensional as a vector space over k.

For a more interesting example, consider $X = \mathbb{A}^2 \backslash \{(0,0)\}$ and compute $H^1(X, \mathcal{O}_X)$. Using the affine open cover $X = U_1 \cup U_2$ with $U_i = \{x_i \neq 0\}$ for $i = 1,2$, we get

$$\begin{aligned} H^1(X, \mathcal{O}_X) &= \mathcal{O}_X(U_1 \cap U_2)/(\mathcal{O}_X(U_1) + \mathcal{O}_X(U_2)) \\ &= \left\langle x_1^i x_2^j \,;\, i,j \in \mathbb{Z} \right\rangle / \left\langle x_1^i x_2^j \,;\, j \geq 0 \text{ or } i \geq 0 \right\rangle \\ &= \left\langle x_1^i x_2^j \,;\, i,j < 0 \right\rangle, \end{aligned}$$

which is not finite-dimensional. So we conclude that X is not projective (which is obvious). But we have also reproven the statement that X is not affine (see remark 2.3.17), as otherwise we would have a contradiction to example 8.1.6 (ii).

8.5. **Proof of the independence of the affine cover.** To make our discussion of sheaf cohomology rigorous it remains to be proven that the cohomology groups as of definition 8.1.4 do not depend on the choice of affine open cover. So let us go back to the original definitions 8.1.2 and 8.1.4 that (seem to) depend on this choice. For simplicity let us assume that all affine covers involved are finite.

Lemma 8.5.1. *Let $\mathcal{F}$ be a quasi-coherent sheaf on an affine scheme X. Then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$ and every choice of affine open cover $\{U_i\}$.*

Proof. Let us define a "sheafified version" of the Čech complex as follows: we set

$$\mathcal{C}^p(\mathcal{F}) = \prod_{i_0 < \cdots < i_p} i_* \mathcal{F}|_{U_{i_0} \cap \cdots \cap U_{i_p}}$$

where $i : U_{i_0} \cap \cdots \cap U_{i_p} \to X$ denotes the various inclusion maps. Then the $\mathcal{C}^p(\mathcal{F})$ are quasi-coherent sheaves on X by proposition 7.2.9. Their spaces of global sections are $\Gamma(\mathcal{C}^p(\mathcal{F})) = C^p(\mathcal{F})$ by definition. There are boundary morphisms $d^p : \mathcal{C}^p(\mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{F})$ defined by the same formula as in definition 8.1.2, giving rise to a complex

$$\mathcal{C}^0(\mathcal{F}) \to \mathcal{C}^1(\mathcal{F}) \to \mathcal{C}^2(\mathcal{F}) \to \cdots . \qquad (*)$$

Note that it suffices to prove that this sequence is exact: as taking global sections of quasi-coherent sheaves on affine schemes preserves exact sequences by proposition 7.2.2 (ii) it then follows that the sequence

$$C^0(\mathcal{F}) \to C^1(\mathcal{F}) \to C^2(\mathcal{F}) \to \cdots$$

is exact as well, which by definition means that $H^i(X, \mathcal{F}) = 0$ for $i > 0$.

The exactness of $(*)$ can be checked on the stalks. So let $P \in X$ be any point, and let U_j be an affine open subset of the given cover that contains P. We define a morphism of stalks of sheaves at P

$$k : \mathcal{C}_P^k \to \mathcal{C}_P^{k-1}, \quad \alpha \mapsto k\alpha$$

by $(k\alpha)_{i_0,\dots,i_{p-1}} = \alpha_{j,i_0,\dots,i_{p-1}}$, where we make the following convention: if the indices $j, i_0, \dots, i_{p-1}$ are not in sorted order and $\sigma \in S_{p+1}$ is the permutation such that $\sigma(j) < \sigma(i_0) < \cdots < \sigma(i_{p-1})$ then by $\alpha_{j,i_0,\dots,i_{p-1}}$ we mean $(-1)^\sigma \cdot \alpha_{\sigma(j),\sigma(i_0),\dots,\sigma(i_{p-1})}$.

We claim that $kd + dk : \mathcal{C}_P^k \to \mathcal{C}_P^k$ is the identity. In fact, we have

$$(dk\alpha)_{i_0,\dots,i_p} = \alpha_{i_0,\dots,i_p} - \sum_{k=1}^{p} (-1)^k \, \alpha_{j,i_0,\dots,i_{k-1},i_{k+1},\dots,i_p}$$

and

$$(kd\alpha)_{i_0,\dots,i_p} = \sum_{k=1}^{p} (-1)^k \, \alpha_{j,i_0,\dots,i_{k-1},i_{k+1},\dots,i_p}$$

from which the claim follows.

Finally we can now prove that the sequence $(*)$ is exact at any point P: we know already that $\operatorname{im} d^{k-1} \subset \ker d^k$ as $d^k \circ d^{k-1} = 0$. Conversely, if $\alpha \in \ker d^k$, i. e. $d\alpha = 0$, then $\alpha = (kd + dk)(\alpha) = d(k\alpha)$, i. e. $\alpha \in \operatorname{im} d^{k-1}$. □

Lemma 8.5.2. *Let $\mathcal{F}$ be a quasi-coherent sheaf on a scheme X. Pick an affine open cover $\mathcal{U} = \{U_1, \dots, U_k\}$. Let $U_0 \subset X$ be any other affine open subset, and denote by $\tilde{\mathcal{U}}$ the affine open cover $\{U_0, \dots, U_k\}$. Then the cohomology groups determined by the open covers $\mathcal{U}$ and $\tilde{\mathcal{U}}$ are the same.*

Proof. Let $C^p(\mathcal{F})$ and $H^p(X, \mathcal{F})$ be the groups of Čech cycles and the cohomology groups for the cover $\mathcal{U}$, and denote by $\tilde{C}^p(\mathcal{F})$ and $\tilde{H}^p(X, \mathcal{F})$ the corresponding groups for the cover $\tilde{\mathcal{U}}$.

Note that there are natural morphisms $\tilde{C}^p(\mathcal{F}) \to C^p(\mathcal{F})$ and $\tilde{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$ given by "forgetting the data that involves the open subset U_0", i. e. by

$$(\alpha_{i_0,\dots,i_p})_{0 \le i_0 < i_1 < \dots < i_p \le k} \mapsto (\alpha_{i_0,\dots,i_p})_{1 \le i_0 < i_1 < \dots < i_p \le k}.$$

More precisely, an element $\tilde{\alpha} \in \tilde{C}^p(\mathcal{F})$ can be thought of as a pair $\tilde{\alpha} = (\alpha, \alpha^0)$, where $\alpha \in C^p(\mathcal{F})$ is given by $\alpha_{i_0,\dots,i_p} = \tilde{\alpha}_{i_0,\dots,i_p}$ (for $i_0 > 0$) and $\alpha^0 \in C^{p-1}(U_0, \mathcal{F}|_{U_0})$ is given by $\alpha^0_{i_0,\dots,i_{p-1}} = \tilde{\alpha}_{0,i_0,\dots,i_{p-1}}$. Moreover, $d\tilde{\alpha} = 0$ if and only if

$$d\alpha = 0 \tag{1}$$

(these are the equations $(d\tilde{\alpha})_{i_0,\dots,i_{p+1}} = 0$ for $i_0 > 0$) and

$$\alpha|_{U_0} - d\alpha^0 = 0 \tag{2}$$

(these are the equations $(d\tilde{\alpha})_{i_0,\dots,i_{p+1}} = 0$ for $i_0 = 0$).

We have to show that the morphism $\tilde{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$ is injective and surjective.

(i) $\tilde{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$ is surjective: Let $\alpha \in H^p(X, \mathcal{F})$ be a cohomology cycle, i. e. $d\alpha = 0$. We have to find an $\alpha^0 \in C^{p-1}(U_0, \mathcal{F}|_{U_0})$ such that $\tilde{\alpha} = (\alpha, \alpha^0)$ satisfies $d\tilde{\alpha} = 0$, i. e. by (2) such that $d\alpha^0 = \alpha|_{U_0}$. But $d(\alpha|_{U_0}) = (d\alpha)|_{U_0} = 0$, so by lemma 8.5.1 $\alpha|_{U_0} = d\alpha^0$ for some α^0.

(ii) $\tilde{H}^p(X, \mathcal{F}) \to H^p(X, \mathcal{F})$ is injective: Let $\tilde{\alpha} \in \tilde{H}^p(X, \mathcal{F})$ be a cohomology cycle (i. e. $d\tilde{\alpha} = 0$) such that $\alpha = 0 \in H^p(X, \mathcal{F})$, i. e. $\alpha = d\beta$ for some $\beta \in C^{p-1}(\mathcal{F})$. We have to show that $\tilde{\alpha} = 0 \in \tilde{H}^p(X, \mathcal{F})$, i. e. we have to find a $\tilde{\beta} = (\beta, \beta^0) \in \tilde{C}^{p-1}(\mathcal{F})$ such that $d\tilde{\beta} = \tilde{\alpha}$. By (2) this means that we need $\beta|_{U_0} - d\beta^0 = \alpha^0$. But $d(\beta|_{U_0} - \alpha^0) = \alpha|_{U_0} - \alpha|_{U_0} = 0$, so by lemma 8.5.1 there is a β^0 such that $\beta|_{U_0} - \alpha^0 = d\beta^0$.

□

Corollary 8.5.3. *The cohomology groups of quasi-coherent sheaves on any scheme do not depend on the choice of open affine cover.*

Proof. Let $\mathcal{F}$ be a quasi-coherent sheaf on a scheme X. Let $\mathcal{U} = \{U_1, \dots, U_k\}$ and $\mathcal{U}' = \{U'_1, \dots, U'_m\}$ be two affine open covers of X. Then the cohomology groups $H^i(X, \mathcal{F})$ determined by $\mathcal{U}$ are the same as those determined by $\mathcal{U} \cup \mathcal{U}'$ by (a repeated application of) lemma 8.5.2, which in turn are equal to those determined by $\mathcal{U}'$ by the same lemma. □

8.6. **Exercises.**

Exercise 8.6.1. Let X be a smooth projective curve. For any point $P \in X$ consider the exact skyscraper sequence of sheaves on X

$$0 \to \omega_X \to \omega_X \otimes \mathcal{O}_X(P) \to k_P \to 0$$

as in exercise 7.8.4. Show that the induced sequence of global sections is not exact, i. e. the last map $\Gamma(\omega_X \otimes \mathcal{O}_X(P)) \to \Gamma(k_P)$ is not surjective.

Exercise 8.6.2. Complete the proof of lemma 8.2.2, i. e. show that the sequence of morphisms of cohomology groups

$$\cdots \to H^{p-1}(E) \to H^p(C) \to H^p(D) \to H^p(E) \to H^{p+1}(C) \to \cdots$$

associated to an exact sequence of complexes $0 \to C \to D \to E \to 0$ is exact at the $H^p(C)$ and $H^p(E)$ positions.

Exercise 8.6.3. Compute the cohomology groups $H^i(\mathbb{P}^1 \times \mathbb{P}^1, p^*\mathcal{O}_{\mathbb{P}^1}(a) \otimes q^*\mathcal{O}_{\mathbb{P}^1}(b))$ for all $a, b \in \mathbb{Z}$, where p and q denote the two projection maps from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$.

Exercise 8.6.4. Give an example of a smooth projective curve X and line bundles $\mathcal{L}_1, \mathcal{L}_2$ on X *of the same degree* such that $h^0(X, \mathcal{L}_1) \neq h^0(X, \mathcal{L}_2)$.

Exercise 8.6.5. Let $X \subset \mathbb{P}^r$ be a **complete intersection** of dimension $n \geq 1$, i. e. it is the scheme-theoretic zero locus of $r-n$ homogeneous polynomials. Show that X is connected.

(Hint: Prove by induction on n that the natural map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(d)) \to H^0(X, \mathcal{O}_X(d))$ is surjective for all $d \in \mathbb{Z}$.)

9. INTERSECTION THEORY

A k-cycle on a scheme X (that is always assumed to be separated and of finite type over an algebraically closed field in this section) is a finite formal linear combination $\sum_i n_i[V_i]$ with $n_i \in \mathbb{Z}$, where the V_i are k-dimensional subvarieties of X. The group of k-cycles is denoted $Z_k(X)$. A rational function φ on any subvariety $Y \subset X$ of dimension $k+1$ determines a cycle $\operatorname{div}(\varphi) \in Z_k(X)$, which is just the zeroes of φ minus the poles of φ, counted with appropriate multiplicities. The subgroup $B_k(X) \subset Z_k(X)$ generated by cycles of this form is called the group of k-cycles that are rationally equivalent to zero. The quotient groups $A_k(X) = Z_k(X)/B_k(X)$ are the groups of cycle classes or Chow groups. They are the main objects of study in intersection theory. The Chow groups of a scheme should be thought of as being analogous to the homology groups of a topological space.

A morphism $f : X \to Y$ is called proper if inverse images of compact sets (in the classical topology) are compact. Any proper morphism f gives rise to push-forward homomorphisms $f_* : A_*(X) \to A_*(Y)$ between the Chow groups. On the other hand, some other morphisms $f : X \to Y$ (e. g. inclusions of open subsets or projections from vector bundles) admit pull-back maps $f^* : A_*(Y) \to A_*(X)$.

If X is a purely n-dimensional scheme, a Weil divisor is an element of $Z_{n-1}(X)$. In contrast, a Cartier divisor is a global section of the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$. Every Cartier divisor determines a Weil divisor. On smooth schemes, Cartier and Weil divisors agree. On almost any scheme, Cartier divisors modulo linear equivalence correspond exactly to line bundles.

We construct bilinear maps $\operatorname{Pic} X \times A_k(X) \to A_{k-1}(X)$ that correspond geometrically to taking intersections of the divisor (a codimension-1 subset of X) with the k-dimensional subvariety. If one knows the Chow groups of a space and the above intersection products, one arrives at Bézout style theorems that allow to compute the number of intersection points of k divisors on X with a k-dimensional subspace.

9.1. **Chow groups.** Having discussed the basics of scheme theory, we will now start with the foundations of intersection theory. The idea of intersection theory is the same as that of homology in algebraic topology. Roughly speaking, what one does in algebraic topology is to take e. g. a real differentiable manifold X of dimension n and an integer $k \geq 0$, and consider formal linear combinations of real k-dimensional submanifolds (with boundary) on X with integer coefficients, called cycles. If $Z_k(X)$ is the group of closed cycles (those having no boundary) and $B_k(X) \subset Z_k(X)$ is the group of those cycles that are boundaries of $(k+1)$-dimensional cycles, then the homology group $H_k(X,\mathbb{Z})$ is the quotient $Z_k(X)/B_k(X)$.

There are (at least) two main applications of this. First of all, the groups $H_k(X,\mathbb{Z})$ are (in contrast to the $Z_k(X)$ and $B_k(X)$) often finitely generated groups and provide invariants of the manifold X that can be used for classification purposes. Secondly, there are intersection products: homology classes in $H_{n-k}(X,\mathbb{Z})$ and $H_{n-l}(X,\mathbb{Z})$ can be "multiplied" to give a class in $H_{n-k-l}(X,\mathbb{Z})$ that geometrically corresponds to taking intersections of submanifolds. Hence if we are for example given submanifolds V_i of X whose codimensions sum up to n (so that we expect a finite number of points in the intersection $\bigcap_i V_i$), then this number can often be computed easily by taking the corresponding products in homology.

Our goal is to establish a similar theory for schemes. For any scheme of finite type over a ground field and any integer $k \geq 0$ we will define the so-called Chow groups $A_k(X)$ whose elements are formal linear combinations of k-dimensional closed subvarieties of X, modulo "boundaries" in a suitable sense. The formal properties of these groups $A_k(X)$ will be similar to those of homology groups; if the ground field is $\mathbb{C}$ you might even want to think of the $A_k(X)$ as being "something like" $H_{2k}(X,\mathbb{Z})$, although these groups are usually different. But there is always a map $A_k(X) \to H_{2k}(X,\mathbb{Z})$ (at least if one uses the "right" homology theory, see [F] chapter 19 for details), so you can think of elements in the Chow groups as something that determines a homology class, but this map is in general neither injective nor surjective.

Another motivation for the Chow groups $A_k(X)$ is that they generalize our notions of divisors and divisor classes. In fact, if X is a smooth projective curve then $A_0(X)$ will be by definition the same as $\operatorname{Pic} X$. In general, the definition of the groups $A_k(X)$ is very similar to our definition of divisors: we consider the free Abelian groups $Z_k(X)$ generated by the k-dimensional subvarieties of X. There is then a subgroup $B_k(X) \subset Z_k(X)$ that corresponds to those linear combinations of subvarieties that are zeros minus poles of rational functions. The Chow groups are then the quotients $A_k(X) = Z_k(X)/B_k(X)$.

To make sense of this definition, the first thing we have to do is to define the divisor of a rational function (see definition 6.3.4) in the higher-dimensional case. This is essentially a problem of commutative algebra, so we will only sketch it here. The important ingredient is the notion of the length of a module.

Remark 9.1.1. (For the following facts we refer to [AM] chapter 6 and [F] section A.1.) Let M be a finitely generated module over a Noetherian ring R. Then there is a so-called **composition series**, i. e. a finite chain of submodules

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r = M \qquad (*)$$

such that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for some prime ideals $\mathfrak{p}_i \in R$. The series is not unique, but for any prime ideal $\mathfrak{p} \subset R$ the number of times $\mathfrak{p}$ occurs among the $\mathfrak{p}_i$ does not depend on the series.

The geometric meaning of this composition series is easiest explained in the case where R is an integral domain and $M = R/I$ for some ideal $I \subset R$. In this case $\operatorname{Spec} M$ is a closed subscheme of the irreducible scheme $\operatorname{Spec} R$ (see examples 5.2.3 and 7.2.10). The prime ideals $\mathfrak{p}_i$ are then precisely the ideals of the irreducible (and maybe embedded) components of $\operatorname{Spec} M$, or in other words the prime ideals associated to all primary ideals in the primary decomposition of I. The number of times $\mathfrak{p}$ occurs among the $\mathfrak{p}_i$ can be thought of as the “multiplicity” of the corresponding component in the scheme. For example, if I is a radical ideal (so $\operatorname{Spec} M$ is reduced) then the $\mathfrak{p}_i$ are precisely the ideals of the irreducible components of $\operatorname{Spec} M$, all occurring once.

We will need this construction mainly in the case where $I = (f) \subset R$ is the ideal generated by a single (non-zero) function. In this case all irreducible components of $\operatorname{Spec} M$ have codimension 1. If $\mathfrak{p} \subset R$ is a prime ideal corresponding to any codimension-1 subvariety of $\operatorname{Spec} R$ we can consider a composition series as above for the localized module $M_\mathfrak{p}$ over $R_\mathfrak{p}$. As the only prime ideals in $R_\mathfrak{p}$ are (0) and $\mathfrak{p}R_\mathfrak{p}$ (corresponding geometrically to $\operatorname{Spec} R$ and $\operatorname{Spec} M$, respectively) and f does not vanish identically on $\operatorname{Spec} M$, the only prime ideal that can occur in the composition series of $M_\mathfrak{p}$ is $\mathfrak{p}R_\mathfrak{p}$. The number of times it occurs, i. e. the length r of the composition series, is then called the **length** of the module $M_\mathfrak{p}$ over $R_\mathfrak{p}$, denoted $l_{R_\mathfrak{p}}(M_\mathfrak{p})$. It is equal to the number of times $\mathfrak{p}$ occurs in the composition series of M over R. By what we have said above, we can interpret this number geometrically as the multiplicity of the subvariety corresponding to $\mathfrak{p}$ in the scheme $\operatorname{Spec} R/(f)$, or in other words as the order of vanishing of f at this codimension-1 subvariety.

We should rephrase these ideas in terms of general (not necessarily affine) schemes. So let X be a scheme, and let $V \subset X$ be a subvariety of codimension 1. Note that V can be considered as a *point* in the scheme X, so it makes sense to talk about the stalk $\mathcal{O}_{X,V}$ of the structure sheaf $\mathcal{O}_X$ at V. If $U = \operatorname{Spec} R \subset X$ is any affine open subset with non-empty intersection with V then $\mathcal{O}_{X,V}$ is just the localized ring $R_\mathfrak{p}$ where $\mathfrak{p}$ is the prime ideal corresponding to the subvariety $V \cap U$ of U (see proposition 5.1.12 (i)). So if $f \in \mathcal{O}_{X,V}$ is a local function around V then its order of vanishing at the codimension-1 subvariety V is simply the length $l_{\mathcal{O}_{X,V}}(\mathcal{O}_{X,V}/(f))$. To define the order of a possibly rational function φ on X we just have to observe that the field of fractions of the ring $\mathcal{O}_{X,V}$ is equal to the field of rational functions on X. So we can write φ as $\frac{f}{g}$ for some $f,g \in \mathcal{O}_{X,V}$ and simply define the order of φ at V to be the difference of the orders of f and g at V.

With these prerequisites we can now define the Chow groups in complete analogy to the Picard group of divisor classes in section 6.3. For the rest of this section by a scheme we will always mean a scheme of finite type over some algebraically closed field (that is not necessarily smooth, irreducible, or reduced). A variety is a reduced and irreducible (but not necessarily smooth) scheme.

Definition 9.1.2. Let X be a variety, and let $V \subset X$ be a subvariety of codimension 1, and set $R = \mathcal{O}_{X,V}$. For every non-zero $f \in R \subset K(X)$ we define the **order** of f at V to be the integer $\operatorname{ord}_V(f) := l_R(R/(f))$. If $\varphi \in K(X)$ is a non-zero rational function we write $\varphi = \frac{f}{g}$ with $f,g \in R$ and define the order of φ at V to be

$$\operatorname{ord}_V(\varphi) := \operatorname{ord}_V(f) - \operatorname{ord}_V(g).$$

To show that this is well-defined, i. e. that $\operatorname{ord}_V \frac{f}{g} = \operatorname{ord}_V \frac{f'}{g'}$ whenever $fg' = gf'$, one uses the exact sequence

$$0 \to R/(a) \xrightarrow{\cdot b} R/(ab) \to R/(b) \to 0$$

and the fact that the length of modules is additive on exact sequences. From this it also follows that the order function is a homomorphism of groups $\operatorname{ord}_V : K(X)^* := K(X) \backslash \{0\} \to \mathbb{Z}$.

Example 9.1.3. Let $X = \mathbb{A}^1 = \operatorname{Spec} k[x]$ and let $V = \{0\} \subset X$ be the origin. Consider the function $\varphi = x^r$ for $r \geq 0$. Then $R = \mathcal{O}_{X,V} = k[x]_{(x)}$, and $R/(x) \cong k$. So as $R/(x^r) = \{a_0 + a_1 x + \cdots + a_{r-1}x^{r-1}\}$ has vector space dimension r over k we conclude that $\operatorname{ord}_0(x^r) = r$, as expected. By definition, we then have the equality $\operatorname{ord}_0(x^r) = r$ for all $r \in \mathbb{Z}$.

Definition 9.1.4. Let X be a scheme. For $k \geq 0$ denote by $Z_k(X)$ the free Abelian group generated by the k-dimensional subvarieties of X. In other words, the elements of $Z_k(X)$ are finite formal sums $\sum_i n_i [V_i]$, where $n_i \in \mathbb{Z}$ and the V_i are k-dimensional (closed) subvarieties of X. The elements of $Z_k(X)$ are called **cycles** of dimension k.

