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CHROMATIC NUMBER OF GRAPHS AND EDGE FOLKMAN NUMBERS

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Abstract

We consider only simple graphs. The graph $G_1 + G_2$ consists of vertex disjoint copies of G_1 and G_2 and all possible edges between the vertices of G_1 and G_2. The chromatic number of the graph G will be denoted by $\chi(G)$ and the clique number of G by $\mathrm{cl}(G)$. The graphs G for which $\chi(G) - \mathrm{cl}(G) \geq 3$ are considered. For these graphs the inequality $|V(G)| \geq \chi(G) + 6$ was proved in [12], where $V(G)$ is the vertex set of G. In this paper we prove that equality $|V(G)| = \chi(G) + 6$ can be achieved only for the graphs $K_{\chi(G)-7} + Q$, $\chi(G) \geq 7$ and $K_{\chi(G)-9} + C_5 + C_5 + C_5$, $\chi(G) \geq 9$, where graph Q is given in Fig. 1 and K_n and C_5 are complete graphs on n vertices and simple 5-cycle, respectively. With the help of this result we prove the new facts for some edge Folkman numbers (Theorem 4.2).

Key words: vertex Folkman numbers, edge Folkman numbers
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1. Introduction. We consider only finite, non-oriented graphs without loops and multiple edges. We call a p-clique of the graph G a set of p vertices each two of which are adjacent. The largest positive integer p such that G contains a p-clique is denoted by $\mathrm{cl}(G)$ (clique number of G). We shall use also the following notations:

- $V(G)$ is the vertex set of G;
- $E(G)$ is the edge set of G;

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- $\overline{G}$ is the complement of G;
- $G - V$, $V \subseteq V(G)$ is the subgraph of G induced by $V(G) \setminus V$;
- $\alpha(G)$ is the vertex independence number of G;
- $\chi(G)$ is the chromatic number of G;
- $f(G) = \chi(G) - \mathrm{cl}(G)$;
- K_n is the complete graph on n vertices;
- C_n is the simple cycle on n vertices;
- $N_G(v)$ is the set of neighbours of a vertex v in G.

Let G_1 and G_2 be two graphs. We denote by $G_1 + G_2$ the graph G for which $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[x,y], x \in V(G_1), y \in V(G_2)\}$.

We will use the following theorem by Dirac [2]:

Theorem 1.1. *Let G be a graph such that $f(G) \geq 1$. Then $|V(G)| \geq \chi(G)+2$ and $|V(G)| = \chi(G) + 2$ only when $G = K_{\chi(G)-3} + C_5$.*

If $f(G) \geq 2$, then we have [12] (see also [16]).

Theorem 1.2. *Let $f(G) \geq 2$. Then*

(a) $|V(G)| \geq \chi(G) + 4$*;*

(b) $|V(G)| = \chi(G) + 4$ *only when* $\chi(G) \geq 6$ *and* $G = K_{\chi(G)-6} + C_5 + C_5$.

In the case $\chi(G) = 4$ and $\chi(G) = 5$ we have the following better inequalities:

(1.1) if $f(G) \geq 2$ and $\chi(G) = 4$ then $|V(G)| \geq 11$, [1];

(1.2) if $f(G) \geq 2$ and $\chi(G) = 5$ then $|V(G)| \geq 11$, [13] (see also [14]).

For the case $f(G) \geq 3$ it was known that [12] (see also [17, 18])

Theorem 1.3. *Let G be a graph such that $f(G) \geq 3$. Then $|V(G)| \geq \chi(G) + 6$.*

In this paper we consider the case $|V(G)| = \chi(G)+6$. We prove the following main theorem.

