Galois groups of co-abelian ball quotient covers

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Abstract

If $X' = (\mathbb{B}/\Gamma)'$ is a torsion free toroidal compactification of a discrete ball quotient $X_o = \mathbb{B}/\Gamma$ and $\xi : (X', T = X' \setminus X_o) \to (X, D = \xi(T))$ is the blow-down of the (-1)-curves to the corresponding minimal model, then $G' = Aut(X', T)$ coincides with the finite group $G = Aut(X, D)$. In particular, for an elliptic curve E with endomorphism ring $R = End(E)$ and a split abelian surface $X = A = E \times E$, G is a finite subgroup of $Aut(A) = \mathcal{T}_A \leftthreetimes GL(2, R)$, where $(\mathcal{T}_A, +) \simeq (A, +)$ is the translation group of A and $GL(2, R) = \{g \in R_{2\times 2} \mid \det(g) \in R^*\}$.

The present work classifies the finite subgroups H of $Aut(A = E \times E)$ for an arbitrary elliptic curve E. By the means of the geometric invariants theory, it characterizes the Kodaira-Enriques types of $A/H \simeq (\mathbb{B}/\Gamma)'/H$, in terms of the fixed point sets of H on A. The abelian and the K3 surfaces A/H are elaborated in [7]. The first section provides necessary and sufficient conditions for A/H to be a hyper-elliptic, ruled with elliptic base, Enriques or a rational surface. In such a way, it depletes the Kodaira-Enriques classification of the finite Galois quotients A/H of a split abelian surface $A = E \times E$. The second section derives a complete list of the conjugacy classes of the linear automorphisms $g \in GL(2, R)$ of A of finite order, by the means of their eigenvalues. The third section classifies the finite subgroups H of $GL(2, R)$. The last section provides explicit generators and relations for the finite subgroups H of $Aut(A)$ with K3, hyper-elliptic, rules with elliptic base or Enriques quotients $A/H \simeq (\mathbb{B}/\Gamma)'/H$.

Let

$$\mathbb{B} = \{z = (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\} \simeq SU_{2,1}/S(U_2 \times U_1).$$

be the complex 2-ball. In [4] Holzapfel settled the problem of the characterization of the projective surfaces, which are birational to an eventually singular ball quotient $\mathbb{B}/\Gamma$ by a lattice Γ of $SU_{2,1}$. Note that if $\gamma \in \Gamma$ is a torsion element with isolated fixed points on $\mathbb{B}$ then $\mathbb{B}/\Gamma$ has isolated cyclic quotient singularity, which ought to be resolved in order to obtain a smooth open surface. The aforementioned resolution creates smooth rational curves of self-intersection ≤ -2, which alter the local differential geometry of $\mathbb{B}/\Gamma$, modeled by $\mathbb{B}$. That is why we split the problem to the description of the minimal models X_o of the smooth toroidal compactifications $X'_o = (\mathbb{B}/\Gamma_o)'$ of torsion free Γ_o and to the characterization of the birational equivalence classes of

X_o/H for appropriate finite automorphism groups H. This reduction is based on the fact that any finitely generated lattice Γ in the simple Lie group $SU_{2,1}$ has a torsion free normal subgroup Γ_o of finite index $[\Gamma : \Gamma_o]$. Therefore $\mathbb{B}/\Gamma = (\mathbb{B}/\Gamma_o)\,/\,(\Gamma/\Gamma_o)$ and the classification of $\mathbb{B}/\Gamma$ is attempted by the classification of $\mathbb{B}/\Gamma_o$ and the finite automorphism groups $H = \Gamma/\Gamma_o$ of $\mathbb{B}/\Gamma_o$.

According to the next proposition, for any torsion free ball lattice Γ_o and any $\Gamma < SU_{2,1}$, containing Γ_o as a normal subgroup of finite index, the quotient group Γ/Γ_o acts on the toroidal compactifying divisor $T = (\mathbb{B}/\Gamma_o)' \setminus (\mathbb{B}/\Gamma_o)$ and provides a compactification $\overline{\mathbb{B}/\Gamma} = (\mathbb{B}/\Gamma_o)'\,/\,(\Gamma/\Gamma_o)$ of $\mathbb{B}/\Gamma$ with at worst isolated cyclic quotient singularities. Therefore $H = \Gamma/\Gamma_o$ is a subgroup of $Aut(X_o', T)$. The birational equivalence classes of $\overline{\mathbb{B}/\Gamma}$ are to be described by the numerical invariants of the minimal resolutions Y of the singularities of $\overline{\mathbb{B}/\Gamma}$. These can be computed by the means of the geometric invariant theory, applied to X_o and a finite subgroup H of the biholomorphism group $Aut(X_o)$.

Proposition 1. *Let Γ be a lattice of $SU_{2,1}$ and Γ_o be a normal torsion free subgroup of Γ with finite index $[\Gamma : \Gamma_o]$. Then the group $G = \Gamma/\Gamma_o$ acts on the toroidal compactifying divisor $T = (\mathbb{B}/\Gamma_o)' \setminus (\mathbb{B}/\Gamma_o)$ and the quotient $(\mathbb{B}/\Gamma_o)'/G = (\mathbb{B}/\Gamma) \cup (T/G) = \overline{\mathbb{B}/\Gamma}$ is a compactification of $\mathbb{B}/\Gamma$ with at worst isolated cyclic quotient singularities.*

Proof. Recall that $p \in \partial_\Gamma \mathbb{B}$ is a Γ-rational boundary point exactly when the intersection $\Gamma \cap Stab_{SU_{2,1}}(p)$ is a lattice of $Stab_{SU_{2,1}}(p)$. Since $[\Gamma : \Gamma_o] < \infty$, the quotient

$$Stab_{SU_{2,1}}(p)/[\Gamma \cap Stab_{SU_{2,1}}(p)] =$$

$$= \left\{Stab_{SU_{2,1}}(p)/\left[\Gamma_o \cap Stab_{SU_{2,1}}(p)\right]\right\} / \left\{[\Gamma \cap Stab_{SU_{2,1}}(p)/[\Gamma_o \cap Stab_{SU_{2,1}}(p)]\right\}$$

has finite invariant volume exactly when $Stab_{SU_{2,1}}(p)/[\Gamma_o \cap Stab_{SU_{2,1}}(p)]$ has finite invariant volume. Therefore the Γ-rational boundary points coincide with the Γ_o-rational boundary points, $\partial_\Gamma \mathbb{B} = \partial_{\Gamma_o} \mathbb{B}$. It suffices to establish that the Γ-action on $\mathbb{B}$ admits local extensions on neighborhoods of the liftings of T_i to complex lines through $p_i \in \partial_{\Gamma_o}\mathbb{B}$ with $Orb_{\Gamma_o}(p_i) = \kappa_i$. According to [**?**], the cusp κ_i, associated with the smooth elliptic curve T_i has a neighborhood $N(\kappa_i) = T_i \times \Delta^*(0,\varepsilon) \subset (\mathbb{B}/\Gamma_o)$ for a sufficiently small punctured disc $\Delta^*(0,\varepsilon) = \{z \in \mathbb{C} \mid |z| < \varepsilon\}$. The biholomorphisms $\gamma : \mathbb{B} \to \mathbb{B}$ from Γ extend to $\gamma : \mathbb{B} \cup \partial_{\Gamma_o}\mathbb{B} \to \mathbb{B} \cup \partial_{\Gamma_o}\mathbb{B}$, as far as $\partial_{\Gamma_o}\mathbb{B}$ consists of isolated points. If $p_i \in \partial_{\Gamma_o}\mathbb{B}$, $\gamma(p_i) = p_j \in \partial_{\Gamma_o}\mathbb{B}$ and $\kappa_j = Orb_{\Gamma_o}(p_j)$ then there is a biholomorphism

$$\gamma : N(\kappa_i) \cap \gamma^{-1} N(\kappa_j) \longrightarrow \gamma N(\kappa_i) \cap N(\kappa_j).$$

For any $q_i \in T_i$ let $\Delta_{T_i}(q_i, \eta)$ be a sufficiently small disc on T_i, centered at q_i, which is contained in a $\pi_1(T_i)$-fundamental domain, centered at q_i. One can view $\Delta_{T_i}(q_i,\eta) = \Delta_{\widetilde{T_i}}(q_i,\eta)$ as a disc on the lifting $\widetilde{T_i}$ of T_i to a complex line through p_i.

Then $N(\kappa_i, q_i) := \Delta_{\widetilde{T_i}}(q_i, \eta) \times \Delta^*(0, \varepsilon)$ is a bounded neighborhood of $q_i \in T_i$ on $\mathbb{B}/\Gamma_o$ and the holomorphic map

$$\gamma : N(\kappa_i, q_i) \cap \gamma^{-1} N(\kappa_j, q_j) \to \gamma N(\kappa_i, q_i) \cap N(\kappa_j, q_j) \subseteq N(\kappa_j, q_j) = \Delta_{\widetilde{T_j}}(q_j, \eta) \times \Delta^*(0, \varepsilon)$$

is bounded. Thus, $\gamma : \mathbb{B} \to \mathbb{B}$ is locally bounded around $\widetilde{T} = \sum_{p_i \in \partial_{\Gamma_o}\mathbb{B}} \widetilde{T}_i(p_i)$ and admits a holomorphic extension $\gamma : \mathbb{B} \cup \widetilde{T} \to \mathbb{B} \cup \widetilde{T}$. This induces a biholomorphism $\gamma\Gamma_o : (\mathbb{B}/\Gamma_o)' \to (\mathbb{B}/\Gamma_o)'$.

$\square$

The next proposition establishes that an arbitrary torsion free toroidal compactification $(\mathbb{B}/\Gamma_o)'$ has finitely many Galois quotients $(\mathbb{B}/\Gamma_o)'/H = \overline{\mathbb{B}/\Gamma_H}$ with $\Gamma_H/\Gamma_o = H$. For torsion free $(\mathbb{B}/\Gamma_o)'$ with an abelian minimal model $X_o = A$, the result is proved in [6]. Note also that [9] constructs an infinite series $\left\{(\mathbb{B}/\Gamma_n)'\right\}_{n=1}^{\infty}$ of mutually non-birational torsion free toroidal compactifications with abelian minimal models, which are finite Galois covers of a fixed $\left(\overline{\mathbb{B}/\Gamma_{H_1}}, T(1)/H\right) = \left((\mathbb{B}/\Gamma_n)', T(n)\right)/H_n$, $H_n \leq Aut\left((\mathbb{B}/\Gamma_n)', T(n)\right)$.

Proposition 2. *Let $X' = (\mathbb{B}/\Gamma)' = (\mathbb{B}/\Gamma) \cup T$ be a torsion free toroidal compactification and $\xi : (X', T) \to (X = \xi(X'), D = \xi(T))$ be the blow-down of the (-1)-curves to the minimal model X of X'. Then $Aut(X', T)$ is a finite group, which coincides with $Aut(X, D)$.*

Proof. Let us denote $G = Aut(X, D)$, $G' = Aut(X', T)$ and observe that X' is the blow-up of X at the singular locus D^{sing} of D. Since $D = \sum_{i=1}^{h} D_i$ has smooth elliptic irreducible components D_i, the singular locus $D^{\text{sing}} = \sum_{1 \leq i < j \leq h} D_i \cap D_j$ and its complement $X \setminus D^{\text{sing}}$ are G-invariant. We claim that the G-action extends to the exceptional divisor $E = \xi^{-1}(D^{\text{sing}})$ of ξ, so that $X' = (X \setminus D^{\text{sing}}) \cup E$ is G-invariant. Indeed, for any $g \in G$ and $p \in D^{\text{sing}}$ with $q = g(p)$, let us choose local holomorphic coordinates $x = (x_1, x_2)$, respectively, $y = (y_1, y_2)$ on sufficiently small neighborhoods $N(p)$, $N(q)$ of p and q on X with $gN(p) \subseteq N(q)$. Then $g : N(p) \to N(q) \subset \mathbb{C}^2$ consists of two local holomorphic functions $g = (g_1, g_2)$ on $N(p)$. By the very definition of a blow-up,

$$\xi^{-1}N(p) = \{(x_1, x_2) \times [x_1 : x_2] \ \mid \ (x_1, x_2) \in N(p)\} \ \mid \text{and}$$

$$\xi^{-1}N(q) = \{(g_1(x), g_2(x)) \times [g_1(x) : g_2(x)] \ \mid \ g(x) = (g_1(x), g_2(x)) \in N(q)\},$$

so that

$$g : \xi^{-1}N(p) \to \xi^{-1}N(q),$$

$$(x_1, x_2) \times [x_1 : x_2] \mapsto (g_1(x), g_2(x)) \times [g_1(x) : g_2(x)]$$

extends the action of $g \in G$ to $\xi^{-1}(D^{\rm sing})$ and $G \subset Aut(X')$. Towards the G-invariance of T, note that the birational maps $\xi : T_i \to \xi(T_i) = D_i$ of the smooth irreducible components T_i of T are biregular. Thus, the G-invariance of $D = \sum\limits_{i=1}^{h} D_i$ implies the G-invariance of $T = \sum\limits_{i=1}^{h} T_i$ and $G \subseteq G' = Aut(X', T)$. For the opposite inclusion $G' = Aut(X', T) \subseteq G = Aut(X, D)$ observe that an arbitrary $g' \in G'$ acts on the union E of the (-1)-curves on X' and permutes the finite set $\xi(E) = D^{\rm sing}$. In such a way, g' turns to be a biregular morphism of $X = (X' \setminus E) \cup D^{\rm sing}$. The restriction of g' on T_i has image $g'(T_i) = T_j$ for some $1 \leq j \leq h$ and induces a biholomorphism $g' : D_i \to D_j$. As a result, $g' \in G'$ acts on D and $g' \in G = Aut(X, D)$.

In order to justify that $G = \mathrm{Aut}(X, D)$ is a finite group, let us consider the natural representation

$$\varphi : G \to \mathrm{Sym}(D_1, \dots, D_h)$$

in the permutation group of the irreducible components $D_1, \dots, D_h$ of D. As far as the image $\varphi(G)$ is a finite group, it suffices to prove that the kernel $\ker \varphi$ is finite. Fix $p \in D^{\rm sing}$ and two local irreducible branches U_o and V_o of D through p. If $U_o \subset D_i$ and $V_o \subset D_j$ for $i \neq j$ then consider the natural representation

$$\varphi_o : \ker \varphi \to \mathrm{Sym}(D_i \cap D_j).$$

The group homomorphism φ_o has a finite image, so that the problem reduces to the finiteness of $G_o := \ker (\varphi_o|_{\ker \varphi})$. By its very definition, $G_o \leq \mathrm{Stab}_G(p)$. Let us move the origin of D_i at p and realize G_o as a subgroup of the finite cyclic group $\mathrm{End}^*(D_i)$. After an eventual shrinking, U_o is contained in a coordinate chart of X. Then $U = \cap_{g_o \in G_o} [g_o(U_o)]$ is a G_o-invariant neighborhood of p on D_i. Similarly, pass to a G_o-invariant neighborhood $V \subset V_o$ of p on D_j, intersecting transversally U. Through any point $v \in V$ there is a local complex line $U(v)$, parallel to U. The union $W = \cup_{v \in V} U(v)$ is a neighborhood of p on X, biholomorphic to $U \times V$. In holomorphic coordinates $(u, v) \in W$, one gets $G_o \leq \mathrm{End}^*(U) \times \mathrm{End}^*(V)$. Note that $\mathrm{End}^*(U) \subseteq \mathrm{End}^*(D_i)$ and $\mathrm{End}^*(D_i)$ is a finite cyclic group of order $1, 2, 3, 4$ or 6, so that $|G_o| < \infty$.

□

1 Kodaira-Enriques classification of the finite Galois quotients of a split abelian surface

Let $A = E \times E$ be the Cartesian square of an elliptic curve E. For an arbitrary finite automorphism group $H \leq \mathrm{Aut}(A)$, we characterize the Kodaira-Enriques classification type of A/H in terms of the fixed point set $Fix_A(H)$ of H on A. Partial results for this problem are provided by [7]. Namely, any A/H is a finite cyclic Galois quotient of a smooth abelian surface A/K or a normal model A/K of a K3 surface. The surface A/K is abelian exactly when $K = \mathcal{T}(H)$ is a translation group. The note [7] specifies that a necessary and sufficient condition for $A/[\mathcal{T}(H)\langle h \rangle]$ to have irregularity $q(Y) = h^{1,0}(Y) = 1$ is the presence of an entire elliptic curve in the fixed point set $Fix_A(h)$ of h. This result is similar to S. Tokunaga and M. Yoshida's study [11] of the discrete subgroups $\Lambda \leq \mathbb{C}^n \leftthreetimes U(n)$ with compact quotient $\mathbb{C}^n/\Lambda$. Namely, [11] establishes that if the linear part $\mathcal{L}(\Lambda)$ of such Λ does not fix pointwise a complex line on $\mathbb{C}^2$, then $\mathbb{C}^n/\Lambda$ has vanishing irregularity. Further, [7] observes that if some $h \in H$ fixes pointwise an entire elliptic curve on A, then the Kodaira dimension $\kappa(A/H) = -\infty$ drops down. Tokunaga and Yoshida prove the same statement for discrete subgroups $\Lambda \leq \mathbb{C}^n \leftthreetimes U(n)$ with compact quotient $\mathbb{C}^n/\Lambda$. The note [7] proves also that if A/K is a K3 double cover of A/H then A/H is birational to an Enriques surface if and only if $A/K \to A/H$ is unramified.

The present note establishes that an arbitrary cyclic cover $\zeta_H^K : A/K \to A/H$ of degree ≥ 3 by a K3 surfaces A/K with isolated cyclic quotient singularities is ramified over a finite set of points and A/H is a rational surface. If a K3 surface A/K is a double cover $\zeta_H^K : A/K \to A/H$ of A/H then A/H is birational to an Enriques surface exactly when ζ_H^K is unramified. The quotients A/H with ramified K3 double covers $\zeta_H^K : A/K \to A/H$ are rational surfaces. If $H = \mathcal{T}(H)\langle h \rangle$ and the fixed points of $\mathcal{L}(h)$ on A contain an elliptic curve then A/H is hyper-elliptic (respectively, ruled with an elliptic base) if and only if H has not a fixed point on A (respectively, H has a fixed point on A, whereas H has a pointwise fixed elliptic curve on A). If $H = \mathcal{T}(H)\langle h \rangle$ and $\mathcal{L}(h)$ has isolated fixed points on A then A/H is a rational surface.

In order to construct the normal subgroup K of H, let us recall that the automorphism group $\mathrm{Aut}(A) = \mathcal{T}_A \rightthreetimes \mathrm{Aut}_{\check{o}_A}(A)$ of A is a semi-direct product of the translation group $\mathcal{T}_A \simeq (A, +)$ and the stabilizer $\mathrm{Aut}_{\check{o}_A}(A)$ of the origin $\check{o}_A \in A$. Each $g \in \mathrm{Aut}_{\check{o}_A}(A)$ is a linear transformation

$$g = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GL_2(\mathbb{C}),$$

leaving invariant the fundamental group $\pi_1(A) = \pi_1(E) \times \pi_1(E)$ of $A = E \times E$. Therefore $a_{ij}\pi_1(E) \subseteq \pi_1(E)$ for all $1 \leq i, j \leq 2$ and $a_{ij} \in R$ for the endomorphism ring R of E. The same holds for the entries of the inverse matrix

$$g^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \in \mathrm{Aut}_{\check{o}_A}(A). \quad (1)$$

Now, $\det(g) \in R$ and $\det(g^{-1}) = (\det(g))^{-1} \in R$ imply that $\det(g) \in R^*$ is a unit. Thus, $\mathrm{Aut}_{\check{o}_A}(A)$ is contained in

$$Gl(2,R) := \{g \in (R)_{2\times 2} \ \ | \ \ \det(g) \in R^*\}.$$

The opposite inclusion $Gl(2,R) \subseteq \mathrm{Aut}_{\check{o}_A}(A)$ is clear from (1) and $\mathrm{Aut}_{\check{o}_A}(A) = Gl(2,R)$.

The map $\mathcal{L} : \mathrm{Aut}(A) \to Gl(2,R)$, associating to $g \in \mathrm{Aut}(A)$ its linear part $\mathcal{L}(g) \in Gl(2,R)$ is a group homomorphism with kernel $\ker(\mathcal{L}) = \mathcal{T}_A$. Denote by $\mathcal{O}_{-d}$ the integers ring of an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$. The determinant $\det : Gl(2,R) \to R^*$ is a group homomorphism in the cyclic units group

$$R^* = \langle \zeta_{-d} \rangle \simeq \begin{cases} \mathbb{C}_2 & \text{for } R \neq \mathbb{Z}[i], \mathcal{O}_{-3}, \\ \mathbb{C}_4 & \text{for } R = \mathbb{Z}[i] = \mathcal{O}_{-1}, \\ \mathbb{C}_6 & \text{for } R = \mathcal{O}_{-3} \end{cases}$$

of order $o(R)$. For an arbitrary subgroup H of $\mathrm{Aut}(A)$, let us denote by $K = K_H$ the kernel of the group homomorphism $\det \mathcal{L} : H \to R^*$. The image $\det \mathcal{L}(H) \leq (R^*, .)$ is a cyclic group of order m, dividing $o(R^*)$, i.e., $\det \mathcal{L}(H) = \langle \zeta_{-d}^k \rangle$ for some natural divisor $k = \frac{o(R^*)}{m}$ of $o(R^*)$. For an arbitrary $h_0 \in H$ with $\det \mathcal{L}(h_0) = \zeta_{-d}^k$ the first homomorphism theorem reads as

$$\{K_H, h_0 K_H, \ldots, h_0^{m-1} K_H\} = H/K_H \simeq \langle \zeta_{-d}^k \rangle = \{1, \zeta_{-d}^k, \zeta_{-d}^{2k}, \ldots, \zeta_{-d}^{(m-1)k}\}.$$

Therefore $H = K_H \langle h_0 \rangle$ is a product of $K_H = \ker(\det \mathcal{L}|_H)$ and the cyclic subgroup $\langle h_0 \rangle$ of H.

Denote by $E_1(H)$ the set of $h \in H$, whose linear parts $\mathcal{L}(h) \in GL_2(R)$ have eigenvalue 1 of multiplicity 1. In other words, $h \in E_1(H)$ exactly when $\mathcal{L}(h)$ fixes pointwise an elliptic curve on A through the origin $\check{o}_A$. Put $E_0(H)$ for the set of $h \in H$, whose linear parts have no eigenvalue 1. Observe that $h \in E_0(H)$ if and only if $\mathcal{L}(h) \in GL(2,R)$ has isolated fixed points on A.

An automorphism $h \in H \setminus \{\mathrm{Id}\}$ is called a reflection if fixes pointwise an elliptic curve on A. We claim that $h \in H$ is a reflection if and only if $h \in E_1(H)$ and h has a fixed point on A. Indeed, if h fixes an elliptic curve F on A, then one can move the origin $\check{o}_A$ of A on F, in order to represent h by a linear transformation $h = \mathcal{L}(h) \in GL(2,R) \setminus \{\mathrm{Id}\} = E_1(GL(2,R)) \cup E_0(GL(2,R))$. Any $h = \mathcal{L}(h) \in E_0(GL(2,R))$ has isolated fixed points on A, so that $h = \mathcal{L}(h) \in E_1(H)$ and $Fix_A(h) \neq \emptyset$. Conversely, if $h \in E_1(H)$ and $Fix_A(h) \neq \emptyset$, then after moving the origin of A at $\check{o}_A \in Fix_A(h)$, one attains $h = \mathcal{L}(h)$. Thus, h fixes pointwise an elliptic curve on A or h is a reflection.

Towards the complete classification of the Kodaira-Enriques type of A/H, we use the following results from [7]:

Proposition 3. *(i)* (cf. Corollary 5 from [7]) *The quotient A/H of $A = E \times E$ by a finite automorphism group H is an abelian surface if and only if $H = \ker(\mathcal{L}|_H) = \mathcal{T}(H)$ is a translation group.*

(ii) (Lemma 7 from [7]) *The quotient A/H is birational to a K3 surface if and only if $H = \ker(\det \mathcal{L}|_H)$ and $H \supsetneqq \ker(\mathcal{L}|_H) = \mathcal{T}(H)$.*

Proposition 4. *(i)*(cf. Lemma 11 from [7]) *If a finite automorphism group $H \leq \mathrm{Aut}(A)$ contains a reflection then A/H is of Kodaira dimension $\kappa(A/H) = -\infty$.*

(ii) (cf. Proposition 12 from [7]) *A smooth model Y of A/H is of irregularity $q(Y) = h^{1,0}(Y) = 1$ if and only if $H = \mathcal{T}(H)\langle h\rangle$ is a product of its normal translation subgroup $\mathcal{T}(H) = \ker(\mathcal{L}|_H)$ and a cyclic group $\langle h\rangle$, generated by $h \in E_1(H)$.*

From now on, we consider only subgroups $H \leq \mathrm{Aut}(A, T)$ with $\det \mathcal{L}(H) \neq \{1\}$ and distinguish between translation $K = \ker(\det \mathcal{L}|_H) = \ker(\mathcal{L}|_H) = \mathcal{T}(H)$ and non-translation $K = \ker(\det \mathcal{L}|_H) \supsetneqq \ker(\mathcal{L}|_H) = \mathcal{T}(H)$. Any $h \notin K = \ker(\det \mathcal{L}|_H)$ belongs to $E_1(H)$ or to $E_0(H)$.

Proposition 5. *Let $H = \mathcal{T}(H)\langle h\rangle$ be a product of its (normal) translation subgroup $\mathcal{T}(H) = \ker(\mathcal{L}|_H)$ and a cyclic group $\langle h\rangle$, generated by $h \in E_1(H)$. Then:*

(i) the fixed point set $Fix_A(H) = \emptyset$ of H on A is empty if and only if A/H is a smooth hyper-elliptic surface;

(ii) the fixed point set $Fix_A(H) \neq \emptyset$ is non-empty if and only if A/H is a smooth ruled surface with an elliptic base. If so, then $Fix_A(H)$ is of codimension 1 in A.

Proof. According to Proposition 4 (ii), $H = \mathcal{T}(H)\langle h\rangle$ with $h \in E_1(H)$ if and only if any smooth model Y of A/H has irregularity $q(Y) = h^{1,0}(Y) = 1$. More precisely, Y is a hyper-elliptic surface or a ruled surface with an elliptic base.

If $Fix_A(H) = \emptyset$ then $A \to A/H$ is an unramified cover and the Kodaira dimension $\kappa(A/H) = \kappa(A) = 0$. Therefore A/H is hyper-elliptic.