For any $(k+1)$-dimensional subvariety W of X and any non-zero rational function φ on W we define a cycle of dimension k on X by

$$\operatorname{div}(\varphi) = \sum_V \operatorname{ord}_V(\varphi)[V] \in Z_k(X),$$

called the **divisor** of φ, where the sum is taken over all codimension-1 subvarieties V of W. Note that this sum is always finite: it suffices to check this on a finite affine open cover $\{U_i\}$ of W and for $\varphi \in \mathcal{O}_{U_i}(U_i)$, where it is obvious as $Z(\varphi)$ is closed and U_i is Noetherian.

Let $B_k(X) \subset Z_k(X)$ be the subgroup generated by all cycles of the form $\operatorname{div}(\varphi)$ for all $W \subset X$ and $\varphi \in K(W)^*$ as above. We define the group of k-dimensional **cycle classes** to be the quotient $A_k(X) = Z_k(X)/B_k(X)$. These groups are usually called the **Chow groups** of X. Two cycles in $Z_k(X)$ that determine the same element in $A_k(X)$ are said to be **rationally equivalent**.

We set $Z_*(X) = \bigoplus_{k \geq 0} Z_k(X)$ and $A_*(X) = \bigoplus_{k \geq 0} A_k(X)$.

Example 9.1.5. Let X be a scheme of pure dimension n. Then $B_n(X)$ is trivially zero, and thus $A_n(X) = Z_n(X)$ is the free Abelian group generated by the irreducible components of X. In particular, if X is an n-dimensional variety then $A_n(X) \cong \mathbb{Z}$ with $[X]$ as a generator. In the same way, $Z_k(X)$ and $A_k(X)$ are trivially zero if $k > n$.

Example 9.1.6. Let X be a smooth projective curve. Then $Z_0(X) = \operatorname{Div} X$ and $A_0(X) = \operatorname{Pic} X$ by definition. In fact, the 1-dimensional subvariety W of X in definition 9.1.4 can only be X itself, so we arrive at precisely the same definition as in section 6.3.

Example 9.1.7. Let $X = \{x_1x_2 = 0\} \subset \mathbb{P}^2$ be the union of two projective lines $X = X_1 \cup X_2$ that meet in a point. Then $A_1(X) = \mathbb{Z}[X_1] \oplus \mathbb{Z}[X_2]$ by example 9.1.5. Moreover, $A_0(X) \cong \mathbb{Z}$ is generated by the class of any point in X. In fact, any two points on X_1 are rationally equivalent by example 9.1.6, and the same is true for X_2. As both X_1 and X_2 contain the intersection point $X_1 \cap X_2$ we conclude that all points in X are rationally equivalent. So $A_0(X) \cong \mathbb{Z}$.

Now let $P_1 \in X_1 \backslash X_2$ and $P_2 \in X_2 \backslash X_1$ be two points. Note that the line bundles $\mathcal{O}_X(P_1)$ and $\mathcal{O}_X(P_2)$ (defined in the obvious way: $\mathcal{O}_X(P_i)$ is the sheaf of rational functions that are regular away from P_i and have at most a simple pole at P_i) are *not* isomorphic: if $i : X_1 \to X$ is the inclusion map of the first component, then $i^*\mathcal{O}_X(P_1) \cong \mathcal{O}_{\mathbb{P}^1}(1)$, whereas $i^*\mathcal{O}_X(P_2) \cong \mathcal{O}_{\mathbb{P}^1}$. So we see that for singular curves the one-to-one correspondence between $A_0(X)$ and line bundles no longer holds.

Example 9.1.8. Let $X = \mathbb{A}^n$. We claim that $A_0(X) = 0$. In fact, if $P \in X$ is any point, pick a line $W \cong \mathbb{A}^1 \subset \mathbb{A}^n$ through P and a linear function φ on W that vanishes precisely at P. Then $\operatorname{div}(\varphi) = [P]$. It follows that the class of any point is zero in $A_0(X)$. Therefore $A_0(X) = 0$.

Example 9.1.9. Now let $X = \mathbb{P}^n$; we claim that $A_0(X) \cong \mathbb{Z}$. In fact, if P and Q are any two distinct points in X let $W \cong \mathbb{P}^1 \subset \mathbb{P}^n$ be the line through P and Q, and let φ be a rational function on W that has a simple zero at P and a simple pole at Q. Then $\operatorname{div}(\varphi) = [P] - [Q]$, i. e. the classes in $A_0(X)$ of any two points in X are the same. It follows that $A_0(X)$ is generated by the class $[P]$ of any point in X.

On the other hand, if $W \subset X = \mathbb{P}^n$ is any curve and φ a rational function on W then we have seen in remark 6.3.5 that the degree of the divisor of φ is always zero. It follows that the class $n \cdot [P] \in A_0(X)$ for $n \in \mathbb{Z}$ can only be zero if $n = 0$. We conclude that $A_0(X) \cong \mathbb{Z}$ with the class of any point as a generator.

Example 9.1.10. Let X be a scheme, and let $Y \subset X$ be a closed subscheme with inclusion morphism $i : Y \to X$. Then there are canonical **push-forward** maps $i_* : A_k(Y) \to A_k(X)$ for any k, given by $[Z] \mapsto [Z]$ for any k-dimensional subvariety $Z \subset Y$. It is obvious from the definitions that this respects rational equivalence.

Example 9.1.11. Let X be a scheme, and let $U \subset X$ be an open subset with inclusion morphism $i : U \to X$. Then there are canonical **pull-back** maps $i^* : A_k(X) \to A_k(U)$ for any k, given by $[Z] \mapsto [Z \cap U]$ for any k-dimensional subvariety $Z \subset X$. This respects rational equivalence as $i^* \operatorname{div}(\varphi) = \operatorname{div}(\varphi|_U)$ for any rational function φ on a subvariety of X.

Remark 9.1.12. If $f : X \to Y$ is any morphism of schemes it is an important part of intersection theory to study whether there are push-forward maps $f_* : A_*(X) \to A_*(Y)$ or pull-back maps $f^* : A_*(Y) \to A_*(X)$ and which properties they have. We have just seen two easy examples of this. Note that neither example can be reversed (at least not in an obvious way):

(i) if $Y \subset X$ is a closed subset, then a subvariety of X is in general not a subvariety of Y, so there is no pull-back morphism $A_*(X) \to A_*(Y)$ sending $[V]$ to $[V]$ for any subvariety $V \subset X$.
(ii) if $U \subset X$ is an open subset, there are no push-forward maps $A_*(U) \to A_*(X)$: if $U = \mathbb{A}^1$ and $X = \mathbb{P}^1$ then the class of a point is zero in $A_*(U)$ but non-zero in $A_*(\mathbb{P}^1)$ by examples 9.1.8 and 9.1.9.

We will construct more general examples of push-forward maps in section 9.2, and more general examples of pull-back maps in proposition 9.1.14.

Lemma 9.1.13. *Let X be a scheme, let $Y \subset X$ be a closed subset, and let $U = X \backslash Y$. Denote the inclusion maps by $i : Y \to X$ and $j : U \to X$. Then the sequence*

$$A_k(Y) \xrightarrow{i_*} A_k(X) \xrightarrow{j^*} A_k(U) \to 0$$

is exact for all $k \geq 0$. The homomorphism i_* is in general not injective however.

Proof. This follows more or less from the definitions. If $Z \subset U$ is any k-dimensional subvariety then the closure $\bar{Z}$ of Z in X is a k-dimensional subvariety of X with $j^*[\bar{Z}] = [Z]$. So j^* is surjective.

If $Z \subset Y$ then $Z \cap U = 0$, so $j^* \circ i_* = 0$. Conversely, assume that we have a cycle $\sum a_r[V_r] \in A_k(X)$ whose image in $A_k(U)$ is zero. This means that there are rational functions φ_s on $(k+1)$-dimensional subvarieties W_s of U such that $\sum \operatorname{div}(\varphi_s) = \sum a_r[V_r \cap U]$ on U. Now the φ_s are also rational functions on the closures of W_s in X, and as such their divisors can only differ from the old ones by subvarieties V_r' that are contained in $X \backslash U = Y$. We conclude that $\sum \operatorname{div}(\varphi_s) = \sum a_r[V_r] - \sum b_r[V_r']$ on X for some b_r. So $\sum a_r[V_r] = i_* \sum b_r[V_r']$.

As an example that i_* is in general not injective let Y be a smooth cubic curve in $X = \mathbb{P}^2$. If P and Q are two distinct points on Y then $[P] - [Q] \neq 0 \in A_0(Y) = \operatorname{Pic} X$ by proposition 6.3.13, but $[P] - [Q] = 0 \in A_0(X) \cong \mathbb{Z}$ by example 9.1.9. □

Proposition 9.1.14. *Let X be a scheme, and let $\pi : E \to X$ be a vector bundle of rank r on X (see remark 7.3.2). Then for all $k \geq 0$ there is a well-defined surjective pull-back homomorphism $\pi^* : A_k(X) \to A_{k+r}(E)$ given on cycles by $\pi^*[V] = [\pi^{-1}(V)]$.*

Proof. It is clear that π^* is well-defined: it obviously maps k-dimensional cycles to $(k+r)$-dimensional cycles, and $\pi^* \operatorname{div}(\varphi) = \operatorname{div}(\pi^*\varphi)$ for any rational function φ on a $(k+1)$-dimensional subvariety of X.

We will prove the surjectivity by induction on $\dim X$. Let $U \subset X$ be an affine open subset over which E is of the form $U \times \mathbb{A}^r$, and let $Y = X \backslash U$. By lemma 9.1.13 there is a commutative diagram

$$\begin{tikzcd}
A_k(Y) \arrow[r] \arrow[d] & A_k(X) \arrow[r] \arrow[d, "\pi^*"] & A_k(U) \arrow[r] \arrow[d] & 0 \\
A_{k+r}(E|_Y) \arrow[r] & A_{k+r}(E) \arrow[r] & A_{k+r}(U \times \mathbb{A}^r) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. A diagram chase (similar to that of the proof of lemma 8.2.2) shows that in order for π^* to be surjective it suffices to prove that the left and right vertical arrows are surjective. But the left vertical arrow is surjective by the induction assumption since $\dim Y < \dim X$. So we only have to show that the right vertical arrow is surjective. In other words, we have reduced to the case where $X = \operatorname{Spec} R$ is affine and $E = X \times \mathbb{A}^r$ is the trivial bundle. As π then factors as a sequence

$$E = X \times \mathbb{A}^r \to X \times \mathbb{A}^{r-1} \to \cdots \to X \times \mathbb{A}^1 \to X$$

we can furthermore assume that $r = 1$, so that $E = X \times \mathbb{A}^1 = \operatorname{Spec} R[t]$.

We have to show that $\pi^* : A_k(X) \to A_k(X \times \mathbb{A}^1)$ is surjective. So let $V \subset X \times \mathbb{A}^1$ be a $(k+1)$-dimensional subvariety, and let $W = \overline{\pi(V)}$. There are now two cases to consider:

- $\dim W = k$. Then $V = W \times \mathbb{A}^1$, so $[V] = \pi^*[W]$.
- $\dim W = k+1$. As it suffices to show that $[V]$ is in the image of the pull-back map $A_k(W) \to A_{k+r}(W \times \mathbb{A}^1)$ we can assume that $W = X$. Consider the ideal $I(V) \otimes_R K \subset K[t]$, where $K = K(W)$ denotes the quotient field of R. It is not the unit ideal as otherwise we would be in case (i). On the other hand $K[t]$ is a principal ideal domain, so $I(V) \otimes_R K$ is generated by a single polynomial $\varphi \in K[t]$.

Considering φ as a rational function on $X \times \mathbb{A}^1$ we see that the divisor of φ is precisely $[V]$ by construction, plus maybe terms of the form $\sum a_i \pi^*[W_i]$ for some $W_i \subset X$ corresponding to our tensoring with the field of rational functions $K(X)$. So $[V] = \pi^*(\sum a_i [W_i])$ (plus the divisor of a rational function), i. e. $[V]$ is in the image of π^*.

□

Remark 9.1.15. Note that the surjectivity part of proposition 9.1.14 is obviously false on the cycle level, i. e. for the pull-back maps $Z_k(X) \to Z_k(E)$: not every subvariety of a vector bundle E over X is the inverse image of a subvariety in X. So this proposition is an example of the fact that working with Chow groups (instead of with the subvarieties themselves) often makes life a little easier. In fact one can show (see [F] theorem 3.3 (a)) that the pull-back maps $\pi^* : A_k(X) \to A_{k+r}(E)$ are always isomorphisms.

Corollary 9.1.16. *The Chow groups of affine spaces are given by*

$$A_k(\mathbb{A}^n) = \begin{cases} \mathbb{Z} & \text{for } k = n, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. The statement for $k \geq n$ follows from example 9.1.5. For $k < n$ note that the homomorphism $A_0(\mathbb{A}^{n-k}) \to A_k(\mathbb{A}^n)$ is surjective by proposition 9.1.14, so the statement of the corollary follows from example 9.1.8. □

Corollary 9.1.17. *The Chow groups of projective spaces are $A_k(\mathbb{P}^n) \cong \mathbb{Z}$ for all $0 \leq k \leq n$, with an isomorphism given by $[V] \mapsto \deg V$ for all k-dimensional subvarieties $V \subset \mathbb{P}^n$.*

Proof. The statement for $k \geq n$ follows again from example 9.1.5, so let us assume that $k < n$. We prove the statement by induction on n. By lemma 9.1.13 there is an exact sequence

$$A_k(\mathbb{P}^{n-1}) \to A_k(\mathbb{P}^n) \to A_k(\mathbb{A}^n) \to 0.$$

We have $A_k(\mathbb{A}^n) = 0$ by corollary 9.1.16, so we conclude that $A_k(\mathbb{P}^{n-1}) \to A_k(\mathbb{P}^n)$ is surjective. By the induction hypothesis this means that $A_k(\mathbb{P}^n)$ is generated by the class of a k-dimensional linear subspace. As the morphism $Z_k(\mathbb{P}^{n-1}) \to Z_k(\mathbb{P}^n)$ trivially preserves degrees it only remains to be shown that any cycle $\sum a_i [V_i]$ that is zero in $A_k(\mathbb{P}^n)$ must satisfy $\sum a_i \deg V_i = 0$. But this is clear from Bézouts theorem, as $\deg \operatorname{div}(\varphi) = 0$ for all rational functions on any subvariety of $\mathbb{P}^n$. □

Remark 9.1.18. There is a generalization of corollary 9.1.17 as follows. Let X be a scheme that has a stratification by affine spaces, i. e. X has a filtration by closed subschemes $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_n = X$ such that $X_k \backslash X_{k-1}$ is a disjoint union of a_k affine spaces $\mathbb{A}^k$ for all k. For example, $X = \mathbb{P}^n$ has such a stratification with $a_k = 1$ for $0 \leq k \leq n$, namely $\emptyset \subset \mathbb{P}^0 \subset \mathbb{P}^1 \subset \cdots \subset \mathbb{P}^n = X$.

We claim that then $A_k(X)$ is isomorphic to $\mathbb{Z}^{a_k}$ modulo some (possibly trivial) subgroup, where $\mathbb{Z}^{a_k}$ is generated by the classes of the closures of the a_k copies of $\mathbb{A}^k$ mentioned above. We will prove this by induction on $\dim X$, the case of dimension 0 being obvious. In fact, consider the exact sequence of lemma 9.1.13

$$A_k(X_{n-1}) \to A_k(X) \to \oplus_{i=1}^{a_n} A_k(\mathbb{A}^n) \to 0.$$

Note that X_{n-1} itself is a scheme with a filtration $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_{n-1}$ as above. So it follows that:

(i) For $k < n$ we have $A_k(\mathbb{A}^n) = 0$, so $A_k(X)$ is generated by $A_k(X_{n-1})$. Hence the claim follows from the induction hypothesis.
(ii) For $k \geq n$ we have $A_k(X_{n-1}) = 0$, so $A_n(X) \cong \oplus_{i=1}^{a_n} A_k(\mathbb{A}^n)$ is generated by the classes of the closures of the a_n copies of $\mathbb{A}^n$ in $X \backslash X_{n-1}$.

This proves the claim. In fact, one can show that $A_k(X)$ is always isomorphic to $\mathbb{Z}^{a_k}$ if X has a stratification by affine spaces as above (see [F] example 1.9.1).

In particular, this construction can be applied to compute the Chow groups of products of projective spaces and Grassmannian varieties (see exercise 3.5.4).

Remark 9.1.19. Using Chow groups, Bézout's theorem can obviously be restated as follows: we have seen in corollary 9.1.17 that $A_k(\mathbb{P}^n) \cong \mathbb{Z}$ for all $k \le n$, with the class of a k-dimensional linear subspace as a generator. Using this isomorphism, define a product map

$$A_{n-k}(\mathbb{P}^n) \times A_{n-l}(\mathbb{P}^n) \to A_{n-k-l}(\mathbb{P}^n), \quad (a,b) \mapsto ab$$

for $k+l \le n$. This "intersection pairing" has the following property: if $X, Y \subset \mathbb{P}^n$ are two subvarieties that intersect in the expected dimension (i. e. $\operatorname{codim}(X \cap Y) = \operatorname{codim} X + \operatorname{codim} Y$) then $[X \cap Y] = [X] \cdot [Y]$. So "intersections of subvarieties can be performed on the level of cycle classes". As we have mentioned in the introduction to this section, the existence of such intersection pairing maps between the Chow groups will generalize to arbitrary smooth varieties. It is one of the most important properties of the Chow groups.

9.2. **Proper push-forward of cycles.** We now want to generalize the push-forward maps of example 9.1.10 to more general morphisms, i. e. given a morphism $f : X \to Y$ of schemes we will study the question under which conditions there are induced push-forward maps $f_* : A_k(X) \to A_k(Y)$ for all k that are (roughly) given by $f_*[V] = [f(V)]$ for a k-dimensional subvariety V of X.

Remark 9.2.1. We have seen already in remark 9.1.12 (ii) that there are no such push-forward maps for the open inclusion $\mathbb{A}^1 \to \mathbb{P}^1$. The reason for this is precisely that the point $P = \mathbb{P}^1 \backslash \mathbb{A}^1$ is "missing" in the domain of the morphism: a rational function on $\mathbb{A}^1$ (which is then also a rational function on $\mathbb{P}^1$) may have a zero and / or pole at the point P which is then present on $\mathbb{P}^1$ but not on $\mathbb{A}^1$. As the class of P is not trivial in the Chow group of $\mathbb{P}^1$, this will change the rational equivalence class. Therefore there is no well-defined push-forward map between the Chow groups.

Another example of a morphism for which there is no push-forward for Chow groups is the trivial morphism $f : \mathbb{A}^1 \to \mathrm{pt}$: again the class of a point is trivial in $A_0(\mathbb{A}^1)$ but not in $A_0(\mathrm{pt})$. In contrast, the morphism $f : \mathbb{P}^1 \to \mathrm{pt}$ admits a well-defined push-forward map $f_* : A_0(\mathbb{P}^1) \cong \mathbb{Z} \to A_0(\mathrm{pt}) \cong \mathbb{Z}$ sending the class of a point in $\mathbb{P}^1$ to the class of a point in pt.

These counterexamples can be generalized by saying that in general there should be no points "missing" in the domain of the morphism $f : X \to Y$ for which we are looking for a push-forward f_*. For example, if Y is the one-pointed space, by "no points missing" we mean exactly that X should be compact (in the classical topology), i. e. complete in the sense of remark 3.4.5. For general Y we need a "relative version" of this compactness (resp. completeness) condition. Morphisms satisfying this condition are called *proper*. We will give both the topological definition (corresponding to "compactness") and the algebraic definition (corresponding to "completeness").

Definition 9.2.2. (Topological definition:) A continuous map $f : X \to Y$ of topological spaces is called **proper** if $f^{-1}(Z)$ is compact for every compact set $Z \subset Y$.

(Algebraic definition:) Let $f : X \to Y$ be a morphism of "nice" schemes (separated, of finite type over a field). For every morphism $g : Z \to Y$ from a third scheme Z form the fiber diagram

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] & Y. \end{tikzcd}$$

The morphism f is said to be **proper** if the induced morphism f' is closed for every such $g : Z \to Y$, i. e. if f' maps closed subsets of $X \times_Y Z$ to closed subsets of Z. This condition is sometimes expressed by saying that f is required to be "universally closed".

Remark 9.2.3. Note that the two definitions look quite different: whereas the topological definition places a condition on *inverse images* of (compact) subsets by some morphism, the algebraic definition places a condition on *images* of (closed) subsets by some morphism. Yet one can show that for varieties over the complex numbers the two definitions agree if we apply the topological definition to the classical (not the Zariski) topology. We will only illustrate this by some examples below. Note however that both definitions are "obvious" generalizations of their absolute versions, i. e. properness of a map in topology is a straightforward generalization of compactness of a space, whereas properness of a morphism in algebraic geometry is the expected generalization of completeness of a variety (see remark 3.4.5). In particular, if $Y = \mathrm{pt}$ is a point then the (trivial) morphism $f : X \to \mathrm{pt}$ is proper if and only if X is complete (resp. compact).

Example 9.2.4. If X is complete (resp. compact) then any morphism $f : X \to Y$ is proper. We will prove this both in the topological and the algebraic setting:

(i) In topology, let $Z \subset Y$ be a compact subset of Y. In particular Z is closed, hence so is the inverse image $f^{-1}(Z)$ as f is continuous. It follows that $f^{-1}(Z)$ is a closed subset of a compact space X, hence compact.

(ii) In algebra, the fiber product $X \times_Y Z$ in definition 9.2.2 is isomorphic to the closed subscheme $p^{-1}(\Delta_Y)$ of $X \times Z$, where $p = (f,g) : X \times Z \to Y \times Y$ and $\Delta_Y \subset Y \times Y$ is the diagonal. So if $V \subset X \times_Y Z$ is any closed subset, then V is also closed in $X \times Z$, and hence its image in Z is closed as X is complete.

This is the easiest criterion to determine that a morphism is proper. Some more can be found in exercise 9.5.5.

Example 9.2.5. Let $U \subset X$ be a non-empty open subset of a (connected) scheme X. Then the inclusion morphism $i : U \to X$ is not proper. This is obvious for the algebraic definition, as i is not even closed itself (it maps the closed subset $U \subset U$ to the non-closed subset $U \subset X$). In the topological definition, let $Z \subset X$ be a small closed disc around a point $P \in X \backslash U$. Its inverse image $i^{-1}(Z) = Z \cap U$ is Z minus a closed non-empty subset, so it is not compact.

Example 9.2.6. If $f : X \to Y$ is proper then every fiber $f^{-1}(P)$ is complete (resp. compact). Again this is obvious for the topological definition, as $\{P\} \subset Y$ is compact. In the algebraic definition let $P \in Y$ be a point, let Z be any scheme, and form the fiber diagram

$$\begin{tikzcd}
Z \times f^{-1}(P) \arrow[r] \arrow[d, "f'"] & f^{-1}(P) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
Z \arrow[r] & P \arrow[r] & Y.
\end{tikzcd}$$

If f is proper then by definition the morphism f' is closed for all choices of P and Z. By definition this means exactly that all fibers $f^{-1}(P)$ of f are complete.

The converse is not true however: every fiber of the morphism $\mathbb{A}^1 \to \mathbb{P}^1$ is complete (resp. compact), but the morphism is not proper.