Theorem 1.4. *Let G be a graph such that $f(G) \geq 3$ and $|V(G)| = \chi(G)+6$. Then $\chi(G) \geq 7$ and $G = K_{\chi(G)-7}+Q$ or $\chi(G) \geq 9$ and $G = K_{\chi(G)-9} + C_5 + C_5 + C_5$, where Q is the graph, whose complementary graph $\overline{Q}$ is given in Fig. 1.*

Obviously, if $f(G) \geq 3$ then $\chi(G) \geq 5$. Therefore we will consider only the cases $\chi(G) \geq 5$. If $\chi(G) = 5$ or $\chi(G) = 6$ then by Theorem 1.3 and Theorem 1.4

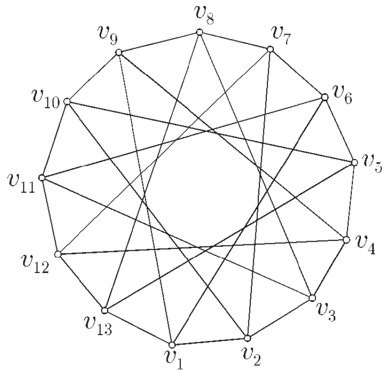

Fig. 1. Graph $\overline{Q}$

we see that $|V(G)| \geq \chi(G) + 7$. In these two cases we can state the following stronger results:

$$\text{if } f(G) \geq 3 \text{ and } \chi(G) = 5 \text{ then } |V(G)| \geq 22, [6]; \tag{1.3}$$

$$\text{if } f(G) \geq 3 \text{ and } \chi(G) = 6 \text{ then } |V(G)| \geq 16, [9]. \tag{1.4}$$

The inequalities (1.3) and (1.4) are exact. LATHROP and RADZISZOWSKI [9] proved that there are only two 16-vertex graphs for which (1.4) holds.

At the end of this paper we obtain by Theorem 1.4 new results about some edge-Folkman numbers (Theorem 4.2).

2. Auxiliary results. A graph G is defined to be vertex-critical chromatic if $\chi(G-v) < \chi(G)$ for all $v \in V(G)$. We shall use the following results of GALLAI [4] (see also [5]).

Theorem 2.1. *Let G be a vertex-critical chromatic graph and $\chi(G) \geq 2$. If $|V(G)| < 2\chi(G) - 1$ then $G = G_1 + G_2$, where $V(G_i) \neq \varnothing$, $i = 1, 2$.*

Theorem 2.2. *Let G be a vertex-critical k-chromatic graph, $|V(G)| = n$ and $k \geq 3$. Then there exist $\geq \left\lceil \frac{3}{2}\left(\frac{5}{3}k - n\right)\right\rceil$ vertices with the property that each of them is adjacent to all the other $n-1$ vertices.*

Remark 2.1. The formulations of Theorem 2.1 and Theorem 2.2 given above are obviously equivalent to the original ones in [4] (see Remark 1 and Remark 2 in [16]).

Proposition 2.1. *Let G be a graph such that $f(G) \geq 3$ and $|V(G)| = \chi(G) + 6$. Then G is a vertex-critical chromatic graph.*

Proof. Assume the opposite. Then $\chi(G - v) = \chi(G)$ for some $v \in V(G)$. Let $G' = G - v$. Since $\mathrm{cl}(G') \leq \mathrm{cl}(G)$ we have $f(G') \geq f(G) \geq 3$. By Theorem 1.3

$$|V(G')| \geq \chi(G') + 6 = \chi(G) + 6 = |V(G)|,$$

which is a contradiction. □

The following result by KERRY [7] will be used later.

Theorem 2.3. *Let G be a 13-vertex graph such that $\alpha(G) \leq 2$ and $\mathrm{cl}(G) \leq 4$. Then G is isomorphic to the graph Q, whose complementary graph $\overline{Q}$ is given in Fig. 1.*

Definition 2.1. The graph G is called a Sperner graph if $N_G(u) \subseteq N_G(v)$ for some $u, v \in V(G)$.

Obviously if $N_G(u) \subseteq N_G(v)$ then $\chi(G-u) = \chi(G)$. Thus we have

Proposition 2.2. *Every vertex-critical chromatic graph is not a Sperner graph.*

The following lemmas are used in the proof of Theorem 1.4.

Lemma 2.1. *Let G be a graph and $f(G) \geq 2$. Then*

(a) $|V(G)| \geq 10$*;*

(b) $|V(G)| = 10$ *only when* $G = C_5 + C_5$.