Suppose that there is an H-fixed point $p \in Fix_A(H)$ and move the origin $\check{o}_A$ of A at p. For any $h_1 \in Stab_H(\check{o}_A) \setminus \{Id_A\}$ one has $\check{o}_A = h_1(\check{o}_A) = \tau(h_1)\mathcal{L}(h_1)(\check{o}_A) = \tau(h_1)(\check{o}_A)$, so that h_1 has trivial translation part $\tau(h_1) = \tau_{\check{o}_A}$. As a result, $h_1 = \mathcal{L}(h_1) \in E_1(H) \setminus \{Id_A\}$ is a reflection and fixes pointwise an elliptic curve on A. In particular, $Fix_A(H)$ is of complex codimension 1. If

$$\mathcal{L}(h) = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_2(h) \end{pmatrix} \quad \text{with} \quad \lambda_2(h) \neq 1$$

then

$$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_2(h)^i \end{pmatrix} \quad \text{with} \quad i \in \mathbb{Z}, \ \ \lambda_2(h)^i \neq 1.$$

By Proposition 4 (i), the quotient $A/\langle h_1\rangle$ by the cyclic group $\langle h_1\rangle$, generated by the reflection $h_1 = \mathcal{L}(h_1) \in E_1(H)$ is of Kodaira dimension $\kappa(A/\langle h_1\rangle) = -\infty$. Along the finite (not necessarily Galois) cover $A/\langle h_1\rangle \to A/H$, one has $\kappa(A/\langle h_1\rangle) \geq \kappa(A/H)$, whereas $\kappa(A/H) = -\infty$ and A/H is birational to a ruled surface with an elliptic base. Note that all $h \in H$ with $Fix_A(h) \neq \emptyset$ are reflections, so that the quotient A/H is a smooth surface by a result of Chevalley [5].

That proves the proposition, as far as the assumption $Fix_A(H) \neq \emptyset$ for a hyper-elliptic A/H leads to a contradiction, as well as the assumption $Fix_A(H) = \emptyset$ for a ruled A/H with an elliptic base.

□

Proposition 6. *Let $H = \mathcal{T}(H)\langle h \rangle$ for some*

$$h \in E_0(H) = \{h \in H \ \ | \ \ \lambda_j \mathcal{L}(h) \neq 1, \quad 1 \leq j \leq 2\}$$

with $\det \mathcal{L}(h) \neq 1$. *Then A/H is a rational surface.*

Proof. We claim that A/H with $A = E \times E$ is simply connected. To this end, let us denote by R the endomorphism ring of E and lift H to a subgroup $\widetilde{H}$ of the affine-linear group $Aff(\mathbb{C}^2, R) = (\mathbb{C}^2, +) \rtimes GL(2, R)$, containing $(\pi_1(A), +)$ as a normal subgroup with quotient $\widetilde{H}/\pi_1(A) = H$. Then

$$A/H = \left[\mathbb{C}^2/\pi_1(A)\right] / \left[\widetilde{H}/\pi_1(A)\right] \simeq \mathbb{C}^2/\widetilde{H}.$$

The universal cover $\widetilde{A} = \mathbb{C}^2$ of A is a path connected, simply connected locally compact metric space and $\widetilde{H}$ is a discontinuous group of homeomorphisms of $\mathbb{C}^2$. That allows to apply Armstrong's result [1] and conclude that

$$\pi_1(A/H) = \pi_1\left(\mathbb{C}^2/\widetilde{H}\right) \simeq \widetilde{H}/\widetilde{N},$$

where $\widetilde{N}$ is the normal subgroup of $\widetilde{H}$, generated by $\widetilde{h} \in \widetilde{H}$ with $Fix_{\mathbb{C}^2}(\widetilde{h}) \neq \emptyset$. There remains to be shown the coincidence $\widetilde{H} = \widetilde{N}$. In the case under consideration, let us choose generators $\tau_{(P_i,Q_i)}$ of $\mathcal{T}(H)$, $1 \leq i \leq m$ and fix liftings $(p_i, q_i) \in \mathbb{C}^2 = \widetilde{A}$ of $(p_i + \pi_1(E), q_i + \pi_1(E)) = (P_i, Q_i)$. If $\pi_1(E) = \lambda_1\mathbb{Z} + \lambda_2\mathbb{Z}$ for some $\lambda_1, \lambda_2 \in \mathbb{C}^*$ with $\frac{\lambda_2}{\lambda_1} \in \mathbb{C} \setminus \mathbb{R}$, then $\pi_1(A) = \pi_1(E) \times \pi_1(E)$ is generated by

$$\Lambda_{11} = (\lambda_1, 0), \quad \Lambda_{12} = (\lambda_2, 0), \quad \Lambda_{21} = (0, \lambda_1) \quad \text{and} \quad \Lambda_{22} = (0, \lambda_2).$$

Let $\widetilde{h} = \tau_{(u,v)}\mathcal{L}(h) \in \widetilde{H}$ be a lifting of $h = \tau_{(U,V)}\mathcal{L}(h) \in H$, i.e., $(u+\pi_1(E), v+\pi_1(E)) = (U, V)$. Then $\widetilde{H}$ is generated by its subset

$$S = \left\{\Lambda_{ij}, \quad \tau_{(p_k,q_k)}, \quad \widetilde{h} \ \ | \ \ 1 \leq i, j \leq 2, \quad 1 \leq k \leq m\right\}.$$

Since $\mathcal{L}(h)$ has eigenvalues $\lambda_1\mathcal{L}(h) \neq 1$, $\lambda_2\mathcal{L}(h) \neq 1$, for any $(a, b) \in \mathbb{C}^2$ the automorphism $\tau_{(a,b)}\mathcal{L}(h) \in Aut(\mathbb{C}^2)$ has a fixed point on $\mathbb{C}^2$. One can replace the generators Λ_{ij} and $\tau_{(p_k,q_k)}$ of $\widetilde{H}$ by $\Lambda_{ij}\widetilde{h}$, respectively, $\tau_{(p_k,q_k)}\widetilde{h}$, since

$$\langle S \rangle \supseteq \{\Lambda_{ij}\widetilde{h}, \quad \tau_{(p_k,q_k)}\widetilde{h}, \quad \widetilde{h} \ \ | \ \ 1 \leq i, j \leq 2, \quad 1 \leq k \leq m\}$$

and $\Lambda_{ij}, \tau_{(p_k,q_k)} \in \langle \{\Lambda_{ij}\widetilde{h}, \ \ \tau_{(p_k,q_k)}\widetilde{h}, \ \ \widetilde{h} \ \mid \ 1 \leq i,j \leq 2, \ \ 1 \leq k \leq m\}\rangle$. Thus

$$\widetilde{H} = \langle \Lambda_{ij}\widetilde{h}, \ \ \tau_{(p_k,q_k)}\widetilde{h}, \ \ \widetilde{h} \ \mid \ 1 \leq i,j \leq 2, \ \ 1 \leq k \leq m\rangle$$

coincides with $\widetilde{N}$, because $\widetilde{H}$ is generated by elements with fixed points. As a result, $\pi_1(A/H) = \{1\}$.

Note that the simply connected surfaces A/H are either rational or K3. According to $\det \mathcal{L}(h) \neq 1$, the quotient A/H is not birational to a K3 surface, so that A/H is a rational surface with isolated cyclic quotient singularities.

□

Proposition 7. *Let $H < Aut(A)$ be a finite subgroup of the form $H = K\langle h\rangle$ with $\mathcal{L}(K) < SL(2,R)$ and $\det \mathcal{L}(H) = \langle \det \mathcal{L}(h)\rangle \neq \{1\}$.*

(i) The complement $H \setminus K$ has fixed points on A, $Fix_A(H \setminus K) \neq \emptyset$ if and only if A/H is a rational surface;

(ii) The complement $H \setminus K$ has no fixed points on A, $Fix_A(H \setminus K) = \emptyset$ if and only if A/H is birational to an Enriques surface Y. If so, then the K3 universal cover $\widetilde{Y}$ of Y is birational to A/K and the index $[H : K] = 2$.

Proof. First of all, the H/K-Galois cover $\zeta : A/K \to A/H$ is ramified if and only if the complement $H \setminus K$ has a fixed point on A. More precisely, a point $Orb_K(p) \in A/K$, $p \in A$ is fixed by $hK \in H/K \setminus \{K\}$ exactly when $hOrb_K(p) = Orb_K(p)$ or

$$\{hk(p) \ \mid \ k \in K\} = \{k(p) \ \mid \ k \in K\}. \tag{2}$$

The condition (2) implies the existence of $k_o \in K$ with $h(p) = k_o(p)$. Therefore $h_1 = k_o^{-1}h \in Stab_H(p) \setminus K$ has a fixed point and

$$h_1K = (k_o^{-1}h)K = k_o^{-1}(hK) = k_o^{-1}Kh = Kh = hK,$$

as far as K is a normal subgroup of H. Conversely, if $h_1(p) = p$ for some $h_1 \in H \setminus K$ then $K_p = Kh_1(p) = h_1K(p)$ and the point $Orb_K(p) \in A/K$ is fixed by $h_1K \in H/K$.

Note that the presence of a covering $\zeta : A/K \to A/H$ by a (singular) K3 model A/K implies the vanishing $q(X) = h^{1,0}(X)$ of the irregularity of any smooth model X of A/H, as far as $q(X) \leq q(Y) = 0$ for any smooth H/K-Galois cover Y of X, birational to A/K. The smooth projective surfaces S with irregularity $q(S) = 0$ and Kodaira dimension $\kappa(S) \leq 0$ are the rational, K3 and Enriques S. Due to $\mathcal{L}(h) \neq 1$, the smooth model X of A/H is not a K3 surface. Thus, X is either an Enriques or a rational surface.

If $Fix_A(H \setminus K) = \emptyset$ and $\zeta : A/K \to A/H$ in unramified, then $\kappa(X) = \kappa(Y) = 0$ by [10] and X is an Enriques surface.

Let us assume that $Fix_A(H \setminus K) \neq \emptyset$ and the minimal resolution Y of the singularities of A/H is an Enriques surface. Consider the minimal resolution $\rho_1 : Y \to A/K$

of the singularities of A/K and the resolution $\nu_2 : X_2 \to A/H$ of $\zeta(A/H)^{\text{sing}}$. Then there is a commutative diagram

$$\begin{array}{ccc} A/K & \xleftarrow{\rho_1} & Y \\ \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \zeta_1} \\ A/H & \xleftarrow{\nu_2} & X_2 \end{array} \tag{3}$$

with H/K-Galois cover ζ_1, ramified over the pull-back $\nu_2^{-1}B(\zeta)$ of the branch locus $B(\zeta) \subset A/H$ of ζ. The minimal resolution $\mu_2 : X \to X_2$ of the singularities $X_2^{\text{sing}} = (A/H)^{\text{sing}} \setminus \zeta(A/K)^{\text{sing}}$ of X_2 and $\zeta_1 : Y \to X_2$ give rise to the fibered product commutative diagram

$$\begin{array}{ccc} Y & \xleftarrow{\text{pr}_1} & Z = Y \times_{X_2} X \\ \downarrow{\scriptstyle \zeta_1} & & \downarrow{\scriptstyle \zeta_2} \\ X_2 & \xleftarrow{\mu_2} & X \end{array} , \tag{4}$$

with ramified H/K-Galois cover ζ_2 and birational pr_1. Note that Z is a smooth surface, since otherwise $\emptyset \neq \text{pr}_1(Z^{\text{sing}}) \subseteq X^{\text{sing}} = \emptyset$. Moreover, Z is of type K3. Let us consider the universal double covering $U_X : \widetilde{X} \to X$ of X by a K3 surface $\widetilde{X}$. Since Z is simply connected and $U_X : \widetilde{X} \to X$ is unramified, the finite cover $\zeta_2 : Z \to X$ lifts to a morphism $\widetilde{\zeta} : Z\widetilde{X}$, closing the commutative diagram

$$\begin{array}{ccc} & & \widetilde{X} \\ & \nearrow^{\widetilde{\zeta}} & \downarrow{\scriptstyle U_X} \\ Z & \xrightarrow{\zeta_2} & X \end{array} \tag{5}$$

The finite ramified morphism $\zeta_2 = U_X\widetilde{\zeta}$ has finite ramified factor $\widetilde{\zeta}$, as far as the universal covering $U_X : \widetilde{X} \to X$ is unramified. If $B(\widetilde{\zeta}) \subset Z$ is the branch locus of $\widetilde{\zeta}$ then the canonical divisor

$$\mathcal{O}_Z = \mathcal{K}_Z = \widetilde{\zeta}^* \mathcal{K}_{\widetilde{X}} + B(\widetilde{\zeta}) = \widetilde{\zeta}^* \mathcal{O}_{\widetilde{X}} + B(\widetilde{\zeta}),$$

which is an absurd. Therefore, $Fix_A(H \setminus K) \neq \emptyset$ implies that A/H is a rational surface.

If $\zeta : A/K \to A/H$ is unramified and A/H is an Enriques surface then $\zeta_1 : Y \to X_2$ from diagram (3) and $\zeta_2 : Z \to X$ from (4) are unramified. As a result, $\widetilde{\zeta} : Z \to \widetilde{X}$ from diagram (5) is a finite ramified cover of smooth simply connected surfaces, whereas $\deg(\widetilde{\zeta}) = 1$ and Z coincides with the universal cover $\widetilde{X}$ of X. Thus, $\widetilde{X}$ is birational to A/K and

$$\deg(\zeta) = \deg(\zeta_1) = \deg(\zeta_2) = \deg(U_X) = 2,$$

so that $[H : K] = |H/K| = \deg(\zeta) = 2$.

□

By the very construction, the surfaces A/H and $\overline{\mathbb{B}/\Gamma_H} = (\mathbb{B}/\Gamma)'/H$ are simultaneously singular. The classical work [5] of Chevalley establishes that A/H is singular if and only if there is $h \in H$, whose linear part $\mathcal{L}(h) \in GL(2, R)$ has eigenvalues $\{\lambda_1 \mathcal{L}(h), \lambda_2 \mathcal{L}(h)\} \not\ni 1$. Thus, A/H and $\overline{\mathbb{B}/\Gamma_H}$ are smooth exactly when birational to a hyper-elliptic or a ruled surface with an elliptic base.

Let T_i be an irreducible component of $T = (\mathbb{B}/\Gamma)' \setminus (\mathbb{B}/\Gamma)$ of $\mathbb{B}/\Gamma$. Then the irreducible component $Orb_H(T_i)/H$ of $T/H = \left(\overline{\mathbb{B}/\Gamma_H}\right) \setminus (\mathbb{B}/\Gamma_H)$ is elliptic (respectively, rational) if and only if $Fix_A(H) \cap D_i = \emptyset$ (respectively, $Fix_A(H) \cap D_i \neq \emptyset$) for the image $D_i = \xi(T_i)$ of T_i under the blow-down $\xi : (\mathbb{B}/\Gamma)' \to A$ of the (-1)-curves.

2 Linear automorphisms of finite order

Throughout this section, let R be the endomorphism ring of an elliptic curve E. It is well known that $R = \mathbb{Z} + f\mathcal{O}_{-d}$ for a natural number $f \in \mathbb{N}$, called the conductor of E and integers ring $\mathcal{O}_{-d}$ of an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$. More precisely, $\mathcal{O}_{-d} = \mathbb{Z} + \omega_{-d}\mathbb{Z}$ with

$$\omega_{-d} = \begin{cases} \sqrt{-d} & \text{for } -d \not\equiv 1 (\text{mod } \ 4), \\ \frac{1+\sqrt{-d}}{2} & \text{for } -d \equiv 1 (\text{mod } \ 4). \end{cases}$$

and $R = \mathbb{Z} + f\omega_{-d}\mathbb{Z}$ for $R \neq \mathbb{Z}$. In particular, R is a subring of $\mathbb{Q}(\sqrt{-d})$. We write $R \subset \mathbb{Q}(\sqrt{-d})$ both, for the case of $R = \mathbb{Z} + f\omega_{-d}\mathbb{Z}$ or $R = \mathbb{Z}$, without specifying the presence of a complex multiplication on E. (For $R = \mathbb{Z}$ one hat $R \subset \mathbb{Q}(\sqrt{-d})$ for $\forall d \in \mathbb{N}$.)

The automorphism group of the abelian surface $A = E \times E$ is a semi-direct product

$$\text{Aut}(A) = (A, +) \rtimes GL(2, R)$$

of its translation subgroup $(A, +)$ and the isotropy group

$$\text{Aut}_{\check{o}_A}(A) = GL(2, R) = \{g \in R_{2\times 2} \ \ | \ \ \det(g) \in R^*\}$$

of the origin $\check{o}_A \in A$.

Lemma 8. *Let R be the endomorphism ring of an elliptic curve E. If R is different from $\mathcal{O}_{-1} = \mathbb{Z}[i]$ and $\mathcal{O}_{-3}$ then*

$$R^* = \langle -1 \rangle = \{\pm 1\} = \mathbb{C}_2$$

is the cyclic group of the square roots of the unity.

If $R = \mathbb{Z}[i]$ is the ring of the Gaussian integers then

$$R^* = \langle i \rangle = \{\pm 1, \pm i\} = \mathbb{C}_4$$

is the cyclic group of the roots of unity of order 4.

The units group of Eisensten integers $R = \mathcal{O}_{-3}$ is the cyclic group

$$R^* = \langle e^{\frac{2\pi i}{6}} \rangle = \{\pm 1, \ \ e^{\pm\frac{2\pi i}{3}}, \ \ e^{\pm\frac{\pi i}{3}}\} = \mathbb{C}_6$$

of the sixth roots of unity.

Proof. Recall that the units group $\mathcal{O}_{-d}^*$ of the integers ring $\mathcal{O}_{-d}$ of an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$ is

$$\mathcal{O}_{-d}^* = \langle -1 \rangle \simeq \mathbb{C}_2 \quad \text{for} \quad d \neq 1, 3 \quad \text{and}$$

$$\mathcal{O}_{-1}^* = \mathbb{Z}[i]^* = \langle i \rangle = \mathbb{C}_4,$$
$$\mathcal{O}_{-3}^* = \langle e^{\frac{2\pi i}{6}} \rangle = \mathbb{C}_6.$$

The units group R^* of the subring $R = \mathbb{Z} + f\mathcal{O}_{-d}$ of $\mathcal{O}_{-d}$ is a subgroup of $\mathcal{O}_{-d}^*$, so that $R^* = \langle -1 \rangle \simeq \mathbb{C}_2$ for $R = \mathbb{Z}$ or $R = \mathbb{Z} + f\mathcal{O}_{-d}$ with $d \in \mathbb{N} \setminus \{1, 3\}$, $f \in \mathbb{N}$. In the case of $R = \mathbb{Z} + f\mathcal{O}_{-1}$, the assumption $i \in R^*$ implies $R = \mathcal{O}_{-1}$ and happens only for the conductor $f = 1$. Similarly, the existence of $e^{\frac{2\pi i}{3}} \in R^* \setminus \{\pm 1\}$ for $R = \mathbb{Z} + f\mathcal{O}_{-3}$ forces
$$e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}i}{2} = -1 + \frac{1 + \sqrt{-3}}{2} = -1 + \omega_{-3} \in R^*,$$
whereas $\omega_{-3} \in R$ and $R = \mathcal{O}_{-3}$.

□

Towards the description of $g \in GL(2, R)$ of finite order, let us recall that the polynomials
$$f(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n \in \mathbb{Z}[x]$$
with leading coefficient 1 are called monic.

Definition 9. *If A is a subring with unity of a ring B then $b \in B$ is integral over A if annihilates a monic polynomial*

$$f(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n \in A[x]$$

with coefficients from A.

It is well known (cf. [2]) that $b \in B$ is integral over A if and only if the polynomial ring $A[b]$ is a finitely generated A-module.

Definition 10. *The complex numbers $c \in \mathbb{C}$, which are integral over $\mathbb{Z}$ are called algebraic integers.*

Any algebraic integer c is algebraic over $\mathbb{Q}$. If $g(x) \in \mathbb{Q}[x] \setminus \mathbb{Q}$ is a polynomial of minimal degree k with a root c then $g(x)$ divides any $h(x) \in \mathbb{Q}[x] \setminus \mathbb{Q}$ with $h(c) = 0$. An arbitrary $g'(x) \in \mathbb{Q}[x]$ of degree k with a root c is of the form $g'(x) = qg(x)$ for some $\mathbb{Q}^*$. The polynomials $qg(x)$ with arbitrary $q \in \mathbb{Q}^*$ are referred to as minimal polynomials of c over $\mathbb{Q}$. If c is algebraic over $\mathbb{Q}$ then the ring of the polynomials $\mathbb{Q}[c]$ of c with rational coefficients coincides with the field $\mathbb{Q}(c)$ of the rational functions of c, $\mathbb{Q}[c] = \mathbb{Q}(c)$ and the degree $[\mathbb{Q}(c) : \mathbb{Q}]$ equals the degree of a minimal polynomial of c over $\mathbb{Q}$.

Definition 11. *If $c \in \mathbb{C}$ is algebraic over $\mathbb{Q}$, then $[\mathbb{Q}(c) : \mathbb{Q}] = \dim_{\mathbb{Q}} \mathbb{Q}(c)$ is called the degree of c over $\mathbb{Q}$.*

Let c be an algebraic integer and $f(x) \in \mathbb{Z}[x] \backslash \mathbb{Z}$ be a monic polynomial of minimal degree with a root c. Then any $h(x) \in \mathbb{Z}[x]$ with $h(c) = 0$ is divisible by $f(x)$. Thus, $f(x)$ is unique and referred to as the minimal integral relation of c. If $f(x) \in \mathbb{Z}[x] \setminus \mathbb{Z}$ is the minimal integral relation of $c \in \mathbb{C}$ and $g(x) \in \mathbb{Q}[x] \setminus \mathbb{Q}$ is a minimal polynomial of c over $\mathbb{Q}$, then $g(x) = qf(x)$ for the leading coefficient $q = LC(g) \in \mathbb{Q}^*$ of $g(x)$. More precisely, $g(x)$ divides $f(x)$ and $f(x)$ is indecomposable over $\mathbb{Q}$, as far as it is indecomposable over $\mathbb{Z}$. In such a way, one obtains the following

Lemma 12. *If $c \in \mathbb{C}$ is an algebraic integer, then the degree $\deg_{\mathbb{Q}}(c) = [\mathbb{Q}(c) : \mathbb{Q}]$ of c over $\mathbb{Q}$ equals the degree of the minimal integral relation*

$$f(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n \in \mathbb{Z}[x] \quad \textit{of} \quad c.$$

Lemma 13. *Let E be an elliptic curve, $R = End(E)$ and $g \in GL(2, R)$. Then any eigenvalue λ_1 of g is an algebraic integer of degree $1, 2$ or 4 over $\mathbb{Q}$.*

Proof. It suffices to observe that if $A \subset B$ are subrings with unity of a ring C, A is a Noetherian ring, B is a finitely generated A-module and $c \in C$ is integral over B, then c is integral over A. Indeed, let $f \in \mathbb{N}$ be the conductor of E and

$$\omega_{-d} = \begin{cases} \sqrt{-d} & \text{for } -d \not\equiv 1(\text{mod}4), \\ \frac{1+\sqrt{-d}}{2} & \text{for } -d \equiv 1(\text{mod}4). \end{cases} \tag{6}$$

Then the integers ring $\mathbb{Z}$ is Noetherian and the endomorphism ring

$$R = \mathbb{Z} + f\mathcal{O}_{-d} = \mathbb{Z} + f\omega_{-d}\mathbb{Z}$$

of E is a free $\mathbb{Z}$-module of rank 2. The eigenvalue $\lambda_1 \in \mathbb{C}$ of $g \in GL(2, R)$ is a root of the characteristic polynomial

$$\mathcal{X}_g(\lambda) = \lambda^2 - \text{tr}(g)\lambda + \det(g) \in R[\lambda]$$

of g, so that λ_1 is integral over R. According to the claim, λ_1 is integral over $\mathbb{Z}$ or $\lambda_1 \in \mathbb{C}$ is an algebraic integer. On one hand, the degree of λ_1 over $\mathbb{Q}(\sqrt{-d})$ is

$$\deg_{\mathbb{Q}(\sqrt{-d})}(\lambda_1) = [\mathbb{Q}(\sqrt{-d}, \lambda_1) : \mathbb{Q}(\sqrt{-d})] = 1 \quad \text{or} \quad 2,$$

so that

$$[\mathbb{Q}(\sqrt{-d}, \lambda_1) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{-d}, \lambda_1) : \mathbb{Q}(\sqrt{-d})][\mathbb{Q}(\sqrt{-d}) : \mathbb{Q}] = 2 \quad \text{or} \quad 4.$$

On the other hand, the inclusions

$$\mathbb{Q} \subseteq \mathbb{Q}(\lambda_1) \subseteq \mathbb{Q}(\sqrt{-d}, \lambda_1)$$

of subfields imply that

$$[\mathbb{Q}(\lambda_1):\mathbb{Q}] = \frac{[\mathbb{Q}(\sqrt{-d},\lambda_1):\mathbb{Q}]}{[\mathbb{Q}(\sqrt{-d},\lambda_1):\mathbb{Q}(\lambda_1)]}.$$

Therefore, the degree $\deg_{\mathbb{Q}}(\lambda_1) = [\mathbb{Q}(\lambda_1):\mathbb{Q}]$ of λ_1 over $\mathbb{Q}$ is a divisor of the degree $[\mathbb{Q}(\sqrt{-d},\lambda_1):\mathbb{Q}]$ or $\deg_{\mathbb{Q}}(\lambda_1) \in \{1,2,4\}$.

In order to justify the claim, recall that $c \in C$ is integral over B if and only if the polynomial ring $B[c] = B + Bc + \ldots + Bc^{n-1}$ is a finitely generated B-module. If $B = A\beta_1 + \ldots + A\beta_s$ is a finitely generated A-module, then

$$B[c] = \sum_{i=1}^{s}\sum_{j=0}^{n-1} A\beta_i c^j$$

is a finitely generated A-module. Since A is a Noetherian ring, the A-submodule $A[c]$ of $B[c]$ is a finitely generated A-module.

□

Note that if $h = \tau_{(U,V)}\mathcal{L}(h) \in H \leq Aut(A)$ is an automorphism of $A = E \times E$ of finite order r then

$$h^r = \tau_{\sum\limits_{s=0}^{r-1}\mathcal{L}(h)^s\left(\begin{smallmatrix} U \\ V \end{smallmatrix}\right)}\mathcal{L}(h)^r = Id$$

implies that $\sum\limits_{s=0}^{r-1}\mathcal{L}(h)^s\left(\begin{smallmatrix} U \\ V \end{smallmatrix}\right) = \check{o}_A$ and $\mathcal{L}(h)^r = I_2$. In other words, the automorphisms $h \in Aut(A)$ of finite order have linear parts $\mathcal{L}(h) \in GL(2,R)$ of finite order.

From now on, we concentrate on $g \in GL(2,R)$ of finite order.

Proposition 14. *If R is the endomorphism ring of an elliptic curve E and $g \in GL(2,R)$ is of finite order r, then g is diagonalizable and the eigenvalues λ_j of g are primitive roots of unity of degree $r_j = 1,2,3,4,6,8$ or 12.*

Proof. Let us assume that $g \in GL(2,R)$ of finite order r is not diagonalizable. Then there exists $S \in GL(2,\mathbb{C})$, reducing g to its Jordan normal form

$$J = S^{-1}gS = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}.$$

By an induction on n, one verifies that

$$J^n = \begin{pmatrix} \lambda_1^n & (n-1)\lambda_1^{n-1} \\ 0 & \lambda_1^n \end{pmatrix} \quad \text{for} \quad \forall n \in \mathbb{N}.$$

In particular,

$$I_2 = S^{-1}I_2S = S^{-1}g^rS = (S^{-1}gS)^r = J^r = \begin{pmatrix} \lambda_1^r & (r-1)\lambda_1^{r-1} \\ 0 & \lambda_1^r \end{pmatrix}$$

is an absurd, justifying the diagonalizability of g.