Remark 9.2.7. It turns out that the condition of properness of a morphism $f : X \to Y$ is enough to guarantee the existence of well-defined push-forward maps $f_* : A_k(X) \to A_k(Y)$. To construct them rigorously however we have to elaborate further on our idea that f_* should map any k-dimensional cycle $[V]$ to $[f(V)]$, as the following two complications can occur:

(i) The image $f(V)$ of V may have dimension smaller than k, so that $f(V)$ does not define a k-dimensional cycle. It turns out that we can consistently define $f_*[V]$ to be zero in this case.
(ii) It may happen that $\dim f(V) = \dim V$ and the morphism f is a *multiple covering map*, i. e. that a general point in $f(V)$ has $d > 1$ inverse image points. In this case the image $f(V)$ is "covered d times by V", so we would expect that we have to set $f_*[V] = d \cdot [f(V)]$. Let us define this "order of the covering" d rigorously:

Proposition 9.2.8. *Let $f : X \to Y$ be a morphism of varieties of the same dimension such that $f(X)$ is dense in Y. Then:*

(i) *$K(X)$ is a finite-dimensional vector space over $K(Y)$. Its dimension is called the **degree** of the morphism f, denoted* $\deg f$. *(One also says that $K(X) : K(Y)$ is a **field extension** of dimension $[K(X) : K(Y)] = \deg f$.)*
(ii) *The degree of f is equal to the number of points in a general fiber of f. (This means: there is a non-empty open set $U \subset Y$ such that the fibers of f over U consist of exactly* $\deg f$ *points.)*
(iii) *If moreover f is proper then* every zero-dimensional *fiber of f consists of exactly* $\deg f$ *points if the points are counted with their scheme-theoretic multiplicities.*

Proof. (i): We begin with a few reduction steps. As the fields of rational functions do not change when we pass to an open subset, we can assume that $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are affine. Next, we factor the morphism $f : X \to Y$ as $f = \pi \circ \gamma$ with $\gamma : X \to \Gamma \subset X \times Y$ the graph morphism and $\pi : X \times Y \to Y$ the projection. As γ is an isomorphism it is sufficient to show the statement of the proposition for the projection map π. Finally, we can factor the projection π (which is the restriction of the obvious projection map $\mathbb{A}^{n+m} \to \mathbb{A}^m$ to $X \times Y$) into n projections that are given by dropping one coordinate at a time. Hence we can assume that $X \subset \mathbb{A}^{n+1}$ and $Y \subset \mathbb{A}^n$, and prove the statement for the map $\pi : X \to Y$ that is the restriction of the projection map $(x_0, \dots, x_n) \mapsto (x_1, \dots, x_n)$ to X.

In this case the field $K(X)$ is generated over $K(Y)$ by the single element x_0. Assume that $x_0 \in K(X)$ is transcendental over $K(Y)$, i. e. there is no polynomial relation of the form

$$F_d x_0^d + F_{d-1} x_0^{d-1} \cdots + F_0 = 0, \qquad (*)$$

for $F_i \in K(Y)$ and $F_d \neq 0$. Then for every choice of $(x_1, \dots, x_n) \in Y$ the value of x_0 in X is not restricted, i. e. the general fiber of f is not finite. But then $\dim X > \dim Y$ in contradiction to our assumption. So $x_0 \in K(X)$ is algebraic over $K(Y)$, i. e. there is a relation $(*)$. It follows that $K(X)$ is a vector space over $K(Y)$ with basis $\{1, x_0, \dots, x_0^{d-1}\}$.

(ii): Continuing the proof of (i), note that on the non-empty open subset of Y where all F_i are regular and F_d is non-zero every point in the target has exactly d inverse image points (counted with multiplicity). Restricting the open subset further to the open subset where the discriminant of the polynomial $(*)$ is non-zero, we can in fact show that there is an open subset of Y on which the inverse images of f consist set-theoretically of exactly d points that all count with multiplicity 1.

(iii): We will only sketch this part, using the topological definition of properness. By (ii) there is an open subset $U \subset Y$ on which all fibers of f consist of exactly n points. Let $P \in Y$ be any point, and choose a small closed disc $\Delta \subset U \cup \{P\}$ around P. If Δ is small enough then the inverse image $f^{-1}(\Delta \backslash \{P\})$ will be a union of d copies of $\Delta \backslash \{P\}$. As f is proper, the inverse image $f^{-1}(\Delta)$ has to be compact, i. e. all the holes in the d copies of $\Delta \backslash \{P\}$ have to be filled in by inverse image points of P. So the fiber $f^{-1}(P)$ must contain at least d points (counted with multiplicities). But we see from $(*)$ above that every fiber contains at most d points unless it is infinite (i. e. all F_i are zero at P). This shows part (iii). $\square$

We are now ready to construct the push-forward maps $f_* : A_k(X) \to A_k(Y)$ for proper morphisms $f : X \to Y$.

Construction 9.2.9. Let $f : X \to Y$ be a proper morphism of schemes. Then for every subvariety $Z \subset X$ the image $f(Z)$ is a closed subvariety of dimension at most $\dim Z$. On the cycle level we define homomorphisms $f_* : Z_k(X) \to Z_k(Y)$ by

$$f_*[Z] = \begin{cases} [K(Z) : K(f(Z))] \cdot [f(Z)] & \text{if } \dim f(Z) = \dim Z, \\ 0 & \text{if } \dim f(Z) < \dim Z. \end{cases}$$

By proposition 9.2.8 this is well-defined and corresponds to the ideas mentioned in remark 9.2.7.

Remark 9.2.10. By the multiplicativity of degrees of field extensions it follows that the push-forwards are functorial, i. e. $(g \circ f)_* = g_* f_*$ for any two morphisms $f : X \to Y$ and $g : Y \to Z$.

Of course we want to show that these homomorphisms pass to the Chow groups, i. e. give rise to well-defined homomorphisms $f_* : A_k(X) \to A_k(Y)$. For this we have to show by definition that divisors of rational functions are pushed forward to divisors of rational functions.

Theorem 9.2.11. *Let $f : X \to Y$ be a proper surjective morphism of varieties, and let $\varphi \in K(X)^*$ be a non-zero rational function on X. Then*

$$f_* \operatorname{div}(\varphi) = \begin{cases} 0 & \textit{if } \dim Y < \dim X \\ \operatorname{div}(N(\varphi)) & \textit{if } \dim Y = \dim X \end{cases}$$

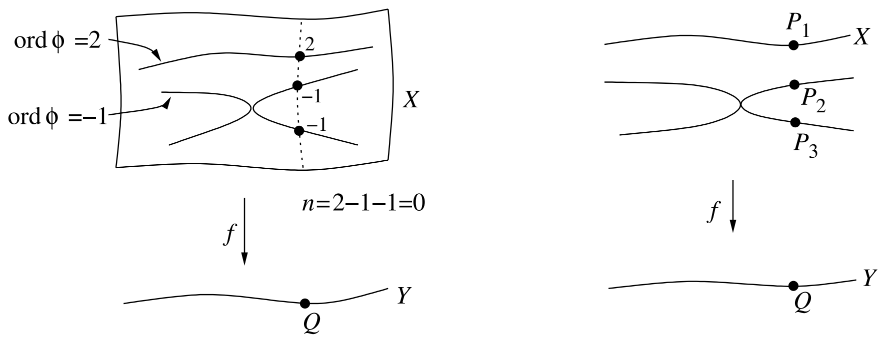
*in $Z_*(Y)$, where $N(\varphi) \in K(Y)$ denotes the determinant of the endomorphism of the $K(Y)$-vector space $K(X)$ given by multiplication by φ (this is usually called the **norm** of φ).*

Proof. The complete proof of the theorem with all algebraic details is beyond the scope of these notes; it can be found in [F] proposition 1.4. We will only sketch the idea of the proof here.

Case 1: $\dim Y < \dim X$ (see the picture below). We can assume that $\dim Y = \dim X - 1$, as otherwise the statement is trivial for dimensional reasons. Note that we must have $f_* \operatorname{div}(\varphi) = n \cdot [Y]$ for some $n \in \mathbb{Z}$ by example 9.1.5. So it only remains to determine the number n. By our interpretation of remark 9.2.7 (ii) we can compute this number on a general fiber of f by counting all points in this fiber with the multiplicity with which they occur in the restriction of φ to this fiber. In other words, we have $n = \sum_{P : f(P) = Q} \operatorname{ord}_P(\varphi|_{f^{-1}(Q)})$ for any point $Q \in Y$ over which the fiber of f is finite. But this number is precisely the degree of $\varphi|_{f^{-1}(Q)}$ on the complete curve $f^{-1}(Q)$, which must be zero. (Strictly speaking we have only shown this for smooth projective curves in remark 6.3.5, but it is true in the general case as well. The important ingredient is here that the fiber is complete.)

Case 2: $\dim Y = \dim X$ (see the picture below). We will restrict ourselves here to showing the stated equation *set-theoretically*, i. e. we will assume that φ is (locally around a fiber) a regular function and show that $f(Z(\varphi)) = Z(N(\varphi))$, where $Z(\cdot)$ denotes as usual the zero locus of a function.

Note first that we can neglect the fibers of f that are not finite: these fibers can only lie over a subset of Y of codimension at least 2 (otherwise the non-zero-dimensional fibers would form a component of X for dimensional reasons, in contrast to X being irreducible). So as $f_* \operatorname{div}(\varphi)$ is a cycle of codimension 1 in Y these higher-dimensional fibers cannot contribute to the push-forward.



Case 1 Case 2

Now let $Q \in Y$ be any point such that the fiber $f^{-1}(Q)$ is finite. Then $f^{-1}(Q)$ consists of exactly $d = [K(X) : K(Y)]$ points by proposition 9.2.8 (iii). Let us assume for simplicity that all these points are distinct (although this is not essential), so $f^{-1}(Q) = \{P_1, \dots, P_d\}$. The space of functions on this fiber is then just k^d, corresponding to the value at the d points. In this basis, the restriction of the function φ to this fiber is then obviously given by the diagonal matrix with entries $\varphi(P_1), \dots, \varphi(P_d)$, so its determinant is $N(\varphi)(Q) = \prod_{i=1}^d \varphi(P_i)$. Now it is clear that

$$\begin{aligned} Q \in f(Z(\varphi)) &\iff \text{there is a } P_i \text{ over } Q \text{ with } \varphi(P_i) = 0 \\ &\iff Q \in Z(N(\varphi)). \end{aligned}$$

We can actually see the multiplicities arising as well: if there are k points among the P_i where φ vanishes, then the diagonal matrix $\varphi|_{f^1(Q)}$ contains k zeros on the diagonal, hence its determinant is a product that contains k zeros, so it should give rise to a zero of order k, in accordance with our interpretation of remark 9.2.7 (ii). □

Corollary 9.2.12. *Let $f : X \to Y$ be a proper morphism of schemes. Then there are well-defined push-forward maps $f_* : A_k(X) \to A_k(Y)$ for all $k \geq 0$ given by the definition of construction 9.2.9.*

Proof. This follows immediately from theorem 9.2.11 applied to the morphism from a $(k+1)$-dimensional subvariety of X to its image in Y. □

Example 9.2.13. Let X be a complete scheme, and let $f : X \to \mathrm{pt}$ be the natural (proper) map. For any 0-dimensional cycle class $\alpha \in A_0(X)$ we define the **degree** of α to be the integer $f_*\alpha \in A_0(\mathrm{pt}) \cong \mathbb{Z}$. This is well-defined by corollary 9.2.12. More explicitly, if $\alpha = \sum_i n_i[P_i]$ for some points $P_i \in X$ then $\deg \alpha = \sum_i n_i$.

Example 9.2.14. Let $X = \tilde{\mathbb{P}}^2$ be the blow-up of $\mathbb{P}^2$ with coordinates $(x_0 : x_1 : x_2)$ in the point $P = (1 : 0 : 0)$, and denote by $E \subset X$ the exceptional hypersurface. In this example we will compute the Chow groups of X using remark 9.1.18.

Note that $\mathbb{P}^2$ has a stratification by affine spaces as $\mathbb{A}^2 \cup \mathbb{A}^1 \cup \mathbb{A}^0$. Identifying $\mathbb{A}^0$ with P and recalling that the blow-up $\tilde{\mathbb{P}}^2$ is obtained from $\mathbb{P}^2$ by "replacing the point P with a line $\mathbb{P}^1$" we see that X has a stratification $\mathbb{A}^2 \cup \mathbb{A}^1 \cup \mathbb{A}^1 \cup \mathbb{A}^0$. By remark 9.1.18 it follows that the closures of these four strata generate $A_*(X)$. More precisely, these four classes are $[X] \in A_2(X)$, $[L] \in A_1(X)$ where L is the strict transform of a line in $\mathbb{P}^2$ through P, the exceptional hypersurface $[E] \in A_1(X)$, and the class of a point in $A_0(X)$. It follows immediately that $A_2(X) \cong \mathbb{Z}$ and $A_0(X) \cong \mathbb{Z}$. Moreover we see that $A_1(X)$ is generated by $[L]$ and $[E]$.

We have already stated without proof in remark 9.1.18 that $[L]$ and $[E]$ form in fact a basis of $A_1(X)$. Let us now prove this in our special case at hand. So assume that there is a relation $n[L] + m[E] = 0$ in $A_1(X)$. Consider the following two morphisms:

(i) Let $\pi : X \to \mathbb{P}^2$ be the projection to the base of the blow-up. This is a proper map, and we have $\pi_*[L] = [H]$ and $\pi_*[E] = 0$ where $[H] \in A_1(\mathbb{P}^2)$ is the class of a line. So we see that

$$0 = \pi_*(0) = \pi_*(n[L] + m[E]) = n[H] \in A_1(\mathbb{P}^2),$$

from which we conclude that $n = 0$.

(ii) Now let $p : X \to \mathbb{P}^1$ be the morphism that is the identity on E, and sends every point $Q \in X \backslash E$ to the unique intersection point of E with the strict transform of the line through P and Q. Again this is a proper map, and we have $p_*[L] = 0$ and $p_*[E] = [\mathbb{P}^1]$. So again we see that

$$0 = p_*(0) = p_*(n[L] + m[E]) = m[\mathbb{P}^1] \in A_1(\mathbb{P}^1),$$

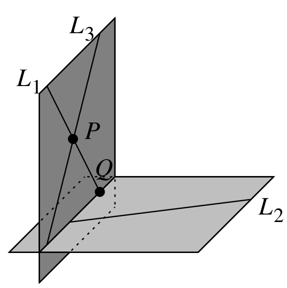
from which we conclude that $m = 0$ as well.

Combining both parts we see that there is no non-trivial relation of the form $n[L] + m[E] = 0$ in $A_1(X)$.

Now let $[H]$ be the class of a line in X that does not intersect the exceptional hypersurface. We have just shown that $[H]$ must be a linear combination of $[L]$ and $[E]$. To compute which one it is, consider the rational function $\frac{x_1}{x_0}$ on X. It has simple zeros along L and E, and a simple pole along H (with coordinates for L and H chosen appropriately). So we conclude that $[H] = [L] + [E]$ in $A_1(X)$.

9.3. **Weil and Cartier divisors.** Our next goal is to describe intersections on the level of Chow groups as motivated in the beginning of section 9.1. We will start with the easiest case, namely with the intersection of a variety with a subset of codimension 1. To put it more precisely, given a subvariety $V \subset X$ of dimension k and another one $D \subset X$ of codimension 1, we want to construct an *intersection cycle* $[V] \cdot [D] \in A_{k-1}(X)$ with the property that $[V] \cdot [D] = [V \cap D]$ if this intersection $V \cap D$ actually has dimension $k-1$. Of course these intersection cycles should be well-defined on the Chow groups, i. e. the product cycle $[V] \cdot [D] \in A_{k-1}(X)$ should only depend on the classes of V and D in $A_*(X)$.

Example 9.3.1. Here is an example showing that this is too much to hope for in the generality as we stated it. Let $X = \mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2$ be the union of two projective planes glued together along a common line. Let $L_1, L_2, L_3 \subset X$ be the lines as in the following picture.



Their classes in $A_1(X)$ are all the same since $A_1(X) \cong \mathbb{Z}$ by remark 9.1.18. But note that $L_1 \cap L_2$ is empty, whereas $L_1 \cap L_3$ is a single point P. But $0 \neq [P] \in A_0(X)$, so there can be no well-defined product map $A_1(X) \times A_1(X) \to A_0(X)$ that describes intersections on this space X.

The reason why this construction failed is quite a subtle one: we have to distinguish between codimension-1 subspaces and spaces that can locally be written as the zero locus of a single function. In general the intersection product exists only for intersections with spaces that are locally the zero locus of a single function. For most spaces this is the same thing as codimension-1 subspaces, but notably not in example 9.3.1 above: neither of the three lines L_i can be written as the zero locus of a single function *on* X: there is a (linear) function on the vertical $\mathbb{P}^2$ that vanishes precisely on L_1, but we cannot extend it to a function on all of X that vanishes at the point Q but nowhere else on the horizontal $\mathbb{P}^2$. (We can write the L_i as the zero locus of a single function *on a component of* X, but this is not what we need.)

So for intersection-theoretic purposes we have to make a clear distinction between codimension-1 subspaces and spaces that are locally the zero locus of a single function. Let us make the corresponding definitions.

Definition 9.3.2. Let X be a scheme.

(i) If X has pure dimension n a **Weil divisor** on X is an element of $Z_{n-1}(X)$. Obviously, the Weil divisors form an Abelian group. Two Weil divisors are called **linearly equivalent** if they define the same class in $A_{n-1}(X)$. The quotient group $A_{n-1}(X)$ is called the group of **Weil divisor classes**.

(ii) Let $\mathcal{K}_X$ be the sheaf of rational functions on X, and denote by $\mathcal{K}_X^*$ the subsheaf of invertible elements (i. e. of those functions that are not identically zero on any component of X). Note that $\mathcal{K}_X^*$ is a sheaf of Abelian groups, with the group structure given by multiplication of rational functions. Similarly, let $\mathcal{O}_X^*$ be the sheaf of invertible elements of $\mathcal{O}_X$ (i. e. of the regular functions that are nowhere zero). Note that $\mathcal{O}_X^*$ is a sheaf of Abelian groups under multiplication as well. In fact, $\mathcal{O}_X^*$ is a subsheaf of $\mathcal{K}_X^*$.

A **Cartier divisor** on X is a global section of the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$. Obviously, the Cartier divisors form an Abelian group under multiplication, denoted $\operatorname{Div} X$. In analogy to Weil divisors the group structure on $\operatorname{Div} X$ is usually written additively however. A Cartier divisor is called linearly equivalent to zero if it is induced by a global section of $\mathcal{K}_X^*$. Two Cartier divisors are **linearly equivalent** if their difference (i. e. quotient, see above) is linearly equivalent to zero. The quotient group $\operatorname{Pic} X := \Gamma(\mathcal{K}_X^*/\mathcal{O}_X^*)/\Gamma(\mathcal{K}_X^*)$ is called the group of **Cartier divisor classes**.

Remark 9.3.3. Let us analyze the definition of Cartier divisors. There is an obvious exact sequence of sheaves on X

$$0 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^*/\mathcal{O}_X^* \to 0.$$

Note that these are *not* sheaves of $\mathcal{O}_X$-modules, so their flavor is slightly different from the ones we have considered so far. But it is still true that we get an exact sequence of global sections

$$0 \to \Gamma(\mathcal{O}_X^*) \to \Gamma(\mathcal{K}_X^*) \to \Gamma(\mathcal{K}_X^*/\mathcal{O}_X^*)$$

that is in general not exact on the right. More precisely, recall that the quotient sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$ is *not* just the sheaf that is $\mathcal{K}_X^*(U)/\mathcal{O}_X^*(U)$ for all open subsets $U \subset X$, but rather the sheaf associated to this presheaf. Therefore $\Gamma(\mathcal{K}_X^*/\mathcal{O}_X^*)$ is in general not just the quotient $\Gamma(\mathcal{K}_X^*)/\Gamma(\mathcal{O}_X^*)$.

To unwind the definition of sheafification, an element of $\operatorname{Div} X = \Gamma(\mathcal{K}_X^*/\mathcal{O}_X^*)$ can be given by a (sufficiently fine) open covering $\{U_i\}$ and elements of $\mathcal{K}_X^*(U_i)/\mathcal{O}_X^*(U_i)$ represented by rational functions φ_i for all i such that their quotients $\frac{\varphi_i}{\varphi_j}$ are in $\mathcal{O}_X^*(U_i \cap U_j)$ for all i, j. So a Cartier divisor is an object that is locally a (non-zero) rational function modulo a nowhere-zero regular function. Intuitively speaking, the only data left from a rational function if we mod out locally by nowhere-zero regular functions is the locus of its zeros and poles together with their multiplicities. So one can think of Cartier divisors as objects that are (linear combinations of) zero loci of functions.

A Cartier divisor is linearly equivalent to zero if it is *globally* a rational function, just the same as for Weil divisors. From cohomology one would expect that one can think of the quotient group $\operatorname{Pic} X$ as the cohomology group $H^1(X, \mathcal{O}_X^*)$. We cannot say this rigorously because we have only defined cohomology for quasi-coherent sheaves (which $\mathcal{O}_X^*$ is not).

But there is a more general theory of cohomology of arbitrary sheaves of Abelian groups on schemes, and in this theory the statement that $\operatorname{Pic} X = H^1(X, \mathcal{O}_X^*)$ is correct.

Lemma 9.3.4. *Let X be a purely n-dimensional scheme. Then there is a natural homomorphism* $\operatorname{Div} X \to Z_{n-1}(X)$ *that passes to linear equivalence to give a homomorphism* $\operatorname{Pic} X \to A_{n-1}(X)$. *In other words, every Cartier divisor (class) determines a Weil divisor (class).*

Proof. Let $D \in \operatorname{Div} X$ be a Cartier divisor on X, represented by an open covering $\{U_i\}$ of X and rational functions φ_i on U_i. For any $(n-1)$-dimensional subvariety V of X define the order of D at V to be $\operatorname{ord}_V D := \operatorname{ord}_{V \cap U_i} \varphi_i$, where i is an index such that $U_i \cap V \neq \emptyset$. This does not depend on the choice of i as the quotients $\frac{\varphi_i}{\varphi_j}$ are nowhere-zero regular functions, so the orders of φ_i and φ_j are the same where they are both defined. So we get a well-defined map $\operatorname{Div} X \to Z_{n-1}(X)$ defined by $D \mapsto \sum_V \operatorname{ord}_V D \cdot [V]$. It is obviously a homomorphism as $\operatorname{ord}_V(\varphi_i \cdot \varphi_i') = \operatorname{ord}_V \varphi_i + \operatorname{ord}_V \varphi_i'$.

It is clear from the definition that a Cartier divisor that is linearly equivalent to zero, i. e. a global rational function, determines a Weil divisor in $B_{n-1}(X)$. Hence the homomorphism passes to linear equivalence. □

Lemma 9.3.5. *Let X be a smooth projective curve. Then Cartier divisors (resp. Cartier divisor classes) on X are the same as Weil divisors (resp. Weil divisor classes). In particular, our definition 9.3.2 (ii) of* $\operatorname{Div} X$ *and* $\operatorname{Pic} X$ *agrees with our earlier one from section 6.3.*

Proof. The idea of the proof is lemma 7.5.6 which tells us that every point of X is locally the scheme-theoretic zero locus of a single function, hence a Cartier divisor.

To be more precise, let $\sum_{i=1}^n a_i P_i \in Z_0(X)$ be a Weil divisor. We will construct a Cartier divisor $D \in \operatorname{Div} X$ that maps to the given Weil divisor under the correspondence of lemma 9.3.4. To do so, pick an open neighborhood U_i of P_i for all $i = 1, \dots, n$ such that

(i) $P_j \notin U_i$ for $j \neq i$, and
(ii) there is a function φ_{P_i} on U_i such that $\operatorname{div} \varphi_{P_i} = 1 \cdot P_i$ on U_i (see lemma 7.5.6).

Moreover, set $U = X \backslash \{P_1, \dots, P_n\}$. Then we define a Cartier divisor D by the open cover $\{U, U_1, \dots, U_n\}$ and the rational functions

(i) 1 on U,
(ii) $\varphi_{P_i}^{a_i}$ on U_i.