Proof. The inequality (a) follows from (1.1), (1.2) and Theorem 1.2(a). Let $|V(G)| = 10$. Then by (1.1), (1.2) and Theorem 1.2(a) we see that $\chi(G) = 6$. From Theorem 1.2(b) we obtain $G = C_5 + C_5$. $\square$

Lemma 2.2. *Let G be a graph such that $f(G) \geq 3$ and G is not a Sperner graph. Then*

$$|V(G)| \geq 11 + \alpha(G).$$

Proof. Assume the opposite, i.e.

$$|V(G)| \leq 10 + \alpha(G). \tag{2.1}$$

Let $A \subseteq V(G)$ be an independent set of vertices of G such that $|A| = \alpha(G)$. Consider the subgraph $G' = G - A$. From (2.1) we see that $|V(G')| \leq 10$. Since A is independent from $f(G) \geq 3$ it follows $f(G') \geq 2$. According to Lemma 2.1(b), $G' = C_5^{(1)} + C_5^{(1)}$, where $C_5^{(i)}$, $i = 1, 2$, are 5-cycles. Hence $\chi(G') = 6$, $\chi(G) \leq 7$ and $\mathrm{cl}(G) \leq 4$. Thus if $a \in A$, then $N_G(a) \cap V(C_5^{(1)})$ or $N_G(a) \cap V(C_5^{(2)})$ is an independent set. Let $N_G(a) \cap V(C_5^{(1)})$ be independent set and $C_5^{(1)} = v_1v_2v_3v_4v_5v_1$. Then we may assume that $N_G(a) \cap V(C_5^{(1)}) \subseteq \{v_1, v_3\}$. We obtain that $N_G(a) \subseteq N_G(v_2)$ which contradicts the assumption of Lemma 2.2. $\square$

Lemma 2.3. *Let G be a graph such that $f(G) \geq 3$ and $|V(G)| = \chi(G) + 6$. Then $\chi(G) \geq 7$ and:*

(a) $G = Q$ *if* $\chi(G) = 7$*;*

(b) $G = K_1 + Q$ *if* $\chi(G) = 8$*;*

(c) $G = K_2 + Q$ *or* $G = C_5 + C_5 + C_5$ *if* $\chi(G) = 9$.

Proof. Since $\chi(G) \neq \mathrm{cl}(G)$ we have $\mathrm{cl}(G) \geq 2$. Thus, from $f(G) \geq 3$ it follows $\chi(G) \geq 5$. By (1.3) and (1.4) we see that $\chi(G) \neq 5$ and $\chi(G) \neq 6$. Hence, $\chi(G) \geq 7$.

CASE 1. $\chi(G) = 7$. In this case $|V(G)| = 13$. From $\chi(G) = 7$ and $f(G) \geq 3$ we see that $\mathrm{cl}(G) = 4$. It follows from Lemma 2.2 that $\alpha(G) \leq 2$. Thus, by Theorem 2.3, $G = Q$.

CASE 2. $\chi(G) = 8$. In this situation we have $|V(G)| = 14$. By Proposition 2.1, G is a vertex-critical chromatic graph. Since $|V(G)| < 2\chi(G) - 1$, from Theorem 2.1 we obtain that $G = G_1 + G_2$. Clearly,

$$|V(G)| = |V(G_1)| + |V(G_2)|; \tag{2.2}$$

$$\chi(G) = \chi(G_1) + \chi(G_2); \tag{2.3}$$

$$f(G) = f(G_1) + f(G_2); \tag{2.4}$$

$$G_1 \text{ and } G_2 \text{ are vertex-critical chromatic graphs.} \tag{2.5}$$

SUBCASE 2.A. $G = K_1 + G'$. Since $\chi(G') = 7$ and $f(G') = f(G) \geq 3$, by the Case 1 we obtain $G' = Q$ and $G = K_1 + Q$.