If

$$D = S^{-1} g S = \left(\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array} \right)$$

is a diagonal form of g then

$$I_2 = S^{-1} I_2 S = S^{-1} g^r S = (S^{-1} g S)^r = \left(\begin{array}{cc} \lambda_1^r & 0 \\ 0 & \lambda_2^r \end{array} \right)$$

reveals that λ_1 and λ_2 are r-th roots of unity.

Thus, λ_j are of finite order r_j, dividing r and the least common multiple $m = LCM(r_1, r_2) \in \mathbb{N}$ divides r. Conversely,

$$I_2 = \left(\begin{array}{cc} \lambda_1^m & 0 \\ 0 & \lambda_2^m \end{array} \right) = (S^{-1} g S)^m = S^{-1} g^m S$$

implies that $g^m = S I_2 S^{-1} = I_2$, so that $r \in \mathbb{N}$ divides $m \in \mathbb{N}$ and $r = m$.

Let $\lambda_j \in \mathbb{C}^*$ be a primitive r_j-th root of unity. Then the cyclotomic polynomials $\Phi_{r_j}(x) \in \mathbb{Z}[x]$ are the minimal integral relations of λ_j. More precisely, the minimal integral relations $f_j(x) \in \mathbb{Z}[x] \setminus \mathbb{Z}$ of λ_j are monic polynomials of degree $\deg_{\mathbb{Q}}(\lambda_j)$. On the other hand, $\Phi_{r_j}(x) \in \mathbb{Z}[x] \setminus \mathbb{Z}$ are irreducible over $\mathbb{Z}$ and $\mathbb{Q}$. Therefore $\Psi_{r_j}(x)$ are minimal polynomials of λ_j over $\mathbb{Q}$ and $\Psi_{r_j}(x) = q f_j(x)$ for some $q \in \mathbb{Q}^*$. As far as $\Phi_{r_j}(x)$ and $f_j(x)$ are monic, there follows $q = 1$ and $\Phi_{r_j}(x) \equiv f_j(x) \in \mathbb{Z}[x]$.

Recall Euler's function

$$\varphi : \mathbb{N} \longrightarrow \mathbb{N},$$

associating to each $n \in \mathbb{N}$ the number of the residues $0 \leq r \leq n-1$ modulo n, which are relatively prime to n. The degree of $\Phi_{r_j}(x)$ is $\varphi(r_j)$. If $r_j = p_1^{a_1} \ldots p_m^{a_m}$ is the unique factorization of $r_j \in \mathbb{N}$ into a product of different prime numbers p_s, then

$$\varphi\left(p_1^{a_1} \ldots p_m^{a_m}\right) = \varphi\left(p_1^{a_1}\right) \ldots \varphi\left(p_m^{a_m}\right) = p_1^{a_1 - 1}(p_1 - 1) \ldots p_m^{a_m - 1}(p_m - 1).$$

According to Lemma 13, the algebraic integers λ_j are of degree

$$\deg_{\mathbb{Q}}(\lambda_j) = \deg \Phi_{r_j}(x) = \varphi(r_j) = 1, 2, \text{ or } 4.$$

If r_j has a prime divisor $p \geq 7$ then $\varphi(r_j)$ has a factor $p - 1 \geq 6$, so that $\varphi(r_j) > 4$. Therefore $r_j = 2^a 3^b 5^c$ for some non-negative integers a, b, c. If $c \geq 1$ then

$$\varphi(r_j) = \varphi(2^a 3^b) \varphi(5^c) = \varphi(2^a 3^b) 5^{c-1}.4 \in \{1, 2, 4\}$$

exactly when $\varphi(r_j) = 4$, $c = 1$ and $\varphi(2^a 3^b) = 1$. For $b \geq 1$ one has

$$\varphi(2^a 3^b) = \varphi(2^a) 3^{b-1}.2 > 1,$$

so that $\varphi(2^a 3^b) = 1$ requires $b = 0$ and $\varphi(2^a) = 1$. As a result, $a = 0$ or 1 and $r_j = 5$ or 10, if 5 divides r_j. From now on, let us assume that $r_j = 2^a 3^b$ with $a, b \in \mathbb{N} \cup \{0\}$. If $b \geq 2$ then $\varphi(r_j) = \varphi(2^a).3^{b-1}.2$ with $b - 1 \geq 1$ is divisible by 3 and cannot equal $1, 2$ or 4. Therefore $r_j = 2^a.3$ or $r_j = 2^a$ with $a \geq 0$. Straightforwardly,

$$\varphi(2^a.3) = 2\varphi(2^a) \in \{1, 2, 4\}$$

exactly when $\varphi(2^a) = 1$ or $\varphi(2^a) = 2$. These amount to $a \in \{0, 1, 2\}$ and reveal that $3, 6, 12$ are possible values for r_j. Finally, $\varphi(r_j) = \varphi(2^a) \in \{1, 2, 4\}$ for $r_j = 1, 2, 4$ or 8. Thus, $\varphi(r_j) \in \{1, 2, 4\}$ if and only if

$$r_j \in \{1, 2, 3, 4, 5, 6, 8, 10, 12\}.$$

In order to exclude $r_j = 5$ and $r_j = 10$ with $\varphi(5) = \varphi(10) = 4$, recall that λ_j is of degree $\deg_{\mathbb{Q}(\sqrt{-d})}(\lambda_j) = [\mathbb{Q}(\sqrt{-d}, \lambda_j) : \mathbb{Q}(\sqrt{-d})] \leq 2$ over $\mathbb{Q}(\sqrt{-d})$, so that

$$[\mathbb{Q}(\sqrt{-d}, \lambda_j) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{-d}, \lambda_j) : \mathbb{Q}(\sqrt{-d})][\mathbb{Q}(\sqrt{-d}) : \mathbb{Q}] \leq 4.$$

On the other hand,

$$\mathbb{Q} \subset \mathbb{Q}(\lambda_j) \subseteq \mathbb{Q}(\sqrt{-d}, \lambda_j)$$

implies that

$$[\mathbb{Q}(\sqrt{-d}, \lambda_j) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{-d}, \lambda_j) : \mathbb{Q}(\lambda_j)][\mathbb{Q}(\lambda_j) : \mathbb{Q}] = 4[\mathbb{Q}(\sqrt{-d}, \lambda_j) : \mathbb{Q}(\lambda_j)] \geq 4,$$

whereas $[\mathbb{Q}(\sqrt{-d}, \lambda_j) : \mathbb{Q}] = [\mathbb{Q}(\lambda_j) : \mathbb{Q}] = 4$ and $[\mathbb{Q}(\sqrt{-d}, \lambda_j) : \mathbb{Q}(\lambda_j)] = 1$. Therefore $\mathbb{Q}(\sqrt{-d}, \lambda_j) = \mathbb{Q}(\lambda_j)$, so that $\sqrt{-d} \in \mathbb{Q}(\lambda_j)$ and $\mathbb{Q}(\sqrt{-d}) \subset \mathbb{Q}(\lambda_j)$ with

$$[\mathbb{Q}(\lambda_j) : \mathbb{Q}(\sqrt{-d})] = \frac{[\mathbb{Q}(\lambda_j) : \mathbb{Q}]}{[\mathbb{Q}(\sqrt{-d}) : \mathbb{Q}]} = \frac{4}{2} = 2.$$

As far as $\mathbb{Q}(\sqrt{-d})$ and $\mathbb{Q}(\lambda_j)$ are finite Galois extensions of $\mathbb{Q}$ (i.e., normal and separable), the subfield $\mathbb{Q}(\sqrt{-d})$ of $\mathbb{Q}(\lambda_1)$ of index $[\mathbb{Q}(\lambda_1) : \mathbb{Q}(\sqrt{-d})] = 2$ is the fixed point set of a subgroup H of the Galois group $Gal(\mathbb{Q}(\lambda_j)/\mathbb{Q})$ with $|H| = 2$. The minimal polynomial of λ_j over $\mathbb{Q}$ is the cyclotomic polynomial $\Phi_{r_j}(x) \in \mathbb{Z}[x]$ of degree $\deg(\Phi_{r_j}) = \varphi(r_j) = 4$ for $r_j \in \{5, 10\}$ and the Galois group

$$Gal(\mathbb{Q}(\lambda_j)/\mathbb{Q}) \simeq \mathbb{Z}_{r_j}^*$$

coincides with the multiplicative group $\mathbb{Z}_{r_j}^*$ of the congruence ring $\mathbb{Z}_{r_j}$ modulo r_j. More precisely, the roots of $\Phi_{r_j}(x)$ are $\{\lambda_j^s \ | \ s \in \mathbb{Z}_{r_j}^*\}$ and for any $s \in \mathbb{Z}_{r_j}^*$ the correspondence $\lambda_j \mapsto \lambda_j^s$ extends to an automorphism of $\mathbb{Q}(\lambda_j)$, fixing $\mathbb{Q}$. The groups

$$\mathbb{Z}_5^* = \{\pm 1(\mathrm{mod} 5), \ \ \pm 3(\mathrm{mod} 5)\} = \langle 3(\mathrm{mod} 5)\rangle = \langle -3(\mathrm{mod} 5)\rangle \simeq \mathbb{C}_4$$

and

$$\mathbb{Z}_{10}^* = \{\{\pm 1(\mathrm{mod} 10), \quad \pm 3(\mathrm{mod} 10)\} = \langle 3(\mathrm{mod} 10)\rangle = \langle -3(\mathrm{mod} 10)\rangle \simeq \mathbb{C}_4$$

are cyclic and contain unique subgroups $H_5 = \langle -1(\mathrm{mod}5)\rangle$, respectively, $H_{10} = \langle -1(\mathrm{mod}10)\rangle$ or order 2. Denote by h the generator of H_5 or H_{10} with $h(\lambda_j) = \lambda_j^{-1}$, $h|\mathbb{Q} = Id_{\mathbb{Q}}$. In both cases, the degree

$$\deg_{\mathbb{Q}(\sqrt{-d})}(\lambda_j) = [\mathbb{Q}(\lambda_j, \sqrt{-d}) : \mathbb{Q}(\sqrt{-d})] = [\mathbb{Q}(\lambda_j) : \mathbb{Q}(\sqrt{-d})] = 2,$$

so that the characteristic polynomial

$$\mathcal{X}_g(\lambda) = \lambda^2 - \mathrm{tr}(g)\lambda + \det(g) \in R[\lambda] \subset \mathbb{Q}(\sqrt{-d})[\lambda]$$

of g is irreducible over $\mathbb{Q}(\sqrt{-d})$. In fact, $\mathcal{X}_g(\lambda)$ is a minimal polynomial of λ_j over $\mathbb{Q}(\sqrt{-d})$ and divides the cyclotomic polynomial $\Phi_{r_j}(\lambda) \in \mathbb{Z}[\lambda] \subset \mathbb{Q}(\sqrt{-d})[\lambda]$ with $\Phi_{r_j}(\lambda_j) = 0$. In particular, the other eigenvalue λ_{3-j} of g is a root of $\Phi_{r_j}(\lambda)$ or a primitive r_j-th root of unity. That allows to express $\lambda_{3-j} = \lambda_j^t$ by some $t \in \mathbb{Z}_{r_j}^*$. According to

$$\lambda_j^{t+1} = \lambda_j \lambda_j^t = \lambda_j \lambda_{3-j} = \det(g) \in R^* \subset \mathbb{Q}(\sqrt{-d}) = \mathbb{Q}(\lambda_j)^{\langle h\rangle},$$

one has

$$\lambda_j^{t+1} = h(\lambda_j^{t+1}) = \lambda_j^{-t-1} \quad \text{or} \quad \lambda_j^{2(t+1)} = 1.$$

If λ_j is a primitive fifth root of unity then $\lambda_j^{2(t+1)} = 1$ requires that $2(t+1)$ to be divisible by 5. Since $GCD(2,5) = 1$, 5 is to divide $t+1$ or $t \equiv -1(\mathrm{mod}5)$. Similarly, if λ_j is a primitive tenth root of unity then 10 divides $2(t+1)$, i.e., $2(t+1) = 10z$ for some $z \in \mathbb{Z}$. As a result, 5 divides $t+1$ and $t \equiv -1(\mathrm{mod}10)$. Thus, for any $r_1 \in \{5, 10\}$ there follows $\lambda_{3-j} = \lambda_j^t = \lambda_j^{-1}$. Expressing $\lambda_j = e^{\frac{2\pi i s}{r_j}}$ for some natural number $1 \leq s \leq r_j - 1$, relatively prime to r_j, one observes that

$$\mathrm{tr}(g) = \lambda_j + \lambda_{3-j} = \lambda_j + \lambda_j^{-1} = e^{\frac{2\pi i s}{r_j}} + e^{-\frac{2\pi i s}{r_j}} = 2\cos\left(\frac{2\pi s}{r_j}\right) \in R \cap \mathbb{R}.$$

We claim that $R \cap \mathbb{R} = \mathbb{Z}$. The inclusion $\mathbb{Z} \subseteq R \cap \mathbb{R}$ is clear. Conversely, let

$$r \in \mathbb{R} \cap R = \mathbb{R} \cap (\mathbb{Z} + f\omega_{-d}\mathbb{Z})$$

for the conductor $f \in \mathbb{N}$ of E and ω_{-d} from (6). In the case of $-d \not\equiv 1(\mathrm{mod}4)$ there exist $a, b \in \mathbb{Z}$ with $r = a + f\sqrt{-d}b$. The complex number $a - r + f\sqrt{-d}b = 0$ vanishes exactly when its real part $a - r = 0$ and its imaginary part $f\sqrt{d}b = 0$ are zero. Therefore $b = 0$ and $r = a \in \mathbb{Z}$, i.e., $\mathbb{R} \cap R \subseteq \mathbb{Z}$ for $-d \not\equiv 1(\mathrm{mod}4)$.

If $-d \equiv 1(\mathrm{mod}4)$ then

$$r = a + fb\frac{(1 + \sqrt{-d})}{2} \quad \text{for some} \quad a, b \in \mathbb{Z}$$

yields

$$\left| \begin{array}{l} r = a + \frac{fb}{2} \\ \frac{f\sqrt{d}}{2}b = 0 \end{array} \right.$$

by comparison of the real and imaginary parts. As a result, again $b = 0$ and $r = a \in \mathbb{Z}$, i.e., $\mathbb{R} \cap R \subseteq \mathbb{Z}$ for $-d \equiv 1(\mathrm{mod} 4)$. That justifies $\mathbb{R} \cap R = \mathbb{Z}$ and implies that $\mathrm{tr}(g) = 2\cos\left(\frac{2\pi s}{r_j}\right) \in \mathbb{Z}$. Bearing in mind the $\cos\left(\frac{2\pi s}{r_j}\right) \in [-1, 1]$, one concludes

$$\mathrm{tr}(g) = 2\cos\left(\frac{2\pi s}{r_j}\right) \in [-2, 2] \cap \mathbb{Z} = \{0, \pm 1, \pm 2\} \quad \text{or} \tag{7}$$

$$\cos\left(\frac{2\pi s}{r_j}\right) \in \left\{0, \pm\frac{1}{2}, \pm 1\right\}.$$

For a natural number $1 \leq s \leq r_j - 1$, one has $\frac{2\pi s}{r_1} \in [0, 2\pi)$. The solutions of $\cos(x) = 0$ in $[0, 2\pi)$ are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, while $\cos(x) = \pm 1$ holds for $x \in \{0, \pi\}$. Finally, $\cos(x) = \pm\frac{1}{2}$ is satisfied by $x \in \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$, so that (7) implies

$$\frac{2\pi s}{r_j} \in \left\{0, \quad \frac{\pi}{2}, \pi, \quad \frac{3\pi}{2}, \quad \frac{\pi}{3}, \quad \frac{2\pi}{3}, \quad \frac{4\pi}{3}, \frac{5\pi}{3}\right\}. \tag{8}$$

For $r_j = 5$ or 10 this is an absurd, so that

$$r_j \in \{1, 2, 3, 4, 6, 8, 12\}.$$

□

Now we are ready to describe the elements of $GL(2, R)$ of finite order, by specifying their eigenvalues λ_1, λ_2. The roots λ_1, λ_2 of the characteristic polynomial

$$\mathcal{X}_g(\lambda) = \lambda^2 - \mathrm{tr}(g)\lambda + \det(g) \in R[\lambda]$$

of g are in a bijective correspondence with the trace $\mathrm{tr}(g) = \lambda_1 + \lambda_2 \in R$ and the determinant $\det(g) = \lambda_1\lambda_2 \in R^*$ of g. Making use of Lemma 8, we subdivide the problem to the description of finite order $g \in GL(2, R)$ with a fixed determinant $\det(g) \in R^*$. The traces of such g take finitely many values and allow to list explicitly the eigenvalues of all $g \in GL(2, R)$ of finite order. The classification of the unordered pairs of eigenvalues λ_1, λ_2 of $g \in GL(2, R)$ of finite order is a more specific result than Proposition 14. Note that the next classification of λ_1, λ_2 is derived independently of Proposition 14.

Let us start with the case of $\det(g) = 1$. The next proposition puts in a bijective correspondence the traces $\mathrm{tr}(g)$ of $g \in SL(2, R)$ with the orders r of g.

Proposition 15. *If $g \in SL(2,R)$ is of finite order r then the trace*

$$\mathrm{tr}(g) \in \{\pm 2, \quad \pm 1, \quad 0\}. \tag{9}$$

The eigenvalues λ_1, λ_2 of g are of order

$$r_1 = r_2 = r \in \{1,2,3,4,6\}. \tag{10}$$

More precisely,

(i) $\mathrm{tr}(g) = 2$ *or* $\lambda_1 = \lambda_2 = 1$, $g = I_2$ *if and only if g is of order* 1;
(ii) $\mathrm{tr}(g) = -2$ *or* $\lambda_1 = \lambda_2 = -1$, $g = -I_2$ *if and only if g is of order* 2;
(iii) $\mathrm{tr}(g) = 1$ *or* $\lambda_1 = e^{\frac{\pi i}{3}}$, $\lambda_2 = e^{-\frac{\pi i}{3}}$ *if and only if g is of order* 6;
(iv) $\mathrm{tr}(g) = -1$ *or* $\lambda_1 = e^{\frac{2\pi i}{3}}$, $\lambda_2 = e^{-\frac{2\pi i}{3}}$ *if and only if g is of order* 3;
(v) $\mathrm{tr}(g) = 0$ *or* $\lambda_1 = i$, $\lambda_2 = -i$ *if and only if g is of order* 4.

Proof. If $g \in SL(2,R)$ is of order r then the eigenvalues λ_j of g are of finite order r_j, dividing $r = LCM(r_1, r_2)$. According to

$$1 = \det(g) = \lambda_1 \lambda_2,$$

one has $\lambda_1 = e^{\frac{2\pi i s}{r_1}}$, $\lambda_2 = e^{-\frac{2\pi i s}{r_1}}$ for some natural number $1 \leq s \leq r_1 - 1$, relatively prime to r_1. Thus, λ_2 is a primitive r_1-th root and $r_1 = r_2 = LCM(r_1, r_2) = r$. As in the proof of Proposition 14,

$$\mathrm{tr}(g) = \lambda_1 + \lambda_2 = e^{\frac{2\pi i s}{r_1}} + e^{-\frac{2\pi i s}{r_1}} = 2\cos\left(\frac{2\pi s}{r_1}\right) \in \mathbb{R} \cap R = \mathbb{Z}$$

and $\cos\left(\frac{2\pi s}{r_1}\right) \in [-1, 1]$ specify (9). Consequently,

$$\cos\left(\frac{2\pi s}{r_1}\right) \in \left\{0, \quad \pm\frac{1}{2}, \quad \pm 1\right\} \quad \text{and}$$

$$\frac{2\pi s}{r_1} \in \left\{0, \quad \frac{\pi}{2}, \quad \pi, \quad \frac{3\pi}{2}, \quad \frac{\pi}{3}, \quad \frac{2\pi}{3}, \quad \frac{4\pi}{3}, \quad \frac{5\pi}{3}\right\},$$

as in (8). Straightforwardly, $\lambda_1 = e^0 = 1$ is of order 1, $\lambda_1 = e^{\pi i} = -1$ is of order 2, $\lambda_1 \in \left\{e^{\frac{\pi i}{2}}, e^{\frac{3\pi i}{2}}\right\}$ are of order 4, $\lambda_1 \in \left\{e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\right\}$ are of order 3 and $\lambda_1 \in \left\{e^{\frac{\pi i}{3}}, e^{\frac{5\pi i}{3}}\right\}$ are of order 6. That justifies (10).

If g is of order $r = 1$ then $\lambda_1 \in \mathbb{C}^*$ is of order $r_1 = 1$, so that $\lambda_1 = 1$. Consequently, $\lambda_2 = 1$ and $g = I_2$, as far as I_2 is the only conjugate of the scalar matrix I_2. The trace $\mathrm{tr}(g) = \mathrm{tr}(I_2) = 2$. Conversely, if $\lambda_1 = \lambda_2 = 1$, then $g = I_2$ is of order 1.

An automorphism $g \in SL(2,R)$ of order $r = 2$ has eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}^*$ of order 2, or $\lambda_1 = \lambda_2 = -1$. Consequently, $g = -I_2$ and $\mathrm{tr}(g) = -2$. Conversely, for $\lambda_1 = \lambda_2 - 1$ the matrix $g = -I_2$ is of order 2.

Let us suppose that $g \in SL(2,R)$ is of order 3. Then the eigenvalues λ_1, λ_2 of g are of order 3 or $\lambda_1 = e^{\frac{2\pi i}{3}}$, $\lambda_2 = e^{-\frac{2\pi i}{3}}$, up to a transposition. The trace $\mathrm{tr}(g) = \lambda_1 + \lambda_2 = -1$. Conversely, if $\lambda_1 = e^{\frac{2\pi i}{3}}$, $\lambda_2 = e^{-\frac{2\pi i}{3}}$ then $r = r_1 = r_2 = 3$.

For $g \in SL(2,R)$ of order 4 one has $\lambda_1, \lambda_2 \in \mathbb{C}^*$ of order 4 or $\lambda_1 = i$, $\lambda_2 = -i$, up to a transposition. The trace $\mathrm{tr}(g) = \lambda_1 + \lambda_2 = 0$. Conversely, for $\lambda_1 = i$, $\lambda_2 = -i$ there follows $r = r_1 = r_2 = 4$.

Suppose that $g \in SL(2,R)$ is of order 6. Then $\lambda_1, \lambda_2 \in \mathbb{C}^*$ are of order 6 or $\lambda_1 = e^{\frac{\pi i}{3}}$, $\lambda_2 = e^{-\frac{\pi i}{3}}$, up to a transposition. The trace $\mathrm{tr}(g) = \lambda_1 + \lambda_2 = 1$. Conversely, the assumption $\lambda_1 = e^{\frac{\pi i}{3}}$, $\lambda_2 = e^{-\frac{\pi i}{3}}$ implies $r = r_1 = r_2 = 6$.

Note that

$$g_1 = \begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix} \in SL(2,\mathbb{Z}) \subseteq SL(2,R)$$

with $\mathrm{tr}(g_1) = -1$, $\mathrm{tr}(g_2) = 0$, $\mathrm{tr}(g_3) = 1$ realize all the possibilities, listed in the statement of the proposition.

□

If E is an elliptic curve with complex multiplication by an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$ and conductor $f \in \mathbb{N}$ then we denote the endomorphism ring of E by

$$R_{-d,f} = \mathbb{Z} + f\mathcal{O}_{-d} = \mathbb{Z} + f\omega_{-d}\mathbb{Z},$$

where ω_{-d} is the non-trivial generator of $\mathcal{O}_{-d}$ as a $\mathbb{Z}$-module, given in (6). If E has no complex multiplication, we put

$$R_{0,1} := \mathbb{Z}.$$

Proposition 16. *Let $g \in GL(2, R_{-d,f})$ be a linear automorphism of $A = E \times E$ of order r, with $\det(g) = -1$ and eigenvalues $\lambda_1(g), \lambda_2(g) \in \mathbb{C}^*$.*

(i) The automorphism g is of order 2 if and only if its trace is $\mathrm{tr}(g) = 0$ or, equivalently, $\lambda_1(g) = -1$, $\lambda_2(g) = 1$.

(ii) If $R_{-d,f} \neq \mathbb{Z}[i], \mathcal{O}_{-2}, \mathcal{O}_{-3}, R_{-3,2}$ then any $g \in GL(2, R_{-d,f}) \setminus SL(2,R)$ is of order 2.

(iii) If $g \in GL(2, \mathcal{O}_{-2})$ is of order $r > 2$ and $\det(g) = -1$ then $r = 8$ and the trace $\mathrm{tr}(g) \in \{\pm\sqrt{-2}\}$.

More precisely,

(a) $\mathrm{tr}(g) = \sqrt{-2}$ if and only if $\lambda_1(g) = e^{\frac{\pi i}{4}}$, $\lambda_2(g) = e^{\frac{3\pi i}{4}}$;

(b) $\mathrm{tr}(g) = -\sqrt{-2}$ if and only if $\lambda_1(g) = e^{\frac{5\pi i}{4}}$, $\lambda_2(g) = e^{-\frac{\pi i}{4}}$.

(iv) If $g \in GL(2, \mathbb{Z}[i])$ is of order $r > 2$ and $\det(g) = -1$, then $r \in \{4, 12\}$ and the trace $\mathrm{tr}(g) \in \{\pm i, \pm 2i\}$.

More precisely,

(a) $\mathrm{tr}(g) = 2i$ exactly when $g = iI_2$;

(b) $\mathrm{tr}(g) = -2i$ exactly when $g = -iI_2$;

(c) $\mathrm{tr}(g) = i$ *exactly when* $\lambda_1(g) = e^{\frac{\pi i}{6}}$, $\lambda_2(g) = e^{\frac{5\pi i}{6}}$;

(d) $\mathrm{tr}(g) = -i$ *exactly when* $\lambda_1(g) = e^{\frac{7\pi i}{6}}$, $\lambda_2(g) = e^{-\frac{\pi i}{6}}$.

(v) If $g \in GL(2, R_{-3,f})$ *with* $R_{-3,f} \in \{R_{-3,1} = \mathcal{O}_{-3}, R_{-3,2} = \mathbb{Z} + \sqrt{-3}\mathbb{Z}\}$ *is of order* $r > 2$ *and* $\det(g) = -1$ *then* $r = 6$ *and the trace* $\mathrm{tr}(g) \in \{\pm\sqrt{-3}\}$.

More precisely,

(a) $\mathrm{tr}(g) = \sqrt{-3}$ *if and only if* $\lambda_1(g) = e^{\frac{\pi i}{3}}$, $\lambda_2(g) = e^{\frac{2\pi i}{3}}$;

(b) $\mathrm{tr}(g) = -\sqrt{-3}$ *if and only if* $\lambda_1(g) = e^{-\frac{2\pi i}{3}}$, $\lambda_2(g) = e^{-\frac{\pi i}{3}}$.