Note that these data define a Cartier divisor: no intersection of two elements of the open cover contains one of the points P_i, and the functions given on the elements of the open cover are regular and non-vanishing away from the P_i. By construction, the Weil divisor associated to D is precisely $\sum_{i=1}^n a_i P_i$, as desired. □

Example 9.3.6. In general, the map from Cartier divisors (resp. Cartier divisor classes) to Weil divisors (resp. Cartier divisor classes) is neither injective nor surjective. Here are examples of this:

(i) not injective: This is essentially example 9.1.7. Let $X = X_1 \cup X_2$ be the union of two lines $X_i \cong \mathbb{P}^1$ glued together at a point $P \in X_1 \cap X_2$. Let Q be a point on $X_1 \backslash X_2$. Consider the open cover $X = U \cup V$ with $U = X \backslash Q$ and $V = X_1 \backslash P$.

   We define a Cartier divisor D on X by choosing the following rational functions on U and V: the constant function 1 on U, and the linear function on $V \cong \mathbb{A}^1$ that has a simple zero at Q. Note that the quotient of these two functions is regular and nowhere zero on $U \cap V$, so D is well-defined. Its associated Weil divisor $[D]$ is $[Q]$.

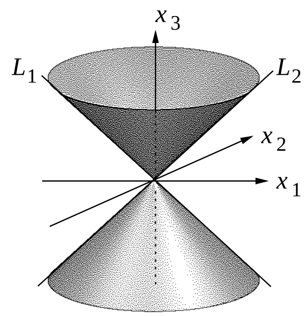
By symmetry, we can construct a similar Cartier divisor D' whose associated Weil divisor is the class of a point $Q' \in X_2 \backslash X_1$.

Now note that the Cartier divisor classes of D and D' are different (because $D - D'$ is not the divisor of a rational function), but their associated Weil divisors $[Q]$ and $[Q']$ are the same by example 9.1.7.

(ii) not surjective: This is essentially example 9.3.1. The classes $[L_i]$ of this example are Weil divisors but not Cartier divisors.

Another example on an irreducible space X is the cone

$$X = \{x_3^2 = x_1^2 + x_2^2\} \subset \mathbb{P}^3.$$



Let $L_1 = Z(x_2, x_1 + x_3)$ and $L_2 = Z(x_2, x_1 - x_3)$ be the two lines as in the picture. We claim that there is no Cartier divisor on X corresponding to the Weil divisor $[L_1]$. In fact, if there was such a Cartier divisor, defined locally around the origin by a function φ, we must have an equality of ideals

$$(x_1^2 + x_2^2 - x_3^2, \varphi) = (x_2, x_1 + x_3)$$

in the local ring $\mathscr{O}_{\mathbb{P}^3,0}$. This is impossible since the right ideal contains two linearly independent linear parts, whereas the left ideal contains only one. But note that the section x_2 of the line bundle $\mathscr{O}_X(1)$ defines a Cartier divisor $\operatorname{div}(x_2)$ on X whose associated Weil divisor is $[L_1] + [L_2]$, and the section $x_1 + x_3$ defines a Cartier divisor whose associated Weil divisor is $2[L_1]$. So $[L_1]$ and $[L_2]$ are not Cartier divisors, whereas $[L_1] + [L_2]$, $2[L_1]$, and $2[L_2]$ are. In particular, there is in general no "decomposition of a Cartier divisor into its irreducible components" as we have it by definition for Weil divisors.

There is quite a deep theorem however that the two notions agree on smooth schemes:

Theorem 9.3.7. *Let X be a smooth n-dimensional scheme. Then* $\operatorname{Div} X \cong Z_{n-1}(X)$ *and* $\operatorname{Pic} X \cong A_{n-1}(X)$.

Proof. We cannot prove this here and refer to [H] remark II.6.11.1.A for details. One has to prove the analogue of lemma 7.5.6, i. e. that every codimension-1 subvariety of X is locally the scheme-theoretic zero locus of a single function. This is a commutative algebra statement as it can be shown on the local ring of X at the subvariety.

(To be a little more precise, the property of X that we need is that its local rings are unique factorization domains: if this is the case and $V \subset X$ is an subvariety of codimension 1, pick any non-zero (local) function $f \in \mathscr{O}_{X,V}$ that vanishes on V. As $\mathscr{O}_{X,V}$ is a unique factorization domain we can decompose f into its irreducible factors $f = f_1 \cdots f_n$. Of course one of the f_i has to vanish on V. But as f_i is irreducible, its ideal must be the ideal of V, so V is locally the zero locus of a single function. The problem with this is that it is almost impossible to check that a ring (that one does not know very well) is a unique factorization domain. So one uses the result from commutative algebra that every *regular* local ring (i. e. "the local ring of a scheme at a *smooth* point") is a unique factorization domain. Actually, we can see from the above argument that it is enough that X is "smooth in codimension 1", i. e. that its set of singular points has codimension at least 2 — or to express it algebraically, that its local rings $\mathcal{O}_{X,V}$ at codimension-1 subvarieties V are regular.) □

Example 9.3.8. Finally let us discuss the relation between divisors and line bundles as observed for curves in section 7.5. Note that we have in fact used such a correspondence already in example 9.3.6 where we defined a Cartier divisor by giving a section of a line bundle. The precise relation between line bundles and Cartier divisors is as follows.

Lemma 9.3.9. *For any scheme X there are one-to-one correspondences*

$$\{\text{Cartier divisors on } X\} \leftrightarrow \{(\mathcal{L},s)\ ;\ \mathcal{L} \text{ a line bundle on } X \text{ and } s \text{ a rational section of } \mathcal{L}\}$$

and

$$\{\text{Cartier divisor classes on } X\} \leftrightarrow \{\text{line bundles on } X \text{ that admit a rational section}\}.$$

Proof. The proof of this is essentially the same as the correspondence between divisor classes and line bundles on a smooth projective curve in proposition 7.5.9. Given a Cartier divisor $D = \{(U_i, \varphi_i)\}$ on X, we get an associated line bundle $\mathcal{O}(D)$ by taking the subsheaf of $\mathcal{O}_X$-modules of $\mathcal{K}_X$ generated by the functions $\frac{1}{\varphi_i}$ on U_i. Conversely, given a line bundle with a rational section, this section immediately defines a Cartier divisor. The proof that the same correspondence holds for divisor classes is the same as in proposition 7.5.9. □

Remark 9.3.10. We should note that almost any line bundle on any scheme X admits a rational section. In fact, this is certainly true for irreducible X (as the line bundle is then isomorphic to the structure sheaf on a dense open subset of X by definition), and one can show that it is true in most other cases as well (see [H] remark 6.14.1 for more information). Most books actually *define* the group $\operatorname{Pic} X$ to be the group of line bundles on X.

Summarizing our above discussions we get the following commutative diagram:

$$\begin{tikzcd}
& \text{line bundles} \arrow[dl, "\text{together with a rational section}"'] \\
\text{Cartier divisors } \operatorname{Div} X \arrow[r] \arrow[d] & \text{Cartier divisor classes } \operatorname{Pic} X \arrow[u] \arrow[d] \\
\text{Weil divisors } Z_{n-1}(X) \arrow[r] & \text{Weil divisor classes } A_{n-1}(X)
\end{tikzcd}$$

where

(i) the bottom row (the Weil divisors) exists only if X is purely n-dimensional,
(ii) the upper right vertical arrow is an isomorphism in most cases, at least if X is irreducible,
(iii) the lower vertical arrows are isomorphisms at least if X is smooth (in codimension 1).

Remark 9.3.11. Although line bundles, Cartier divisor classes, and Weil divisor classes are very much related and even all the same thing on many schemes (e. g. smooth varieties), note that their "functorial properties" are quite different: if $f: X \to Y$ is a morphism then for line bundles and Cartier divisors the pull-back f^* is the natural operation, whereas for Weil divisors (i. e. elements of the Chow groups) the push-forward f_* as in section 9.2 is more natural. In algebraic topology this can be expressed by saying that Weil divisors correspond to *homology* cycles, whereas Cartier divisors correspond to *cohomology* cycles. On nice spaces this is the same by Poincaré duality, but this is a non-trivial statement. The natural operation for homology (resp. cohomology) is the push-forward (resp. pull-back). Intersection products are defined between a cohomology and a homology class, yielding a homology class. This corresponds to our initial statement of this section that intersection products of Chow cycles ("homology classes") with divisors will usually only be well-defined with Cartier divisors ("cohomology classes") and not with Weil divisors.

9.4. **Intersections with Cartier divisors.** We are now ready to define intersection products of Chow cycles with Cartier divisors, as motivated in the beginning of section 9.3. Let us give the definition first, and then discuss some of its features.

Definition 9.4.1. Let X be a scheme, let $V \subset X$ be a k-dimensional subvariety with inclusion morphism $i : V \to X$, and let D be a Cartier divisor on X. We define the **intersection product** $D \cdot V \in A_{k-1}(X)$ to be

$$D \cdot V = i_*[i^* O_X(D)],$$

where $O_X(D)$ is the line bundle on X associated to the Cartier divisor D by lemma 9.3.9, i^* denotes the pull-back of line bundles, $[i^* O_X(D)]$ is the Weil divisor class associated to the line bundle $i^* O_X(D)$ by remark 9.3.10 (note that V is irreducible), and i_* denotes the proper push-forward of corollary 9.2.12.

Note that by definition the intersection product depends only on the divisor class of D, not on D itself. So using our definition we can construct bilinear intersection products

$$\operatorname{Pic} X \times Z_k(X) \to A_{k-1}(X), \quad \big(D, \sum a_i[V_i]\big) \mapsto \sum a_i(D \cdot V_i).$$

If X is smooth and pure-dimensional (so that Weil and Cartier divisors agree) and W is a codimension-1 subvariety of X, we denote by $W \cdot V \in A_{k-1}(X)$ the intersection product $D \cdot V$, where D is the Cartier divisor corresponding to the Weil divisor $[W]$.

Example 9.4.2. Let X be a smooth n-dimensional scheme, and let V and W be subvarieties of dimensions k and $n-1$, respectively. If $V \not\subset W$, i. e. if $\dim(W \cap V) = k-1$, then the intersection product $W \cdot V$ is just the cycle $[W \cap V]$ with possibly some scheme-theoretic multiplicities. In fact, in this case the Weil divisor $[W]$ corresponds by remark 9.3.10 to a line bundle $O_X(W)$ together with a section f whose zero locus is precisely W. By definition of the intersection product we have to pull back this line bundle to V, i. e. restrict the section f to V. The cycle $W \cdot V$ is then the zero locus of $f|_V$, with possibly scheme-theoretic multiplicities if f vanishes along V with higher order.

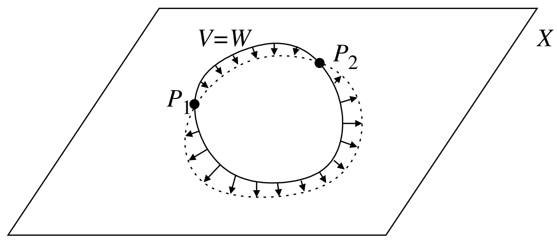
As a concrete example, let C_1 and C_2 be two curves in $\mathbb{P}^2$ of degrees d_1 and d_2, respectively, that intersect in finitely many points $P_1, \dots, P_n$. Then the intersection product $C_1 \cdot C_2 \in A_0(\mathbb{P}^2)$ is just $\sum_i a_i[P_i]$, where a_i is the scheme-theoretic multiplicity of the point P_i in the intersection scheme $C_1 \cap C_2$. Using that all points in $\mathbb{P}^2$ are rationally equivalent, i. e. that $A_0(\mathbb{P}^2) \cong \mathbb{Z}$ is generated by the class of any point, we see that $C_1 \cdot C_2$ is just the Bézout number $d_1 \cdot d_2$.

Example 9.4.3. Again let X be a smooth n-dimensional scheme, and let V and W be subvarieties of dimensions k and $n-1$, respectively. This time let us assume that $V \subset W$, so that the intersection $W \cap V = V$ has dimension k and thus does not define a $(k-1)$-dimensional cycle. There are two ways to interpret the intersection product $W \cdot V$ in this case:

(i) Recall that the intersection product $W \cdot V$ depends only on the divisor *class* of W, not on W itself. So if we can replace W by a linearly equivalent divisor W' such that $V \not\subset W'$ then the intersection product $W \cdot V$ is just $W' \cdot V$ which can now be constructed as in example 9.4.2. For example, let $H \subset \mathbb{P}^2$ be a line and assume that we want to compute the intersection product $H \cdot H \in A_0(\mathbb{P}^2) \cong \mathbb{Z}$. The intersection $H \cap H$ has dimension 1, but we can move the first H to a different line H' which is linearly equivalent to H. So we see that $H \cdot H = H' \cdot H = 1$, as

$H' \cap H$ is just one point. Note however that it may not always be possible to find such a linearly equivalent divisor that makes the intersection have the expected dimension.

(ii) If the strategy of (i) does not work or one does not want to apply it, there is also a different description of the intersection product for which no moving of W is necessary. Let us assume for simplicity that W is smooth. By the analogue of remark 7.4.17 for general hypersurfaces the bundle $i^* O_X(W)$ (where $i : V \to X$ is the inclusion morphism) is precisely the restriction to V of the normal bundle $N_{W/X}$ of W in X. By definition 9.4.1 the intersection product $W \cdot V$ is then the Weil divisor associated to this bundle, i. e. the locus of zeros minus poles of a rational section of the normal bundle $N_{W/X}$ restricted to V.



$W \cdot V = [P_1] + [P_2]$

Note that we can consider this procedure as an infinitesimal version of (i): the section of the normal bundle describes an "infinitesimal deformation" of W in X, and the deformed W meets V precisely in the locus where the section vanishes.

Proposition 9.4.4. *(Commutativity of the intersection product) Let X be an n-dimensional variety, and let D_1, D_2 be Cartier divisors on X with associated Weil divisors $[D_1], [D_2]$. Then $D_1 \cdot [D_2] = D_2 \cdot [D_1] \in A_{n-2}(X)$.*

Proof. We will only sketch the proof in two easy cases (that cover most applications however). For the general proof we refer to [F] theorem 2.4.

Case 1: D_1 and D_2 intersect in the expected dimension, i. e. the locus where the defining equations of both D_1 and D_2 have a zero or pole has codimension 2 in X. Then one can show that both $D_1 \cdot [D_2]$ and $D_2 \cdot [D_1]$ is simply the sum of the components of the geometric intersection $D_1 \cap D_2$, counted with their scheme-theoretic multiplicities. In other words, if $V \subset X$ is a codimension-2 subvariety and if we assume for simplicity that the local defining equations f_1, f_2 for D_1, D_2 around V are regular, then $[V]$ occurs in both intersection products with the coefficient $l_A(A/(f_1, f_2))$, where $A = O_{X,V}$ is the local ring of X at V.

Case 2: X is a smooth scheme, so that Weil and Cartier divisors agree on X. Then it suffices to compare the intersection products $W \cdot V$ and $V \cdot W$ for any two $(n-1)$-dimensional subvarieties V, W of X. But the two products are obviously equal if $V = W$, and they are equal by case 1 if $V \neq W$. □

Corollary 9.4.5. *The intersection product passes to rational equivalence, i. e. there are well-defined bilinear intersection maps* $\operatorname{Pic} X \times A_k(X) \to A_{k-1}(X)$ *determined by* $D \cdot [V] = [D \cdot V]$ *for all* $D \in \operatorname{Pic} X$ *and all k-dimensional subvarieties V of X.*

Proof. All that remains to be shown is that $D \cdot \alpha = 0$ for any Cartier divisor D if the cycle α is zero in the Chow group $A_k(X)$. But this follows from proposition 9.4.4, as for any rational function φ on a $(k+1)$-dimensional subvariety W of X we have

$$D \cdot [\operatorname{div}(\varphi)] = \operatorname{div}(\varphi) \cdot [D] = 0$$

(note that $\operatorname{div}(\varphi)$ is a Cartier divisor on W that is linearly equivalent to zero). □

Remark 9.4.6. Obviously we can now iterate the process of taking intersection products with Cartier divisors: if X is a scheme and $D_1,\dots,D_m$ are Cartier divisors (or divisor classes) on X then there are well-defined commutative intersection products

$$D_1 \cdot D_2 \cdots D_m \cdot \alpha \in A_{k-m}(X)$$

for any k-cycle $\alpha \in A_k(X)$. If X is an n-dimensional variety and $\alpha = [X]$ is the class of X we usually omit $[X]$ from the notation and write the intersection product simply as $D_1 \cdot D_2 \cdots D_m \in A_{n-m}(X)$. If $m = n$ and X is complete, the notation $D_1 \cdot D_2 \cdots D_m$ is moreover often used to denote the *degree* of the 0-cycle $D_1 \cdot D_2 \cdots D_m \in A_0(X)$ (see example 9.2.13) instead of the cycle itself. If a divisor D occurs m times in the intersection product we will also write this as D^m.

Example 9.4.7. Let $X = \mathbb{P}^2$. Then $\operatorname{Pic} X = A_1(X) = \mathbb{Z} \cdot [H]$, and the intersection product is determined by $H^2 = 1$ ("two lines intersect in one point"). In the same way, $H^n = 1$ on $\mathbb{P}^n$.

Example 9.4.8. Let $X = \tilde{\mathbb{P}}^2$ be the blow-up of $\mathbb{P}^2$ in a point P. By example 9.2.14 we have $\operatorname{Pic} X = \mathbb{Z}[H] \oplus \mathbb{Z}[E]$, where E is the exceptional divisor, and H is a line in $\mathbb{P}^2$ not intersecting E. The strict transform L of a line in $\mathbb{P}^2$ through P has class $[L] = [H] - [E] \in \operatorname{Pic} X$.

The intersection products on X are therefore determined by computing the three products H^2, $H \cdot E$, and E^2. Of course, $H^2 = 1$ and $H \cdot E = 0$ (as $H \cap E = \emptyset$). To compute E^2 we use the relation $[E] = [H] - [L]$ and the fact that E and L meet in one point (with multiplicity 1):

$$E^2 = E \cdot (H - L) = E \cdot H - E \cdot L = 0 - 1 = -1.$$

By our interpretation of example 9.4.3 (ii) this means that the normal bundle of $E \cong \mathbb{P}^1$ in X is $\mathscr{O}_{\mathbb{P}^1}(-1)$. In particular, this normal bundle has no global sections. This means that E *cannot* be deformed in X as in the picture of example 9.4.3 (ii): one says that the curve E is *rigid* in X.

We can consider the formulas $H^2 = 1$, $H \cdot E = 0$, $E^2 = -1$, together with the existence of the intersection product $\operatorname{Pic} X \times \operatorname{Pic} X \to \mathbb{Z}$ as a Bézout style theorem for the blow-up $X = \tilde{\mathbb{P}}^2$. In the same way, we get Bézout style theorems for other (smooth) surfaces and even higher-dimensional varieties.

Example 9.4.9. As a more complicated example, let us reconsider the question of exercise 4.6.6: how many lines are there in $\mathbb{P}^3$ that intersect four general given lines $L_1,\dots,L_4 \subset \mathbb{P}^3$? Recall from exercise 3.5.4 that the space of lines in $\mathbb{P}^3$ is the smooth four-dimensional Grassmannian variety $X = G(1,3)$ that can be described as the set of all rank-2 matrices

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix}$$

modulo row transformations. By the Gaussian algorithm it follows that $G(1,3)$ has a stratification by affine spaces $X_4, X_3, X_2, X_2', X_1, X_0$ (where the subscript denotes the dimension and the stars denote arbitrary complex numbers)

$$\underset{X_4}{\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}} \qquad \underset{X_3}{\begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}} \qquad \underset{X_2}{\begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

$$\underset{X_2'}{\begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}} \qquad \underset{X_1}{\begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \qquad \underset{X_0}{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

If we denote by $\sigma_4,\dots,\sigma_0$ the classes in $A_*(X)$ of the closures of $X_4,\dots,X_0$, we have seen in remark 9.1.18 that $A_*(X)$ is generated by the classes $\sigma_4,\dots,\sigma_0$. These classes actually all have a geometric interpretation:

(i) $\sigma_4 = [X]$.
(ii) σ_3 is the class of all lines that intersect the line $\{x_0 = x_1 = 0\} \subset \mathbb{P}^3$. Note that this is precisely the zero locus of $a_0b_1 - a_1b_0$. In particular, if $L \subset \mathbb{P}^3$ is any other line then the class σ_3^L of all lines in $\mathbb{P}^3$ meeting L is also a quadratic function q in the entries of the matrix that is invariant under row transformations (in fact a 2×2 minor in a suitable choice of coordinates of $\mathbb{P}^3$). The quotient $\frac{a_0b_1-a_1b_0}{q}$ is then a rational function on X whose divisor is $\sigma_3 - \sigma_3^L$. It follows that the class σ_3^L does not depend on L. So we can view σ_3 as the class that describes all lines intersecting *any* given line in $\mathbb{P}^3$.
(iii) σ_2 is the class of all lines passing through the point $(0:0:0:1)$. By an argument similar to that in (ii) above, we can view σ_2 as the class of all lines passing through *any* given point in $\mathbb{P}^3$.
(iv) σ_2' is the class of all lines that are contained in a plane (namely in the plane $x_0 = 0$ for the cycle X_2' given above).
(v) σ_1 is the class of all lines that are contained in a plane and pass through a given point in this plane.
(vi) σ_0 is the class of all lines passing through two given points in $\mathbb{P}^3$.

Hence we see that the intersection number we are looking for is just $\sigma_3^4 \in A_0(X) \cong \mathbb{Z}$ — the number of lines intersecting any four given lines in $\mathbb{P}^3$. So let us compute this number.

Step 1. Let us compute $\sigma_3^2 \in A_2(X)$, i. e. class of all lines intersecting two given lines L_1, L_2 in $\mathbb{P}^3$. We have seen above that it does not matter which lines we take, so let us choose L_1 and L_2 such that they intersect in a point $P \in \mathbb{P}^3$. A line that intersects both L_1 and L_2 has then two possibilities:

(i) it is any line in the plane spanned by L_1 and L_2,
(ii) it is any line in $\mathbb{P}^3$ passing through P.

As (i) corresponds to σ_2' and (ii) to σ_2 we see that $\sigma_3^2 = \sigma_2 + \sigma_2'$. To be more precise, we still have to show that σ_3^2 contains both X_2 and X_2' with multiplicity 1 (and not with a higher multiplicity). As an example, we will show that σ_3^2 contains σ_2 with multiplicity 1; the proof for σ_2' is similar. Consider the open subset $X_4 \subset G(1,3)$; it is isomorphic to an affine space $\mathbb{A}^4$ with coordinates a_2, a_3, b_2, b_3. On this open subset, the space of lines intersecting the line $\{x_0 = x_2 = 0\}$ is given scheme-theoretically by the equation $b_2 = 0$, whereas the space of lines intersecting the line $\{x_0 = x_3 = 0\}$ is given scheme-theoretically by the equation $b_3 = 0$. The scheme-theoretic intersection of these two spaces (i. e. the product σ_3^2) is then given by $b_2 = b_3 = 0$, which is precisely the locus of lines through the point $(0:1:0:0)$ (with multiplicity 1), i. e. the cycle σ_2.

Step 2. In the same way we compute that

(i) $\sigma_3 \cdot \sigma_2 = \sigma_1$ (lines meeting a line L and a point P are precisely lines in the plane spanned by L and P passing through P),
(ii) $\sigma_3 \cdot \sigma_2' = \sigma_1$ (lines meeting a line L and contained in a plane H are precisely lines in the plane H passing through the point $H \cap L$),
(iii) $\sigma_3 \cdot \sigma_1 = \sigma_0$.