SUBCASE 2.B. G_1 and G_2 are not complete graphs. In this subcase, by (2.5), we have $\chi(G_i) \geq 3$ and $\chi(G_i) \neq \mathrm{cl}(G_i)$, $i = 1, 2$. Thus $f(G_i) \geq 1$, $i = 1, 2$. According to Theorem 1.1, $|V(G_i)| \geq 5$, $i = 1, 2$. From these inequalities and (2.2) it follows

$$|V(G_i)| \leq 9, \quad i = 1, 2. \tag{2.6}$$

Let $f(G_1) \leq f(G_2)$. Then, by (2.4), $f(G_2) \geq 2$. From Lemma 2.1 we obtain $|V(G_2)| \geq 10$. This contradicts (2.6).

CASE 3. $\chi(G) = 9$. In this case $|V(G)| = 15$. By Proposition 2.1, G is a vertex-critical chromatic graph. Since $|V(G)| < 2\chi(G) - 1$, from Theorem 2.1 it follows that $G = G_1 + G_2$.

SUBCASE 3.A. $G = K_1 + G'$. Since $|V(G')| = 14$, $\chi(G') = 8$ and $f(G') = f(G) \geq 3$, by Case 2 we have $G' = K_1 + Q$. Hence $G = K_2 + Q$.

SUBCASE 3.B. G_1 and G_2 are not complete graphs. By (2.5) it follows $|V(G_i)| \geq 5$, $i = 1, 2$. From these inequalities and (2.2) we obtain

$$|V(G_i)| \leq 10, \quad i = 1, 2. \tag{2.7}$$

Let $f(G_1) \leq f(G_2)$. Then according to (2.4) we have $f(G_2) \geq 2$. From (2.7) and Theorem 2.1 we obtain $G_2 = C_5 + C_5$. Since $|V(G_2)| = 10$ and $\chi(G_2) = 6$ we see from (2.2) and (2.3) that $|V(G_1)| = 5$ and $\chi(G_1) = 3$. Thus, by (2.5), we conclude that $G_1 = C_5$. Hence $G_1 = C_5 + C_5 + C_5$. □

3. Proof of Theorem 1.4. By Lemma 2.3 we have that $\chi(G) \geq 7$. If $\chi(G) = 7$ or $\chi(G) = 8$ Theorem 1.4 follows from Lemma 2.3. Let $\chi(G) \geq 9$. We prove Theorem 1.4 by induction on $\chi(G)$. The inductive base $\chi(G) = 9$ follows from Lemma 2.3(c). Let $\chi(G) \geq 10$. Then $\frac{5}{3}\chi(G) - |V(G)| > 0$. By Proposition 2.1 G is vertex-critical chromatic graph. Thus, according to Theorem 2.2, we have $G = K_1 + G'$. As $\chi(G') = \chi(G) - 1$, $f(G') = f(G) \geq 3$ and $|V(G')| = \chi(G') + 6$, we can now use the inductive assumption and obtain

$$G' = K_{\chi(G')-7} + Q \quad \text{or} \quad G' = K_{\chi(G')-9} + C_5 + C_5 + C_5.$$

Hence $G = K_{\chi(G)-7} + Q$ or $G = K_{\chi(G)-9} + C_5 + C_5 + C_5$.

4. Edge Folkman numbers $\boldsymbol{F_e(a_1, \dots, a_r; R(a_1, \dots, a_r) - 2)}$. Let $a_1, \dots, a_r$ be integers, $a_i \geq 2$, $i = 1, \dots, r$. The symbol $G \overset{e}{\to} (a_1 \dots, a_r)$ means that in every r-coloring

$$E(G) = E_1 \cup \dots \cup E_r, \quad E_i \cap E_j = \varnothing, \quad i \neq j,$$

of the edge set $E(G)$there exists a monochromatic a_i-clique Q of colour i for some $i \in \{1, \dots, r\}$, that is $E(Q) \subseteq E_i$. The Ramsey number $R(a_1, \dots, a_r)$ is defined as $\min\{n : K_n \overset{e}{\to} (a_1, \dots, a_r)\}$. Define

$$H_e(a_1, \dots, a_r; q) = \{G : G \overset{e}{\to} (a_1 \dots, a_r) \text{ and } \mathrm{cl}(G) < q\};$$
$$F_e(a_1, \dots, a_r; q) = \min\{|V(G)| : G \in H_e(a_1, \dots, a_r; q)\}.$$