Proof. The eigenvalues $\lambda_1(g), \lambda_2(g) \in \mathbb{C}^*$ of $g \in GL(2, R_{-d,f})$ with $\det(g) = -1$ are subject to $\lambda_2(g) = -\lambda_1(g)^{-1}$. More precisely, if $\lambda_1(g) = e^{\frac{2\pi si}{r_1}}$ is a primitive r_1-th root of unity then $\lambda_2(g) = -e^{-\frac{2\pi si}{r_1}}$. The trace

$$\mathrm{tr}(g) = \lambda_1(g) + \lambda_2(g) = e^{\frac{2\pi si}{r_1}} - e^{-\frac{2\pi si}{r_1}} = 2i \sin\left(\frac{2\pi s}{r_1}\right) \in R_{-d,f} \cap i\mathbb{R}. \qquad (11)$$

We claim that

$$R_{-d,f} \cap i\mathbb{R} = \begin{cases} f\sqrt{-d}\mathbb{Z} & \text{for } -d \not\equiv 1(\mathrm{mod}4) \text{ or } -d \equiv 1(\mathrm{mod}4),\ f \equiv 1(\mathrm{mod}2), \\ \frac{f}{2}\sqrt{-d}\mathbb{Z} & \text{for } -d \equiv 1(\mathrm{mod}4),\ f \equiv 0(\mathrm{mod}2). \end{cases}$$

Indeed, if $-d \not\equiv 1(\mathrm{mod}4)$ then $\mathcal{O}_{-d} = \mathbb{Z} + \sqrt{-d}\mathbb{Z}$ and $R_{-d,f} = \mathbb{Z} + f\sqrt{-d}\mathbb{Z}$ contains $f\sqrt{-d}$, i.e., $f\sqrt{-d}\mathbb{Z} \subseteq R_{-d,f} \cap i\mathbb{R}$. Any $ir = a + bf\sqrt{-d} \in i\mathbb{R} \cap R_{-d,f}$ with $r \in \mathbb{R}$, $a, b \in \mathbb{Z}$ has imaginary part $r = bf\sqrt{d}$, so that $i\mathbb{R} \cap R_{-d,f} \subseteq f\sqrt{-d}\mathbb{Z}$ and $i\mathbb{R} \cap R_{-d,f} = f\sqrt{-d}\mathbb{Z}$.

Suppose that $-d \equiv 1(\mathrm{mod}4)$ and the conductor $f = 2k+1 \in \mathbb{N}$ is odd. Then $R_{-d,2k+1} = \mathbb{Z} + f\frac{(1+\sqrt{-d})}{2}\mathbb{Z}$ contains $f\sqrt{-d} = -f + (2f)\frac{(1+\sqrt{-d})}{2}$, so that $f\sqrt{-d}\mathbb{Z} \subseteq R_{-d,2k+1} \cap i\mathbb{R}$. Any $ir = a + \frac{bf}{2}(1+\sqrt{-d})$ with $r \in \mathbb{R}$, $a, b \in \mathbb{Z}$ has real part $a + \frac{bf}{2} = 0$ and imaginary part $r = \frac{bf}{2}\sqrt{d}$. Note that $\frac{bf}{2} = \frac{b(2k+1)}{2} = -a \in \mathbb{Z}$ is an integer only for an even $b = 2b_1$, $b_1 \in \mathbb{Z}$, so that $r = b_1 f\sqrt{d}$ and $i\mathbb{R} \cap R_{-d,2k+1} \subseteq f\sqrt{-d}\mathbb{Z}$. That justifies $i\mathbb{R} \cap R_{-d,2k+1} = f\sqrt{-d}\mathbb{Z}$ for $-d \equiv 1(\mathrm{mod}4)$, $f \equiv 1(\mathrm{mod}2)$.

Finally, for $-d \equiv 1(\mathrm{mod}4)$ and an even conductor $f = 2k \in \mathbb{N}$ the endomorphism ring $R_{-d,2k} = \mathbb{Z} + k(1+\sqrt{-d})\mathbb{Z}$ contains $k\sqrt{-d}$, so that $k\sqrt{-d}\mathbb{Z} \subseteq i\mathbb{R} \cap R_{-d,2k}$. Note that $ir = a + bk(1+\sqrt{-d})$ with $r \in \mathbb{R}$, $a, b \in \mathbb{Z}$ has real part $a + bk = 0$ and imaginary part $r = bk\sqrt{d}$, so that $i\mathbb{R} \cap R_{-d,2k} \subseteq k\sqrt{-d}\mathbb{Z}$ and $i\mathbb{R} \cap R_{-d,2k} = k\sqrt{-d}\mathbb{Z}$.

Now, (11) implies that

$$2\sin\left(\frac{2\pi s}{r_1}\right) \in [-2, 2] \cap i(R_{-d,f} \cap i\mathbb{R}) =$$

$$= \begin{cases} [-2, 2] \cap f\sqrt{d}\mathbb{Z} & \text{for } -d \not\equiv 1(\mathrm{mod}4) \text{ or } -d \equiv 1(\mathrm{mod}4),\ f \equiv 1(\mathrm{mod}2), \\ [-2, 2] \cap \frac{f}{2}\sqrt{d}\mathbb{Z} & \text{for } -d \equiv 1(\mathrm{mod}4),\ f \equiv 0(\mathrm{mod}2). \end{cases}$$

If $d \geq 5$ then $\sqrt{d} \geq \sqrt{5} > 2$ and $[-2,2] \cap f\sqrt{d}\mathbb{Z} = \{0\}$ for $\forall f \in \mathbb{N}$ and $[-2,2] \cap \frac{f}{2}\sqrt{d}\mathbb{Z} = \{0\}$ for $\forall f \in 2\mathbb{N}$. Note that $\sin\left(\frac{2\pi s}{r_1}\right) = 0$ for some natural number $1 \leq s \leq r_1 - 1$ with $GCD(s, r_1) = 1$ has unique solution $\frac{2\pi s}{r_1} = \pi$, since $\frac{2\pi s}{r_1} \in (0, 2\pi)$. That implies $2s = r_1$, whereas s divides r_1 and $s = GCD(s, r_1) = 1$, $r_1 = 2$. Thus, $\lambda_1 = e^{\frac{2\pi i}{2}} = e^{\pi i} = -1$, $\lambda_2 = -(-1) = 1$ and g is conjugate to

$$D_2 = \left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right).$$

In particular, g is of order 2. Note that the case of $g \in GL(2, R)$ with $\lambda_1 = -1$, $\lambda_2 = 1$ is realized by the diagonal matrix $D_2 \in GL(2, \mathbb{Z}) \leq GL(2, R_{-d,f})$.

If $d = 1$ and $f \geq 3$ then $2\sin\left(\frac{2\pi s}{r_1}\right) \in [-2,2] \cap f\mathbb{Z} = \{0\}$ and D_2 is the only diagonal form for g. For $d = 2$ and $f \geq 2$ the intersection $[-2,2] \cap f\sqrt{2}\mathbb{Z} = \{0\}$, so that any $g \in GL(2, R_{-2,f})$ with $f \geq 2$ and $\det(g) = -1$ is conjugate to D_2. If $d = 3$ and $f = 2k+1 \geq 3$ then $[-2,2] \cap f\sqrt{3}\mathbb{Z} = \{0\}$. Similarly, for $d = 3$ and $f = 2k \geq 4$ one has $[-2,2] \cap k\sqrt{3}\mathbb{Z} = \{0\}$. In such a way, the existence of $g \in GL(2, R_{-d,f})$ with $\det(g) = -1$, $\mathrm{tr}(g) \neq 0$ requires $R_{-d,f}$ to be among

$$R_{-1,1} = \mathcal{O}_{-1} = \mathbb{Z}[i], \quad R_{-1,2} = \mathbb{Z} + 2i\mathbb{Z}, \quad R_{-2,1} = \mathcal{O}_{-2} = \mathbb{Z} + \sqrt{-2}\mathbb{Z},$$

$$R_{-3,1} = \mathcal{O}_{-3} = \mathbb{Z} + \frac{1+\sqrt{-3}}{2}\mathbb{Z} \quad \text{or} \quad R_{-3,2} = \mathbb{Z} + 2\left(\frac{1+\sqrt{-3}}{2}\right)\mathbb{Z} = \mathbb{Z} + \sqrt{-3}\mathbb{Z}.$$

The next considerations exploit the following simple observation: If a, b are relatively prime natural numbers and s, r_1 are relatively prime natural numbers then $as = br_1$ if and only if $s = b$ and $r_1 = a$. Namely, b divides as and $GCD(a, b) = 1$ requires b to divide s. Thus, $s = bs_1$ for some $s_1 \in \mathbb{N}$ and $as_1 = r_1$. Now s_1 is a natural common divisor of the relatively prime s, r_1, so that $s_1 = 1$, $s = b$ and $r_1 = a$.

For $d = 1$ and $f = 2$ one has $2\sin\left(\frac{2\pi s}{r_1}\right) \in [-2,2] \cap f\mathbb{Z} = \{0, \pm 2\}$. Let $\mathrm{tr}(g) = 2i$ or $\sin\left(\frac{2\pi s}{r_1}\right) = 1$ for $r_1 \in \mathbb{N}$ and some natural number $1 \leq s \leq r_1 - 1$, $GCD(s, r_1) = 1$. Then $\frac{2\pi s}{r_1} = \frac{\pi}{2}$ or $4s = r_1$. As a result, $s = 1$, $r_1 = 4$ and $\lambda_1 = e^{\frac{\pi i}{2}} = i$, $\lambda_2 = -e^{-\frac{\pi i}{2}} = i$. Now $g = iI_2$ as the unique matrix, conjugate to the scalar matrix iI_2. However, $iI_2 \notin GL(2, R_{-1,2}) = GL(2, \mathbb{Z} + 2i\mathbb{Z})$, so that $g = iI_2$ is not a solution of the problem. For $\mathrm{tr}(g) = -2i$ one has $\sin\left(\frac{2\pi s}{r_1}\right) = -1$, whereas $\frac{2\pi s}{r_1} = \frac{3\pi}{2}$ and $4s = 3r_1$. Thus, $s = 3$, $r_1 = 4$ and $\lambda_1 = e^{\frac{3\pi i}{3}} = -i$, $\lambda_2 = -e^{-\frac{3\pi i}{3}} = -i$. That determines a unique $g = -iI_2$. But $-iI_2 \notin GL(2, R_{-1,2}) = GL(2, \mathbb{Z} + 2i\mathbb{Z})$, so that $\lambda_1 = 1$, $\lambda_2 = -1$ are the only possible eigenvalues for $g \in GL(2, R_{-1,2})$ of finite order with $\det(g) = -1$.

In the case of $d = 1$ and $f = 1$, note that $2\sin\left(\frac{2\pi s}{r_1}\right) \in [-2,2] \cap \mathbb{Z} = \{0, \pm 1, \pm 2\}$. Besides $g \in GL(2, \mathbb{Z}[i])$ with $\det(g) = -1$, $\mathrm{tr}(g) = 0$, one has $g = iI_2 \in GL(2, \mathbb{Z}[i])$ and $g = -iI_2 \in GL(2, \mathbb{Z}[i])$. The case of $\mathrm{tr}(g) = i$ corresponds to $\sin\left(\frac{2\pi s}{r_1}\right) = \frac{1}{2}$

and holds for $\frac{2\pi s}{r_1} = \frac{\pi}{6}$ or $\frac{2\pi s}{r_1} = \frac{5\pi}{6}$. Note that $12s = r_1$ implies $s = 1$, $r_1 = 12$ and $\lambda_1 = e^{\frac{\pi i}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, $\lambda_2 = -e^{-\frac{\pi i}{6}} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i = e^{\frac{5\pi i}{6}}$. Thus, g is of order $r = LCM(12, 12) = 12$. This possibility is realized, for instance, by

$$g(i) = \begin{pmatrix} 1 & 1 \\ i & (-1+i) \end{pmatrix} \in GL(2, \mathbb{Z}[i]) \quad \text{with} \quad \det(g(i)) = -1, \quad \text{tr}(g(i)) = i.$$

If $12s = 5r_1$ then $s = 5$, $r_1 = 12$ and $\lambda_1 = e^{\frac{5\pi i}{6}} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$, $\lambda_2 = -e^{-\frac{5\pi i}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i = e^{\frac{\pi i}{6}}$, which was already obtained. Note that $\text{tr}(g) = -i$ amounts to $\sin\left(\frac{2\pi s}{r_1}\right) = -\frac{1}{2}$ and holds for $\frac{2\pi s}{r_1} = \frac{7\pi}{6}$ or $\frac{2\pi s}{r_1} = \frac{11\pi}{6}$. If $12s = 7r_1$ then $s = 7$, $r_1 = 12$ and $\lambda_1 = e^{\frac{7\pi i}{6}} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$, $\lambda_2 = -e^{-\frac{7\pi i}{6}} = \frac{\sqrt{3}}{2} - \frac{1}{2}i = e^{-\frac{\pi i}{6}}$ and g is of order $r = LCM(12, 12) = 12$. Note that

$$g(-i) = \begin{pmatrix} 1 & 1 \\ -i & (-1-i) \end{pmatrix} \in GL(2, \mathbb{Z}[i]) \quad \text{with} \quad \det(g(-i)) = -1, \quad \text{tr}(g(-i)) = -i$$

realizes the aforementioned possibility.

In the case of $12s = 11r_1$ one has $s = 11$, $r_1 = 12$ and $\lambda_1 = e^{\frac{11\pi i}{6}} = \frac{\sqrt{3}}{2} - \frac{1}{2}i$, $\lambda_2 = -e^{\frac{\pi i}{6}} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$, which is already listed as a solution. That concludes the considerations for $g \in GL(2, \mathbb{Z}[i])$ with $\det(g) = -1$.

If $d = 2$ and $f = 1$ then $2\sin\left(\frac{2\pi s}{r_1}\right) \in [-2, 2] \cap \sqrt{2}\mathbb{Z} = \{0, \pm\sqrt{2}\}$. Note that $\sin\left(\frac{2\pi s}{r_1}\right) = \frac{\sqrt{2}}{2}$ holds for $\frac{2\pi s}{r_1} = \frac{\pi}{4}$ or $\frac{2\pi s}{r_1} = \frac{3\pi}{4}$. The equality $r_1 = 8s$ implies $s = 1$ and $r_1 = 8$. As a result, $\lambda_1 = e^{\frac{\pi i}{4}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $\lambda_2 = -e^{-\frac{\pi i}{4}} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i = e^{\frac{3\pi i}{4}}$. Observe that

$$g(\sqrt{-2}) = \begin{pmatrix} 1 & 1 \\ \sqrt{-2} & (-1+\sqrt{-2}) \end{pmatrix} \in GL(2, \mathcal{O}_{-2}), \mathcal{O}_{-2} = \mathbb{Z} + \sqrt{-2}\mathbb{Z}$$

with $\det(g(\sqrt{-2})) = -1$, $\text{tr}(g(\sqrt{-2})) = \sqrt{-2}$ realizes the aforementioned possibility. If $8s = 3r_1$ then $s = 3$, $r_1 = 8$ and $\lambda_1 = e^{\frac{3\pi i}{4}} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $\lambda_2 = -e^{-\frac{3\pi i}{4}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i = e^{\frac{\pi i}{4}}$. These eigenvalues have been already mentioned.

For $\sin\left(\frac{2\pi s}{r_1}\right) = -\frac{\sqrt{2}}{2}$ there follows $\frac{2\pi s}{r_1} = \frac{5\pi}{4}$ or $\frac{2\pi s}{r_1} = \frac{7\pi}{4}$. If $8s = 5r_1$ then $s = 5$, $r_1 = 8$ and $\lambda_1 = e^{\frac{5\pi i}{4}} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$, $\lambda_2 = -e^{-\frac{5\pi i}{4}} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i = e^{-\frac{\pi i}{4}}$. The corresponding automorphism g is of order $r = LCM(8, 8) = 8$. Note that

$$g(-\sqrt{-2}) = \begin{pmatrix} 1 & 1 \\ -\sqrt{-2} & (-1-\sqrt{-2}) \end{pmatrix} \in GL(2, \mathcal{O}_{-2})$$

with $\det(g(-\sqrt{-2})) = -1$, $\text{tr}(g(-\sqrt{-2})) = -\sqrt{-2}$. realizes this possibility. In the case of $8s = 7r_1$, one has $s = 7$, $r_1 = 8$. The eigenvalues $\lambda_1 = e^{\frac{7\pi i}{4}} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$,

$\lambda_2 = -e^{-\frac{7\pi i}{4}} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ were already obtained. That concludes the considerations for $d = 2$.

If $d = 3$ and $f = 1$, note that $2\sin\left(\frac{2\pi s}{r_1}\right) \in [-2, 2] \cap \sqrt{3}\mathbb{Z} = \{0, \pm\sqrt{3}\}$. Similarly, for $d = 3$ and $f = 2$ one has $2\sin\left(\frac{2\pi s}{r_1}\right) \in [-2, 2] \cap \sqrt{3}\mathbb{Z} = \{0, \pm\sqrt{3}\}$. If $\sin\left(\frac{2\pi s}{r_1}\right) = \frac{\sqrt{3}}{2}$ then $\frac{2\pi s}{r_1} = \frac{\pi}{3}$ or $\frac{2\pi s}{r_1} = \frac{2\pi}{3}$. In the case of $6s = r_1$ there follows $s = 1$, $r_1 = 6$. The eigenvalues $\lambda_1 = e^{\frac{\pi i}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\lambda_2 = -e^{-\frac{\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{\frac{2\pi i}{3}}$ and g is of order $r = LCM(6, 3) = 6$. The automorphism

$$g(\sqrt{-3}) = \left(\begin{array}{cc} 1 & 1 \\ \sqrt{-3} & (-1+\sqrt{-3}) \end{array}\right) \in GL(2, R_{-3,2}) \leq GL(2, \mathcal{O}_{-3})$$

with $\det(g(\sqrt{-3})) = -1$, $\mathrm{tr}(g(\sqrt{-3})) = \sqrt{-3}$ realizes the aforementioned possibility. If $3s = r_1$ then $s = 1$, $r_1 = 3$ and $\lambda_1 = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\lambda_2 = -e^{-\frac{2\pi i}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{\frac{\pi i}{3}}$, which was already obtained.

If $\sin\left(\frac{2\pi s}{r_1}\right) = -\frac{\sqrt{3}}{2}$ then $\frac{2\pi s}{r_1} = \frac{4\pi}{3}$ or $\frac{2\pi s}{r_1} = \frac{5\pi}{3}$. In the case of $3s = 2r_1$ note that $s = 2$, $r_1 = 3$ and $\lambda_1 = e^{\frac{4\pi i}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, $\lambda_2 = -e^{-\frac{4\pi i}{3}} = \frac{1}{2} - \frac{\sqrt{3}}{2}i = e^{-\frac{\pi i}{3}}$. The automorphisms g with such eigenvalues are of order $r = LCM(3, 6) = 6$. In particular,

$$g(-\sqrt{-3}) = \left(\begin{array}{cc} 1 & 1 \\ -\sqrt{-3} & (-1-\sqrt{-3}) \end{array}\right) \in GL(2, R_{-3,2}) \leq GL(2, \mathcal{O}_{-3})$$

with $\det(g(-\sqrt{-3})) = -1$, $\mathrm{tr}(g(-\sqrt{-3})) = -\sqrt{-3}$ realizes the aforementioned possibility.

If $6s = 5r_1$ then $s = 5$, $r_1 = 6$ and $\lambda_1 = e^{\frac{5\pi i}{3}} = e^{-\frac{\pi i}{3}} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$, $\lambda_2 = -e^{\frac{\pi i}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i = e^{\frac{4\pi i}{3}}$. These eigenvalues are already obtained. That concludes the considerations for $d = 3$ and the description of all $g \in GL(2, R_{-d,f})$ with $\det(g) = -1$. $\square$

Proposition 17. *If $g \in GL(2, \mathbb{Z}[i])$ is of finite order r and $\det(g) = i$ then*

$$\mathrm{tr}(g) \in \{0, \pm(1+i)\}, \quad r \in \{4, 8\}.$$

More precisely,

(i) $\mathrm{tr}(g) = 0$ or $\lambda_1 = e^{\frac{3\pi i}{4}}$, $\lambda_2 = e^{-\frac{\pi i}{4}}$ if and only if g is of order 8;

(ii) if $\mathrm{tr}(g) = 1 + i$ or $\lambda_1 = i$, $\lambda_2 = 1$ then g is of order 4;

(iii) if $\mathrm{tr}(g) = -1 - i$ or $\lambda_1 = -i$, $\lambda_2 = -1$ then g is of order 4.

Proof. If $\lambda_1 = e^{\frac{2\pi s i}{r_1}}$ for the order $r_1 \in \mathbb{N}$ of $\lambda_1 \in \mathbb{C}^*$ and some natural number $1 \leq s < r_1$, $GCD(s, r_1) = 1$, then $\lambda_2 = \det(g)\lambda_1^{-1} = ie^{-\frac{2\pi s i}{r_1}}$. Therefore, the trace

$$\mathrm{tr}(g) = \lambda_1 + \lambda_2 = \left[\cos\left(\frac{2\pi s}{r_1}\right) + \sin\left(\frac{2\pi s}{r_1}\right)\right](1+i) =$$

$$= \sqrt{2}\sin\left(\frac{2\pi s}{r_1}+\frac{\pi}{4}\right)(1+i)\in \mathbb{Z}[i]=\mathbb{Z}+i\mathbb{Z}$$

if and only if the real part

$$\sqrt{2}\sin\left(\frac{2\pi s}{r_1}+\frac{\pi}{4}\right)\in \mathbb{Z}\cap[-\sqrt{2},\sqrt{2}]=\{0,\pm 1\}.$$

As a result, $\mathrm{tr}(g)\in\{0,\pm(1+i)\}$. If $\mathrm{tr}(g)=0$ or, equivalently, $\sin\left(\frac{2\pi s}{r_1}+\frac{\pi}{4}\right)=0$ for $\frac{2\pi s}{r_1}+\frac{\pi}{4}\in\left(\frac{\pi}{4},\frac{9\pi}{4}\right)$ then $\frac{2\pi s}{r_1}+\frac{\pi}{4}=\pi$ or $\frac{2\pi s}{r_1}+\frac{\pi}{4}=2\pi$. For $\frac{2s}{r_1}=\frac{3}{4}$ there follows $8s=3r_1$ and $s=3$, $r_1=8$. As a result, $\lambda_1=e^{\frac{3\pi i}{4}}=-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$, $\lambda_2=ie^{-\frac{3\pi i}{4}}=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i=e^{-\frac{\pi i}{4}}$ and g is of order $r=LCM(8,8)=8$. For instance,

$$g_i(0)=\left(\begin{array}{cc} i & i \\ (-1-i) & -i\end{array}\right)\in GL(2,\mathbb{Z}[i])$$

with $\det(g_i(0))=i$, $\mathrm{tr}(g_i(0))=0$ attains this possibility.

If $\frac{2s}{r_1}=\frac{7}{4}$ then $8s=7r_1$ and $s=7$, $r_1=8$. The eigenvalues $\lambda_1=e^{\frac{7\pi i}{4}}=e^{-\frac{\pi i}{4}}=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$, $\lambda_2=ie^{\frac{\pi i}{4}}=-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i=e^{\frac{3\pi i}{4}}$ are already obtained.

In the case of $\mathrm{tr}(g)=1+i$, one has $\sin\left(\frac{2\pi s}{r_1}+\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$, which is equivalent to $\frac{2\pi s}{r_1}+\frac{\pi}{4}=\frac{3\pi}{4}$ for $\frac{2\pi s}{r_1}+\frac{\pi}{4}\in\left(\frac{\pi}{4},\frac{9\pi}{4}\right)$. Now, $\frac{2s}{r_1}=\frac{1}{2}$, whereas $4s=r_1$ and $s=1$, $r_1=4$. The eigenvalues are $\lambda_1=e^{\frac{\pi i}{2}}=i$, $\lambda_2=ie^{-\frac{\pi i}{2}}=1$ and g is of order $r=LCM(4,1)=4$. Note that

$$g_i(1+i)=\left(\begin{array}{cc} i & 0 \\ 0 & 1\end{array}\right)\in GL(2,\mathbb{Z}[i])$$

with $\det(g_i(1+i))=i$, $\mathrm{tr}(g_i(1+i))=1+i$ realizes this case.

Finally, for $\mathrm{tr}(g)=-1-i$ there follows $\sin\left(\frac{2\pi s}{r_1}+\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$. Consequently, $\frac{2\pi s}{r_1}+\frac{\pi}{4}=\frac{5\pi}{4}$ or $\frac{2\pi s}{r_1}+\frac{\pi}{4}=\frac{7\pi}{4}$ for $\frac{2\pi s}{r_1}+\frac{\pi}{4}\in\left(\frac{\pi}{4},\frac{9\pi}{4}\right)$. In the case of $\frac{2s}{r_1}=1$ one has $s=1$, $r_1=2$. The eigenvalues of g are $\lambda_1=e^{\pi i}=-1$, $\lambda_2=ie^{-\pi i}=-i$, so that g is of order $r=LCM(2,4)=4$. This possibility is realized by

$$g_i(-1-i)=\left(\begin{array}{cc} -i & 0 \\ 0 & -1\end{array}\right)\in GL(2,\mathbb{Z}[i])$$

with $\det(g_i(-1-i))=i$, $\mathrm{tr}(g_i(-1-i))=-1-i$.

If $\frac{2s}{r_1}=\frac{3}{2}$ then $4s=3r_1$ and $s=3$, $r_1=4$. The eigenvalues $\lambda_1=e^{\frac{3\pi i}{2}}=-i$, $\lambda_2=ie^{-\frac{3\pi i}{2}}=-1$ are already obtained. That concludes the description of the eigenvalues of all $g\in GL(2,\mathbb{Z}[i])$ of finite order with $\det(g)=i$.

□

Proposition 18. *If $g\in GL(2,\mathbb{Z}[i])$ is of finite order r and* $\det(g)=-i$ *then*

$$\mathrm{tr}(g)\in\{0,\pm(1-i)\},\quad r\in\{4,8\}.$$

More precisely,

(i) $\mathrm{tr}(g) = 0$ *or* $\lambda_1 = e^{\frac{\pi i}{4}}$, $\lambda_2 = e^{\frac{5\pi i}{4}}$ *if and only if* g *is of order* 8;

(ii) if $\mathrm{tr}(g) = 1 - i$ *or* $\lambda_1 = -i$, $\lambda_2 = 1$ *then* g *is of order* 4;

(iii) if $\mathrm{tr}(g) = -1 + i$ *or* $\lambda_1 = i$, $\lambda_2 = -1$ *then* g *is of order* 4.