So we conclude that

$$\sigma_3^4 = \sigma_3^2(\sigma_2 + \sigma_2') = 2\sigma_3\sigma_1 = 2,$$

i. e. there are exactly two lines in $\mathbb{P}^3$ meeting four other general given lines.

We should note that similar decompositions into affine spaces exist for all Grassmannian varieties, as well as rules how to intersect the corresponding Chow cycles. These rules are usually called *Schubert calculus*. They can be used to answer almost any question of the form: how many lines in $\mathbb{P}^n$ satisfy some given conditions?

Finally, let us prove a statement about intersection products that we will need in the next section. It is based on the following set-theoretic idea: let $f : X \to Y$ be any map of sets, and let $V \subset X$ and $W \subset Y$ be arbitrary subsets. Then it is checked immediately that

$$f(f^{-1}(W) \cap V) = W \cap f(V).$$

This relation is called a **projection formula**. There are projection formulas for many other morphisms and objects that can be pushed forward and pulled back along a morphism. We will prove an intersection-theoretic version here.

Lemma 9.4.10. *Let $f : X \to Y$ be a proper surjective morphism of schemes. Let $\alpha \in A_k(X)$ be a k-cycle on X, and let $D \in \operatorname{Pic} Y$ be a Cartier divisor (class) on Y. Then*

$$f_*(f^*D \cdot \alpha) = D \cdot f_*\alpha \in A_{k-1}(Y).$$

Proof. (Note that this is precisely the set-theoretic intersection formula from above, together with the statement that the scheme-theoretic multiplicities match up in the right way.)

By linearity we may assume that $\alpha = [V]$ for a k-dimensional subvariety $V \subset X$. Let $W = f(V)$, and denote by $g : V \to W$ the restriction of f to V. Then the left hand side of the equation of the lemma is $g_*[g^*D']$, where D' is the Cartier divisor on W associated to the line bundle $\mathcal{O}_Y(D)|_W$. The right hand side is $[K(V) : K(W)] \cdot [D']$ by construction 9.2.9, with the convention that $[K(V) : K(W)] = 0$ if $\dim W < \dim V$. We will prove that these expressions actually agree in $Z_{k-1}(W)$ for any given Cartier divisor D'. This is a local statement (as we just have to check that every codimension-1 subvariety of W occurs on both sides with the same coefficient), so passing to an open subset we can assume that D' is the divisor of a rational function φ on W. But then by theorem 9.2.11 the left hand side is equal to

$$g_* \operatorname{div}(g^*\varphi) = \operatorname{div} N(g^*\varphi) = \operatorname{div}(\varphi^{[K(V):K(W)]}) = [K(V) : K(W)] \cdot \operatorname{div}(\varphi),$$

which equals the right hand side. □

9.5. Exercises.

Exercise 9.5.1. Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d. Compute the Chow group $A_{n-1}(\mathbb{P}^n \backslash X)$.

Exercise 9.5.2. Compute the Chow groups of $X = \mathbb{P}^n \times \mathbb{P}^m$ for all $n, m \geq 1$. Assuming that there are "intersection pairing homomorphisms"

$$A_{n+m-k}(X) \times A_{n+m-l}(X) \to A_{n+m-k-l}(X), \quad (\alpha, \alpha') \mapsto \alpha \cdot \alpha'$$

such that $[V \cap W] = [V] \cdot [W]$ for all subvarieties $V, W \subset X$ that intersect in the expected dimension, compute these homomorphisms explicitly. Use this to state a version of Bézout's theorem for products of projective spaces.

Exercise 9.5.3. (This is a generalization of example 9.1.7.) If X_1 and X_2 are closed subschemes of a scheme X show that there are exact sequences

$$A_k(X_1 \cap X_2) \to A_k(X_1) \oplus A_k(X_2) \to A_k(X_1 \cup X_2) \to 0$$

for all $k \geq 0$.

Exercise 9.5.4. Show that for any schemes X and Y there are well-defined product homomorphisms
$$A_k(X) \times A_l(Y) \to A_{k+l}(X \times Y), \quad [V] \times [W] \mapsto [V \times W].$$
If X has a stratification by affine spaces as in remark 9.1.18 show that the induced homomorphisms
$$\bigoplus_{k+l=m} A_k(X) \times A_l(Y) \to A_m(X \times Y)$$
are surjective. (In general, they are neither injective nor surjective).

Exercise 9.5.5. Prove the following criteria to determine whether a morphism $f : X \to Y$ is proper:

(i) The composition of two proper morphisms is proper.
(ii) Properness is "stable under base change": if $f : X \to Y$ is proper and $g : Z \to Y$ is any morphism, then the induced morphism $f' : X \times_Y Z \to Z$ is proper as well.
(iii) Properness is "local on the base": if $\{U_i\}$ is any open cover of Y and the restrictions $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ are proper for all i then f is proper.
(iv) Closed immersions (see 7.2.10) are proper.

Exercise 9.5.6. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be the morphism given in homogeneous coordinates by $(x_0 : x_1) \mapsto (x_0^2 : x_1^2)$. Let $P \in \mathbb{P}^1$ be the point $(1 : 1)$, and consider the restriction $\tilde{f} : \mathbb{P}^1 \backslash \{P\} \to \mathbb{P}^1$. Show that $\tilde{f}$ is not proper, both with the topological and the algebraic definition of properness.

Exercise 9.5.7. For any $n > 0$ compute the Chow groups of $\mathbb{P}^2$ blown up in n points.

Exercise 9.5.8. Let k be an algebraically closed field. In this exercise we will construct an example of a variety that is complete (i. e. compact if $k = \mathbb{C}$) but not projective.

Consider $X = \mathbb{P}^3$ and the curves $C_1 = \{x_3 = x_2 - x_1 = 0\}$ and $C_2 = \{x_3 = x_0 x_2 - x_1^2 = 0\}$ in X. Denote by $P_1 = (1 : 0 : 0 : 0)$ and $P_2 = (1 : 1 : 1 : 0)$ their two intersection points.

Let $\tilde{X}_1' \to X$ be the blow-up at C_1, and let $\tilde{X}_1 \to \tilde{X}_1'$ be the blow-up at the strict transform of C_2. Denote by $\pi_1 : \tilde{X}_1 \to X$ the projection map. Similarly, let $\pi_2 : \tilde{X}_2 \to X$ be the composition of the two blow-ups in the opposite order; first blow up C_2 and then the strict transform of C_1. Obviously, $\tilde{X}_1$ and $\tilde{X}_2$ are isomorphic away from the inverse image of $\{P_1, P_2\}$, so we can glue $\pi_1^{-1}(X \backslash \{P_1\})$ and $\pi_2^{-1}(X \backslash \{P_2\})$ along the isomorphism $\pi_1^{-1}(X \backslash \{P_1, P_2\}) \cong \pi_2^{-1}(X \backslash \{P_1, P_2\})$ to get a variety Y. This variety will be our example. From the construction there is an obvious projection map $\pi : Y \to X$.

(i) Show that Y is proper over k.
(ii) For $i = 1, 2$ we know that C_i is isomorphic to $\mathbb{P}^1$. Hence we can choose a rational function φ_i on C_i with divisor $P_1 - P_2$. Compute the divisor of the rational function $\varphi_i \circ \pi$ on the variety $\pi^{-1}(C_i)$, as an element in $Z_1(Y)$.
(iii) From (ii) you should have found two irreducible curves $D_1, D_2 \subset Y$ such that $[D_1] + [D_2] = 0 \in A_1(Y)$. Deduce that Y is not a projective variety.

Exercise 9.5.9. Let X be a smooth projective surface, and let $C, D \subset X$ be two curves in X that intersect in finitely many points.

(i) Prove that there is an exact sequence of sheaves on X
$$0 \to \mathcal{O}_X(-C-D) \to \mathcal{O}_X(-C) \oplus \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_{C \cap D} \to 0.$$
(ii) Conclude that the intersection product $C \cdot D \in \mathbb{Z}$ is given by the formula
$$C \cdot D = \chi(X, \mathcal{O}_X) + \chi(X, \mathcal{O}_X(-C-D)) - \chi(X, \mathcal{O}_X(-C)) - \chi(X, \mathcal{O}_X(D)))$$
where $\chi(X, \mathcal{F}) = \sum_i (-1)^i h^i(X, \mathcal{F})$ denotes the Euler characteristic of the sheaf $\mathcal{F}$.

(iii) Show how the idea of (ii) can be used to define an intersection product of divisors on a smooth complete surface (even if the divisors do not intersect in dimension zero).

10. Chern classes

For any vector bundle $\pi : F \to X$ of rank r on a scheme X we define an associated projective bundle $p : \mathbb{P}(F) \to X$ whose fibers $p^{-1}(P)$ are just the projectivizations of the affine fibers $\pi^{-1}(P)$. We construct natural line bundles $O_{\mathbb{P}(F)}(d)$ on $\mathbb{P}(F)$ for all $d \in \mathbb{Z}$ that correspond to the standard line bundles $O(d)$ on projective spaces. As in the case of vector bundles there are pull-back homomorphisms $A_*(X) \to A_*(\mathbb{P}(F))$ between the Chow groups.

For a bundle as above we define the i-th Segre class $s_i(F) : A_*(X) \to A_{*-i}(X)$ by $s_i(F) \cdot \alpha = p_*(D_F^{r-1+i} \cdot p^*\alpha)$, where D_F denotes the Cartier divisor associated to the line bundle $O_{\mathbb{P}(F)}(1)$. The Chern classes $c_i(F)$ are defined to be the inverse of the Segre classes. Segre and Chern classes are commutative; they satisfy the projection formula for proper push-forwards and are compatible with pull-backs. They are multiplicative on exact sequences. Moreover, $c_i(F) = 0$ for $i > r$. The top Chern class $c_r(F)$ has the additional geometric interpretation as the zero locus of a section of F. Using the technique of Chern roots one can compute the Chern classes of almost any bundle that is constructed from known bundles in some way (e. g. by means of direct sums, tensor products, dualizing, exact sequences, symmetric and exterior products).

The Chern character $\operatorname{ch}(F)$ and Todd class $\operatorname{td}(F)$ are defined to be certain polynomial combinations of the Chern classes of F. The Hirzebruch-Riemann-Roch theorem states that $\sum_i h^i(X,F) = \deg(\operatorname{ch}(F) \cdot \operatorname{td}(T_X))$ for any vector bundle F on a smooth projective scheme X. We study some examples and applications of this theorem and give a sketch of proof.

10.1. **Projective bundles.** Recall that for any line bundle $\mathcal{L}$ on a variety X there is a Cartier divisor on X corresponding to $\mathcal{L}$ that in turn defines intersection homomorphisms $A_k(X) \to A_{k-1}(X)$. These homomorphisms can be thought of as intersecting a k-cycle on X with the divisor of any rational section of $\mathcal{L}$. We now want to generalize this idea from line bundles to vector bundles. To do so, we need some preliminaries on projective bundles first.

Roughly speaking, the projective bundle $\mathbb{P}(E)$ associated to a vector bundle E of rank r on a scheme X is simply obtained by replacing the fibers (that are all isomorphic to $\mathbb{A}^r$) by the corresponding projective spaces $\mathbb{P}^{r-1} = (\mathbb{A}^r \backslash \{0\})/k^*$. Let us give the precise definition.

Definition 10.1.1. Let $\pi : F \to X$ be a vector bundle of rank r on a scheme X (see remark 7.3.2). In other words, there is an open covering $\{U_i\}$ of X such that

(i) there are isomorphisms $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{A}^r$ over U_i,
(ii) on the overlaps $U_i \cap U_j$ the compositions

$$\psi_i \circ \psi_j^{-1} : (U_i \cap U_j) \times \mathbb{A}^r \to (U_i \cap U_j) \times \mathbb{A}^r$$

are linear in the coordinates of $\mathbb{A}^r$, i. e. they are of the form

$$(P,x) \mapsto (P, \Psi_{i,j} x)$$

where $P \in U$, $x = (x_1,\dots,x_r) \in \mathbb{A}^r$, and the $\Psi_{i,j}$ are $r \times r$ matrices with entries in $O_X(U_i \cap U_j)$.

Then the **projective bundle** $\mathbb{P}(F)$ is defined by glueing the patches $U_i \times \mathbb{P}^{r-1}$ along the same transition functions, i. e. by glueing $U_i \times \mathbb{P}^{r-1}$ to $U_j \times \mathbb{P}^{r-1}$ along the isomorphisms

$$(U_i \cap U_j) \times \mathbb{P}^{r-1} \to (U_i \cap U_j) \times \mathbb{P}^{r-1}, \quad (P,x) \mapsto (P, \Psi_{i,j} x)$$

for all i, j, where $P \in U_i \cap U_j$ and $x = (x_1 : \dots : x_r) \in \mathbb{P}^{r-1}$. We say that $\mathbb{P}(F)$ is a projective bundle of rank $r-1$ on X.

Note that in the same way as for vector bundles there is a natural projection morphism $p : \mathbb{P}(F) \to X$ that sends a point (P,x) to P. In contrast to the vector bundle case however the morphism p is *proper* (which follows easily from exercise 9.5.5).

Example 10.1.2. Let $X = \mathbb{P}^1$, and let F be the vector bundle (i. e. locally free sheaf) $\mathcal{O}_X \oplus \mathcal{O}_X(-1)$ on X. Then $\mathbb{P}(F)$ is a projective bundle of rank 1 on X, so it is a scheme of dimension 2. We claim that $\mathbb{P}(F)$ is isomorphic to the blow-up $\tilde{\mathbb{P}}^2$ of the projective plane in a point P. In fact, this can be checked directly: by definition 10.1.1 $\mathbb{P}(F)$ is obtained by glueing two copies U_1, U_2 of $\mathbb{A}^1 \times \mathbb{P}^1$ along the isomorphism

$$(\mathbb{A}^1 \backslash \{0\}) \times \mathbb{P}^1 \to (\mathbb{A}^1 \backslash \{0\}) \times \mathbb{P}^1, \quad (z, (x_1 : x_2)) \mapsto (\frac{1}{z}, (x_1 : zx_2)).$$

On the other hand, $\tilde{\mathbb{P}}^2$ is given by

$$\tilde{\mathbb{P}}^2 = \{((x_0 : x_1 : x_2), (y_1 : y_2)) \,;\, x_1 y_2 = x_2 y_1\} \subset \mathbb{P}^2 \times \mathbb{P}^1$$

(see example 4.3.4). Now an isomorphism is given by

$$\begin{aligned} U_1 \cong \mathbb{A}^1 \times \mathbb{P}^1 \to \tilde{\mathbb{P}}^2, &\quad (z, (x_1 : x_2)) \mapsto ((x_1 : zx_2 : x_2), (z : 1)), \\ U_2 \cong \mathbb{A}^1 \times \mathbb{P}^1 \to \tilde{\mathbb{P}}^2, &\quad (z, (x_1 : x_2)) \mapsto ((x_1 : x_2 : zx_2), (1 : z)) \end{aligned}$$

(note that this is compatible with the glueing isomorphism above).

To see geometrically that $\tilde{\mathbb{P}}^2$ is a projective bundle of rank 1 over $\mathbb{P}^1$ let $p : \tilde{\mathbb{P}}^2 \to E \cong \mathbb{P}^1$ be the projection morphism onto the exceptional divisor as of example 9.2.14 (ii). The fibers of this morphism are the strict transforms of lines through P, so they are all isomorphic to $\mathbb{P}^1$.

Remark 10.1.3. If F is a vector bundle and L a line bundle on X then $\mathbb{P}(F) \cong \mathbb{P}(F \otimes L)$. In fact, tensoring F with L just multiplies the transition matrices $\Psi_{i,j}$ of definition 10.1.1 with a scalar function, which does not affect the morphism as the x_i are projective coordinates.

Example 10.1.4. Let $p : \mathbb{P}(F) \to X$ be a projective bundle over a scheme X, given by an open cover $\{U_i\}$ of X and transition matrices $\Psi_{i,j}$ as in definition 10.1.1. In this example we want to construct line bundles $\mathcal{O}_{\mathbb{P}(F)}(d)$ for all $d \in \mathbb{Z}$ on $\mathbb{P}(F)$ that are relative versions of the ordinary bundles $\mathcal{O}_{\mathbb{P}^{r-1}}(d)$ on projective spaces.

The construction is simple: on the patches $U_i \times \mathbb{P}^{r-1}$ of $\mathbb{P}(F)$ we take the line bundles $\mathcal{O}_{\mathbb{P}^{r-1}}(d)$. On the overlaps $U_i \cap U_j$ these line bundles are glued by $\varphi \mapsto \varphi \circ \Psi_{i,j}$, where $\varphi = \frac{f}{g}$ is (locally) a quotient of homogeneous polynomials $f, g \in k[x_1, \dots, x_r]$ with $\deg f - \deg g = d$. Note that the $\varphi \circ \Psi_{i,j}$ satisfies the same degree conditions as the $\Psi_{i,j}$ are linear functions.

Summarizing, we can say that sections of the line bundle $\mathcal{O}_{\mathbb{P}(F)}(d)$ are locally given by quotients of two polynomials which are homogeneous in the fiber coordinates and whose degree difference is d.

Construction 10.1.5. Again let $p : \mathbb{P}(F) \to X$ be a projective bundle over a scheme X, given by an open cover $\{U_i\}$ of X and transition matrices $\Psi_{i,j}$. Consider the vector bundle p^*F on $\mathbb{P}(F)$. It is given by glueing the patches $U_i \times \mathbb{P}^{r-1} \times \mathbb{A}^r$ along the isomorphisms

$$(U_i \cap U_j) \times \mathbb{P}^{r-1} \times \mathbb{A}^r \to (U_i \cap U_j) \times \mathbb{P}^{r-1} \times \mathbb{A}^r, \quad (P, x, y) \mapsto (P, \Psi_{i,j} x, \Psi_{i,j} y),$$

where $x = (x_1 : \cdots : x_r)$ are projective coordinates on $\mathbb{P}^{r-1}$, and $y = (y_1, \dots, y_r)$ are affine coordinates on $\mathbb{A}^r$. Now consider the subbundle S of p^*F given locally by the equations $x_i y_j = x_j y_i$ for all $i, j = 1, \dots, r$, i. e. the subbundle of p^*F consisting of those $(y_1, \dots, y_r)$ that are scalar multiples of $(x_1 : \cdots : x_r)$. Obviously, S is a line bundle on $\mathbb{P}(F)$ contained in p^*F. Geometrically, the fiber of S over a point $(P,x) \in \mathbb{P}(F)$ is precisely the line in the fiber F_P whose projectivization is the point x. The line bundle $S \subset p^*F$ is called the **tautological subbundle** on $\mathbb{P}(F)$.

We can actually identify the subbundle S in the language of example 10.1.4: we claim that S is isomorphic to $\mathscr{O}_{\mathbb{P}(F)}(-1)$. In fact, an isomorphism is given by

$$\mathscr{O}_{\mathbb{P}(F)}(-1) \to S, \quad \varphi \mapsto (y_i = \varphi \cdot x_i),$$

where φ is (locally) the quotient of two polynomials homogeneous in the x_i of degree difference -1. It is obvious that the $\varphi \cdot x_i$ are then quotients of two polynomials homogeneous in the x_i of the same degree, so that the y_i are well-defined.

Example 10.1.6. One place where projective bundles occur naturally is in blow-ups. Recall from construction 4.3.2 that the blow-up $\tilde{X}$ of an affine variety $X \subset \mathbb{A}^n$ at a subvariety $Y \subset X$ with ideal $I(Y) = (f_1, \dots, f_r)$ is defined to be the closure of the graph

$$\Gamma = \{(P, (f_1(P) : \cdots : f_r(P))) \; ; \; P \in X \backslash Y\} \subset X \times \mathbb{P}^{r-1}.$$

The exceptional hypersurface of the blow-up must be contained in $Y \times \mathbb{P}^{r-1}$, which has dimension $\dim Y + r - 1$. So if Y has dimension $\dim X - r$ (which is the expected dimension as its ideal has r generators) then the exceptional hypersurface must be all of $Y \times \mathbb{P}^{r-1}$ for dimensional reasons.

Let us now sketch how this construction can be generalized to blow-ups of arbitrary (not necessarily affine) varieties X in a subvariety Y. For simplicity let us assume that there are r line bundles $\mathscr{L}_1, \dots, \mathscr{L}_r$ on X together with global sections $s_i \in H^0(X, \mathscr{L}_i)$ such that Y is scheme-theoretically the zero locus $s_1 = \cdots = s_r = 0$. Then the straightforward generalization of the above construction is to define the blow-up of X in Y to be the closure of the graph

$$\Gamma = \{(P, (s_1(P) : \cdots : s_r(P)) \; ; \; P \in X \backslash Y\} \subset \mathbb{P}(\mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_r).$$

As above, if Y has codimension r in X then the exceptional hypersurface of the blow-up is the projective bundle $\mathbb{P}((\mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_r)|_Y)$ over Y.

Now recall from remark 7.4.17 and example 9.4.3 (ii) that the normal bundle of a smooth codimension-1 hypersurface Y in a smooth variety X that is given as the zero locus of a section of a line bundle $\mathscr{L}$ is just the restriction of this line bundle $\mathscr{L}$ to Y. If we iterate this result r times we see that the normal bundle of a smooth codimension-r hypersurface Y in a smooth variety X that is given as the zero locus of sections of r line bundles $\mathscr{L}_1, \dots, \mathscr{L}_r$ is just $(\mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_r)|_Y$. Combining this with what we have said above we conclude that *the exceptional hypersurface of the blow-up of a smooth variety X in a smooth variety Y is just the projectivized normal bundle $\mathbb{P}(N_{Y/X})$ over Y.* This is a relative version of our earlier statement that the exceptional hypersurface of the blow-up of a variety in a smooth point is isomorphic to the projectivized tangent space at this point.

In the above argument we have used for simplicity that the codimension-r subvariety Y is globally the zero locus of r sections of line bundles. Actually we do not need this. We only need that Y is *locally* around every point the zero locus of r regular functions, as we can then make the above construction locally and finally glue the local patches together. Using techniques similar to those in theorem 9.3.7 one can show that *every* smooth subvariety Y of codimension r in a smooth variety X is locally around every point the zero locus of r regular functions. So it is actually true in general that the exceptional hypersurface of the blow-up of X in Y is $\mathbb{P}(N_{Y/X})$ if X and Y are smooth.

Finally, in analogy to the case of vector bundles in proposition 9.1.14 let us discuss pull-back homomorphisms for Chow groups induced by projective bundles.

Lemma 10.1.7. *Let F be a vector bundle on a scheme X of rank $r+1$, and let $p : \mathbb{P}(F) \to X$ be the associated projective bundle of rank r. Then there are pull-back homomorphisms*

$$p^* : A_k(X) \to A_{k+r}(\mathbb{P}(F)), \quad [V] \mapsto [p^{-1}(V)]$$

for all k, satisfying the following compatibilities with our earlier constructions:

(i) *(Compatibility with proper push-forward) Let $f : X \to Y$ be a proper morphism, and let F be a vector bundle of rank $r+1$ on Y. Form the fiber diagram*

$$\begin{tikzcd} \mathbb{P}(f^*F) \arrow[r, "f'"] \arrow[d, "p'"'] & \mathbb{P}(F) \arrow[d, "p"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Then $p^* f_* = f'_* p'^*$ as homomorphisms $A_k(X) \to A_{k+r}(\mathbb{P}(F))$.

(ii) *(Compatibility with intersection products) Let F be a vector bundle of rank $r+1$ on X, and let $D \in \operatorname{Pic} X$ be a Cartier divisor (class). Then*

$$p^*(D \cdot \alpha) = (p^*D) \cdot (p^*\alpha)$$

in $A_{k+r-1}(\mathbb{P}(F))$ for every k-cycle $\alpha \in A_k(X)$.