It is well known that

$$F_e(a_1, \dots, a_r; q) \text{ exists} \iff q > \max\{a_1, \dots, a_r\}. \tag{4.1}$$

In the case $r = 2$ this was proved in [3] and the general case in [19]. The numbers $F_e(a_1, \dots, a_r; q)$ are called edge Folkman numbers. An exposition of the known edge Folkman numbers is given in [8]. In this section we consider the numbers $F_e(a_1, \dots, a_r; R(a_1 \dots, a_r) - 2)$, where $a_i \geq 3$, $i = 1, \dots, r$. We know only one Folkman number of this kind, namely $F_e(3, 3, 3, 3; 15) = 23$ (see [11]).

In [12] we prove the following statement.

Theorem 4.1. *Let* $a_1, \dots, a_r$ *be integers and* $a_i \geq 3$, $i = 1, \dots, r$, $r \geq 2$. *Then*

$$F_e(a_1, \dots, a_r; R(a_1 \dots, a_r) - 2) \geq R(a_1 \dots, a_r) + 6. \tag{4.2}$$

Remark 4.1. It follows from $a_i \geq 3$ and $r \geq 2$ that $R(a_1, \dots, a_r) > 2 + \max\{a_1, \dots, a_r\}$. Thus, by (4.1), the numbers $F_e(a_1, \dots, a_r; R(a_1, \dots, a_r) - 2)$ exist.

The aim of this section is to prove the following result.

Theorem 4.2. *Let* $a_1, \dots, a_r$ *be integers and* $a_i \geq 3$, $i = 1, \dots, r$, $r \geq 2$. *Then*

$$F_e(a_1, \dots, a_r; R(a_1, \dots, a_r) - 2) = R(a_1, \dots, a_r) + 6$$

if and only if $K_{R-7} + Q \overset{e}{\to} (a_1, \dots, a_r)$ *or* $K_{R-9} + C_5 + C_5 + C_5 \overset{e}{\to} (a_1, \dots, a_r)$, *where* $R = R(a_1, \dots, a_r)$.

We shall use the following result obtained by Lin [10]:

$$G \overset{e}{\to} (a_1, \dots, a_r) \Rightarrow \chi(G) \geq R(a_1, \dots, a_r). \tag{4.3}$$

Proof of Theorem 4.2. I. Let $F_e(a_1, \dots, a_r; R-2) = R + 6$. Let $G \in H_e(a_1, \dots, a_r; R-2)$ and

$$|V(G)| = R + 6. \tag{4.4}$$

Since $\mathrm{cl}(G) \leq R - 3$, from (4.3) it follows $f(G) \geq 3$. By Theorem 1.3, we have

$$|V(G)| \geq \chi(G) + 6. \tag{4.5}$$

From (4.3), (4.4) and (4.5) we see that $\chi(G) = R$ and $|V(G)| = \chi(G) + 6$. Thus, according to Theorem 1.4, $G = K_{\chi(G)-7} + Q = K_{R-7} + Q$ or $G = K_{\chi(G)-9} + C_5 + C_5 + C_5 = K_{R-9} + C_5 + C_5 + C_5$. This implies $K_{R-7} + Q \overset{e}{\to} (a_1, \dots, a_r)$ or $K_{R-9} + C_5 + C_5 + C_5 \overset{e}{\to} (a_1, \dots, a_r)$ because $G \in H_e(a_1, \dots, a_r; R-2)$.

II. Let $K_{R-7} + Q \overset{e}{\to} (a_1, \dots, a_r)$. Then $K_{R-7} + Q \in H_e(a_1, \dots, a_r; R-2)$ because $\mathrm{cl}(K_{R-7} + Q) = R - 3$. Hence

$$F_e(a_1, \dots, a_r; R-2) \leq |V(K_{R-7} + Q)| = R + 6.$$

This inequality and (4.2) imply that $F_e(a_1, \dots, a_r; R-2) = R + 6$.