Proof. If one of the eigenvalues of g is $\lambda_1 = e^{\frac{2\pi si}{r_1}}$ then the other one is $\lambda_2 = -ie^{-\frac{2\pi si}{r_1}}$. Thus, the trace

$$\mathrm{tr}(g) = \lambda + \lambda_2 = \left[\cos\left(\frac{2\pi s}{r_1}\right) - \sin\left(\frac{2\pi s}{r_1}\right)\right](1-i) = \sqrt{2}\cos\left(\frac{2\pi s}{r_1} + \frac{\pi}{4}\right)(1-i)$$

belongs to $\mathbb{Z}[i] = \mathbb{Z} + \mathbb{Z}i$ if and only if $\sqrt{2}\cos\left(\frac{2\pi s}{r_1} + \frac{\pi}{4}\right) \in \mathbb{Z}$. As a result,

$$\sqrt{2}\cos\left(\frac{2\pi s}{r_1} + \frac{\pi}{4}\right) \in \mathbb{Z} \cap [-\sqrt{2}, \sqrt{2}] = \{0, \pm 1\}$$

or $\mathrm{tr}(g) \in \{0, \pm(1-i)\}$. Note that $\mathrm{tr}(g) = 0$ reduces to $\cos\left(\frac{2\pi s}{r_1} + \frac{\pi}{4}\right) = 0$ with solutions $\frac{2\pi s}{r_1} + \frac{\pi}{4} = \frac{\pi}{2}$ or $\frac{2\pi s}{r_1} + \frac{\pi}{4} = \frac{3\pi}{2}$. If $\frac{2s}{r_1} = \frac{1}{4}$ then $8s = r_1$ and $s = 1$, $r_1 = 8$. The eigenvalues of g are $\lambda_1 = e^{\frac{\pi i}{4}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $\lambda_2 = -ie^{-\frac{\pi i}{4}} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$ and g is of order $r = LCM(8,8) = 8$. Note that

$$g_{-i}(0) = \begin{pmatrix} -i & -i \\ (-1+i) & i \end{pmatrix} \in GL(2, \mathbb{Z}[i])$$

with $\det(g_{-i}(0)) = -i$, $\mathrm{tr}(g_{-i}(0)) = 0$ realizes the aforementioned possibility. In the case of $\frac{2\pi s}{r_1} = \frac{5}{4}$ there holds $8s = 5r_1$, whereas $s = 5$, $r_1 = 8$ and $\lambda_1 = e^{\frac{5\pi i}{4}} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$, $\lambda_2 = -ie^{-\frac{5\pi i}{4}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i = e^{\frac{\pi i}{4}}$. This case has been already discussed.

For $\mathrm{tr}(g) = 1 - i$ one has $\cos\left(\frac{2\pi s}{r_1} + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, which reduces to $\frac{2\pi s}{r_1} + \frac{\pi}{4} = \frac{7\pi}{4}$ for $\frac{2\pi s}{r_1} + \frac{\pi}{4} \in \left(\frac{\pi}{4}, \frac{9\pi}{4}\right)$. Now $\frac{2s}{r_1} = \frac{3}{2}$ reads as $4s = 3r_1$ and determines $s = 3$, $r_1 = 4$. The eigenvalues of g are $\lambda_1 = e^{\frac{3\pi i}{2}} = -i$, $\lambda_2 = -ie^{-\frac{3\pi i}{2}} = 1$ and g is of order $r = LCM(4,1) = 4$. This possibility is realized by

$$g_{-i}(1-i) = \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{Z}[i])$$

with $\det(g_{-i}(1-i)) = -i$, $\mathrm{tr}(g_{-i}(1-i)) = 1 - i$.

Finally, $\mathrm{tr}(g) = -1 + i$ is equivalent to $\cos\left(\frac{2\pi s}{r_1} + \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$ and holds for $\frac{2\pi s}{r_1} + \frac{\pi}{4} = \frac{3\pi}{4}$ or $\frac{2\pi s}{r_1} + \frac{\pi}{4} = \frac{5\pi}{4}$. In the case of $\frac{2s}{r_1} = \frac{1}{2}$, one has $4s = r_1$ and $s = 1$, $r_1 = 4$. The eigenvalues of g are $\lambda_1 = e^{\frac{\pi i}{2}} = i$, $\lambda_2 = -ie^{-\frac{\pi i}{2}} = -1$ and g is of order $r = LCM(4,2) = 4$. The automorphism

$$g_{-i}(-1+i) = \begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix} \in GL(2, \mathbb{Z}[i])$$

realizes the case under discussion. For $\frac{2s}{r_1} = 1$ there follow $s = 1$, $r_1 = 2$ and $\lambda_1 = e^{\pi i} = -1$, $\lambda_2 = -ie^{-\pi i} = i$, which was already discussed. That concludes the description of the automorphisms $g \in GL(2, \mathbb{Z}[i])$ with $\det(g) = -i$.

$\square$

Proposition 19. *If $g \in GL(2, \mathcal{O}_{-3})$ is of finite order r and $\det(g) = e^{\frac{\pi i}{3}}$ then*

$$r = 6 \quad \textit{and} \quad \mathrm{tr}(g) \in \left\{0, \pm\left(\frac{3}{2} + \frac{\sqrt{-3}}{2}\right)\right\}.$$

More precisely,

(i) $\mathrm{tr}(g) = 0$ *exactly when* $\lambda_1 = e^{\frac{2\pi i}{3}}$, $\lambda_2 = e^{-\frac{\pi i}{3}}$;

(ii) $\mathrm{tr}(g) = \frac{3}{2} + \frac{\sqrt{-3}}{2}$ *exactly when* $\lambda_1 = e^{\frac{\pi i}{3}}$, $\lambda_2 = 1$;

(iii) $\mathrm{tr}(g) = -\frac{3}{2} - \frac{\sqrt{-3}}{2}$ *exactly when* $\lambda_1 = e^{-\frac{2\pi i}{3}}$, $\lambda_2 = -1$.

Proof. If $\lambda_1 = e^{\frac{2\pi si}{r_1}}$ then $\lambda_2 = e^{\frac{\pi i}{3}} e^{-\frac{2\pi si}{r_1}}$ and the trace

$$\mathrm{tr}(g) = \lambda_1 + \lambda_2 = (\sqrt{3} + i)\sin\left(\frac{2\pi s}{r_1} + \frac{\pi}{3}\right)$$

belongs to $\mathcal{O}_{-3} = \mathbb{Z} + \frac{1+\sqrt{-3}}{2}\mathbb{Z}$ if and only if $\sin\left(\frac{2\pi s}{r_1} + \frac{\pi}{3}\right) \in \frac{\sqrt{3}}{2}\mathbb{Z}$. Combining with $\sin\left(\frac{2\pi si}{r_1} + \frac{\pi}{3}\right) \in [-1, 1]$, one gets $\sin\left(\frac{2\pi s}{r_1} + \frac{\pi}{3}\right) \in \frac{\sqrt{3}}{2}\mathbb{Z} \cap [-1, 1] = \left\{0, \pm\frac{\sqrt{3}}{2}\right\}$ and, respectively, $\mathrm{tr}(g) \in \left\{0, \pm\left(\frac{3}{2} + \frac{\sqrt{-3}}{2}\right)\right\}$.

If $\sin\left(\frac{2\pi s}{r_1} + \frac{\pi}{3}\right) = 0$ then $\frac{2\pi s}{r_1} + \frac{\pi}{3} = \pi$ or $\frac{2\pi s}{r_1} + \frac{\pi}{3} = 2\pi$. For $\frac{2s}{r_1} = \frac{2}{3}$ there follows $s = 1$, $r_1 = 3$ and $\lambda_1 = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$, $\lambda_2 = e^{\frac{\pi i}{3}} e^{-\frac{2\pi i}{3}} = e^{-\frac{\pi i}{3}} = \frac{1}{2} - \frac{\sqrt{-3}}{2}$. The automorphisms $g \in GL(2, \mathcal{O}_{-3})$ with such eigenvalues are of order $r = LCM(3, 6) = 6$. For instance,

$$\begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{pmatrix} \in GL(2, \mathcal{O}_{-3})$$

attains the aforementioned possibility.

In the case of $\frac{2s}{r_1} = \frac{5}{3}$ one has $s = 5$, $r_1 = 6$ and $\lambda_1 = e^{-\frac{\pi i}{3}}$, $\lambda_2 = e^{\frac{\pi i}{3}} e^{\frac{\pi i}{3}} = e^{\frac{2\pi i}{3}}$, which was already obtained.

Note that $\sin\left(\frac{2\pi s}{r_1} + \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ for $\frac{2\pi s}{r_1} + \frac{\pi}{3} \in \left(\frac{\pi}{3}, \frac{7\pi}{3}\right)$ implies $\frac{2\pi s}{r_1} + \frac{\pi}{3} = \frac{2\pi}{3}$, whereas $6s = r_1$ and $s = 1$, $r_1 = 6$. The corresponding eigenvalues are $\lambda_1 = e^{\frac{\pi i}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\lambda_2 = e^{\frac{\pi i}{3}} e^{-\frac{\pi i}{3}} = 1$ and g is of order $r = LCM(6, 1) = 6$. Note that

$$\begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathcal{O}_{-3})$$

realizes this possibility.

The equality $\sin\left(\frac{2\pi s}{r_1}+\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ holds for $\frac{2\pi s}{r_1}+\frac{\pi}{3} = \frac{4\pi}{3}$ or $\frac{2\pi s}{r_1}+\frac{\pi}{3} = \frac{5\pi}{3}$. If $2s = r_1$ then $s = 1$, $r_1 = 2$ and $\lambda_1 = e^{\pi i} = -1$, $\lambda_2 = e^{\frac{\pi i}{3}} e^{-\pi i} = e^{-\frac{2\pi i}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$. The automorphism g is of order $r = LCM(2,3) = 6$. Note that

$$\left(\begin{array}{cc} e^{-\frac{2\pi i}{3}} & 0 \\ 0 & -1 \end{array}\right) \in GL(2, \mathcal{O}_{-3})$$

attains this possibility and concludes the proof of the proposition.

□

Proposition 20. *If $g \in GL(2, \mathcal{O}_{-3})$ is of finite order r and $\det(g) = e^{-\frac{\pi i}{3}}$ then*

$$r = 6 \quad \textit{and} \quad \mathrm{tr}(g) \in \left\{0, \pm\left(\frac{3}{2} - \frac{\sqrt{-3}}{2}\right)\right\}.$$

More precisely,

(i) $\mathrm{tr}(g) = 0$ *exactly when* $\lambda_1 = e^{\frac{\pi i}{3}}$, $\lambda_2 = e^{-\frac{2\pi i}{3}}$*;*

(ii) $\mathrm{tr}(g) = \frac{3}{2} - \frac{\sqrt{3}}{2} i$ *exactly when* $\lambda_1 = e^{-\frac{\pi i}{3}}$, $\lambda_2 = 1$*;*

(iii) $\mathrm{tr}(g) = -\frac{3}{2} + \frac{\sqrt{3}}{2} i$ *exactly when* $\lambda_1 = \frac{2\pi i}{3}$, $\lambda_2 = -1$*.*

Proof. If $\lambda_1 = e^{\frac{2\pi s i}{r_1}}$ then $\lambda_2 = e^{-\frac{\pi i}{3}} e^{-\frac{2\pi s i}{r_1}}$ and the trace

$$\mathrm{tr}(g) = \lambda_1 + \lambda_2 = (-\sqrt{3} + i) \sin\left(\frac{2\pi s}{r_1} - \frac{\pi}{3}\right)$$

belongs to $\mathcal{O}_{-3} = \mathbb{Z} + \frac{1+\sqrt{3}i}{2}\mathbb{Z}$ if and only if $\sin\left(\frac{2\pi s}{r_1} - \frac{\pi}{3}\right) \in \frac{\sqrt{3}}{2}\mathbb{Z}$. As a result, $\sin\left(\frac{2\pi s}{r_1} = \frac{\pi}{3}\right) \in \frac{\sqrt{3}}{2}\mathbb{Z} \cap [-1,1] = \left\{0, \pm\frac{\sqrt{3}}{2}\right\}$ and $\mathrm{tr}(g) \in \left\{0, \pm\left(\frac{3}{2} - \frac{\sqrt{3}}{2} i\right)\right\}$.

The equation $\sin\left(\frac{2\pi s}{r_1} - \frac{\pi}{3}\right) = 0$ for $\frac{2\pi s}{r_1} - \frac{\pi}{3} \in \left(-\frac{\pi}{3}, \frac{5\pi}{3}\right)$ has solutions $\frac{2\pi s}{r_1} - \frac{\pi}{3} = 0$ and $\frac{2\pi s}{r_1} - \frac{\pi}{3} = \pi$.

If $6s = r_1$ then $s = 1$, $r_1 = 6$ and $\lambda_1 = e^{\frac{\pi i}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2} i$, $\lambda_2 = e^{-\frac{\pi i}{3}} e^{-\frac{\pi i}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$. The automorphisms $g \in GL(2, \mathcal{O}_{-3})$ with such eigenvalues are of order $r = LCM(6,3) = 6$. For instance,

$$\left(\begin{array}{cc} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{array}\right) \in GL(2, \mathcal{O}_{-3})$$

attains this case.

If $\sin\left(\frac{2\pi s}{r_1} - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ then $\frac{2\pi s}{r_1} - \frac{\pi}{3} = \frac{\pi}{3}$ or $\frac{2\pi s}{r_1} - \frac{\pi}{3} = \frac{2\pi}{3}$. For $3s = r_1$ one has $s = 1$, $r_1 = 3$ and $\lambda_1 = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$, $\lambda_2 = e^{-\frac{\pi i}{3}} e^{-\frac{2\pi i}{3}} = e^{-\pi i} = -1$, attained by

$$\left(\begin{array}{cc} e^{\frac{2\pi i}{3}} & 0 \\ 0 & -1 \end{array}\right) \in GL(2, \mathcal{O}_{-3}).$$

All $g \in GL(2, \mathcal{O}_{-3})$ with such eigenvalues are of order $r = LCM(3, 2) = 6$.

In the case of $2s = r_1$ there follows $s = 1$, $r_1 = 2$ and $\lambda_1 = e^{\pi i} = -1$, $\lambda_2 = e^{-\frac{\pi i}{3}} e^{-\frac{\pi i}{3}} = e^{-\frac{2\pi i}{3}}$, which is already discussed.

The equation $\sin\left(\frac{2\pi s}{r_1} - \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ for $\frac{2\pi s}{r_1} - \frac{\pi}{3} \in \left(-\frac{\pi}{3}, \frac{5\pi}{3}\right)$ has solution $\frac{2\pi s}{r_1} - \frac{\pi}{3} = \frac{5\pi}{3}$. Therefore $6s = 5r_1$ and $s = 5$, $r_1 = 6$, As a result, $\lambda_1 = e^{\frac{5\pi i}{3}} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$, $\lambda_2 = e^{-\frac{\pi i}{3}} e^{\frac{\pi i}{3}} = 1$ and g is of order $r = LCM(6, 1) = 6$. Note that

$$\left(\begin{array}{cc} e^{-\frac{\pi i}{3}} & 0 \\ 0 & 1 \end{array}\right) \in GL(2, \mathcal{O}_{-3})$$

attains this possibility and concludes the proof of the proposition.

□

Proposition 21. *If $g \in GL(2, \mathcal{O}_{-3})$ is of finite order r and $\det(g) = e^{\frac{2\pi i}{3}}$ then*

$$\mathrm{tr}(g) \in \left\{0, \pm\frac{(1+\sqrt{-3})}{2}, \pm(1+\sqrt{-3})\right\}, \quad r \in \{3, 6, 12\}.$$

More precisely,

(i) $\mathrm{tr}(g) = 0$ or $\lambda_1 = e^{\frac{5\pi i}{6}}$, $\lambda_2 = e^{-\frac{\pi i}{6}}$ if and only if g is of order 12;

(ii) if $\mathrm{tr}(g) = \frac{1+\sqrt{3}i}{2}$ or $\lambda_1 = e^{\frac{2\pi i}{3}}$, $\lambda_2 = 1$ then g is of order 3;

(iii) if $\mathrm{tr}(g) = -1 - \sqrt{3}i$ or $g = e^{-\frac{2\pi i}{3}} I_2$ then g is of order 3;

(iv) if $\mathrm{tr}(g) = \frac{-1-\sqrt{3}i}{2}$ or $\lambda_1 = e^{-\frac{\pi i}{3}}$, $\lambda_2 = -1$ then g is of order 6;

(v) if $\mathrm{tr}(g) = 1 + \sqrt{3}i$ or $g = e^{\frac{\pi i}{3}} I_2$ then g is of order 6.

Proof. If $\lambda_1 = e^{\frac{2\pi s i}{r_1}}$ then $\lambda_2 = e^{\frac{2\pi i}{3}} e^{-\frac{2\pi s i}{r_1}}$ and the trace

$$\mathrm{tr}(g) = \lambda_1 + \lambda_2 = (1 + \sqrt{3}i) \sin\left(\frac{2\pi s}{r_1} + \frac{\pi}{6}\right)$$

belongs to $\mathcal{O}_{-3} = \mathbb{Z} + \frac{1+\sqrt{3}i}{2}\mathbb{Z}$ if and only if $2\sin\left(\frac{2\pi s}{r_1} + \frac{\pi}{6}\right) \in \mathbb{Z}$. Combining with $\sin\left(\frac{2\pi s}{r_1} + \frac{\pi}{6}\right) \in [-1, 1]$, one obtains $2\sin\left(\frac{2\pi s}{r_1} + \frac{\pi}{6}\right) \in \mathbb{Z} \cap [-2, 2] = \{0, \pm 1, \pm 2\}$ and, respectively,

$$\mathrm{tr}(g) \in \left\{0, \pm\frac{(1+\sqrt{3}i)}{2}, \pm(1+\sqrt{3}i)\right\}.$$

If $\sin\left(\frac{2\pi s}{r_1} + \frac{\pi}{6}\right) = 0$ for $\frac{2\pi s}{r_1} + \frac{\pi}{6} \in \left(\frac{\pi}{6}, \frac{13\pi}{6}\right)$ then $\frac{2\pi s}{r_1} + \frac{\pi}{6} = \pi$ or $\frac{2\pi s}{r_1} + \frac{\pi}{6} = 2\pi$.

For $12s = 5r_1$ one has $s = 5$, $r_1 = 12$ and $\lambda_1 = e^{\frac{5\pi i}{6}} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$, $\lambda_2 = e^{\frac{2\pi i}{3}} e^{-\frac{5\pi i}{6}} = e^{-\frac{\pi i}{6}} = \frac{\sqrt{3}}{2} - \frac{1}{2}i$. Therefore g is of order $r = LCM(12, 12) = 12$. Note that

$$\left(\begin{array}{cc} e^{\frac{5\pi i}{6}} & 0 \\ 0 & e^{-\frac{\pi i}{6}} \end{array}\right) \in GL(2, \mathcal{O}_{-3})$$

attains this possibility.

In the case of $12s = 11r_1$ there follows $s = 11$, $r_1 = 12$. As a result, $\lambda_1 = e^{\frac{11\pi i}{6}} = \frac{\sqrt{3}}{2} - \frac{1}{2}i$, $\lambda_2 = e^{\frac{2\pi i}{3}} e^{\frac{\pi i}{6}} = e^{\frac{5\pi i}{6}} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$, which was already obtained.

If $\sin\left(\frac{2\pi s}{r_1} + \frac{\pi}{6}\right) = \frac{1}{2}$ for $\frac{2\pi s}{r_1} + \frac{\pi}{6} \in \left(\frac{\pi}{6}, \frac{13\pi}{6}\right)$ then $\frac{2\pi s}{r_1} + \frac{\pi}{6} = \frac{5\pi}{6}$ and $3s = r_1$. Therefore $s = 1$, $r_1 = 3$ and $\lambda_1 = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\lambda_2 = e^{\frac{2\pi i}{3}} e^{-\frac{2\pi i}{3}} = 1$. The order of g is $r = LCM(3, 1) = 3$. This possibility is attained by

$$\begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathcal{O}_{-3}).$$

The equation $\sin\left(\frac{2\pi s}{r_1} + \frac{\pi}{6}\right) = -\frac{1}{2}$ has solutions $\frac{2\pi s}{r_1} + \frac{\pi}{6} = \frac{7\pi}{6}$ and $\frac{2\pi s}{r_1} + \frac{\pi}{6} = \frac{11\pi}{6}$.

If $2s = r_1$ then $s = 1$, $r_1 = 2$, $\lambda_1 = e^{\pi i} = -1$, $\lambda_2 = e^{\frac{2\pi i}{3}} e^{-\pi i} = e^{-\frac{\pi i}{3}} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$ and g is of order $r = LCM(2, 6) = 6$. For instance,

$$\begin{pmatrix} e^{-\frac{\pi i}{3}} & 0 \\ 0 & -1 \end{pmatrix} \in GL(2, \mathcal{O}_{-3})$$

attains these eigenvalues.

For $6s = 5r_1$ one has $s = 5$, $r_1 = 6$ $\lambda_1 = e^{\frac{5\pi i}{3}} = e^{-\frac{\pi i}{3}} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$, $\lambda_2 = e^{\frac{2\pi i}{3}} e^{\frac{\pi i}{3}} = e^{\pi i} = -1$, which is already obtained.

Note that $\sin\left(\frac{2\pi s}{r_1} + \frac{\pi}{6}\right) = 1$ is equivalent to $\frac{2\pi s}{r_1} + \frac{\pi}{6} = \frac{\pi}{2}$, whereas $6s = r_1$ and $s = 1$, $r_1 = 6$. The eigenvalues $\lambda_1 = e^{\frac{\pi i}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\lambda_2 = e^{\frac{2\pi i}{3}} e^{-\frac{\pi i}{3}} = e^{\frac{\pi i}{3}} = \frac{1}{2} + \frac{sqrt3}{2}i$ are equal, so that $g = e^{\frac{\pi i}{3}} I_2$ and $r = LCM(6, 6) = 6$.

If $\sin\left(\frac{2\pi s}{r_1} + \frac{\pi}{6}\right) = -1$ then $\frac{2\pi s}{r_1} + \frac{\pi}{6} = \frac{3\pi}{2}$ and $3s = 2r_1$, $s = 2$, $r_1 = 3$. Then $\lambda_1 = e^{\frac{4\pi i}{3}} = e^{-\frac{2\pi i}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, $\lambda_2 = e^{\frac{2\pi i}{3}} e^{\frac{2\pi i}{3}} = e^{-\frac{2\pi i}{3}}$ determine uniquely $g = e^{-\frac{2\pi i}{3}} I_2$ of order $r = LCM(3, 3) = 3$. That concludes the description of $g \in GL(2, \mathcal{O}_{-3})$ of finite order and $\det(g) = e^{\frac{2\pi i}{3}}$.

□

Proposition 22. *If $g \in GL(2, \mathcal{O}_{-3})$ is of finite order r and $\det(g) = e^{-\frac{2\pi i}{3}}$ then*

$$\mathrm{tr}(g) \in \left\{0, \pm\frac{(1 - \sqrt{-3})}{2}, \quad \pm(1 - \sqrt{-3})\right\}, \qquad r \in \{3, 6, 12\}.$$

More precisely,

(i) $\mathrm{tr}(g) = 0$ or $\lambda_1 = e^{\frac{\pi i}{6}}$, $\lambda_2 = e^{-5\frac{\pi i}{6}}$ if and only if g is of order 12;

(ii) if $\mathrm{tr}(g) = \frac{1-\sqrt{3}i}{2}$ or $\lambda_1 = e^{\frac{4\pi i}{3}}$, $\lambda_2 = 1$ then g is of order 3;

(iii) if $\mathrm{tr}(g) = -1 + \sqrt{3}i$ or $g = e^{\frac{2\pi i}{3}} I_2$ then g is of order 3;

(iv) if $\mathrm{tr}(g) = \frac{-1+\sqrt{3}i}{2}$ or $\lambda_1 = e^{\frac{\pi i}{3}}$, $\lambda_2 = -1$ then g is of order 6;

(v) if $\mathrm{tr}(g) = 1 - \sqrt{3}i$ or $g = e^{-\frac{\pi i}{3}} I_2$ then g is of order 6.

Proof. If $\lambda_1 = e^{\frac{2\pi s i}{r_1}}$ then $\lambda_2 = e^{-\frac{2\pi i}{3}} e^{-\frac{2\pi s i}{r_1}}$ and the trace

$$\mathrm{tr}(g) = \lambda_1 + \lambda_2 = (-1 + \sqrt{3}i) \sin\left(\frac{2\pi s}{r_1} - \frac{\pi}{6} \right)$$

belongs to $\mathcal{O}_{-3} = \mathbb{Z} + \frac{1+\sqrt{3}i}{2}\mathbb{Z}$ if and only if $2\sin\left(\frac{2\pi s}{r_1} - \frac{\pi}{6}\right) \in \mathbb{Z}$. Combining with $2\sin\left(\frac{2\pi s}{r_1} - \frac{\pi}{6}\right) \in [-2, 2]$, one concludes that $\sin\left(\frac{2\pi s}{r_1} - \frac{\pi}{6}\right) \in \left\{0, \pm\frac{1}{2}, \pm 1\right\}$ and $\mathrm{tr}(g) \in \left\{0, \pm\frac{(1-\sqrt{3}i)}{2}, \pm(1-\sqrt{3}i)\right\}$.

If $\sin\left(\frac{2\pi s}{r_1} - \frac{\pi}{6}\right) = 0$ with $\frac{2\pi s}{r_1} - \frac{\pi}{6} \in \left(-\frac{\pi}{6}, \frac{11\pi}{6}\right)$ then $\frac{2\pi s}{r_1} - \frac{\pi}{6} = 0$ or $\frac{2\pi s}{r_1} - \frac{\pi}{6} = \pi$.

For $12s = r_1$ one has $s = 1$, $r_1 = 12$, $\lambda_1 = e^{\frac{\pi i}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, $\lambda_2 = e^{-\frac{2\pi i}{3}} e^{-\frac{\pi i}{6}} = e^{-\frac{5\pi i}{6}} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$, so that g is of order $r = LCM(12, 12) = 12$. For instance,

$$\begin{pmatrix} e^{\frac{\pi i}{6}} & 0 \\ 0 & e^{-\frac{5\pi i}{6}} \end{pmatrix} \in GL(2, \mathcal{O}_{-3})$$

attains this case.

For $12s = 7r_1$ there follows $s = 7$, $r_1 = 12$, $\lambda_1 = e^{\frac{7\pi i}{6}} = e^{-\frac{5\pi i}{6}}$, $\lambda_2 = e^{-\frac{2\pi i}{3}} e^{\frac{5\pi i}{6}} = e^{\frac{\pi i}{6}}$, which is already discussed.

In the case of $\sin\left(\frac{2\pi s}{r_1} - \frac{\pi}{6}\right) = \frac{1}{2}$ note that $\frac{2\pi s}{r_1} - \frac{\pi}{6} = \frac{\pi}{6}$ or $\frac{2\pi s}{r_1} - \frac{\pi}{6} = \frac{5\pi}{6}$.

If $6s = r_1$ then $s = 1$, $r_1 = 6$, $\lambda_1 = e^{\frac{\pi i}{3}} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\lambda_2 = e^{-\frac{2\pi i}{3}} e^{-\frac{\pi i}{3}} = e^{-\pi i} = -1$ and g is of order $r = LCM(6, 2) = 6$. Note that

$$\begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & -1 \end{pmatrix} \in GL(2, \mathcal{O}_{-3})$$

attains this case.

For $2s = r_1$ there follows $s = 1$, $r_1 = 2$, $\lambda_1 = e^{\pi i} = -1$, $\lambda_2 = e^{-\frac{2\pi i}{3}} e^{-\frac{\pi i}{3}} = e^{-\frac{5\pi i}{3}} = e^{\frac{\pi i}{3}}$, which is already obtained.

Note that $\sin\left(\frac{2\pi s}{r_1} - \frac{\pi}{6}\right) = -\frac{1}{2}$ for $\frac{2\pi s}{r_1} - \frac{\pi}{6} \in \left(-\frac{\pi}{6}, \frac{11\pi}{6}\right)$ implies $\frac{2\pi s}{r_1} - \frac{\pi}{6} = \frac{7\pi}{6}$, whereas $3s = 2r_1$, $s = 2$ and $r_1 = 3$. Then $\lambda_1 = e^{\frac{4\pi i}{3}} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, $\lambda_2 = e^{-\frac{2\pi i}{3}} e^{-\frac{4\pi i}{3}} = e^{-2\pi i} = 1$ and g is of order $r = LCM(3, 1) = 3$, attained by

$$\begin{pmatrix} e^{\frac{4\pi i}{3}} & 0 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathcal{O}_{-3}).$$

If $\sin\left(\frac{2\pi s}{r_1} - \frac{\pi}{6}\right) = 1$ then $\frac{2\pi s}{r_1} - \frac{\pi}{6} = \frac{\pi}{2}$ or $3s = r_1$. As a result, $s = 1$, $r_1 = 3$, $\lambda_1 = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\lambda_2 = e^{-\frac{2\pi i}{3}} e^{-\frac{2\pi i}{3}} = e^{\frac{2\pi i}{3}}$, whereas $g = e^{\frac{2\pi i}{3}} I_2 \in GL(2, \mathcal{O}_{-3})$ is a scalar matrix of order 3.