Proof. (i): Let $V \subset X$ be a k-dimensional subvariety. Then $p^{-1}(f(V)) = f'(p'^{-1}(V)) =: W$, and both $p^* f_*[V]$ and $f'_* p'^*[V]$ are equal to $d \cdot [W]$, where d is the generic number of inverse image points of f (resp. f') on $f(V)$ (resp. $p^{-1}(f(V))$.

(ii): Let $\alpha = [V]$ for a k-dimensional subvariety $V \subset X$. On V the Cartier divisor D is given by a line bundle $\mathcal{L}$. If φ is any rational section of $\mathcal{L}$ then the statement follows from the obvious identity $p^* \operatorname{div}(\varphi) = \operatorname{div}(p^*\varphi)$. □

Remark 10.1.8. We have now constructed pull-back morphisms for Chow groups in three cases:

(i) inclusions of open subsets (example 9.1.11),
(ii) projections from vector bundles (proposition 9.1.14),
(iii) projections from projective bundles (lemma 10.1.7).

These are in fact special cases of a general class of morphisms, called *flat morphisms*, for which pull-back maps exist. See [F] section 1.7 for more details.

10.2. **Segre and Chern classes of vector bundles.** Let X be a scheme, and let F be a vector bundle of rank r on X. Let $p : \mathbb{P}(F) \to X$ be the projection from the corresponding projective bundle. Note that we have the following constructions associated to p:

(i) push-forward homomorphisms $p_* : A_k(\mathbb{P}(F)) \to A_k(X)$ since p is proper (see corollary 9.2.12),
(ii) pull-back homomorphisms $p^* : A_k(X) \to A_{k+r-1}(\mathbb{P}(F))$ by lemma 10.1.7,
(iii) a line bundle $\mathcal{O}_{\mathbb{P}(F)}(1)$ on $\mathbb{P}(F)$ by example 10.1.4 (the dual of the tautological subbundle).

We can now combine these three operations to get homomorphisms of the Chow groups of X that depend on the vector bundle F:

Definition 10.2.1. Let X be a scheme, and let F be a vector bundle of rank r on X. Let $p : \mathbb{P}(F) \to X$ be the projection map from the associated projective bundle. Assume for simplicity that X (and hence $\mathbb{P}(F)$) is irreducible (see below), so that the line bundle $\mathcal{O}_{\mathbb{P}(F)}(1)$ corresponds to a Cartier divisor D_F on $\mathbb{P}(F)$. Now for all $i \geq -r+1$ we define **Segre class homomorphisms** by the formula

$$s_i(F) : A_k(X) \to A_{k-i}(X), \qquad \alpha \mapsto s_i(F) \cdot \alpha := p_*(D_F^{r-1+i} \cdot p^*\alpha).$$

Remark 10.2.2. We will discuss some geometric interpretations of Segre classes (or rather some combinations of them) later in proposition 10.2.3 (i) and (ii), proposition 10.3.12, and remark 10.3.14. For the moment let us just note that every vector bundle F gives rise to these homomorphisms $s_i(F)$ that look like intersections (hence the notation $s_i(F) \cdot \alpha$) with some object of codimension i as they decrease the dimension of cycles by i. (In algebraic topology the Segre class $s_i(F)$ is an object in the cohomology group $H^{2i}(X,\mathbb{Z})$.)

Note also that the condition that X be irreducible is not really necessary: even if $O_{\mathbb{P}(F)}(1)$ does not determine a Cartier divisor on $\mathbb{P}(F)$ it does so on every subvariety of $\mathbb{P}(F)$, and this is all we need for the construction of the intersection product (as we intersect with a cycle in $\mathbb{P}(F)$ which is by definition a formal linear combination of subvarieties).

Proposition 10.2.3. *Let X and Y be schemes.*

(i) *For any vector bundle F on X we have*
   - $s_i(F) = 0$ *for* $i < 0$,
   - $s_0(F) = \mathrm{id}$.

(ii) *For any line bundle L on X we have $s_i(L)\cdot\alpha = (-1)^i D^i \cdot \alpha$ for $i \geq 0$ and all $\alpha \in A_*(X)$, where D is the Cartier divisor class associated to the line bundle L.*

(iii) *(Commutativity) If F_1 and F_2 are vector bundles on X, then*

$$s_i(F_1)\cdot s_j(F_2) = s_j(F_2)\cdot s_i(F_1)$$

as homomorphisms $A_k(X) \to A_{k-i-j}(X)$ for all i,j (where the dot denotes the composition of the two homomorphisms).

(iv) *(Projection formula) If $f: X \to Y$ is proper, F is a vector bundle on Y, and $\alpha \in A_*(X)$, then*

$$f_*(s_i(f^*F)\cdot\alpha) = s_i(F)\cdot f_*\alpha.$$

(v) *(Compatibility with pull-back) If $f: X \to Y$ is a morphism for which a pull-back $f^*: A_*(Y) \to A_*(X)$ exists (see remark 10.1.8), F is a vector bundle on Y, and $\alpha \in A_*(Y)$, then*

$$s_i(f^*F)\cdot f^*\alpha = f^*(s_i(F)\cdot\alpha).$$

Proof. (i): Let $V \subset X$ be a k-dimensional subvariety. By construction we can represent $s_i(F)\cdot[V]$ by a cycle of dimension $k-i$ supported in V. As $Z_{k-i}(V) = 0$ for $i < 0$ and $Z_k(V) = [V]$ we conclude that $s_i(F) = 0$ for $i < 0$ and $s_0(F)\cdot[V] = n\cdot[V]$ for some $n \in \mathbb{Z}$. The computation of the multiplicity n is a local calculation, so we can replace X by an open subset and thus assume that F is a trivial bundle. In this case $\mathbb{P}(F) = X \times \mathbb{P}^{r-1}$ and D_F is a hyperplane in $\mathbb{P}^{r-1}$. So D_F^{r-1} is a point in $\mathbb{P}^{r-1}$, i. e. $D_F^{r-1}\cdot p^*[V] = [V \times \{\mathrm{pt}\}]$ and hence $s_0(F)\cdot[V] = [V]$.

(ii): If L is a line bundle then $\mathbb{P}(L) = X$ and p is the identity. Hence the statement follows from the identity $O_{\mathbb{P}(L)}(-1) = L$.

The proofs of (iii), (iv), and (v) all follow from the various compatibilities between push-forward, pull-back, and intersection products. As an example we give the proof of (iv), see [F] proposition 3.1 for the other proofs.

For (iv) consider the fiber square

$$\begin{tikzcd}
\mathbb{P}(f^*F) \arrow[r, "f'"] \arrow[d, "p'"'] & \mathbb{P}(F) \arrow[d, "p"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

and denote the Cartier divisors associated to the line bundles $\mathscr{O}_{\mathbb{P}(F)}(1)$ and $\mathscr{O}_{\mathbb{P}(f^*F)}(1)$ by D_F and D'_F, respectively. Then

$$\begin{aligned}
f_*(s_i(f^*F)\cdot\alpha) &= f_*p'_*({D'_F}^{r-1+i}\cdot p'^*\alpha) && \text{by definition 10.2.1}\\
&= p_*f'_*({D'_F}^{r-i+1}\cdot p'^*\alpha) && \text{by remark 9.2.10}\\
&= p_*f'_*((f'^*D_F)^{r-i+1}\cdot p'^*\alpha) && \text{as } D'_F = f'^*D_F\\
&= p_*(D_F^{r-i+1}\cdot f'_*p'^*\alpha) && \text{by lemma 9.4.10}\\
&= p_*(D_F^{r-i+1}\cdot p^*f_*\alpha) && \text{by lemma 10.1.7 (i)}\\
&= s_i(E)\cdot f_*\alpha && \text{by definition 10.2.1.}
\end{aligned}$$

□

Corollary 10.2.4. *Let F be a vector bundle on a scheme X, and let $p:\mathbb{P}(F)\to X$ be the projection. Then $p_*: A_*(\mathbb{P}(F))\to A_*(X)$ is surjective and $p^*: A_*(X)\to A_*(\mathbb{P}(F))$ is injective.*

Proof. By proposition 10.2.3 (i) we have

$$\alpha = s_0(F)\cdot\alpha = p_*(D_F^{r-1}\cdot p^*\alpha)$$

for all $\alpha\in A_*(X)$, so p_* is surjective. The same formula shows that $\alpha=0$ if $p^*\alpha=0$, so p^* is injective. □

By proposition 10.2.3 (iii) any polynomial expression in the Segre classes of some vector bundles acts on the Chow groups of X. Although the Segre classes are the characteristic classes of vector bundles that are the easiest ones to define, some others that are polynomial combinations of them have nicer properties and better geometric interpretations. Let us now define these combinations.

Definition 10.2.5. Let X be a scheme, and let F be a vector bundle of rank r on X. The **total Segre class** of F is defined to be the formal sum

$$s(F) = \sum_{i\geq 0} s_i(F): A_*(X)\to A_*(X).$$

Note that:

(i) All $s_i(F)$ can be recovered from the homomorphism $s(F)$ by considering the graded parts.
(ii) Although the sum over i in $s(F)$ is formally infinite, it has of course only finitely many terms as $A_k(X)$ is non-zero only for finitely many k.
(iii) The homomorphism $s(F)$ is in fact an *isomorphism* of vector spaces: by proposition 10.2.3 (i) it is given by a triangular matrix with ones on the diagonal (in the natural grading of $A_*(X)$).

By (iii) it makes sense to define the **total Chern class** of F

$$c(F) = \sum_{i\geq 0} c_i(F)$$

to be the inverse homomorphism of $s(F)$. In other words, the **Chern classes** $c_i(F)$ are the unique homomorphisms $c_i(F): A_k(X)\to A_{k-i}(X)$ such that

$$s(F)\cdot c(F) = (1+s_1(F)+s_2(F)+\cdots)\cdot(c_0(F)+c_1(F)+c_2(F)+\cdots) = \mathrm{id}\,.$$

Explicitly, the first few Chern classes are given by

$$\begin{aligned}
c_0(F) &= 1, \\
c_1(F) &= -s_1(F), \\
c_2(F) &= -s_2(F) + s_1(F)^2, \\
c_3(F) &= -s_3(F) + 2s_1(F)s_2(F) - s_1(F)^3.
\end{aligned}$$

Proposition 10.2.3 translates directly into corresponding statements about Chern classes:

Proposition 10.2.6. *Let X and Y be schemes.*

(i) *For any line bundle L on X with associated Cartier divisor class D we have $c(L) \cdot \alpha = (1+D) \cdot \alpha$. In other words, $c_i(L) = 0$ for $i > 1$, and $c_1(L)$ is the homomorphism of intersection with the Cartier divisor class associated to L. By abuse of notation, the Cartier divisor class associated to L is often also denoted $c_1(L)$.*

(ii) *(Commutativity) If F_1 and F_2 are vector bundles on X, then*

$$c_i(F_1) \cdot c_j(F_2) = c_j(F_2) \cdot c_i(F_1)$$

for all i, j.

(iii) *(Projection formula) If $f : X \to Y$ is proper, F is a vector bundle on Y, and $\alpha \in A_*(X)$, then*

$$f_*(c_i(f^*F) \cdot \alpha) = c_i(F) \cdot f_*\alpha.$$

(iv) *(Pull-back) If $f : X \to Y$ is a morphism for which a pull-back $f^* : A_*(Y) \to A_*(X)$ exists, F is a vector bundle on Y, and $\alpha \in A_*(Y)$, then*

$$c_i(f^*F) \cdot f^*\alpha = f^*(c_i(F) \cdot \alpha).$$

Proof. (i): This follows from proposition 10.2.3, since

$$(1 - D + D^2 - D^3 \pm \cdots)(1+D) = 1.$$

(ii), (iii), (iv): All these statements follow from the corresponding properties of Segre classes in proposition 10.2.3, taking into account that the Chern classes are just polynomials in the Segre classes. □

10.3. **Properties of Chern classes.** In this section we will show how to compute the Chern classes of almost any bundle that is constructed from other known bundles in some way (e. g. by means of direct sums, tensor products, dualizing, exact sequences, symmetric and exterior products). We will also discuss the geometric meaning of Chern classes.

The most important property of Chern classes is that they are multiplicative in exact sequences:

Proposition 10.3.1. *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of vector bundles on a scheme X. Then $c(F) = c(F') \cdot c(F'')$.*

Proof. We prove the statement by induction on the rank of F''.

Step 1: $\operatorname{rank} F'' = 1$. We have to show that $s(F') \cdot [V] = c(F'') \cdot s(F) \cdot [V]$ for all k-dimensional subvarieties $V \subset X$. Consider the diagram

$$\begin{tikzcd}
P' = \mathbb{P}(F'|_V) \arrow[rr, hook, "i"] \arrow[dr, "p'"'] & & \mathbb{P}(F|_V) = P \arrow[dl, "p"] \\
& V &
\end{tikzcd}$$

Then

$$\begin{aligned}
c(F'')\cdot s(F)\cdot[V] &= c(F'')\cdot p_*((1+D_F+D_F^2+\cdots)\cdot[P]) && \text{by definition 10.2.1}\\
&= c(F'')\cdot p_*(s(O_P(-1))\cdot[P]) && \text{by proposition 10.2.3 (ii)}\\
&= (1+c_1(F''))\cdot p_*(s(O_P(-1))\cdot[P]) && \text{by proposition 10.2.6 (i)}\\
&= p_*((1+c_1(p^*F''))\cdot s(O_P(-1))\cdot[P]) && \text{by proposition 10.2.6 (iii).}
\end{aligned}$$

On the other hand, we have a bundle map $O_P(-1) \hookrightarrow p^*F \to p^*F''$ on P, which by construction fails to be injective exactly at the points of P'. In other words, P' in P is the (scheme-theoretic) zero locus of a section of the line bundle $p^*F'' \otimes O_P(-1)^\vee$. So we get

$$\begin{aligned}
s(F')\cdot[V] &= p'_*(s(O_{P'}(-1))\cdot[P'])\\
&= p_*i_*(s(i^*O_P(-1))\cdot[P'])\\
&= p_*(s(O_P(-1))\cdot i_*[P'])\\
&= p_*(s(O_P(-1))\cdot(c_1(p^*F'')-c_1(O_P(-1)))\cdot[P]).
\end{aligned}$$

Subtracting these two equations from each other, we get

$$c(F'')\cdot s(F)\cdot[V]-s(F')\cdot[V] = p_*(s(O_P(-1))\,c(O_P(-1))\,[P]) = p_*[P] = 0$$

for dimensional reasons.

Step 2: $\operatorname{rank} F'' > 1$. Let $Q = \mathbb{P}(F''^\vee)$ with projection map $q : Q \to X$, and let $L^\vee \subset q^*F''^\vee$ be the universal line bundle. Then we get a commutative diagram of vector bundles on Q with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & q^*F' \arrow[r] \arrow[d, equal] & \tilde{F} \arrow[r] \arrow[d] & \tilde{F}'' \arrow[r] \arrow[d] & 0\\
0 \arrow[r] & q^*F' \arrow[r] & q^*F \arrow[r] \arrow[d] & q^*F'' \arrow[r] \arrow[d] & 0\\
 & & L \arrow[r, equal] \arrow[d] & L \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

for some vector bundles $\tilde{F}$ and $\tilde{F}''$ on Q with $\operatorname{rank}\tilde{F}'' = \operatorname{rank} F'' - 1$. Recall that we want to prove the statement that for any short exact sequence of vector bundles the Chern polynomial of the bundle in the middle is equal to the product of the Chern polynomials of the other two bundles. In the above diagram we know that this is true for the columns by step 1 and for the top row by the inductive assumption; hence it must be true for the bottom row as well. So we have shown that

$$c(q^*F) = c(q^*F')\cdot c(q^*F'').$$

It follows that

$$q^*c(F) = q^*(c(F')\cdot c(F''))$$

by proposition 10.2.6 (iv), and finally that

$$c(F) = c(F')\cdot c(F'')$$

as q^* is injective by corollary 10.2.4. □

Remark 10.3.2. Of course proposition 10.3.1 can be split up into graded parts to obtain the equations

$$c_k(F) = \sum_{i+j=k} c_i(F')\cdot c_j(F'')$$

for all $k \geq 0$ and any exact sequence $0 \to F' \to F \to F'' \to 0$ of vector bundles on a scheme X.

Note moreover that by definition the same relation $s(F) = s(F') \cdot s(F'')$ then holds for the Segre classes.

Example 10.3.3. In this example we will compute the Chern classes of the tangent bundle T_X of $X = \mathbb{P}^n$. By lemma 7.4.15 we have an exact sequence of vector bundles on X

$$0 \to O_X \to O_X(1)^{\oplus(n+1)} \to T_X \to 0.$$

Moreover proposition 10.2.6 (i) implies that $c(O_X) = 1$ and $c(O_X(1)) = 1 + H$, where H is (the divisor class of) a hyperplane in X. So by proposition 10.3.1 it follows that

$$c(T_X) = c(O_X(1))^{n+1}/c(O_X) = (1+H)^{n+1},$$

i. e. $c_k(T_X) = \binom{n+1}{k} \cdot H^k$ (where H^k is the class of a linear subspace of X of codimension k).

Remark 10.3.4. Note that proposition 10.3.1 allows us to compute the Chern classes of any bundle F of rank r on a scheme X that has a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{r-1} \subset F_r = F$$

by vector bundles such that the quotients $L_i := F_i/F_{i-1}$ are all line bundles (i. e. F_i has rank i for all i). In fact, in this case a recursive application of proposition 10.3.1 to the exact sequences

$$0 \to F_{i-1} \to F_i \to L_i \to 0$$

yields (together with proposition 10.2.6 (i))

$$c(F) = \prod_{i=1}^{r} (1 + D_i)$$

where $D_i = c_1(L_i)$ is the divisor associated to the line bundle L_i.

Unfortunately, not every vector bundle admits such a filtration. We will see now however that for computations with Chern classes we can essentially pretend that such a filtration always exists.

Lemma 10.3.5. *(Splitting construction) Let F be a vector bundle of rank r on a scheme X. Then there is a scheme Y and a morphism $f: Y \to X$ such that*

(i) *f admits push-forwards and pull-backs for Chow groups (in fact it will be an iterated projective bundle),*
(ii) *the push-forward f_* is surjective,*
(iii) *the pull-back f^* is injective,*
(iv) *f^*F has a filtration by vector bundles*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{r-1} \subset F_r = f^*F$$

such that the quotients F_i/F_{i-1} are line bundles on Y.

In other words, "every vector bundle admits a filtration after pulling back to an iterated projective bundle".

Proof. We construct the morphism f by induction on $\operatorname{rank} F$. There is nothing to do if $\operatorname{rank} F = 1$. Otherwise set $Y' = \mathbb{P}(F^\vee)$ and let $f': Y' \to X$ be the projection. Let $L^\vee \subset f'^* F^\vee$ be the tautological line bundle on Y'. Then we have an exact sequence of vector bundles $0 \to \tilde{F} \to f'^*F \to L \to 0$ on Y', where $\operatorname{rank} \tilde{F} = \operatorname{rank} F - 1$. Hence by the inductive assumption there is a morphism $f'': Y \to Y'$ such that $f''^*\tilde{F}$ has a filtration (F_i) with line bundle quotients. If we set $f = f' \circ f''$ it follows that we have an induced filtration of f^*F on Y

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{r-1} = f''^*\tilde{F} \subset f^*F$$

with line bundle quotients. Moreover, f_* is surjective and f^* is injective, as this is true for f'' by the inductive assumption and for f' by corollary 10.2.4. $\square$

Construction 10.3.6. (**Splitting construction**) Suppose one wants to prove a universal identity among Chern classes of vector bundles on a scheme X, e.g. the statement that $c_i(F) = 0$ whenever $i > \operatorname{rank} F$ (see corollary 10.3.7 below). If the identity is invariant under pull-backs (which it essentially always is because of proposition 10.2.6 (iv)) then one can assume that the vector bundles in question have filtrations with line bundle quotients. More precisely, pick a morphism $f : Y \to X$ as in lemma 10.3.5. We can then show the identity for the pulled-back bundle f^*F on Y, using the filtration. As the pull-back f^* is injective and commutes with the identity we want to show, the identity then follows for F on X as well. (This is the same argument that we used already at the end of the proof of proposition 10.3.1.)

Corollary 10.3.7. *Let F be a vector bundle of rank r on a scheme X. Then $c_i(F) = 0$ for all $i > r$.*

Proof. By the splitting construction 10.3.6 we can assume that F has a filtration with line bundle quotients L_i, $i = 1, \dots, r$. But then $c(F) = \prod_{i=1}^r (1 + c_1(L_i))$ by remark 10.3.4, which obviously has no parts of degree bigger than r. $\square$

Remark 10.3.8. This vanishing of Chern classes beyond the rank of the bundle is a property that is *not* shared by the Segre classes (see e.g. proposition 10.2.3 (ii)). This is one reason why Chern classes are usually preferred over Segre classes in computations (although they carry the same information).

Remark 10.3.9. The splitting construction is usually formalized as follows. Let F be a vector bundle of rank r on a scheme X. We write *formally*

$$c(F) = \prod_{i=1}^{r} (1 + \alpha_i).$$

There are two ways to think of the $\alpha_1, \dots, \alpha_r$:

- The α_i are just formal "variables" such that the k-th elementary symmetric polynomial in the α_i is exactly $c_k(F)$. So any *symmetric* polynomial in the α_i is expressible as a polynomial in the Chern classes of F in a unique way.
- After having applied the splitting construction, the vector bundle F has a filtration with line bundle quotients L_i. Then we can set $\alpha_i = c_1(L_i)$, and the decomposition $c(F) = \prod_{i=1}^r (1 + \alpha_i)$ becomes an actual equation (and not just a formal one).

The α_i are usually called the **Chern roots** of F. Using the splitting construction and Chern roots, one can compute the Chern classes of almost any bundle that is constructed from other known bundles by standard operations:

Proposition 10.3.10. *Let X be a scheme, and let F and F' be vector bundles with Chern roots $(\alpha_i)_i$ and $(\alpha'_j)_j$, respectively. Then:*

(i) *$F^\vee$ has Chern roots $(-\alpha_i)_i$.*
(ii) *$F \otimes F'$ has Chern roots $(\alpha_i + \alpha'_j)_{i,j}$.*
(iii) *$S^k F$ has Chern roots $(\alpha_{i_1} + \cdots + \alpha_{i_k})_{i_1 \le \cdots \le i_k}$.*
(iv) *$\Lambda^k F$ has Chern roots $(\alpha_{i_1} + \cdots + \alpha_{i_k})_{i_1 < \cdots < i_k}$.*

Proof. (i): If F has a filtration $0 = F_0 \subset F_1 \subset \cdots \subset F_r = F$ with line bundle quotients $L_i = F_i / F_{i-1}$, then $F^\vee$ has an induced filtration $0 = (F/F_r)^\vee \subset (F/F_{r-1})^\vee \subset \cdots \subset (F/F_0)^\vee = F^\vee$ with line bundle quotients $L_i^\vee$.

(ii): If F and F' have filtrations

$$0 = F_0 \subset F_1 \subset \cdots \subset F_r = F \quad \text{and} \quad 0 = F'_0 \subset F'_1 \subset \cdots \subset F'_s = F'$$

with line bundle quotients $L_i := F_i/F_{i-1}$ and $L'_i := F'_i/F'_{i-1}$, then $F \otimes F'$ has a filtration

$$0 = F_0 \otimes F' \subset F_1 \otimes F' \subset \cdots \subset F_r \otimes F' = F \otimes F'$$

with quotients $L_i \otimes F'$. But $L_i \otimes F'$ itself has a filtration

$$0 = L_i \otimes F'_0 \subset L_i \otimes F'_1 \subset \cdots \subset L_i \otimes F'_s = L_i \otimes F'$$

with quotients $L_i \otimes L'_j$, so the result follows.