In the same way we see that from $K_{R-9} + C_5 + C_5 + C_5 \overset{e}{\to} (a_1, \dots, a_r)$ it follows that $F_e(a_1, \dots, a_r; R-2) = R + 6$. □

Remark 4.2. We obtain, in [11], the equality $F_e(3,3,3;15) = 23$ proving that $K_8 + C_5 + C_5 + C_5 \overset{e}{\to} (3,3,3)$. We do not know whether $K_{10} + Q \overset{e}{\to} (3,3,3)$.

Remark 4.3. By Theorem 4.1 we have $F_e(3,5;12) \geq 20$ and $F_e(4,4;16) \geq 24$. The exact values of these numbers are not known. Therefore, having in mind Theorem 4.2, it will be interesting to know whether the following statements are true:

$$K_7 + Q \overset{e}{\to} (3,5), \qquad K_5 + C_5 + C_5 + C_5 \overset{e}{\to} (3,5);$$

$$K_{11} + Q \overset{e}{\to} (4,4), \qquad K_9 + C_5 + C_5 + C_5 \overset{e}{\to} (4,4).$$

Remark 4.4. By Theorem 4.1, $F_e(3,4;7) \geq 15$. It was proved in [8] that $F_e(3,4;8) = 16$. Thus $F_e(3,4;7) \geq 17$.

REFERENCES

[1] Chvátal V. Lecture Notes in Math., **406**, 1979, 243–246.
[2] Dirac G. J. London Math. Soc., **31**, 1956, 460–471.
[3] Folkman J. SIAM J. Appl. Math., **18**, 1970, 19–24.
[4] Gallai T. Publ. Math. Inst. Hung. Acad. Sci., Ser. A, **8**, 1963, 373–395.
[5] Gallai T. In: Theory of Graphs and Its Applications, Proceedings of the Symposium held in Smolenice in June 1963, Czechoslovak Acad. Sciences, Prague, 1964, 43–45.
[6] Jensen T., G. Royle. J. Graph Theory, **19**, 1995, 107–116.
[7] Kery G. Mat. Lapok, **15**, 1964, 204–224.
[8] Kolev N., N. Nenov. Compt. rend. Acad. bulg. Sci., **59**, 2006, No 1, 25–30.
[9] Lathrop J., S. Radziszowski. Compute the Folkman Number $F_v(2,2,2,2,2;4)$. 23rd Midwest Conference on Combinatorics, Cryptography and Computing, Oct 3–4, 2009 (to appear in JCMCC).
[10] Lin S. J. Combin. Theory, Ser. B, **12**, 1972, 82–92.
[11] Nenov N. Compt. rend. Acad. bulg. Sci., **34**, 1981, 1209–1212 (in Russian).
[12] Nenov N. Serdica Bulg. Math. Publ., **9**, 1983, 161–167 (in Russian).
[13] Nenov N. Compt. rend. Acad. bulg. Sci., **37**, 1984, 301–304 (in Russian).
[14] Nenov N. Discrete Math., **188**, 1998, 297–298.
[15] Nenov N. Ann. Univ. Sofia Fac. Math. Inform., **95**, 2001, 59–69.
[16] Nenov N. Discrete Math., **271**, 2003, 327–334.
[17] Nenov N. On the vertex Folkman numbers $F_v(\underbrace{2,\dots,2}_{r};r-1)$ and $F_v(\underbrace{2,\dots,2}_{r};r-2)$. Preprint: `arXiv: 0903.3151v1[math. CO]`, 18 Mar 2009, Annuaire Univ. Sofia Fac. Math. Inform. (submitted 2007).
[18] Nenov N. Serdica Math. J., **35**, 2009, 251–272.
[19] Nesetril J., V. Rödl. J. Combin. Theory, Ser. B, **20**, 1976, 243–249.

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