Finally, $\sin\left(\frac{2\pi s}{r_1} - \frac{\pi}{6}\right) = -1$ holds for $\frac{2\pi s}{r_1} - \frac{\pi}{6} = \frac{3\pi}{2}$, i.e., $6s = 5r_1$ and $s = 5$, $r_1 = 6$. Now $\lambda_1 = e^{-\frac{\pi i}{3}} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$, $\lambda_2 = e^{-\frac{2\pi i}{3}} e^{\frac{\pi i}{3}} = e^{-\frac{\pi i}{3}}$, so that $g = e^{-\frac{\pi i}{3}} I_2 \in GL(2, \mathcal{O}_{-3})$ is a scalar matrix of order 6. That concludes the proof of the proposition.

□

3 Finite linear automorphism groups of $E \times E$

The classification of the finite subgroups K of $SL(2,R)$ for an endomorphism ring R of an elliptic curve E starts with a classification of the Sylow subgroups H_{p^k} of K.

Proposition 23. *If K is a finite subgroup of $SL(2,R)$ then K is of order $|K| = 2^a 3^b$ for some integers $0 \leq a \leq 3$, $0 \leq b \leq 1$.*

If K is of even order then the Sylow 2-subgroup H_{2^a} of K is isomorphic to $\mathbb{C}_2$, $\mathbb{C}_4$ or the quaternion group

$$\mathbb{Q}_8 = \langle g_1, g_2 \ \mid \ g_1^2 = g_2^2 = -I_2, \quad g_2 g_1 = -g_1 g_2 \rangle$$

of order 8.

If the order of K is divisible by 3 then the Sylow 3-subgroup H_3 of K is isomorphic to the cyclic group $\mathbb{C}_3$ of the third roots of unity.

Proof. According to the First Sylow Theorem, if $|K| = p_1^{m_1} \dots p_s^{m_s}$ for some rational primes $p_j \in \mathbb{N}$ and some $m_j \in \mathbb{N}$, then for any $1 \leq i \leq k$ there is a subgroup $H_{p_j^i} \leq K$ of order $|H_{p_j^i}| = p_j^i$. In particular, any $H_{p_j} = \langle g_{p_j} \rangle \simeq \mathbb{C}_{p_j}$ of prime order p_j, dividing $|K|$ is cyclic and there is an element $g_{p_j} \in K$ of order p_j. By Proposition 15, the order of an element $g \in SL(2,R)$ is $1, 2, 3, 4, 6$ or ∞. As a result, if $g \in SL(2,R)$ is of prime order p then $p = 2$ or 3. In other words, K is of order $|K| = 2^a 3^b$ for some non-negative integers a, b.

Suppose that $b \geq 1$ and consider the Sylow subgroup $H_{3^b} \leq K$ of order 3^b. Then any $h \in H_{3^b} \setminus \{I_2\}$ is of order 3 since there is no $g \in SL(2,R)$, whose order is divisible by 9. We claim that $H_{3^b} = \langle h_1 \rangle \simeq \mathbb{C}_3$ is a cyclic group of order 3. Otherwise, $b \geq 2$ and there exists $h_2 \in H_{3^b} \setminus \langle h_1 \rangle$. Note that $h_1^j h_2 \in H_{3^b}$ with $1 \leq j \leq 2$ are of order 3, as far as $h_1^j h_2 = I_2$ implies $h_2 = h_1^{-j} \in \langle h_1 \rangle$, contrary to the choice of h_2. We are going to show that if $h_1, h_2, h_1 h_2 \in SL(2,R)$ are of order 3 then $h_1^2 h_2 = I_2$, so that there is no $h_2 \in H_{3^b} \setminus \langle h_1 \rangle$ and $H_{3^b} = \langle h_1 \rangle \simeq \mathbb{C}_3$. According to Proposition 15, $g \in SL(2,R)$ is of order 3 if and only if $\mathrm{tr}(g) = -1$ and g is conjugate to

$$D_g = \left(\begin{array}{cc} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{array} \right).$$

Similarly, $g \in SL(2,R)$ coincides with the identity matrix I_2 exactly when $\mathrm{tr}(g) = 2$. Thus, we have to check that if $h_1, h_2 \in SL(2,R)$ satisfy $\mathrm{tr}(h_1) = \mathrm{tr}(h_2) = \mathrm{tr}(h_1 h_2) = -1$ then $\mathrm{tr}(h_1^2 h_2) = 2$. Let

$$D_1 = S^{-1} h_1 S = \left(\begin{array}{cc} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{array} \right)$$

be a diagonal form of h_1 for some $S \in GL(2,\mathbb{C})$ and

$$D_2 = S^{-1} h_2 S = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in GL(2,\mathbb{C}).$$

(More precisely, if $Q(R) = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-d})$ is the fraction field of R then the eigenvectors of h_1 have entries from $Q(R)(\sqrt{-3})$, so that $S, D_2 \in Q(R)(\sqrt{-3})_{2\times 2}$ have entries from $Q(R)(\sqrt{-3}) = \mathbb{Q}(\sqrt{-3})$ or $\mathbb{Q}(\sqrt{-d}, \sqrt{-3})$.) Since the determinant and the trace of a matrix are invariant under conjugation, the statement is equivalent to the fact that if $\det(D_2) = 1$ and $\mathrm{tr}(D_2) = \mathrm{tr}(D_1D_2) = -1$ then $\mathrm{tr}(D_1^2D_2) = 2$. Indeed, if $d = -a-1$ and $\mathrm{tr}(D_1D_2) = e^{\frac{2\pi i}{3}}a - e^{-\frac{2\pi i}{3}}(a+1) = -1$ then $a = e^{\frac{2\pi i}{3}}$, $d = e^{-\frac{2\pi i}{3}}$, whereas $\mathrm{tr}(D_1^2D_2) = 2$. That proves the non-existence of $h_2 \in H_{3^b} \setminus \langle h_1 \rangle$ and $H_{3^b} = H_3 = \langle h_1 \rangle \simeq \mathbb{C}_3$.

Suppose that K is of even order and denote by H_{2^a} the Sylow 2-subgroup of $K < SL(2,R)$ of order $2^a \geq 2$. Then any $g \in H_{2^a} \setminus \{I_2\}$ is of order

$$r \in \{2^i \ \mid \ i \in \mathbb{N}\} \cap \{1,2,3,4,6\} = \{2,4\}.$$

Recall from Proposition 15 that there is a unique element $-I_2$ of $SL(2,R)$ of order 2 and $g \in SL(2,R)$ is of order 4 if and only if the trace $\mathrm{tr}(g) = 0$. For $a = 1$ the Sylow subgroup $H_2 = \langle -I_2 \rangle \simeq \mathbb{C}_2$ is cyclic of order 2. If $a = 2$ then $H_4 = \langle g \rangle \simeq \mathbb{C}_4$ is cyclic of order 4, since $SL(2,R)$ has a unique element $-I_2$ of order 2. From now on, let us assume that $a \geq 3$ and fix an element $g_1 \in H_{2^a}$ of order 4. Due to $g_1^2 = -I_2 \in \langle g_1 \rangle$, any $g_2 \in H_{2^a} \setminus \langle g_1 \rangle$ is of order 4 and $g_2^2 = -I_2$. Moreover, $g_1g_2 \in H_{2^a}$ is of order 4, as far as $g_1g_2 = \pm I_2$ requires $g_2 = \mp g_1 \in \langle g_1 \rangle$, contrary to the choice of g_2. We claim that if $g_1, g_2 \in SL(2,R)$ of order 4 have product g_1g_2 of order 4 then they generate a quaternion group

$$\langle g_1, g_2 \ \mid \ g_1^2 = g_2^2 = -I_2, \quad g_2g_1 = -g_1g_2 \rangle \simeq \mathbb{Q}_8$$

of order 8. In other words, if $g_1, g_2 \in R_{2\times 2}$ have $\det(g_1) = \det(g_2) = 1$ and $\mathrm{tr}(g_1) = \mathrm{tr}(g_2) = \mathrm{tr}(g_1g_2) = 0$ then $g_2g_1 = -g_1g_2$. In particular, if $g_1, g_2 \in SL(2,R)$ of order 4 have product g_1g_2 of order 4 then $g_2 \notin \langle g_1 \rangle = \{\pm I_2, \ \ \pm g_1\}$. To this end, let

$$D_1 = S^{-1}g_1S = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

be the diagonal form of g_1 and

$$D_2 = S^{-1}g_2S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for appropriate matrices S and D_2 with entries from $Q(R)(\sqrt{-1}) = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{d}, \sqrt{-1})$. The determinant and the trace are invariant under conjugation, so that suffices to show that if $\det(D_2) = 1$ and $\mathrm{tr}(D_2) = \mathrm{tr}(D_1D_2) = 0$ then $D_2D_1 = -D_1D_2$, whereas

$$g_2g_1 = (SD_2S^{-1})(SD_1S^{-1}) = S(D_2D_1)S^{-1} =$$
$$= S(-D_1D_2)S^{-1} = -(SD_1S^{-1})(SD_2S^{-1}) = -g_1g_2.$$

Indeed, $\mathrm{tr}(D_2) = a + d = 0$ and $\mathrm{tr}(D_1 D_2) = i(a - d) = 0$ require $a = d = 0$. Now, $\det(D_2) = -bc = 1$ determines $c = -\frac{1}{b}$ for some $b \in \mathbb{Q}(\sqrt{d}, \sqrt{-1})$ and

$$D_2 D_1 = \begin{pmatrix} 0 & -ib \\ -\frac{i}{b} & 0 \end{pmatrix} = -D_1 D_2.$$

Thus, if $a = 3$ then the Sylow 2-subgroup of K is isomorphic to the quaternion group $\mathbb{Q}_8$ of order 8,

$$H_8 = \langle g_1, g_2 \ \mid \ g_1^2 = g_2^2 = -I_2, \quad g_2 g_1 = -g_1 g_2 \rangle \simeq \mathbb{Q}_8.$$

There remains to be rejected the case of $a \geq 4$. The assumption $a \geq 4$ implies the existence of $g_3 \in H_{2^a} \setminus \langle g_1, g_2 \rangle$. Any such g_3 is of order 4, together with the products $g_1 g_3 \in H_{2^a}$ for $1 \leq j \leq 2$, since $g_j g_3 = \pm I_2$ amounts to $g_3 = \pm g_j^3 \in \langle g_j \rangle$ and contradicts the choice of g_3. Thus, the subgroups

$$\langle g_1, g_3 \ \mid \ g_1^2 = g_3^2 = -I_2, \quad g_3 g_1 = -g_1 g_3 \rangle \simeq$$

$$\langle g_2, g_3, \ \mid \ g_2^2 = g_3^2 = -I_2, \quad g_3 g_2 = -g_2 g_3 \rangle \simeq \mathbb{Q}_8$$

are also isomorphic to $\mathbb{Q}_8$. In particular,

$$D_3 = S^{-1} g_3 S = \begin{pmatrix} 0 & b_3 \\ -\frac{1}{b_3} & 0 \end{pmatrix}$$

with $b_3 \in \mathbb{Q}(\sqrt{d}, \sqrt{-1})^*$ is subject to

$$D_3 D_2 = \begin{pmatrix} -\frac{b_3}{b} & 0 \\ 0 & -\frac{b}{b_3} \end{pmatrix} = \begin{pmatrix} \frac{b}{b_3} & 0 \\ 0 & \frac{b_3}{b} \end{pmatrix} = -D_2 D_3,$$

whereas $b_3^2 = -b^2$ or $b_3 = \pm ib$. As a result, $D_3 = D_1 D_2$ and $g_3 = g_1 g_2$, contrary to the choice of $g_3 \notin \langle g_1, g_2 \rangle$. Therefore $a < 4$ and the Sylow 2-subgroup of a finite group $K < SL(2, R)$ is $H_2 \simeq \mathbb{C}_2$, $H_4 \simeq \mathbb{C}_4$ or $H_8 \simeq \mathbb{Q}_8$.

□

Proposition 24. *Any finite subgroup K of $SL(2, R)$ is isomorphic to one of the following:*

$$K_1 = \{I_2\},$$

$$K_2 = \langle -I_2 \rangle \simeq \mathbb{C}_2,$$

$$K_3 = \langle g_1 \rangle \simeq \mathbb{C}_4 \ \text{ for some } \ g_1 \in SL(2, R) \ \text{ with } \ \mathrm{tr}(g_1) = 0,$$

$$K_4 = \langle g_1, g_2 \ \mid \ g_1^2 = g_2^2 = -I_2, \quad g_2 g_1 g_2 = g_1 \rangle \simeq \mathbb{Q}_8,$$

$$K_5 = \langle g_3 \rangle \simeq \mathbb{C}_3 \quad \text{for some} \quad g_3 \in SL(2, R) \ \text{ with } \ \mathrm{tr}(g_3) = -1,$$

$$K_6 = \langle g_4 \rangle \simeq \mathbb{C}_6 \quad \text{for some} \quad g_4 \in SL(2, R) \ \text{ with } \ \mathrm{tr}(g_4) = 1,$$

$$K_7 = \langle g_1, g_4 \ \mid \ g_1^2 = g_4^3 = -I_2, \quad g_4 g_1 g_4 = g_1 \rangle \simeq \mathbb{Q}_{12}$$

for some $g_1, g_4 \in SL(2,R)$ *with* $\mathrm{tr}(g_1) = 0$, $\mathrm{tr}(g_4) = 1$,

$$K_8 = \langle g_1, g_2, g_3 \ \mid \ g_1^2 = g_2^2 = -I_2, \quad g_3^3 = I_2, \quad g_2 g_1 = -g_1 g_2,$$

$$g_3 g_1 g_3^{-1} = g_2, \quad g_3 g_2 g_3^{-1} = g_1 g_2 \rangle \simeq SL(2, \mathbb{F}_3)$$

for some $g_1, g_2, g_3 \in SL(2,R)$, $\mathrm{tr}(g_1) = \mathrm{tr}(g_2) = 0$, $\mathrm{tr}(g_3) = -1$, *where* $\mathbb{Q}_8$ *denotes the quaternion group of order* 8, $\mathbb{Q}_{12}$ *stands for the dicyclic group of order* 12 *and* $SL(2,\mathbb{F}_3)$ *is the special linear group over the field* $\mathbb{F}_3$ *with three elements.*

Proof. By Proposition 23, K is of order $1, 2, 3, 6, 12$ or 24. The only subgroup $K < SL(2,R)$ of order 1 is $K = K_1 = \{I_2\}$. Since $-I_2$ is the only element of $SL(2,R)$ of order 2, the group $K = K_2 = \langle -I_2 \rangle \simeq \mathbb{C}_2$ is the only cyclic subgroup of $SL(2,R)$ of order 2. Any subgroup $K < SL(2,R)$ of order 4 is cyclic or $K = K_3 = \langle g_1 \rangle$ for some $g_1 \in SL(2,R)$ with $\mathrm{tr}(g_1) = 0$, because $SL(2,R)$ has a unique element $-I_2$ of order 2. Proposition 15 has established the existence of elements $g_1 \in SL(2,\mathbb{Z}) \leq SL(2,R)$ of order 4.

If $K < SL(2,R)$ is a subgroup of order 8 then it coincides with its Sylow 2-subgroup

$$K = H_8 = \langle g_1, g_2 \ \mid \ g_1^2 = g_2^2 = -I_2, \quad g_2 g_1 = -g_1 g_2 \rangle = K_4 \simeq \mathbb{Q}_8,$$

isomorphic to the quaternion group $\mathbb{Q}_8$ of order 8. Note that there is a realization

$$\mathbb{Q}_8 \simeq \langle D_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rangle < SL(2, \mathbb{Z}[i])$$

as a subgroup of $SL(2,\mathbb{Z}[i])$. In general,

$$D_j = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} \in SL(2,R)$$

amount to $a_j^2 + b_j c_j = -1$. The anti-commuting relation $g_2 g_1 = -g_1 g_2$ is equivalent to $2a_1 a_2 + b_1 c_2 + b_2 c_1 = 0$. Therefore $K_4 = \langle g_1, g_2 \rangle < SL(2,R)$ is a realization of $\mathbb{Q}_8$ if and only if $a_j, b_j, c_j \in R$ are subject to

$$\left| \begin{array}{r} a_1^2 + b_1 c_1 = -1 \\ a_2^2 + b_2 c_2 = -1 \\ 2a_1 a_2 + b_1 c_2 + b_2 c_1 = 0 \end{array} \right. . \qquad (12)$$

The existence of a solution of (12) in an arbitrary $R = R_{-d,f} = \mathbb{Z} + f\mathcal{O}_{-d} = \mathbb{Z} + f\omega_{-d}\mathbb{Z}$ is an open problem.

If $|K| = 3$ then $K = K_5 = \langle g_3 \rangle \simeq \mathbb{C}_3$ for some $g_3 \in SL(2,R)$ with $\mathrm{tr}(g_3) = -1$.

From now on, let us assume that K is of order $|K| = 2^a.3$ for some $1 \leq a \leq 3$ and consider some Sylow subgroups $H_2, H_3 = \langle g_4 \rangle \simeq \mathbb{C}_3$ of K. We claim that the product

$$H_{2^a} H_3 = \{ g g_4^i \ \mid \ g \in H_{2^a}, \quad 0 \leq i \leq 2 \}$$

depletes K. More precisely, $H_{2^a} \cap H_3 = \{I_2\}$, because 2^a and 3 are relatively prime. Therefore

$$H_{2^a} H_3 / H_{2^a} = H_{2^a} \cup H_{2^a} g_4 \cup H_{2^a} g_4^2$$

is a right coset decomposition of the subset $H_{2^a} H_3 \subseteq K$ modulo H_{2^a}. Due to the disjointness of this decomposition, one has $|H_{2^a} H_3| = 3 |H_{2^a}| = 3.2^a = |K|$. Therefore, the subset $H_{2^a} H_3$ of K coincides with K and $K = H_{2^a} H_3$ is a product of its Sylow subgroups.

If $K = H_2 H_3 = \langle -I_2 \rangle \langle g_3 \rangle$ for some $g_3 \in SL(2,R)$ with $\mathrm{tr}(g_3) = -1$ then $\pm I_2$ commute with g_3^j for all $0 \leq j \leq 2$ and the group K is abelian. Thus, $K = \langle -g_3 \rangle \simeq \mathbb{C}_6$ is a cyclic group of order 6, generated by $-g_3 \in SL(2,R)$ with $\mathrm{tr}(-g_3) = 1$.

For $K = H_4 H_3 = \langle g_1 \rangle \langle g_3 \rangle$ with $g_1, g_3 \in SL(2,R)$ of $\mathrm{tr}(g_1) = 0$, $\mathrm{tr}(g_3) = -1$, note that $g_4 = -g_3 \in SL(2,R)$ is of order 6. Then $g_4^3 = -I_2 = g_1^2$, because $-I_2 \in SL(2,R)$ is the only element of order 2. We claim that $g_1, g_4 \in SL(2,R)$ are subject to $g_4 g_1 g_4 = g_1$. To this end, let $S \in Q(R)(\sqrt{-3})))_{2 \times 2} \subseteq \mathbb{Q}(\sqrt{-d}, \sqrt{-3})_{2 \times 2}$ be a matrix, whose columns are eigenvectors of g_1. Then

$$D_4 = S^{-1} g_4 S = \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{pmatrix} \quad \text{and}$$

$$D_1 = S^{-1} g_1 S = \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix} \quad \text{with} \quad a_1^2 + b_1 c_1 = -1$$

generate the subgroup $K^o = S^{-1} K S \simeq K$. It suffices to check that $D_4 D_1 D_4 = D_1$, because then $g_4 g_1 g_4 = (S D_4 S^{-1})(S D_1 S^{-1})(S D_4 S^{-1}) = S(D_4 D_1 D_4) S^{-1} = S D_1 S^{-1} = g_1$ and

$$K = \langle g_1, g_3 \rangle = \langle g_1, g_4 = -g_3 \ \mid \ g_1^2 = g_4^3 = -I_2, \quad g_4 g_1 g_4 = g_1 \rangle \simeq \mathbb{Q}_{12}$$

is isomorphic to the dicyclic group $\mathbb{Q}_{12}$ of order 12. The group $K^o = \langle D_1, D_4 \rangle \simeq K$ of order 12 has a cyclic subgroup $\langle D_4 \rangle \simeq \mathbb{C}_6$ of order 6. The index $[K^o : \langle D_4 \rangle] = 2$, so that $\langle D_4 \rangle$ is a normal subgroup of K^o and $D_1 D_4 D_1^{-1} \in \langle D_4 \rangle$ is an element of order 6. More precisely, $D_1 D_4 D_1^{-1} = D_4$ or $D_1 D_4 D_1^{-1} = D_4^{-1} = D_4^5 = -D_4^2$. If $D_1 D_4 = D_4 D_1$ then $D_1 D_4 \in K^o$ is of order 12, as far as $(D_1 D_4)^{12} = (D_1^4)^3 (D_4^6)^2 = I_2^3 I_2^2 = I_1$, $(D_1 D_4)^6 = D_1^2 = -I_2 \neq I_2$, $(D_1 D_4)^4 = D_4^4 = -D_4 \neq I_2$, whereas $D_1 D_4, (D_1 D_4)^2, (D_1 D_4)^3 \not\in \{I_2\}$. Consequently, $D_1 D_4 = -D_4^2 D_1$, so that $D_4 D_1 D_4 = -D_4^3 D_1 = D_1$ and $K \simeq K^o \simeq \mathbb{Q}_{12}$. For instance, the subgroup

$$\langle D_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad D_4 = \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{pmatrix} \ \Bigg| \ D_1^2 = D_4^3 = -I_2, \quad D_1 D_4 D_1^{-1} = D_4^{-1} \rangle$$

of $SL(2, \mathcal{O}_{-3})$ realizes $\mathbb{Q}_{12}$ as a subgroup of $SL(2, \mathcal{O}_{-3})$. The existence of $\mathbb{Q}_{12} \simeq K < SL(2, R)$ for an arbitrary R is an open problem.

There remains to be shown that any subgroup $K = H_8 H_3 = \langle g_1, g_2, g_3 \rangle \simeq \mathbb{Q}_8 \mathbb{C}_3$ of $SL(2, R)$ of order 24 is isomorphic to the special linear group $K_8 \simeq SL(2, \mathbb{F}_3)$ over $\mathbb{F}_3$. In other words, any $K < SL(2, R)$ of order $|K| = 24$ can be generated by such $g_1, g_2, g_3 \in SL(2, R)$ that the subgroup $\langle g_1, g_2 \mid g_1^2 = g_2^2 = -I_2, \ \ g_2 g_1 = -g_1 g_2 \rangle \simeq \mathbb{Q}_8$ is isomorphic to the quaternion group $\mathbb{Q}_8$ of order 8, g_3 is of order 3 and $g_3 g_1 g_3^{-1} = g_2$, $g_3 g_2 g_3^{-1} = g_1 g_2$.

First of all, the Sylow 2-subgroup $H_8 \simeq \mathbb{Q}_8$ of K is normal. More precisely, by the Third Sylow Theorem, the number $n_2 \in \mathbb{N}$ of the Sylow 2-subgroups of K (i.e., the number n_2 of the subgroups of K of order 8) divides $|K| = 24$ and $n_2 \equiv 1 (\mathrm{mod} \ 2)$. Therefore $n_2 = 1$ or $n_2 = 3$. By Second Sylow Theorem, all Sylow 2-subgroups are conjugate to each other, so that $n_2 = 1$ exactly when $H_8 = \langle g_1, g_2 \rangle \simeq \mathbb{Q}_8$ is a normal subgroup of K. Let us assume that $n_2 = 3$ and denote by ν_s the number of the elements $g \in K$ of order s. Due to $-I_2 \in H_8 = \langle g_1, g_2 \rangle < K$, one has $\nu_1 = 1$, $\nu_2 = 1$. Note that $g \in K$ is of order 3 if and only if $-g \in K$ is of order 6, so that $\nu_6 = \nu_3$. By the Third Sylow Theorem, the number $n_3 \in \mathbb{N}$ of the Sylow 3-subgroups of K divides $|K| = 24$ and $n_3 \equiv 1 (\mathrm{mod} \ 3)$. Therefore $n_3 = 1$ or $n_3 = 4$.

If $n_3 = 1$ and there is a unique normal subgroup $H_3 = \langle g_3 \rangle \simeq \mathbb{C}_3$ of K of order 3, then $g_j g_3 g_j^{-1} \in \{g_3, g_3^2\} \subset \langle g_3 \rangle$ for $j = 1$ and $j = 2$. If $g_j g_3 g_j^{-1} = g_3$ then $g_j g_3 = g_3 g_j$ for g_j of order 4 and g_3 of order 3, so that $g_j g_3 \in K$ is of order 12, contrary to the non-existence of an element of $SL(2, R)$ of order 12. Therefore $g_1 g_3 g_1^{-1} = g_3^2$, $g_2 g_3 g_2^{-1} = g_3^2$, whereas

$$(g_1 g_2) g_3 (g_1 g_2)^{-1} = g_1 (g_2 g_3 g_2^{-1}) g_1^{-1} = g_1 g_3^2 g_1^{-1} = (g_1 g_3 g_1^{-1})^2 = (g_3^2)^2 = g_3$$

and $g_1 g_2$ of order 4 commutes with g_3 of order 3. Thus, $(g_1 g_2) g_3 \in K$ is of order 12, which is an absurd. That rejects the assumption $n_3 = 1$ and proves that $n_3 = 4$.

Let $H_{3,j} = \langle g_{3,j} \rangle \simeq \mathbb{C}_3$, $1 \leq j \leq 4$ be the four subgroups of K of order 3. Then $H_{3,i} \cap H_{3,j} = \{I_2\}$ for all $1 \leq i < j \leq 4$, as far as any $g \in H_{3,i} \setminus \{I_2\}$ generates $H_{3,i}$. As a result, $\cup_{i=1}^{4} H_{3,i}$ and K contain 8 different elements $g_{3,i}, g_{3,i}^2$, $1 \leq i \leq 4$ of order 3 and $\nu_6 = \nu_3 = 8$. Thus,

$$24 = |K| = \nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_6 = 18 + \nu_4,$$

so that K has $\nu_4 = 6$ elements of order 4. Since any Sylow 2-subgroup

$$H_8 = \langle g_1, g_2 \mid g_1^2 = g_2^2 = -I_2, \ \ g_2 g_1 = -g_1 g_2 \rangle = \{\pm I_2, \pm g_1, \pm g_2, \pm g_1 g_2\} \simeq \mathbb{Q}_8$$

of K contains six elements $\pm g_1, \pm g_2, \pm g_1 g_2$ of order 4, there cannot be more than one H_8. In other words, $n_2 = 1$ and H_8 is a normal subgroup of K.