(iii) and (iv) follow in the same way. □

Example 10.3.11. The results of proposition 10.3.10 can be restated using Chern classes instead of Chern roots. For example, (i) just says that $c_i(F^\vee) = (-1)^i c_i(F)$. It is more difficult to write down closed forms for the Chern classes in the cases (ii) to (iv). For example, if $F' = L$ is a line bundle, then

$$c(F \otimes L) = \prod_i (1 + (\alpha_i + \alpha')) = \sum_i (1 + c_1(L))^{r-i} c_i(F)$$

where $r = \operatorname{rank} F$. So for $0 \le p \le r$ we have

$$c_p(F \otimes L) = \sum_{i=0}^{p} \binom{r-i}{p-i} c_i(F)\, c_1(L)^{p-i}.$$

Also, from part (iv) it follows immediately that $c_1(F) = c_1(\Lambda^r F)$.

As a more complicated example, assume that F is a rank-2 bundle on a scheme X and let us compute the Chern classes of S^3F. Say F has Chern roots α_1 and α_2, so that $c_1(F) = \alpha_1 + \alpha_2$ and $c_2(F) = \alpha_1\alpha_2$. Then by part (iii) a tedious but easy computation shows that

$$\begin{aligned} c(S^3F) &= (1+3\alpha_1)(1+2\alpha_1+\alpha_2)(1+\alpha_1+2\alpha_2)(1+3\alpha_2) \\ &= 1 + 6c_1(F) + 10c_2(F) + 11c_1(F)^2 + 30c_1(F)c_2(F) \\ &\quad + 6c_1(F)^3 + 9c_2(F)^2 + 18c_1(F)^2c_2(F). \end{aligned}$$

Splitting this up into graded pieces one obtains the individual Chern classes, e. g.

$$c_4(S^3F) = 9c_2(F)^2 + 18c_1(F)^2c_2(F).$$

Now that we have shown how to compute Chern classes let us discuss their geometric meaning. By far the most important property of Chern classes is that the "top Chern class" of a vector bundle (i. e. $c_r(F)$ if $r = \operatorname{rank} F$) is the class of the zero locus of a section:

Proposition 10.3.12. *Let F be a vector bundle of rank r on an n-dimensional scheme X. Let $s \in \Gamma(F)$ be a global section of F, and assume that its scheme-theoretic zero locus $Z(s)$ has dimension $n-r$ (as expected). Then $[Z(s)] = c_r(F) \cdot [X] \in A_{n-r}(X)$.*

Proof. We will only sketch the proof; for details especially about multiplicities we refer to [F] section 14.1.

We prove the statement by induction on r. Applying the splitting principle we may assume that there is an exact sequence

$$0 \to F' \to F \to L \to 0 \qquad (*)$$

of vector bundles on X, where L is a line bundle and $\operatorname{rank} F' = \operatorname{rank} F - 1$. Now let $s \in \Gamma(X,F)$ be a global section of F as in the proposition. Then s induces

(i) a section $l \in \Gamma(X,L)$, and
(ii) a section $s' \in \Gamma(Z(l),F')$ (i. e. "s is a section of F' on the locus where the induced section on L vanishes").

Let us assume that l is not identically zero, and denote by $i : Z(l) \hookrightarrow X$ the inclusion morphism. Note that then $i_*[Z(s')] = c_{r-1}(F) \cdot [Z(l)]$ by the induction hypothesis, and $[Z(l)] = c_1(L) \cdot [X]$ as the Weil divisor associated to a line bundle is just the zero locus of a section. Combining these results we get

$$[Z(s)] = i_*[Z(s')] = c_{r-1}(F) \cdot c_1(L) \cdot [X].$$

But applying proposition 10.3.1 to the exact sequence $(*)$ we get $c_r(F) = c_{r-1}(F') \cdot c_1(L)$, so the result follows. $\square$

Remark 10.3.13. Proposition 10.3.12 is the generalization of our old statement that the first Chern class of a line bundle (i. e. the divisor associated to a line bundle) is the zero locus of a (maybe rational) section of that bundle. In contrast to the line bundle case however, it is not clear that a section of the vector bundle exists that vanishes in the right codimension. This is why proposition 10.3.12 cannot be used as a definition for the top Chern class.

Remark 10.3.14. There are analogous interpretations for the intermediate Chern classes $c_k(F)$ that we state without proof: let F be a vector bundle of rank r on a scheme X. Let $s_1, \dots, s_{r+1-k}$ be global sections of X, and assume that the (scheme-theoretic) locus $Z \subset X$ *where the sections* $s_1, \dots, s_{r+1-k}$ *are linearly dependent* has codimension k in X (which is the expected codimension). Then $[Z] = c_k(F) \cdot [X] \in A_*(X)$. (For a proof of this statement see [F] example 14.4.1).

Two special cases of this property are easy to see however:

(i) In the case $k = r$ we are reduced to proposition 10.3.12.
(ii) In the case $k = 1$ the locus Z is just the zero locus of a section of $\Lambda^r F$, so we have $[Z] = c_1(\Lambda^r F) = c_1(F)$ (the latter equality is easily checked using proposition 10.3.10 (iv)).

Example 10.3.15. As an example of proposition 10.3.12 let us recalculate that there are 27 lines on a cubic surface X in $\mathbb{P}^3$ (see section 4.5). To be more precise, we will not reprove here that the number of lines in X is finite; instead we will assume that it is finite and just recalculate the number 27 under this assumption.

Let $G(1,3)$ be the 4-dimensional Grassmannian variety of lines in $\mathbb{P}^3$. As in construction 10.1.5 there is a *tautological rank-2 subbundle* F of the trivial bundle $\mathbb{C}^4$ whose fiber over a point $[L] \in G(1,3)$ (where $L \subset \mathbb{P}^3$ is a line) is precisely the 2-dimensional subspace of $\mathbb{C}^4$ whose projectivization is L. Dualizing, we get a surjective morphism of vector bundles $(\mathbb{C}^4)^\vee \to F^\vee$ that corresponds to restricting a linear function on $\mathbb{C}^4$ (or $\mathbb{P}^3$) to the line L. Taking the d-th symmetric power of this morphism we arrive at a surjective morphism $S^d(\mathbb{C}^4)^\vee \to S^d F^\vee$ that corresponds to restricting a homogeneous polynomial of degree d on $\mathbb{P}^3$ to L.

Now let $X = \{f = 0\}$ be a cubic surface. By what we have just said the polynomial f determines a section of $S^3 F^\vee$ whose set of zeros in $G(1,3)$ is precisely the set of lines that lie in X (i. e. the set of lines on which f vanishes). So *assuming that this set is finite* we see by proposition 10.3.12 that the number of lines in the cubic surface X is the degree of the cycle $c_4(S^3 F^\vee)$ on $G(1,3)$.

To compute this number note that by example 10.3.11 we have

$$c_4(S^3 F^\vee) = 9c_2(F^\vee)^2 + 18c_1(F^\vee)^2 c_2(F^\vee),$$

so that it remains to compute the numbers $c_2(F^\vee)^2$ and $c_1(F^\vee)^2 c_2(F^\vee)$. There are general rules (called “Schubert calculus”) how to compute such intersection products on Grassmannian varieties, but in this case we can also compute the result directly in a way similar to that in example 9.4.9:

(i) By exactly the same reasoning as above, $c_2(F^\vee) = c_2(S^1 F^\vee)$ is the locus of all lines in $\mathbb{P}^3$ that are contained in a given plane.

(ii) The class $c_1(F^\vee) = c_1(\Lambda^2 F^\vee)$ is (by definition of the exterior product, see also remark 10.3.14) the locus of all lines $L \subset \mathbb{P}^3$ such that two given linear equations f_1, f_2 on $\mathbb{P}^4$ become linearly dependent when restricted to the line. This means that $f_1|_L$ and $f_2|_L$ must have their zero at the same point of L. In other words, L intersects $Z(f_1, f_2)$, which is a line. In summary, $c_1(F^\vee)$ is just the class of lines that meet a given line in $\mathbb{P}^3$.

Using these descriptions we can now easily compute the required intersection products: $c_2(F^\vee)^2$ is the number of lines that are contained in two given planes in $\mathbb{P}^3$, so it is 1 (the line must precisely be the intersection line of the two planes). Moreover, $c_1(F^\vee)^2 c_2(F^\vee)$ is the number of lines intersecting two given lines and lying in a given plane, i. e. the number of lines through two points in a plane, which is 1.

Summarizing, we get that the number of lines on a cubic surface is

$$c_4(S^3 F^\vee) = 9c_2(F^\vee)^2 + 18c_1(F^\vee)^2 c_2(F^\vee) = 9 \cdot 1 + 18 \cdot 1 = 27.$$

Remark 10.3.16. The preceding example 10.3.15 shows very well how enumerative problems can be attacked in general. By an *enumerative problem* we mean that we want to count the number of curves in some space with certain conditions (e. g. lines through two points, lines in a cubic surface, plane conics through 5 points, and so on). Namely:

(i) Find a *complete* (resp. compact) “moduli space” M whose points correspond to the curves one wants to study (in the above example: the Grassmannian $G(1,3)$ that parametrizes lines in $\mathbb{P}^3$).
(ii) Every condition that one imposes on the curves (passing through a point, lying in a given subvariety, ...) corresponds to some intersection-theoretic cycle on M — a divisor, a combination of Chern classes, or something else.
(iii) If the *expected* number of curves satisfying the given conditions is finite then the intersection product of the cycles in (ii) will have dimension 0. As M is complete the degree of this zero-cycle is a well-defined integer. It is called the *virtual solution* to the enumerative problem. Note that this number is well-defined even if the actual number of curves satisfying the given conditions is *not* finite.
(iv) It is now a different (and usually more difficult, in any case not an intersection-theoretic) problem to figure out whether the actual number of curves satisfying the given conditions is finite or not, and if so whether they are counted in the intersection product of (iii) with the scheme-theoretic multiplicity 1. If this is the case then the solution of (iii) is said to be *enumerative* (and not only virtual). For example, we have shown in section 4.5 that the number 27 computed intersection-theoretically in example 10.3.15 is actually enumerative for any smooth cubic surface X.

10.4. **Statement of the Hirzebruch-Riemann-Roch theorem.** As a final application of Chern classes we will now state and sketch a proof of the famous Hirzebruch-Riemann-Roch theorem that is a vast and very useful generalization (yet still not the most general version) of the Riemann-Roch theorem (see section 7.7, in particular remark 7.7.7).

As usual the goal of the Riemann-Roch type theorems is to compute the dimension $h^0(X, \mathcal{F})$ of the space of global sections of a sheaf $\mathcal{F}$ on a scheme X, in the case at hand of a vector bundle on a smooth projective scheme X. As we have already seen in the case where X is a curve and $\mathcal{F}$ a line bundle there is no easy general formula for this number unless you add some “correction term” (that was $-h^1(X, \mathcal{F})$ in the case of curves). The same is true in higher dimensions. Here the Riemann-Roch theorem will compute the Euler characteristic of $\mathcal{F}$:

Definition 10.4.1. Let $\mathcal{F}$ be a coherent sheaf on a projective scheme X. Then the dimensions $h^i(X, \mathcal{F}) = \dim H^i(X, \mathcal{F})$ are all finite by theorem 8.4.7 (i). We define the **Euler**

characteristic of $\mathcal{F}$ to be the integer

$$\chi(X,\mathcal{F}) := \sum_{i\geq 0} (-1)^i h^i(X,\mathcal{F}).$$

(Note that the sum is finite as $h^i(X,\mathcal{F}) = 0$ for $i > \dim X$.)

The "left hand side" of the Hirzebruch-Riemann-Roch theorem will just be $\chi(X,\mathcal{F})$; this is the number that we want to compute. Recall that there were many "vanishing theorems", e. g. $h^i(X,\mathcal{F}\otimes \mathcal{O}_X(d)) = 0$ for $i > 0$ and $d \gg 0$ by theorem 8.4.7 (ii). So in the cases when such vanishing theorems apply the theorem will actually compute the desired number $h^0(X,\mathcal{F})$.

The "right hand side" of the Hirzebruch-Riemann-Roch theorem is an intersection-theoretic expression that is usually easy to compute. It is a certain combination of the Chern (resp. Segre) classes of the bundle F (corresponding to the locally free sheaf $\mathcal{F}$) and the tangent bundle T_X of X. These combinations will have *rational* coefficients, so we have to tensor the Chow groups with $\mathbb{Q}$ (i. e. we consider formal linear combinations of subvarieties with rational coefficients instead of integer ones).

Definition 10.4.2. Let F be a vector bundle of rank r with Chern roots $\alpha_1,\dots,\alpha_r$ on a scheme X. Then we define the **Chern character** $\mathrm{ch}(F) : A_*(X)\otimes\mathbb{Q} \to A_*(X)\otimes\mathbb{Q}$ to be

$$\mathrm{ch}(F) = \sum_{i=1}^{r} \exp(\alpha_i)$$

and the **Todd class** $\mathrm{td}(F) : A_*(X)\otimes\mathbb{Q} \to A_*(X)\otimes\mathbb{Q}$ to be

$$\mathrm{td}(F) = \prod_{i=1}^{r} \frac{\alpha_i}{1-\exp(-\alpha_i)},$$

where the expressions in the α_i are to be understood as formal power series, i. e.

$$\exp(\alpha_i) = 1 + \alpha_i + \frac{1}{2}\alpha_i^2 + \frac{1}{6}\alpha_i^3 + \cdots$$

and

$$\frac{\alpha_i}{1-\exp(-\alpha_i)} = 1 + \frac{1}{2}\alpha_i + \frac{1}{12}\alpha_i^2 + \cdots.$$

Remark 10.4.3. As usual we can expand the definition of $\mathrm{ch}(F)$ and $\mathrm{td}(F)$ to get symmetric polynomials in the Chern roots which can then be written as polynomials (with rational coefficients) in the Chern classes $c_i = c_i(F)$ of F. Explicitly,

$$\mathrm{ch}(F) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots$$

$$\text{and} \quad \mathrm{td}(F) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \cdots.$$

Remark 10.4.4. If $0 \to F' \to F \to F'' \to 0$ is an exact sequence of vector bundles on X then the Chern roots of F are just the union of the Chern roots of F' and F''. So we see that

$$\mathrm{ch}(F) = \mathrm{ch}(F') + \mathrm{ch}(F'')$$

and

$$\mathrm{td}(F) = \mathrm{td}(F')\cdot\mathrm{td}(F'').$$

We can now state the Hirzebruch-Riemann-Roch theorem:

Theorem 10.4.5. ***(Hirzebruch-Riemann-Roch theorem)*** *Let F be a vector bundle on a smooth projective variety X. Then*

$$\chi(X,F) = \deg(\mathrm{ch}(F)\cdot\mathrm{td}(T_X))$$

where $\deg(\alpha)$ *denotes the degree of the dimension-0 part of the (non-homogeneous) cycle* α.

Before we sketch a proof of this theorem in the next section let us consider some examples.

Example 10.4.6. Let $F = L$ be a line bundle on a smooth projective curve X of genus g. Then $\chi(X,L) = h^0(X,L) - h^1(X,L)$. On the right hand side, the dimension-0 part of $\operatorname{ch}(L) \cdot \operatorname{td}(T_X)$, i. e. its codimension-1 part, is equal to

$$\begin{aligned}
\deg(\operatorname{ch}(L) \cdot \operatorname{td}(T_X)) &= \deg((1+c_1(L))(1+\tfrac{1}{2}c_1(T_X))) && \text{by remark 10.4.3} \\
&= \deg(c_1(L) - \tfrac{1}{2}c_1(\Omega_X)) \\
&= \deg L - \tfrac{1}{2}(2g-2) && \text{by corollary 7.6.6} \\
&= \deg L + 1 - g,
\end{aligned}$$

so we are recovering our earlier Riemann-Roch theorem of corollary 8.3.3.

Example 10.4.7. If F is a vector bundle of rank r on a smooth projective curve X then we get in the same way

$$\begin{aligned}
h^0(X,F) - h^1(X,F) &= \deg(\operatorname{ch}(F) \cdot \operatorname{td}(T_X)) \\
&= \deg((r+c_1(F))(1+\frac{1}{2}c_1(T_X))) \\
&= \deg c_1(F) + r(1-g).
\end{aligned}$$

Example 10.4.8. Let $L = \mathcal{O}_X(D)$ be a line bundle on a smooth projective surface X corresponding to a divisor D. Now the dimension-0 part of the right hand side has codimension 2, so the Hirzebruch-Riemann-Roch theorem states that

$$\begin{aligned}
&h^0(X,L) - h^1(X,L) + h^2(X,L) \\
&\qquad = \deg(\operatorname{ch}(F) \cdot \operatorname{td}(T_X)) \\
&\qquad = \deg\left(\left(1 + c_1(L) + \frac{1}{2}c_1(L)^2\right)\left(1 + \frac{1}{2}c_1(T_X) + \frac{1}{12}(c_1(T_X)^2 + c_2(T_X))\right)\right) \\
&\qquad = \frac{1}{2}D \cdot (D - K_X) + \frac{K_X^2 + c_2(T_X)}{12}.
\end{aligned}$$

Note that:

(i) The number $\chi(X, \mathcal{O}_X) = \frac{K_X^2 + c_2(T_X)}{12}$ is an invariant of X that does not depend on the line bundle. The Hirzebruch-Riemann-Roch theorem implies that it is always an integer, i. e. that $K_X^2 + c_2(T_X)$ is divisible by 12 (which is not at all obvious from the definitions).

(ii) If X has degree d and $L = \mathcal{O}_X(n)$ for $n \gg 0$ then $h^1(X,L) = h^2(X,L) = 0$ by theorem 8.4.7 (ii). Moreover we then have $D^2 = dn^2$, so we get

$$h^0(X, \mathcal{O}_X(n)) = \frac{d}{2}n^2 + \frac{1}{2}(H \cdot K_X) \cdot n + \frac{K_X^2 + c_2(T_X)}{12}$$

where H denotes the class of a hyperplane (restricted to X). In other words, we have just recovered proposition 6.1.5 about the Hilbert function of X. Moreover, we have identified the non-leading coefficients of the Hilbert polynomial in terms of intersection-theoretic data.

Example 10.4.9. The computation of example 10.4.8 works for higher-dimensional varieties as well: let X be a smooth projective N-dimensional variety of degree d and consider the line bundle $L = \mathcal{O}_X(n)$ on X for $n \gg 0$. We see immediately that the codimension-N part of $\operatorname{ch}(\mathcal{O}_X(n)) \cdot \operatorname{td}(T_X)$ is a polynomial in n of degree N with leading coefficient

$$\frac{1}{N!}c_1(L)^N = \frac{d}{N!}n^N,$$

which reproves proposition 6.1.5 (for smooth X). Moreover, we can identify the other coefficients of the Hilbert polynomial in terms of intersection-theoretic expressions involving the characteristic classes of the tangent bundle of X.

Example 10.4.10. Let $F = O_X(d)$ be a line bundle on $X = \mathbb{P}^n$. Then we can compute both sides of the Hirzebruch-Riemann-Roch theorem explicitly and therefore prove the theorem in this case:

As for the left hand side, proposition 8.4.1 implies that

$$\chi(X, O_X(d)) = \begin{cases} h^0(X, O_X(d)) = \binom{n+d}{n} & \text{if } d \geq 0, \\ (-1)^n h^n(X, O_X(d)) = (-1)^n \binom{-d-1}{n} & \text{if } d \leq -n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this means in fact in all cases that

$$\chi(X, O_X(d)) = \binom{n+d}{n}.$$

As for the right hand side let us first compute the Todd class of T_X. By the Euler sequence

$$0 \to O_X \to O_X(1)^{\oplus(n+1)} \to T_X \to 0$$

of lemma 7.4.15 together with the multiplicativity of Chern classes (see proposition 10.3.1) we see that the Chern classes (and hence the Todd class) of T_X are the same as those of $O_X(1)^{\oplus(n+1)}$. But the Chern roots of the latter bundle are just $n+1$ times the class H of a hyperplane, so it follows that

$$\operatorname{td}(T_X) = \frac{H^{n+1}}{(1-\exp(-H))^{n+1}}.$$

As the Chern character of $O_X(d)$ is obviously $\exp(dH)$ we conclude that the right hand side of the Hirzebruch-Riemann-Roch theorem is the H^n-coefficient of

$$\frac{H^{n+1}\exp(dH)}{(1-\exp(-H))^{n+1}}.$$

But this is equal to the residue

$$\operatorname{res}_{H=0} \frac{\exp(dH)}{(1-\exp(-H))^{n+1}}\, dH,$$

which we can compute using the substitution $x = 1-\exp(-H)$ (so $\exp(H) = \frac{1}{1-x}$ and $\frac{dH}{dx} = \frac{1}{1-x}$):

$$\operatorname{res}_{H=0} \frac{\exp(dH)}{(1-\exp(-H))^{n+1}}\, dH = \operatorname{res}_{x=0} \frac{(1-x)^{-d-1}}{x^{n+1}}\, dx.$$

This number is equal to the x^n-coefficient of $(1-x)^{-d-1}$, which is simply

$$(-1)^n \binom{-d-1}{n} = \binom{n+d}{n}$$

in agreement with what we had found for the left hand side of the Hirzebruch-Riemann-Roch theorem above. So we have just proven the theorem for line bundles on $\mathbb{P}^n$.

10.5. **Proof of the Hirzebruch-Riemann-Roch theorem.** Finally we now want to give a very short sketch of proof of the Hirzebruch-Riemann-Roch theorem 10.4.5, skipping several subtleties from commutative algebra. The purpose of this section is just to give an idea of the proof, and in particular to show why the rather strange-looking Todd classes come into play. For a more detailed discussion of the proof or more general versions see [F] chapter 15.

The proof of the theorem relies heavily on certain constructions being additive (or otherwise well-behaved) on exact sequences of vector bundles. Let us formalize this idea first.

Definition 10.5.1. Let X be a scheme. The **Grothendieck group of vector bundles** $K^\circ(X)$ on X is defined to be the group of formal finite sums $\sum_i a_i[F_i]$ where $a_i \in \mathbb{Z}$ and the F_i are vector bundles on X, modulo the relations $[F] = [F'] + [F'']$ for every exact sequence $0 \to F' \to F \to F'' \to 0$. (Of course we then also have $\sum_{i=1}^r (-1)^i [F_i] = 0$ for every exact sequence

$$0 \to F_1 \to F_2 \to \cdots \to F_r \to 0.)$$

Example 10.5.2. Definition 10.5.1 just says that every construction that is additive on exact sequences passes to the Grothendieck group. For example:

(i) If X is projective then the Euler characteristic of a vector bundle (see definition 10.4.1) is additive on exact sequences by the long exact cohomology sequence of proposition 8.2.1. Hence the Euler characteristic can be thought of as a *homomorphism of Abelian groups*

$$\chi : K^\circ(X) \to \mathbb{Z}, \quad \chi([F]) = \chi(X, F).$$

(ii) The Chern character of a vector bundle is additive on exact sequences remark 10.4.4. So we get a homomorphism

$$\mathrm{ch} : K^\circ(X) \to A_*(X) \otimes \mathbb{Q}, \quad \mathrm{ch}([F]) = \mathrm{ch}(F).$$

(It can in fact be shown that this homomorphism gives rise to an *isomorphism* $K^\circ(X) \otimes \mathbb{Q} \to A_*(X) \otimes \mathbb{Q}$ if X is smooth; see [F] example 15.2.16(b). We will not need this however in our proof.)