The above considerations show that

$$K = H_8 \rtimes H_3 = \langle g_1, g_2 \mid g_1^2 = g_2^2 = -I_2, \ \ g_2 g_1 = -g_1 g_2 \rangle \rtimes \langle g_3 \mid g_3^3 = I_2 \rangle \simeq \mathbb{Q}_8 \rtimes \mathbb{C}_3$$

is a semi-direct product of $\mathbb{Q}_8$ and $\mathbb{C}_3$. Up to an isomorphism, K is uniquely determined by the group homomorphism

$$\varphi_K : H_3 \longrightarrow Aut(H_8),$$

$$\varphi_K(g_3^j)(\pm g_1^k g_2^l) = g_3^j(\pm g_1^k g_2^l)g_3^{-j} \quad \text{for} \quad \forall \pm g_1^k g_2^l \in H_8, \quad 0 \leq k, l \leq 1.$$

Since $H_3 = \langle g_3 \rangle \simeq \mathbb{C}_3$ is cyclic, φ_K is uniquely determined by $\varphi_K(g_3) \in Aut(H_8)$. On the other hand, H_8 is generated by g_1, g_2, so that suffices to specify $\varphi_K(g_3)(g_j) = g_3 g_j g_3^{-1} \in H_8$ for $1 \leq j \leq 2$, in order to determine φ_K. If the cyclic group $\langle g_1 \rangle \simeq \mathbb{C}_4$ is normalized by g_3 then $g_3 g_1 g_3^{-1} \in \{\pm g_1\}$, as an element of order 4. In the case of $g_3 g_1 g_3^{-1} = g_1$, the element $g_1 \in K$ of order 4 commutes with the element $g_3 \in K$ of order 3 and their product $g_1 g_3 \in K$ is of order 12. The lack of $g \in SL(2, R)$ of order 12 requires $g_3 g_1 g_3^{-1} = -g_1$. Now,

$$g_3^2 g_1 g_3^{-2} = g_3(g_3 g_1 g_3^{-1})g_3^{-1} = g_3(-g_1)g_3^{-1} = g_1$$

is equivalent to $g_3^2 g_1 = g_1 g_3^2$ and the product $g_1 g_3^2 \in K$ of $g_1 \in K$ of order 4 with $g_3^2 \in K$ of order 3 is an element of order 12. The absurd justifies that neither of the cyclic subgroups $\langle g_1 \rangle \simeq \langle g_2 \rangle \simeq \langle g_1 g_2 \rangle \simeq \mathbb{C}_4$ of order 4 of H_8 is normalized by g_3. Thus, an arbitrary $g_1 \in H_8 \simeq \mathbb{Q}_8$ of order 4 is completed by $g_2 := g_3 g_1 g_3^{-1} \in H_8 \setminus \langle g_1 \rangle$ of order 4 to a generating set of $H_8 \simeq \mathbb{Q}_8$. Then

$$g_3^2 g_1 g_3^{-2} = g_3(g_3 g_1 g_3^{-1})g_3^{-1} = g_3 g_2 g_3^{-1} \in H_8 \setminus (\langle g_1 \rangle \cup \langle g_2 \rangle) = \{g_1 g_2, g_2 g_1\}$$

specifies that either $g_3 g_2 g_3^{-1} = g_1 g_2$ or $g_3 g_2 g_3^{-1} = g_2 g_1$. If $g_3 g_2 g_3^{-1} = g_2 g_1$, we replace the generator g_3 of K by $h_3 = g_3^2$ and note that $h_3 g_1 h_3^{-1} = g_2 g_1$. Now, $h_1 := g_1$ and $h_2 := g_2 g_1$ generate $H_8 = \langle h_1, h_2 \mid h_1^2 = h_2^2 = -I_2, \quad h_2 h_1 = -h_1 h_2 \rangle$ and satisfy $h_3 h_1 h_3^{-1} = h_2$,

$$h_3 h_2 h_3^{-1} = g_3[(g_3 g_2 g_3^{-1})(g_3 g_1 g_3^{-1})]g_3^{-1} = g_3(g_2 g_1 g_2)g_3^{-1} = g_3 g_1 g_3^{-1} =$$

$$= g_2 = -(g_2 g_1)g_1 = -h_2 h_1 = h_1 h_2.$$

Thus, the group

$$K' = \langle g_1, g_2, g_3 \,|\, g_1^2 = g_2^2 = -I_2, \; g_2 g_1 = -g_1 g_2, \; g_3^3 = I_2, \; g_3 g_1 g_3^{-1} = g_2, \; g_3 g_2 g_3^{-1} = g_2 g_1 \rangle$$

is isomorphic ro the group

$$K = \langle g_1, g_2, g_3 \,|\, g_1^2 = g_2^2 = -I_2, \; g_2 g_1 = -g_1 g_2, \; g_3^3 = I_2, \; g_3 g_1 g_3^{-1} = g_2, \; g_3 g_2 g_3^{-1} = g_1 g_2 \rangle.$$

We shall realize $SL(2, \mathbb{F}_3)$ as a subgroup $K_8^o = \langle D_1, D_2, D_3 \rangle$ of $SL(2, \mathbb{Q}(\sqrt{-d}, \sqrt{-3}))$. The existence of subgroups $SL(2, \mathbb{F}_3) \simeq K_8 < SL(2, R)$ is an open problem. Towards the construction of K_8^o, let us choose

$$D_j = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} \quad \text{with} \quad a_j^2 + b_j c_j = -1 \quad \text{for} \quad 1 \leq j \leq 2 \quad \text{and}$$

$$D_3 = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix}$$

from $SL(2, \mathbb{Q}(\sqrt{-d}, \sqrt{-3}))$. After computing

$$D_3 D_j D_3^{-1} = \begin{pmatrix} a_j & e^{-\frac{2\pi i}{3}} b_j \\ e^{\frac{2\pi i}{3}} c_j & -a_j \end{pmatrix} \quad \text{for} \quad 1 \leq j \leq 2,$$

observe that $D_3 D_1 D_3^{-1} = D_2$ reduces to

$$\left| \begin{array}{c} a_2 = a_1 \\ b_2 = e^{-\frac{2\pi i}{3}} b_1 \\ c_2 = e^{\frac{2\pi i}{3}} c_1 \end{array} \right. .$$

The relation $D_2 D_1 = -D_1 D_2$ is equivalent to $2a_1 a_2 + b_1 c_2 + b_2 c_1 = 0$ and implies that $2a_1^2 = b_1 c_1$. Now,

$$D_3 D_2 D_3^{-1} = \begin{pmatrix} a_1 & e^{\frac{2\pi i}{3}} b_1 \\ e^{-\frac{2\pi i}{3}} c_1 & -a_1 \end{pmatrix} = \begin{pmatrix} \sqrt{-3} a_1^2 & \sqrt{-3} e^{\frac{2\pi i}{3}} a_1 b_1 \\ \sqrt{-3} e^{-\frac{2\pi i}{3}} a_1 c_1 & -\sqrt{-3} a_1^2 \end{pmatrix} = D_1 D_2$$

is tantamount to

$$\left| \begin{array}{l} a_1(1 - \sqrt{-3} a_1) = 0 \\ b_1(1 - \sqrt{-3} a_1) = 0 \\ c_1(1 - \sqrt{-3} a_1) = 0 \end{array} \right.$$

and specifies that $a_1 = \frac{\sqrt{-3}}{3}$. Namely, the assumption $a_1 \neq -\frac{\sqrt{-3}}{3}$ forces $a_1 = b_1 = c_1 = 0$, whereas $\det(D_1) = 0$, contrary to the choice of $D_1 \in SL(2, \mathbb{Q}(\sqrt{-d}, \sqrt{-3}))$. As a result, $b_1 \neq 0$, $c_1 = -\frac{2}{3b_1}$ and

$$D_1 = \begin{pmatrix} -\frac{\sqrt{-3}}{3} & b_1 \\ -\frac{2}{3b_1} & \frac{\sqrt{-3}}{3} \end{pmatrix}, \quad D_2 = \begin{pmatrix} -\frac{\sqrt{-3}}{3} & e^{-\frac{2\pi i}{3}} b_1 \\ e^{\frac{2\pi i}{3}} c_1 & \frac{\sqrt{-3}}{3} \end{pmatrix}, \quad D_3 = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix}$$

generate a subgroup $SL(2, \mathbb{F}_3) \simeq K_8^o < SL(2, \mathbb{Q}(\sqrt{-d}, \sqrt{-3}))$.

□

Corollary 25. *If the finite subgroup K of $SL(2, R)$ is not isomorphic to the dicyclic group*

$$K_7 = \langle g_1, g_4 \mid g_1^2 = g_4^3 = -I_2, \quad g_4 g_1 g_4 = g_1 \rangle =$$
$$= \langle g_1, g_3 = -g_4 \mid g_1^2 = -I_2, \quad g_3^3 = I_2, \quad g_3 g_1 g_3^{-1} = g_3 g_1 \rangle \simeq \mathbb{Q}_{12}$$

of order 12 then K is isomorphic to a subgroup of the special linear group

$$K_8 = \langle g_1, g_2, g_3 \,|\, g_1^2 = g_2^2 = -I_2, \, g_3^3 = I_2, \, g_2 g_1 = -g_1 g_2, \, g_3 g_1 g_3^{-1} = g_2, g_3 g_2 g_3^{-1} = g_1 g_2 \rangle$$
$$\simeq SL(2, \mathbb{F}_3)$$

over the field $\mathbb{F}_3$ with three elements.

Proof. According to Proposition 24, any finite subgroup $K < SL(2,R)$ is isomorphic to some of the groups $K_1, \ldots, K_8$. Thus, it suffices to establish that any K_j, $1 \leq j \leq 6$ is isomorphic to a subgroup of K_8. Note that $K_1 = \{I_2\} \subset K_8$ and $K_2 = \langle -I_2 \rangle \subset K_8$ are subgroups of K_8. The generator g_1 of K_8 is of order 4, so that any subgroup $K_3 \simeq \mathbb{C}_4$ of $SL(2,R)$ is isomorphic to the subgroup $\langle g_1 \rangle$ of K_8. In the proof of Proposition 24 we have seen that K_8 has a normal Sylow 2-subgroup

$$H_8 = \langle g_1, g_2 \ \mid \ g_1^2 = g_2^2 = -I_2, \quad g_2 g_1 = -g_1 g_2 \rangle \simeq \mathbb{Q}_8,$$

isomorphic to the quaternion group $\mathbb{Q}_8 \simeq K_4$ of order 8. The generator g_3 of K_8 provides a subgroup $\langle g_3 \rangle \simeq \mathbb{C}_3 \simeq K_5$ of K_8. The product $(-I_2)g_3$ of the commuting elements $-I_2 \in K_8$ or order 2 and $g_3 \in K_8$ of order 3 is an element $-g_3 \in K_8$ of order 6, so that $K_6 \simeq \mathbb{C}_6$ is isomorphic to the subgroup $\langle -g_3 \rangle$ of K_8.

□

Towards the classification of the finite subgroups of $GL(2,R)$, we proceed with the following:

Lemma 26. *Let H be a finite subgroup of $GL(2,R)$. Then*

(i) $\det(H)$ is a cyclic subgroup of R^*;

(ii) H is a product $H = [H \cap SL(2,R)]\langle h_o \rangle$ of its normal subgroup $H \cap SL(2,R)$ and any $\mathbb{C}_r \simeq \langle h_o \rangle \subseteq H$ with $\det(H) = \langle \det(h_o) \rangle \simeq \mathbb{C}_s$;

(iii) the order s of $\det(H) = \langle \det(h_o) \rangle$ divides the order r of $h_o \in H$ and

$$[H \cap SL(2,R)] \cap \langle h_o \rangle = \langle h_o^s \rangle \simeq \mathbb{C}_{\frac{r}{s}};$$

(iv) H is of order $s|H \cap SL(2,R)|$;

(v) $s = r$ if and only if $H = [H \cap SL(2,R)] \rtimes \langle h_o \rangle$ is a semi-direct product.

Proof. (i) The image $\det(H)$ of the group homomorphism $\det : H \to R^*$ is a subgroup of R^*. As far as the units group R^* of the endomorphism ring R of E is cyclic, its subgroup $\det(H)$ is cyclic, as well.

(ii) If $\det(h_o)$ is a generator of the cyclic subgroup $\det(H) < R^*$ then one can represent $H = [H \cap SL(2,R)]\langle h_o \rangle$. The inclusion $[H \cap SL(2,R)]\langle h_o \rangle \subseteq H$ is clear by the choice of $h_o \in H$. For the opposite inclusion, note that any $h \in H$ with $\det(h) = \det(h_o)^m$ for some $m \in \mathbb{Z}$ is associated with $hh_o^{-m} \in H \cap SL(2,R)$, so that $h = (hh_o^{-m})h_o^m \in [H \cap SL(2,R)]\langle h_o \rangle$ and $H \subseteq [H \cap SL(2,R)]\langle$.

(iii) If $h_o \in H$ is of order r then $h_o^r = I_2$ and $\det(h_o)^r = 1$. Therefore the order s of $\det(h_o) \in R^*$ divides s. Note that $h_o^s \in [H \cap SL(2,R)] \cap \langle h_o \rangle$, as far as $\det(h_o^s) = \det(h_o)^s = 1$. Therefore $\langle h_o^s \rangle$ is a subgroup of $[H \cap SL(2,R)] \cap \langle h_o \rangle$. Conversely, any $h_o^x \in [H \cap SL(2,R)] \cap \langle h_o \rangle$ has $\det(h_o^x) = \det(h_o)^x = 1$, so that s divides x and $h_o^x \in \langle h_o^s \rangle$. That justifies $[H \cap SL(2,R)] \cap \langle h_o \rangle \subseteq \langle h_o^s \rangle$ and $[H \cap SL(2,R)] \cap \langle h_o \rangle = \langle h_o^s \rangle$. The order of $\langle h_o^s \rangle$ and h_o^s is $\frac{r}{s}$, since s divides r.

(iv) It suffices to show that

$$H = \cup_{i=0}^{s-1}[H \cap SL(2,R)]h_o^j$$

is the coset decomposition of H with respect to its normal subgroup $H \cap SL(2,R)$, in order to conclude that the order $|H|$ of H is s times the order $|H \cap SL(2,R)|$ of $H \cap SL(2,R)$. The inclusion $H \supseteq \cup_{i=0}^{s-1}[H \cap SL(2,R)]h_o^j$ is clear by the choice of $h_o \in H$. According to $H = [H \cap SL(2,R)]\langle h_o \rangle$, any element of H is of the form $h = gh_o^m$ for some $g \in H \cap SL(2,R)$ and $m \in \mathbb{Z}$. If $m = sq + r_o$ is the division of m by s with residue $0 \leq r_o \leq s-1$ then $h = [g(h_o^s)^q]h_o^{r_o} \in [H \cap SL(2,R)]h_o^{r_o}$, due to $h_o^s \in H \cap SL(2,R)$. Therefore $H \subseteq \cup_{j=0}^{s-1}[H \cap SL(2,R)]h_o^j$ and $H = \cup_{j=0}^{s-1}[H \cap SL(2,R)]h_o^j$. The cosets $[H \cap SL(2,R)]h_o^i$ and $[H \cap SL(2,R)]h_o^j$ are mutually disjoint for any $0 \leq i < j \leq s-1$, because the assumption $g_1 h_i = g_2 h_o^j$ for $g_1, g_2 \in H \cap SL(2,R)$ implies that $h_o^{j-i} = g_2^{-1} g_1 \in [H \cap SL(2,R)] \cap \langle h_o \rangle = \langle h_o^s \rangle$. As a result, s divides $0 < j - i < s$, which is an absurd.

(v) According to (iii), the order s of $\det(h_o)$ divides the order r of h_o. On the other hand, $h_o^s = I_2$ exactly when r divides s, so that $h_o^s = I_2$ is equivalent to $r = s$. Thus, $r = s$ exactly when

$$[H \cap SL(2,R)] \cap \langle h_o \rangle = \{I_2\}.$$

As far as the product of the normal subgroup $H \cap SL(2,R)$ and the subgroup $\langle h_o \rangle$ is the entire H, one has a semi-direct product $H = [H \cap SL(2,R)] \rtimes \langle h_o \rangle$ if and only if $r = s$.

□

Lemma 27. *Let $H = [H \cap SL(2,R)]\langle h_o \rangle$ be a finite subgroup of $GL(2,R)$ for $h_o \in H$ of order r with $\det(H) = \langle \det(h_o) \rangle \simeq \mathbb{C}_s$ and $H \cap SL(2,R)$ be generated by $g_0 = h_o^s, g_1, \ldots, g_t$. Then $H \cap SL(2,R)$, r and*

$$h_o g_i h_o^{-1} \in H \cap SL(2,R) \quad \textit{for all} \quad 1 \leq i \leq t$$

determine H up to an isomorphism.

Proof. By the proof of Lemma 26 (iv), H has a coset decomposition

$$H = \cup_{j=0}^{s-1}[H \cap SL(2,R)]h_o^j$$

with respect to its normal subgroup $H \cap SL(2,R)$. Therefore, the group structures of $H \cap SL(2,R)$ and $\langle h_o \rangle \simeq \mathbb{C}_r$, together with the multiplication rule for $h_1 h_o^i, h_2 h_o^j \in H$ with $h_1, h_2 \in H \cap SL(2,R)$ and $0 \leq i, j \leq s-1$ determine the group H up to an isomorphism. Let us represent $h_1 = g_{i_1}^{a_1} g_{i_2}^{a_2} \ldots g_{i_k}^{a_k}$ and $h_2 = g_{j_1}^{b_1} g_{j_2}^{b_2} \ldots g_{j_l}^{b_l}$ as words in the alphabet $g_0 = h_o^s, g_1, \ldots, g_t$ with some integral exponents $a_p, b_q \in \mathbb{Z}$. (The group H is finite, so that any g_i is of finite order r_i and one can reduce the exponent of g_i to a residue modulo r_i.) In order to determine the product $(h_1 h_o^i)(h_2 h_o^j)$ as an element of $H = \cup_{j=0}^{s-1} \langle g_0, g_1, \ldots, g_t \rangle h_o^j$, it suffices to specify $g_i' \in H \cap SL(2,R) = \langle g_0, g_1, \ldots, g_t \rangle$ with $h_o g_i = g_i' h_o$ for all $0 \leq i \leq t$. That allows to move gradually h_o^i to the end of $(h_1 h_o i)(h_2 h_o^j)$, producing $h_1 h_2' h_o^{i+j} \in [H \cap SL(2,R)] h_o^{(i+j)(\text{mod} s)}$ for an appropriate $h_2' \in H \cap SL(2,R)$. In other words, the group structures of $H \cap SL(2,R)$ and $\langle h_o \rangle \simeq \mathbb{C}_r$, together with the conjugates $g_i' = h_o g_i h_o^{-1}$ of g_i determine the group multiplication in H. Note that $h_o g_0 h_o^{-1} = g_0$, since $g_0 = h_o^s$ commutes with h_o. The conjugates $g_i' = h_o g_i h_o^{-1}$ with $1 \leq i \leq t$ belong to the normal subgroup $H \cap SL(2,R) \ni g_i$ of H and have the same orders r_i as g_i.

$\square$

Any finite subgroup $H = [H \cap SL(2,R)] \langle h_o \rangle$ of $GL(2,R)$ with determinant $\det(H) = \langle \det(h_o) \rangle \simeq \mathbb{C}_s$ has a conjugate

$$S^{-1}HS = \{S^{-1}[H \cap SL(2,R)]S\} \langle S^{-1} h_o S \rangle = [S^{-1}HS \cap SL(2,\mathbb{C})] \langle S^{-1} h_o S \rangle$$

with a diagonal matrix $S^{-1} h_o S$. Mote precisely, if R is a subring of the integers ring $\mathcal{O}_{-d}$ of an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$ and $\lambda_1 = \lambda_1(h_o)$, $\lambda_2 = \lambda_2(h_o)$ are the eigenvalues of h_o, then there exists a basis

$$v_1 = \left(\begin{array}{c} s_{11} \\ s_{21} \end{array} \right), \quad v_2 = \left(\begin{array}{c} s_{12} \\ s_{22} \end{array} \right) \quad \text{of} \quad \mathbb{C}^2,$$

consisting of eigenvectors v_j of h_o, associated with the eigenvalues $\lambda_j = \lambda_j(h_o)$. This is due to the finite order of h_o, because the Jordan block

$$J = \left(\begin{array}{cc} \lambda_1 & 1 \\ 0 & \lambda_1 \end{array} \right) \quad \text{with} \quad \lambda_1 \in \mathbb{C}^*$$

is of infinite order in $GL(2,\mathbb{C})$. The matrix $S = (s_{ij})_{i,j=1}^2$ with columns v_1, v_2 is non-singular and its entries belong to the extension $\mathbb{Q}(\sqrt{-d}, \lambda(h_o)) = \mathbb{Q}(\sqrt{-d}, \lambda_2(h_o))$ of $\mathbb{Q}(\sqrt{-d})$ by some of the eigenvalues of h_o. Making use of the classification of $h_o \in GL(2,R)$ of finite order r and $\det(h_o) \in R^*$ of order s, done in section 2, one determines explicitly the field $F_{-d}^{(s,r)} = \mathbb{Q}(\sqrt{-d}, \lambda_1(h_o))$, obtained from $\mathbb{Q}(\sqrt{-d})$ by adjoining an eigenvalue $\lambda_1(h_o)$ of $h_o \in H$. The group

$$S^{-1}HS = [S^{-1}HS \cap SL(2,\mathbb{C})] \langle S^{-1} h_o S \rangle$$

has a diagonal generator $D_o = S^{-1} h_o S$ and the conjugates

$$(S^{-1} h_o S)(S^{-1} g_i S)(S^{-1} h_o S)^{-1} = S^{-1}(h_o g_i h_o^{-1}) S$$

are easier to be computed.

The next lemma collects the fields $F_{-d}^{(s,r)}$.

Lemma 28. *Let $H = [H \cap SL(2,R)]\langle h_o \rangle$ be a finite subgroup of $GL(2,R)$ with $h_o \in H$ of order r, $\det(h_o) \in R^*$ of order s and $F_{-d}^{(s,r)}$ be the number field*

$$F_{-d}^{(s,r)} = \begin{cases} \mathbb{Q}(\sqrt{-d}) & \text{for } s = r = 2, \\ \mathbb{Q}(i) & \text{for } s \in \{2,4\},\ r = 4, \\ \mathbb{Q}(\sqrt{-3}) & \text{for } (s,r) = (2,6) \text{ or } s \in \{3,6\}, \\ \mathbb{Q}(\sqrt{2}, i) & \text{for } s \in \{2,4\},\ r = 8, \\ \mathbb{Q}(\sqrt{3}, i) & \text{for } s = 2,\ r = 12. \end{cases}.$$

Then there exists a matrix $S \in GL(2, F_{-d}^{(s,r)})$ such that

$$D_o = S^{-1} h_o S = \begin{pmatrix} \lambda_1(h_o) & 0 \\ 0 & \lambda_2(h_o) \end{pmatrix}$$

is diagonal and

$$H^o = S^{-1} H S = [S^{-1} H S \cap SL(2, F_{-d}^{(s,r)})]\langle D_o \rangle$$

is a subgroup of $GL(2, F_{-d}^{(s,r)})$, isomorphic to H.

Summarizing the results of section 2, one obtains also the following

Corollary 29. *If $h_o \in GL(2,R) \setminus SL(2,R)$ is of order r with $\det(h_o) \in R^*$ of order s and eigenvalues $\lambda_1(h_o), \lambda_2(h_o)$, then*

$$\frac{\lambda_1(h_o)}{\lambda_2(h_o)} \in \left\{ \pm 1, \quad \pm i, \quad e^{\pm \frac{2\pi i}{3}}, \quad e^{\pm \frac{\pi i}{3}} \right\}.$$

More precisely,

$$\text{(i)} \quad \frac{\lambda_1(h_o)}{\lambda_2(h_o)} = 1 \quad \text{exactly when} \quad h_o \in \left\{ \pm i I_2, \quad e^{\pm \frac{2\pi i}{3}} I_2, \quad e^{\pm \frac{\pi i}{3}} I_2 \right\}$$

is a scalar matrix;

$$\text{(ii)} \quad \frac{\lambda_1(h_o)}{\lambda_2(h_o)} = -1 \quad \text{for}$$

(a) $\lambda_1(h_o) = 1, \quad \lambda_2(h_o) = -1$ *and an arbitrary* $R = R_{-d,f}$;

(b) $\lambda_1(h_o) = e^{\pm \frac{3\pi i}{4}}, \quad \lambda_2(h_o) = e^{\mp \frac{\pi i}{4}}, \quad R = \mathbb{Z}[i], \quad s = 4$;

(c) $\lambda_1(h_o) = e^{\pm \frac{5\pi i}{6}}, \quad \lambda_2(h_o) = e^{\mp \frac{\pi i}{6}}, \quad R = \mathcal{O}_{-3}, \quad s = 3$

(d) $\lambda_1(h_o) = e^{\pm \frac{2\pi i}{3}}, \quad \lambda_2(h_o) = e^{\mp \frac{\pi i}{3}}, R = \mathcal{O}_{-3}, \quad s = 6.$

$$\text{(iii)} \quad \frac{\lambda_1(h_o)}{\lambda_2(h_o)} = \pm i \quad \text{for}$$

$$(a)\quad \lambda_1(h_o) = e^{\pm\frac{3\pi i}{4}},\quad \lambda_2(h_o) = e^{\pm\frac{\pi i}{4}},\quad R = \mathcal{O}_{-2},\quad s = 2;$$

$$(b)\quad \{\lambda_1(h_o), \lambda_2(h_o)\} = \{\pm i, \pm 1\} \quad \text{or } \{\pm i, \mp 1\} \quad \text{with} \quad R = \mathbb{Z}[i],\quad s = 4.$$

$$(iv)\quad \frac{\lambda_1(h_o)}{\lambda_2(h_o)} = e^{\pm\frac{2\pi i}{3}} \quad \text{for}$$

$$(a)\quad \lambda_1(h_o) = e^{\pm\frac{5\pi i}{6}},\quad \lambda_2(h_o) = e^{\pm\frac{\pi i}{6}},\quad R = \mathbb{Z}[i],\quad s = 2;$$

$$(b)\quad \lambda_1(h_o) = e^{\pm\frac{2\pi i}{3}},\quad \lambda_2(h_o) = 1,\quad R = \mathcal{O}_{-3},\quad s = 3;$$

$$(c)\quad \lambda_1(h_o) = e^{\pm\frac{\pi i}{3}},\quad \lambda_2(h_o) = -1,\quad R = \mathcal{O}_{-3},\quad s = 3.$$

$$(v)\quad \frac{\lambda_1(h_o)}{\lambda_2(h_o)} = e^{\pm\frac{\pi i}{3}} \quad \text{for}$$

$$(a)\quad \lambda_1(h_o) = e^{\pm\frac{2\pi i}{3}}, \lambda_2(h_o) = e^{\pm\frac{\pi i}{3}}, R = \mathcal{O}_{-3},\quad s = 2;$$

$$(b)\quad \lambda_1(h_o) = \varepsilon e^{\eta\frac{\pi i}{3}},\quad \lambda_2(h_o) = \varepsilon,\quad R = \mathcal{O}_{-3},\quad s = 6,\quad \varepsilon, \eta \in \{\pm 1\}.$$

Proposition 30. *Let H be a finite subgroup of $GL(2, R)$,*

$$H \cap SL(2, R) = \{I_2\}$$

and $h_o \in H$ be an element of order r with $\det(H) = \langle \det(h_o)\rangle \simeq \mathbb{C}_s$ and eigenvalues $\lambda_1(h_o)$, $\lambda_2(h_o)$. Then $r = s$ and H is isomorphic to $H_{C1}(j) \simeq \mathbb{C}_{s_j}$ for some $1 \leq j \leq 4$, where

$$H_{C1}(1) = \langle h_o\rangle \simeq \mathbb{C}_2 \quad \text{with} \quad \lambda_1(h_o) = 1,\quad \lambda_2(h_o) = -1,$$

$$H_{C1}(2) = \langle h_o\rangle \simeq \mathbb{C}_3 \quad \text{with} \quad R = \mathcal{O}_{-3},\quad h_0 = e^{-\frac{2\pi i}{3}} I_2 \quad \text{or} \quad \lambda_1(h_o) = e^{\frac{2\pi i}{3}},\quad \lambda_2(h_o) = 1,$$

$$H_{C1}(3) = \langle h_o\rangle \simeq \mathbb{C}_4 \quad \text{with} \quad R = \mathbb{Z}[i],\quad \{\lambda_1(h_o), \lambda_2(h_o)\} = \{i, 1\} \quad \text{or} \quad \{-i, -1\},$$

$$H_{C1}(4) = \langle h_o\rangle \simeq \mathbb{C}_6 \quad \text{with} \quad R = \mathcal{O}_{-3},$$

$$\{\lambda_1(h_o), \lambda_2(h_o)\} = \left\{e^{\frac{\pi i}{3}}, 1\right\},\quad \left\{e^{-\frac{2\pi i}{3}}, -1\right\} \quad \text{or} \quad \left\{e^{\frac{2\pi i}{3}}, e^{-\frac{\pi i}{3}}\right\}.$$

Proof. By Lemma 26 (ii), the group $H = \langle h_o\rangle \simeq \mathbb{C}_r$ is cyclic and generated by any $h_o \in H$, whose determinant $\det(h_o)$ generates $\det(H) = \langle \det(h_o)\rangle$. Moreover, Lemma 26 (iii) specifies that $\{I_2\} = [H \cap SL(2, R)] \cap \langle h_o\rangle = \langle h_o^s\rangle$ or the order r of h_o coincides with the order s of $\det(h_o)$. For $s \in \{3, 4, 6\}$ one can assume that $\det(h_o) = e^{\frac{2\pi i}{3}}$, since the generators of $\det(H) = \langle \det(h_o)\rangle \simeq \mathbb{C}_s$ are $e^{\frac{2\pi i}{s}}$ and $e^{-\frac{2\pi i}{s}}$. Making use of the classification of the elements $h_o \in GL(2, R)$ of order s with $\det(h_o) = e^{\frac{2\pi i}{s}}$, done in section 2, one concludes that $H \simeq H_{C1}(j)$ for some $1 \leq j \leq 4$.