(iii) Let X be a smooth projective variety. For the same reason as in (ii) the right hand side of the Hirzebruch-Riemann-Roch theorem gives rise to a homomorphism

$$\tau : K^\circ(X) \to A_*(X) \otimes \mathbb{Q}, \quad \tau(F) = \mathrm{ch}(F) \cdot \mathrm{td}(T_X).$$

In particular, by (i) and (iii) we have checked already that both sides of the Hirzebruch-Riemann-Roch theorem are additive on exact sequences (which is good). So to prove the theorem we only have to check it on a set of generators for $K^\circ(X)$. To use this to our advantage however we first have to gather more information about the structure of the Grothendieck groups. We will need the following lemma of which we can only sketch the proof.

Lemma 10.5.3. *Let X be a* smooth *projective scheme. Then for every coherent sheaf $\mathcal{F}$ on X there is an exact sequence*

$$0 \to F_r \to F_{r-1} \to \cdots \to F_0 \to \mathcal{F} \to 0$$

where the F_i are vector bundles (i. e. locally free sheaves). We say that "every coherent sheaf has a finite locally free resolution". Moreover, if $X = \mathbb{P}^n$ then the F_i can all be chosen to be direct sums of line bundles $\mathcal{O}_X(d)$ for various d.

Proof. By a repeated application of lemma 8.4.6 we know already that there is a (possibly infinite) exact sequence

$$\cdots \to F_r \to \cdots \to F_1 \to F_0 \to \mathcal{F} \to 0.$$

Now one can show that for an n-dimensional *smooth* scheme the kernel K of the morphism $F_{r-1} \to F_{r-2}$ is always a vector bundle (see [F] B.8.3). So we get a locally free resolution

$$0 \to K \to F_{r-1} \to F_{r-2} \to \cdots \to F_0 \to \mathcal{F} \to 0$$

as required.

If $X = \mathbb{P}^n$ with homogeneous coordinate ring $S = k[x_0, \dots, x_n]$ then one can show that a coherent sheaf $\mathcal{F}$ on X is nothing but a *graded* S-module M (in the same way that a coherent sheaf on an affine scheme $\operatorname{Spec} R$ is given by an R-module). By the famous *Hilbert syzygy theorem* (see [EH] theorem III-57) there is a free resolution of M

$$0 \to \bigoplus_i S_{n,i} \to \cdots \to \bigoplus_i S_{1,i} \to \bigoplus_i S_{0,i} \to M \to 0$$

where each $S_{j,i}$ is isomorphic to S, with the grading shifted by some constants $a_{j,i}$. This means exactly that we have a locally free resolution

$$0 \to \bigoplus_i \mathcal{O}_X(a_{n,i}) \to \cdots \to \bigoplus_i \mathcal{O}_X(a_{1,i}) \to \bigoplus_i \mathcal{O}_X(a_{0,i}) \to \mathcal{F} \to 0$$

of $\mathcal{F}$. □

Corollary 10.5.4. *The Hirzebruch-Riemann-Roch theorem 10.4.5 is true for any vector bundle on* $\mathbb{P}^n$.

Proof. By lemma 10.5.3 (applied to $X = \mathbb{P}^n$ and a vector bundle $\mathcal{F}$) the Grothendieck group $K^\circ(\mathbb{P}^n)$ is generated by the classes of the line bundles $\mathcal{O}_{\mathbb{P}^n}(d)$ for $d \in \mathbb{Z}$. As we have already checked the Hirzebruch-Riemann-Roch theorem for these bundles in example 10.4.10 the statement follows by the remark at the end of example 10.5.2. □

Remark 10.5.5. To study the Hirzebruch-Riemann-Roch theorem for general smooth projective X let $i : X \to \mathbb{P}^n$ be an embedding of X in projective space and consider the following diagram:

$$\begin{tikzcd} K^\circ(X) \arrow[r, "i_*"] \arrow[d, "\tau"] & K^\circ(\mathbb{P}^n) \arrow[r, "\chi"] \arrow[d, "\tau"] & \mathbb{Z} \arrow[d, hook] \\ A(X) \otimes \mathbb{Q} \arrow[r, "i_*"] & A(\mathbb{P}^n) \otimes \mathbb{Q} \arrow[r, "\deg"] & \mathbb{Q}. \end{tikzcd}$$

Let us first discuss the right square. The homomorphisms χ and τ are explained in example 10.5.2, and deg denotes the degree of the dimension-0 part of a cycle class. The Hirzebruch-Riemann-Roch theorem for $\mathbb{P}^n$ of corollary 10.5.4 says precisely that this right square is commutative.

Now consider the left square. The homomorphism τ is as above, and the i_* in the bottom row is the proper push-forward of cycles of corollary 9.2.12. We have to explain the push-forward i_* in the top row. Of course we would like to define $i_*[F] = [i_* F]$ for any vector bundle F on X, but we cannot do this directly as $i_* F$ is not a vector bundle but only a coherent sheaf on $\mathbb{P}^n$. So instead we let

$$0 \to F_r \to F_{r-1} \to \cdots \to F_0 \to i_* F \to 0 \qquad (*)$$

be a locally free resolution of the coherent sheaf $i_* F$ on $\mathbb{P}^n$ and set

$$i_* : K^\circ(X) \to K^\circ(\mathbb{P}^n), \quad i_*([F]) = \sum_{k=0}^{r} (-1)^k [F_k].$$

One can show that this is indeed a well-defined homomorphism of groups (i. e. that this definition does not depend on the choice of locally free resolution), see [F] section B.8.3. But in fact we do not really need to know this: we do know by the long exact cohomology sequence applied to $(*)$ that

$$\chi(X, F) = \sum_{k=0}^{r} (-1)^k \chi(\mathbb{P}^n, F_k),$$

so it is clear that at least the composition $\chi \circ i_*$ does not depend on the choice of locally free resolution. The Hirzebruch-Riemann-Roch theorem on X is now precisely the statement that the outer rectangle in the above diagram is commutative.

As we know already that the right square is commutative, it suffices therefore to show that the left square is commutative as well (for any choice of locally free resolution as above), i. e. that

$$\sum_{k=0}^{r} (-1)^k \operatorname{ch}(F_k) \cdot \operatorname{td}(T_{\mathbb{P}^r}) = i_*(\operatorname{ch}(F) \cdot \operatorname{td}(T_X)).$$

As the Todd class is multiplicative on exact sequences by remark 10.4.4 we can rewrite this using the projection formula as

$$\sum_{k=0}^{r} (-1)^k \operatorname{ch}(F_k) = i_* \frac{\operatorname{ch}(F)}{\operatorname{td}(N_{X/\mathbb{P}^n})}.$$

Summarizing our ideas we see that to prove the general Hirzebruch-Riemann-Roch theorem it suffices to prove the following proposition (for $Y = \mathbb{P}^n$):

Proposition 10.5.6. *Let $i : X \to Y$ be a closed immersion of smooth projective schemes, and let F be a vector bundle on X. Then there is a locally free resolution*

$$0 \to F_r \to F_{r-1} \to \cdots \to F_0 \to i_*F \to 0$$

of the coherent sheaf i_*F on Y such that

$$\sum_{k=0}^{r} (-1)^k \operatorname{ch}(F_k) = i_* \frac{\operatorname{ch}(F)}{\operatorname{td}(N_{X/Y})}$$

in $A_*(Y) \otimes \mathbb{Q}$.

Example 10.5.7. Before we give the general proof let us consider an example where both sides of the equation can be computed explicitly: let X be a smooth scheme, E a vector bundle of rank r on X, and $Y = \mathbb{P}(E \oplus O_X)$. The embedding $i : X \to Y$ is given by $X = \mathbb{P}(0 \oplus O_X) \hookrightarrow \mathbb{P}(E \oplus O_X)$. In other words, X is just "the zero section of a projective bundle". The special features of this particular case that we will need are:

(i) There is a projection morphism $p : Y \to X$ such that $p \circ i = \mathrm{id}$.

(ii) X is the zero locus of a section of a vector bundle on Y: consider the exact sequence

$$0 \to S \to p^*(E \oplus O_X) \to Q \to 0 \qquad (*)$$

on Y, where S is the tautological subbundle of construction 10.1.5. The vector bundle Q (which has rank r) is usually called the *universal quotient bundle*. Note that we have a global section of $p^*(E \oplus O_X)$ by taking the point $(0,1)$ in every fiber (i. e. 0 in the fiber of E and 1 in the fiber of O_X). By definition of S the induced section $s \in \Gamma(Q)$ vanishes precisely on $\mathbb{P}(0 \oplus O_X) = X$.

(iii) Restricting $(*)$ to X (i. e. pulling the sequence back by i) we get the exact sequence

$$0 \to i^*S \to E \oplus O_X \to i^*Q \to 0 \qquad (*)$$

on X. Note that the first morphism is given by $\lambda \mapsto (0,\lambda)$ by construction, so we conclude that $i^*Q = E$.

(iv) As X is given in Y as the zero locus of a section of Q, we see from example 10.1.6 that the normal bundle of X in Y is just $N_{X/Y} = i^*Q = E$.

Let us now check proposition 10.5.6 in this case. Note that *away from the zero locus of s* there is an exact sequence

$$0 \to O_Y \stackrel{\cdot s}{\to} Q \stackrel{\wedge s}{\to} \Lambda^2 Q \stackrel{\wedge s}{\to} \Lambda^3 Q \to \cdots \to \Lambda^{r-1} Q \stackrel{\wedge s}{\to} \Lambda^r Q \to 0$$

of vector bundles (which follows from the corresponding statement for vector spaces). Dualizing and tensoring this sequence with p^*F we get the exact sequence

$$0 \to p^*F \otimes \Lambda^r Q^\vee \to p^*F \otimes \Lambda^{r-1} Q^\vee \to \cdots \to p^*F \otimes Q^\vee \to p^*F \to 0$$

again on $Y \backslash Z(s) = Y \backslash X$. Let us try to extend this exact sequence to all of Y. Note that the last morphism $p^*F \otimes Q^\vee \to p^*F$ is just induced by the evaluation morphism $s : Q^\vee \to \mathcal{O}_Y$, so its cokernel is precisely the sheaf $(p^*F)|_{Z(s)} = i_*F$. One can show that the other stages of the sequence remain indeed exact (see [F] B.3.4), so we get a locally free resolution

$$0 \to p^*F \otimes \Lambda^r Q^\vee \to p^*F \otimes \Lambda^{r-1} Q^\vee \to \cdots \to p^*F \otimes Q^\vee \to p^*F \to i_*F \to 0$$

on Y. (This resolution is called the *Koszul complex*.) So what we have to check is that

$$\sum_{k=0}^{r} (-1)^k \operatorname{ch}(p^*F \otimes \Lambda^k Q^\vee) = i_* \frac{\operatorname{ch}(F)}{\operatorname{td}(i^*Q)}.$$

But note that

$$i_* \frac{\operatorname{ch}(F)}{\operatorname{td}(i^*Q)} = \frac{\operatorname{ch}(p^*F)}{\operatorname{td}(Q)} \cdot i_*[X] = \frac{\operatorname{ch}(p^*F) c_r(Q)}{\operatorname{td}(Q)}$$

by the projection formula and proposition 10.3.12. So by the additivity of Chern characters it suffices to prove that

$$\sum_{k=0}^{r} (-1)^k \operatorname{ch}(\Lambda^k Q^\vee) = \frac{c_r(Q)}{\operatorname{td}(Q)}.$$

But this is easily done: if $\alpha_1, \dots, \alpha_r$ are the Chern roots of Q then the left hand side is

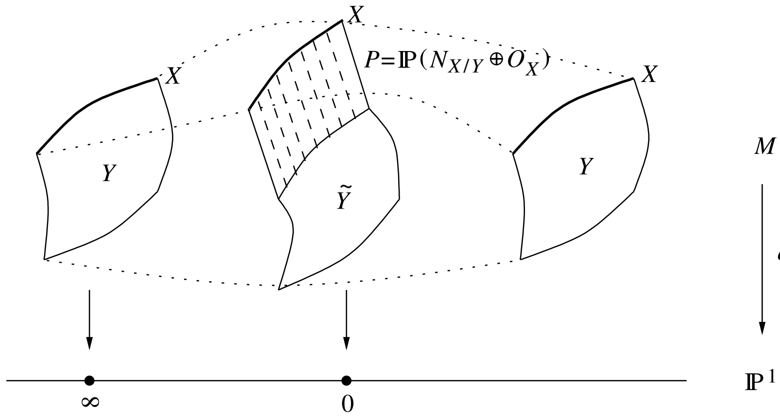
$$\sum_{k=0}^{r} (-1)^k \sum_{i_1 < \cdots < i_k} \exp(-\alpha_{i_1} - \cdots - \alpha_{i_k}) = \prod_{i=1}^{r} (1 - \exp(-\alpha_i)) = \alpha_1 \cdots \alpha_r \cdot \prod_{i=1}^{r} \frac{1 - \exp(-\alpha_i)}{\alpha_i},$$

which equals the right hand side. It is in fact this formal identity that explains the appearance of Todd classes in the Hirzebruch-Riemann-Roch theorem.

Using the computation of this special example we can now give the general proof of the Hirzebruch-Riemann-Roch theorem.

Proof. (of proposition 10.5.6) We want to reduce the proof to the special case considered in example 10.5.7.

Let $i : X \to Y$ be any inclusion morphism of smooth projective varieties. We denote by M be the blow-up of $Y \times \mathbb{P}^1$ in $X \times \{0\}$. The smooth projective scheme M comes together with a projection morphism $q : M \to \mathbb{P}^1$. Its fibers $q^{-1}(P)$ for $P \neq 0$ are all isomorphic to Y. The fiber $q^{-1}(0)$ however is reducible with two smooth components: one of them (the exceptional hypersurface of the blow-up) is the projectivized normal bundle of $X \times \{0\}$ in $Y \times \mathbb{P}^1$ by example 10.1.6, and the other one is simply the blow-up $\tilde{Y}$ of Y in X. We are particularly interested in the first component. As the normal bundle of $X \times \{0\}$ in $Y \times \mathbb{P}^1$ is $N_{X/Y} \oplus \mathcal{O}_X$ this component is just the projective bundle $P := \mathbb{P}(N_{X/Y} \oplus \mathcal{O}_X)$ on X. Note that there is an inclusion of the space $X \times \mathbb{P}^1$ in M that corresponds to the given inclusion $X \subset Y$ in the fibers $q^{-1}(P)$ for $P \neq 0$, and to the “zero section inclusion” $X \subset \mathbb{P}(N_{X/Y} \oplus \mathcal{O}_X) = P$ as in example 10.5.7 in the fiber $q^{-1}(0)$. The following picture illustrates the geometric situation.



The idea of the proof is now simply the following: we have to prove an equality in the Chow groups, i. e. modulo rational equivalence. The fibers $q^{-1}(0)$ and $q^{-1}(\infty)$ are rationally equivalent as they are the zero resp. pole locus of a rational function on the base $\mathbb{P}^1$, so they are effectively "the same" for intersection-theoretic purposes. But example 10.5.7 shows that the proposition is true in the fiber $q^{-1}(0)$, so it should be true in the fiber $q^{-1}(\infty)$ as well.

To be more precise, let F be a sheaf on X as in the proposition. Denote by $p_X : X \times \mathbb{P}^1 \to X$ the projection, and by $i_X : X \times \mathbb{P}^1 \to M$ the inclusion discussed above. Then $i_{X*}p_X^*F$ is a coherent sheaf on M that can be thought of as "the sheaf F on X in every fiber of q". By lemma 10.5.3 we can choose a locally free resolution

$$0 \to F_r \to F_{r-1} \to \cdots \to F_0 \to i_{X*}p_X^*F \to 0 \tag{1}$$

on M.

Note that the divisor $[0]-[\infty]$ on $\mathbb{P}^1$ is equivalent to zero by example 9.1.9. So it follows that

$$\sum_{k=0}^{r}(-1)^k \operatorname{ch}(F_i)\cdot q^*([0]-[\infty]) = 0$$

in $A_*(M)\otimes\mathbb{Q}$. Now by definition of the pull-back we have $q^*[0] = [\tilde{Y}]+[P]$ and $q^*[\infty] = [Y]$, so we get the equality

$$\sum_{k=0}^{r}(-1)^k \operatorname{ch}(F_i|_{\tilde{Y}})\cdot[\tilde{Y}] + \sum_{k=0}^{r}(-1)^k \operatorname{ch}(F_i|_P)\cdot[P] = \sum_{k=0}^{r}(-1)^k \operatorname{ch}(F_i|_Y)\cdot[Y] \tag{2}$$

in $A_*(M)\otimes\mathbb{Q}$. But note that the restriction to $\tilde{Y}$ of the sheaf $i_{X*}p_X^*F$ in (1) is the zero sheaf as $X\times\mathbb{P}^1\cap\tilde{Y} = \emptyset$ in M. So the sequence

$$0 \to F_r|_{\tilde{Y}} \to \cdots \to F_1|_{\tilde{Y}} \to F_0|_{\tilde{Y}} \to 0$$

is exact, which means that the first sum in (2) vanishes. The second sum in (2) is precisely $\frac{\operatorname{ch}(F)}{\operatorname{td}(N_{X/Y})}\cdot[X]$ by example 10.5.7. So we conclude that

$$\sum_{k=0}^{r}(-1)^k \operatorname{ch}(F_i|_Y)\cdot[Y] = \frac{\operatorname{ch}(F)}{\operatorname{td}(N_{X/Y})}\cdot[X]$$

in $A_*(M)\otimes\mathbb{Q}$. Pushing this relation forward by the (proper) projection morphism from M to Y then gives the desired equation. □

This completes the proof of the Hirzebruch-Riemann-Roch theorem 10.4.5.

Remark 10.5.8. Combining proposition 10.5.6 with remark 10.5.5 we see that we have just proven the following statement: let $f : X \to Y$ be a closed immersion of smooth projective schemes, and let F be a coherent sheaf on X. Then there is a locally free resolution

$$0 \to F_r \to F_{r-1} \to \cdots \to F_0 \to f_*F \to 0$$

of the coherent sheaf f_*F on Y such that

$$\sum_{k=0}^{r} (-1)^k \operatorname{ch}(F_k) \cdot \operatorname{td}(T_Y) = f_*(\operatorname{ch}(F) \cdot \operatorname{td}(T_X)) \in A_*(Y) \otimes \mathbb{Q}.$$

This is often written as

$$\operatorname{ch}(f_*F) \cdot \operatorname{td}(T_Y) = f_*(\operatorname{ch}(F) \cdot \operatorname{td}(T_X)).$$

In other words, "the push-forward f_* commutes with the operator τ of example 10.5.2 (iii)".

It is the statement of the **Grothendieck-Riemann-Roch theorem** that this relation is actually true for *any proper* morphism f of smooth projective schemes (and not just for closed immersions). See [F] section 15 for details on how to prove this.

The Grothendieck-Riemann-Roch theorem is probably one of the most general Riemann-Roch type theorems that one can prove. The only further generalization one could think of is to singular schemes. There are some such generalizations to mildly singular schemes; see [F] section 18 for details.

10.6. **Exercises.**

Exercise 10.6.1. Let $X = \mathbb{P}^1$, and for $n \in \mathbb{Z}$ let F_n be the projective bundle $F_n = \mathbb{P}(O_X \oplus O_X(n))$. Let $p : F_n \to X$ be the projection morphism. The surfaces F_n are called **Hirzebruch surfaces**.

(i) Show that $F_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$, and $F_n \cong F_{-n}$ for all n.
(ii) Show that all fibers $p^{-1}(P) \subset F_n$ for $P \in X$ are rationally equivalent as 1-cycles on F_n. Denote this cycle by $D \in A_1(F_n)$.
(iii) Now let $n \geq 0$. Show that the global section $(1, x_0^n)$ of $O_X \oplus O_X(n)$ (where x_0, x_1 are the homogeneous coordinates of X) determines a morphism $s : X \to F_n$. Denote by $C \in A_1(F_n)$ the class of the image curve $s(X)$.
(iv) Again for $n \geq 0$, show that $A_0(F_n) \cong \mathbb{Z}$ and $A_1(F_n) = \mathbb{Z} \cdot [C] \oplus \mathbb{Z} \cdot [D]$. Compute the intersection products C^2, D^2, and $C \cdot D$, arriving at a Bézout style theorem for the surfaces F_n.

Exercise 10.6.2. Let F and F' be two rank-2 vector bundles on a scheme X. Compute the Chern classes of $F \otimes F'$ in terms of the Chern classes of F and F'.

Exercise 10.6.3. Let F be a vector bundle of rank r on a scheme X, and let $p : \mathbb{P}(F) \to X$ be the projection. Prove that

$$D_F^r + D_F^{r-1} \cdot p^* c_1(F) + \cdots + p^* c_r(F) = 0,$$

where D_F is the Cartier divisor associated to the line bundle $O_{\mathbb{P}(F)}(1)$.

Exercise 10.6.4. Let $X \subset \mathbb{P}^4$ be the intersection of two general quadric hypersurfaces.

(i) Show that one expects a finite number of lines in X.
(ii) If there is a finite number of lines in X, show that this number is 16 (as usual counted with multiplicities (which one expects to be 1 for general X)).

Exercise 10.6.5. A **circle** in the plane $\mathbb{P}^2_{\mathbb{C}}$ is defined to be a conic passing through the two points $(1 : \pm i : 0)$.

Why is this called a circle?

How many circles are there in the plane that are tangent to

(i) three circles
(ii) two circles and a line
(iii) one circle and two lines
(iv) three lines

in general position? (Watch out for possible non-enumerative contributions in the intersection products you consider.)

If you are interested, try to find out the answer to the above questions over $\mathbb{R}$ (and the "usual" definition of a circle).

Exercise 10.6.6. Let $X \subset \mathbb{P}^4$ be a smooth quintic hypersurface, i. e. the zero locus of a homogeneous polynomial of degree 5.

(i) Show that one expects a finite number of lines in X, and that this expected number is then 2875.
(ii) Show that the number of lines on the special quintic $X = \{x_0^5 + \cdots + x_4^5 = 0\}$ is *not* finite. This illustrates the fact that the intersection-theoretic computations will only yield virtual numbers in general. (In fact one can show that the number of lines on a *general* quintic hypersurface in $\mathbb{P}^4$ is finite and that the computation of (i) then yields the correct answer.)

Exercise 10.6.7. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. Compute the number $K_X^2 + c_2(T_X)$ directly and check that it is divisible by 12 (see example 10.4.8).

Exercise 10.6.8. Let X and Y be isomorphic smooth projective varieties. Use the Hirzebruch-Riemann-Roch theorem 10.4.5 to prove that the constant coefficients of the Hilbert polynomials of X and Y agree, whereas the non-constant coefficients will in general be different.

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Note: This is a very extensive list of literature of varying usefulness. Here is a short recommendation which of the references you might want to use for what:

- For a general reference on the commutative algebra background, see [AM].
- For commutative algebra problems involving computational aspects, see [GP].
- For motivational aspects, examples, and a generally "fairy-tale" style introduction to the *classical theory* of algebraic geometry (no schemes) without much theoretical background, see [Ha], or maybe [S1] and [S2].
- For motivations and examples concerning scheme theory, see [EH], or maybe [S1] and [S3].
- For a good book that develops the theory, but largely lacks motivations and examples (especially in chapters II and III), see [H]. You should not try to read the "hard-core" parts of this book without some motivational background.
- For intersection theory and Chern classes the best reference is [F].
- For the ultimate reference ("if it is not proven there, it must be wrong"), see [EGA]. Warning: this is unreadable if you do not have a decent background in algebraic geometry yet, and it is close to being unreadable even if you do.