□

Proposition 31. *Let H be a finite subgroup of $GL(2,R)$,*

$$H \cap SL(2,R) = \langle -I_2 \rangle \simeq \mathbb{C}_2$$

and $h_o \in H$ be an element of order r with $\det(H) = \langle \det(h_o) \rangle \simeq \mathbb{C}_s$ and eigenvalues $\lambda_1(h_o)$, $\lambda_2(h_o)$. Then H is isomorphic to $H_{C2}(i)$ for some $1 \leq i \leq 6$, where

$$H_{C2}(1) = \langle iI_2 \rangle \simeq \mathbb{C}_4 \quad \textit{with} \quad R = \mathbb{Z}[i],$$

$$H_{C_2}(2) = \langle -I_2 \rangle \times \langle h_o \rangle \simeq \mathbb{C}_2 \times \mathbb{C}_2 \quad \textit{with} \quad \lambda_1(h_o) = 1, \quad \lambda_2(h_o) = -1,$$

$$H_{C2}(3) = \langle h_o \rangle \simeq \mathbb{C}_6 \quad \textit{with} \quad R = \mathcal{O}_{-3}, \quad h_o = e^{\frac{\pi i}{3}} I_2 \quad \textit{or} \quad \lambda_1(h_o) = e^{-\frac{\pi i}{3}}, \quad \lambda_2(h_o) = -1,$$

$$H_{C2}(4) = \langle h_o \rangle \simeq \mathbb{C}_8 \quad \textit{with} \quad R = \mathbb{Z}[i], \quad \lambda_1(h_o) = e^{\frac{3\pi i}{4}}, \quad \lambda_2(h_o) = e^{-\frac{\pi i}{4}},$$

$$H_{C2}(5) = \langle -I_2 \rangle \times \langle h_o \rangle \simeq \mathbb{C}_2 \times \mathbb{C}_4 \quad \textit{with} \quad R = \mathbb{Z}[i], \quad \lambda_1(h_o) = i, \quad \lambda_2(h_o) = 1,$$

$$H_{C2}(6) = \langle h_o \rangle \simeq \mathbb{C}_8 \quad \textit{with} \quad R = \mathbb{Z}[i], \quad \lambda_1(h_o) = e^{\frac{3\pi i}{4}}, \quad \lambda_2(h_o) = e^{-\frac{\pi i}{4}},$$

$$H_{C2}(7) = \langle -I_2 \rangle \times \langle h_o \rangle \simeq \mathbb{C}_2 \times \mathbb{C}_6 \quad \textit{with} \quad R = \mathcal{O}_{-3},$$

$$\{\lambda_1(h_o), \lambda_2(h_o)\} = \left\{ e^{\frac{2\pi i}{3}}, e^{-\frac{\pi i}{3}} \right\}, \quad \left\{ e^{\frac{\pi i}{3}}, 1 \right\} \quad \textit{or} \quad \left\{ e^{-\frac{2\pi i}{3}}, -1 \right\}. \tag{13}$$

Proof. By Lemma 26 (iii), one has $h_o^s \in H \cap SL(2,R) = \langle -I_2 \rangle$ for some $s \in \{2,3,4,6\}$.

If $h_o^s = I_2$ then $s = r$ and

$$H = \langle -I_2 \rangle \times \langle h_o \rangle \simeq \mathbb{C}_2 \times \mathbb{C}_s$$

is a direct product, as far as the scalar matrix $-I_2$ commutes with h_o. When h_o is of odd order $s = 3$, its opposite matrix $-h_o \in H$ is of order 6 and $H = \langle -h_o \rangle \simeq \mathbb{C}_6$. Without loss of generality, $h_1 := -h_o$ has $\det(h_1) = e^{\frac{2\pi i}{3}}$ and Proposition 21 specifies that either $h_1 = e^{\frac{\pi i}{3}} I_2$ or $\lambda_1(h_1) = e^{-\frac{\pi i}{3}}$, $\lambda_2(h_o) = -1$. For $s = 2$ the group $H = \langle -I_2 \rangle \times \langle h_o \rangle = H_{C2}(2) \simeq \mathbb{C}_2 \times \mathbb{C}_2$, where $h_o \in H$ has eigenvalues $\lambda_1(h_o) = 1$, $\lambda_2(h_o) = -1$. The case $s = 4$ occurs only for $R = \mathbb{Z}[i]$. Assuming $\det(h_o) = i$, one gets $\lambda_1(h_o) = \varepsilon i$, $\lambda_2(h_o) = \varepsilon$ for some $\varepsilon \in \{\pm 1\}$ by Proposition 17. Since $-I_2 \in H$, one can replace h_o by $-h_o$ and reduce to the case of $\varepsilon = 1$. If $s = 6$, then Proposition 19 provides (13).

In the case of $h_o^s = -I_2$, the intersection $\langle h_o \rangle SL(2,R) = \langle -I_2 \rangle = H \cap SL(2,R)$ and the group

$$H = \langle h_o \rangle \simeq \mathbb{C}_{2s}$$

is cyclic. More precisely, for $s = 2$ Proposition 16 implies that $h_o = \pm i I_2$ and $H \simeq H_{C2}(1)$. If $s = 3$ and $\det(h_o) = e^{\frac{2\pi i}{3}}$ then $H \simeq H_{C2}(3)$ by Proposition 21. For $s = 4$ and $\det(h_o) = i$ one has $H \simeq H_{C2}(6)$, according to Proposition 17. Making use of Proposition 19, one observes that there are no $h_o \in GL(2,R)$ of order 12 with $\det(h_o) = e^{\frac{\pi i}{3}}$ and concludes the proof of the proposition.

□

Towards the description of the finite subgroups $H = [H \cap SL(2,R)]\langle h_o \rangle$ of $GL(2,R)$ with $H \cap SL(2,R) \simeq \mathbb{C}_t$ for some $t \in \{3,4,6\}$, one needs the following

Lemma 32. *If $g \in GL(2,\mathbb{C})$ has different eigenvalues $\lambda_1 \neq \lambda_2$ then any $h \in GL(2,\mathbb{C})$ with $hg \neq gh$ and $h^2 g = gh^2$ has vanishing trace* $\mathrm{tr}(h) = 0$.

Proof. The trace is invariant under conjugation, so that

$$g = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

can be assumed to be diagonal. If

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{C}),$$

then $h^2 g = gh^2$ is equivalent to

$$\left| \begin{array}{l} (\lambda_1 - \lambda_2) b(a+d) = 0 \\ (\lambda_1 - \lambda_2) c(a+d) = 0 \end{array} \right. .$$

Due to $\lambda_1 \neq \lambda_2$, there follow $b(a+d) = 0$ and $c(a+d) = 0$. The assumption $\mathrm{tr}(h) = a + d \neq 0$ leads to $b = c = 0$. As a result,

$$h = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

is a diagonal matrix and commutes with g. The contradiction justifies that $\mathrm{tr}(h) = 0$. □

Lemma 33. *Let $H = [H \cap SL(2,R)]\langle h_o \rangle$ be a finite subgroup of $GL(2,R)$ with*

$$H \cap SL(2,R) = \langle g \rangle \simeq \mathbb{C}_t \quad \textit{for some} \quad t \in \{3,4,6\} \quad \textit{and}$$

$$\det(H) = \langle \det(h_o) \rangle = \langle e^{\frac{2\pi i}{s}} \rangle \simeq \mathbb{C}_s, \quad s > 1$$

for some $h_o \in H$ of order r. Then:

$$\textit{(i)} \quad \frac{r}{s} = \begin{cases} 1, 2, 3, 4 \ \textit{or} \ 6 & \textit{for } s = 2, \\ 1, 2 \ \textit{or} \ 4 & \textit{for } s = 3, \\ 1 \ \textit{or} \ 2 & \textit{for } s = 4 \\ 1 & \textit{for } s = 6 \end{cases}$$

divides t;

(ii) $\frac{r}{s} = t$ if and only if $H = \langle h_o \rangle \simeq \mathbb{C}_r$ is cyclic and $H \cap SL(2,R) = \langle h_o^s \rangle$;

(iii) if $\frac{r}{s} < t$ *then* H *is isomorphic to the non-cyclic abelian group*

$$H' = \langle g, h_o \ \mid \ g^t = h_o^r = I_2, \quad h_o g = g h_o \rangle$$

or to the non-abelian group

$$H'' = \langle g, h_o \ \mid \ g^t = h_o^r = I_2, \quad h_o g h_o^{-1} = g^{-1} \rangle;$$

(iv) if $\frac{r}{s} < t$ *and* $H \simeq H''$ *is non-abelian then* h_o *has eigenvalues* $\lambda_1(h_o) = ie^{\frac{\pi i}{s}}$, $\lambda_2(h_o) = -ie^{\frac{\pi i}{s}}$ *and*

$$(r, s) \in \{(2,2), \quad (6,6)\} \quad \textit{for} \quad t = 3,$$

$$(r, s) \in \{(2,2), \quad (8,4), \quad (6,6)\} \quad \textit{for} \quad t = 4,$$

$$(r, s) \in \{(2,2), \quad (8,4), \quad (6,6)\} \quad \textit{for} \quad t = 6.$$

Proof. (i) Note that if $\det(h_o) \in R^*$ is of order s then $\det(h_o^s) = \det(h_o)^s = 1$ and $h_o^s \in H \cap SL(2, R) = \langle g \rangle$ is an element of order $\frac{r}{s}$. Since $\langle g \rangle \simeq \mathbb{C}_t$ is of order t, the ratio $\frac{r}{s} \in \mathbb{N}$ divides t. Proposition 16 provides the list of $\frac{r}{s} = \frac{r}{2}$ for $s = 2$. If $s = 3$ then the values of $\frac{r}{s} = \frac{r}{3}$ are taken from Propositions 21 and 22. Propositions 17 and 18 supply the range of $\frac{r}{s} = \frac{r}{4}$ for $s = 4$, while Propositions 19 and 20 give account for $\frac{r}{s} = \frac{r}{6}$ in the case of $s = 6$.

(ii) Note that $h_o^s \in \langle g \rangle$ is of order $\frac{r}{s} = t$ exactly when $\langle g \rangle = \langle h_o^s \rangle$ and $H = \langle h_o \rangle \simeq \mathbb{C}_r$ is a cyclic group.

(iii) According to Lemma 27, the group $H = [H \cap SL(2, R)]\langle h_o \rangle = \langle g \rangle \langle h_o \rangle$ is completely determined by the order t of g, the order r of h_o and the conjugate $x = h_o g h_o^{-1} \in H \cap SL(2, R) = \langle g \rangle$ of g by h_o. The order t of g is invariant under conjugation, so that $x = g^m$ for some $m \in \mathbb{Z}_t^*$. The Euler function $\varphi(t) = 2$ for $t \in \{3, 4, 6\}$ and $\mathbb{Z}_t^* = \{\pm 1 (\mathrm{mod} t)\}$. Therefore $x = h_o g h_o^{-1} = g$ or $x = h_o g h_o^{-1} = g^{-1}$.

(iv) If $H \simeq H''$ is a non-abelian group then

$$h_o^2 g h_o^{-2} = h_o (h_o g h_o^{-1}) h_o^{-1} = h_o g^{-1} h_o^{-1} = (h_o g h_o^{-1})^{-1} = (g^{-1})^{-1} = g,$$

so that g commutes with h_o^2, but does not commute with h_o. By Lemma 32 there follows $\mathrm{tr}(h_o) = 0$. There exists a matrix $S \in GL\left(2, \mathbb{Q}\left(\sqrt{-d}, e^{\frac{2\pi i}{t}}\right)\right)$, such that

$$D = S^{-1} g S = \left(\begin{array}{cc} e^{\frac{2\pi i}{t}} & 0 \\ 0 & e^{-\frac{2\pi i}{t}} \end{array} \right) \in SL\left(2, \mathbb{Q}\left(\sqrt{-d}, e^{\frac{2\pi i}{t}}\right)\right)$$

is diagonal. Since the trace is invariant under conjugation,

$$D_o := S^{-1} h_o S = \left(\begin{array}{cc} a & b \\ c & -a \end{array} \right) \in GL\left(2, \mathbb{Q}\left(\sqrt{-d}, e^{\frac{2\pi i}{t}}\right)\right).$$

The relation $h_o g = g^{-1} h_o$ implies the vanishing of a. As a result, the characteristic polynomial

$$\mathcal{X}_{h_o}(\lambda) = \lambda^2 + \det(h_o) = \lambda^2 + e^{\frac{2\pi i}{s}} = 0$$

has roots $\lambda_1(h_o) = ie^{\frac{\pi i}{s}}$, $\lambda_2(h_o) = -ie^{\frac{\pi i}{s}}$. More precisely, for $s = 2$ one has $\lambda_1(h_o) = -1$, $\lambda_2(h_o) = 1$, so that h_o and D_o are of order $r = 2$. The ratio $\frac{r}{s} = 1$ divides any $t \in \{3, 4, 6\}$. If $s = 3$ then $\lambda_1(h_o) = e^{\frac{5\pi i}{6}}$, $\lambda_2(h_o) = e^{-\frac{\pi i}{6}}$, so that h_o and D_o are of order $r = 12$. The quotient $\frac{r}{s} = 4$ divides only $t = 4$. Therefore $\frac{r}{s} = t$ and $H = \langle h_o \rangle \simeq \mathbb{C}_{12}$, according to (ii). In the case of $s = 4$, one has $\lambda_1(h_o) = e^{\frac{3\pi i}{4}}$, $\lambda_2(h_o) = e^{-\frac{\pi i}{4}}$, whereas h_o and D_o are of order $r = 8$. The quotient $\frac{r}{s} = 2$ divides only $t \in \{4, 6\}$. Finally, for $s = 6$ the automorphism h_o has eigenvalues $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o) = e^{-\frac{\pi i}{3}}$. Consequently, h_o and D_o are of order $r = 6$ and $\frac{r}{s} = 1$ divides all $t \in \{3, 4, 6\}$.

□

Lemma 34. *(i) For arbitrary $d \in \mathbb{N}$ and $t \in \{3, 4, 6\}$ there is a dihedral subgroup*

$$\mathcal{D}_t = \langle g, h_o \mid g^t = h_o^2 = I_2, \quad h_o g h_o^{-1} = g^{-1} \rangle < GL(2, \mathbb{Q}(\sqrt{-d}))$$

of order $2t$ with $\mathcal{D}_t \cap SL(2, \mathbb{Q}(\sqrt{-d})) = \langle g \rangle \simeq \mathbb{C}_t$, $\det(\mathcal{D}_t) = \langle \det(h_o) \rangle = \langle -1 \rangle \simeq \mathbb{C}_2$ and eigenvalues $\lambda_1(h_o) = -1$, $\lambda_2(h_o) = 1$ of h_o.

(ii) For an arbitrary $t \in \{3, 4, 6\}$ there is a subgroup

$$\mathcal{H}_t = \langle g, h_o \mid g^t = h_o^6 = I_2, \quad h_o g h_o^{-1} = g^{-1} \rangle < GL(2, \mathbb{Q}(\sqrt{-3}))$$

of order $6t$ with $\mathcal{H}_t \cap SL(2, \mathbb{Q}(\sqrt{-3})) = \langle g \rangle \simeq \mathbb{C}_t$, $\det(\mathcal{H}_t) = \langle \det(h_o) \rangle = \langle e^{\frac{\pi i}{3}} \rangle \simeq \mathbb{C}_6$ and eigenvalues $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o) = e^{-\frac{\pi i}{3}}$ of h_o.

(iii) For an arbitrary $t \in \{4, 6\}$ there is a subgroup

$$\mathcal{H}'_t = \langle g, h_o \mid g^{\frac{t}{2}} = h_o^4 = -I_2, \quad h_o g h_o^{-1} = g^{-1} \rangle < GL(2, \mathbb{Q}(\sqrt{2}, i))$$

of order $4t$ with $\mathcal{H}'_t \cap SL(2, \mathbb{Q}(\sqrt{2}, i)) = \langle g \rangle \simeq \mathbb{C}_t$, $\det(\mathcal{H}'_t) = \langle \det(h_o) \rangle = \langle i \rangle \simeq \mathbb{C}_4$ and eigenvalues $\lambda_1(h_o) = e^{\frac{3\pi i}{4}}$, $\lambda_2(h_o) = e^{-\frac{\pi i}{4}}$ of h_o.

Proof. (i) Let us choose a diagonalizing matrix $S \in GL(2, \mathbb{Q}(\sqrt{-d}))$ of h_o, so that

$$D_o = S^{-1} h_o S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Taking into account Proposition 15, one has to show the existence of

$$D = S^{-1} g S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{-d})$$

with

$$D_o D D_o^{-1} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = D^{-1}$$

for any trace $\mathrm{tr}(g) = \mathrm{tr}(D) = a + d \in \{0, \pm 1\}$. More precisely, for $a = d = 0$, $b \neq 0$ and $c = -b^{-1}$, then the matrix

$$D = D_4 = \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}$$

of order 4 and the matrix D_o of order 2 generate a dihedral group $\mathcal{D}_4$ of order 8. If $a = d = -\frac{1}{2}$, $b \neq 0$ and $c = -\frac{3}{4}b^{-1}$ then

$$D = D_3 = \begin{pmatrix} -\frac{1}{2} & b \\ -\frac{3}{4}b^{-1} & -\frac{1}{2} \end{pmatrix}$$

of order 3 and D_o of order 2 generate a symmetric group $\mathcal{D}_3 \simeq S(3)$ of degree 3. In the case of $a = d = \frac{1}{2}$, $b \neq 0$ and $c = -\frac{3}{4}b^{-1}$, the matrix

$$D = D_6 = \begin{pmatrix} \frac{1}{2} & b \\ -\frac{3}{4}b^{-1} & \frac{1}{2} \end{pmatrix}$$

of order 6 and the matrix D_o of order 2 generate a dihedral group $\mathcal{D}_6$ of order 12.

(ii) By Proposition 19, if $h_o \in GL(2, R)$ has eigenvalues $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o) = e^{-\frac{\pi i}{3}}$ then $R = \mathcal{O}_{-3}$. Let us consider

$$D_o = S^{-1} h_o S = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{pmatrix} \in GL(2, \mathbb{Q}(\sqrt{-3}))$$

for some $S \in GL(2, \mathbb{Q}(\sqrt{-3}))$ and

$$D = S^{-1} g S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{-3}))$$

with trace $\mathrm{tr}(g) = \mathrm{tr}(D) = a + d \in \{0, \pm 1\}$. Then

$$D_o D D_o^{-1} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = D^{-1}$$

is equivalent to $a = d$. Consequently, D_3, D_4, D_6 from the proof of (i) satisfy the required conditions.

(iii) Note that

$$D_o = S^{-1} h_o S = \begin{pmatrix} e^{\frac{3\pi i}{4}} & 0 \\ 0 & e^{-\frac{\pi i}{4}} \end{pmatrix} \in GL(2, \mathbb{Q}(\sqrt{2}, i))$$

for some $S \in GL(2, \mathbb{Q}(\sqrt{2}, i))$ and

$$D = S^{-1} g S = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Q}(\sqrt{2}, i))$$

with trace $\mathrm{tr}(g) = \mathrm{tr}(D) = a + d \in \{0, 1\}$ satisfy

$$D_o D D_o^{-1} = \left(\begin{array}{cc} a & -b \\ c & d \end{array} \right) = \left(\begin{array}{cc} d & -b \\ -c & a \end{array} \right) = D^{-1}$$

exactly when $a = d$. In the notations from the proof of (i), one has $\langle D_4, D_o \rangle \simeq \mathcal{H}'_4$ and $\langle D_6, D_o \rangle \simeq \mathcal{H}'_6$.

$\square$

Corollary 35. *Let H be a finite subgroup of $GL(2, R)$,*

$$H \cap SL(2, R) = \langle g \rangle \simeq \mathbb{C}_3$$

and $h_o \in H$ be an element of order r with $\det(H) = \langle \det(h_o) \rangle \simeq \mathbb{C}_s$ and eigenvalues $\lambda_1(h_o)$, $\lambda_2(h_o)$. Then H is isomorphic to some $H_{C3}(i)$, $1 \leq i \leq 5$, where

$$H_{C3}(1) = \langle h_o \rangle \simeq \mathbb{C}_6$$

with $R = R_{-3,f}$, $\lambda_1(h_o) = e^{\frac{\pi i}{3}}$, $\lambda_2(h_o) = e^{\frac{2\pi i}{3}}$,

$$H_{C3}(2) = \langle g, h_o \ \mid \ g^3 = h_o^2 = I_2, \quad h_o g h_o^{-1} = g^{-1} \rangle \simeq S_3$$

is the symmetric group of degree 3, $\lambda_1(h_o) = -1$, $\lambda_2(h_o) = 1$,

$$H_{C3}(3) = \langle g \rangle \times \langle e^{\frac{2\pi i}{3}} I_2 \rangle \simeq \mathbb{C}_3 \times \mathbb{C}_3$$

with $R = \mathcal{O}_{-3}$ and any $g \in SL(2, \mathcal{O}_{-3})$ of trace $\mathrm{tr}(g) = -1$,

$$H_{C3}(4) = \langle g \rangle \times \langle h_o \rangle \simeq \mathbb{C}_3 \times \mathbb{C}_6$$

with $R = \mathcal{O}_{-3}$, $\lambda_1(h_o) = e^{\frac{\pi i}{3}}$, $\lambda_2(h_o) = e^{-\frac{2\pi i}{3}}$,

$$H_{C3}(5) = \langle g, h_o \ \mid \ g^3 = h_o^6 = I_2, \quad h_o g h_o^{-1} = g^{-1} \rangle$$

of order 18 with $R = \mathcal{O}_{-3}$, $\lambda_1(h_o) = E^{\frac{2\pi i}{3}}$, $\lambda_2(h_o) = e^{-\frac{\pi i}{3}}$.

There exist subgroups

$$H_{C3}(1), H_{C3}(3), H_{C3}(4) < GL(2, \mathcal{O}_{-3}),$$

as well as subgroups

$$H^o_{C3}(2) < GL(2, \mathbb{Q}(\sqrt{-d})), \quad H^o_{C3}(5) < GL(2, \mathbb{Q}(\sqrt{-3}))$$

with $H^o_{C3}(j) \simeq H_{C3}(j)$ for $j \in \{2, 5\}$.

Proof. By Lemma 33 (i), the quotient $\frac{r}{s}$ is a divisor of $t = 3$, so that either $r = s$ or $r = 3s = 6$.

For $s = 2$, $r = 6$ one has a cyclic group $H = \langle h_o \rangle \simeq \mathbb{C}_6$ with $\det(h_o) = -1$. Up to an inversion $h_o \mapsto h_o^{-1}$ of the generator, Proposition 16 specifies that $\lambda_1(h_o) = e^{\frac{\pi i}{3}}$, $\lambda_2(h_o) = e^{\frac{2\pi i}{3}}$ and justifies the realization of $H_{C3}(1) = \langle h_o \rangle$ over $\mathcal{O}_{-3}$.

Form now on, let $r = s \in \{2, , 3, 46\}$. According to Lemma 33(iii) and (iv), the group $H = \langle g, h_o \rangle$ is either abelian or isomorphic to some $H_{C3}(j)$ for $j \in \{2, 5\}$.

If $H = \langle g, h_o \ \mid \ g^3 = h_o^r = I_2, \quad gh_o = h_o g \rangle$ is an abelian group of order $3r$, then $H = \langle g \rangle \times \langle h_o \rangle \simeq \mathbb{C}_3 \times \mathbb{C}_r$ is a direct product by Lemma 26 (iv). (Here we use that the semi-direct product $H = [H \cap SL(2, R)] \rtimes \langle h_o \rangle = \langle g \rangle \rtimes \langle h_o \rangle$ is a direct product if and only if $gh_o = h_o g$.)

The order $r = s = 2$ of h_o is relatively prime to the order 3 of g, so that gh_o is an element of order 6 and $\langle g, h_o \rangle = \langle gh_o \rangle \simeq \mathbb{C}_6 \simeq H_{C3}(1)$.

The order $r = s = 4$ of h_o is relatively prime to the order 3 of g and gh_o is of order 12. By the classification of $x \in GL(2, R)$ of finite order, done in section 2, one has $\det(gh_o) = -1$. Therefore $\det(h_o) = -1$ and $s = 2$, contrary to the assumption $s = 4$.

For $r = s = 3$ one can assume $\det(h_o) = e^{-\frac{2\pi i}{3}}$, after an eventual inversion $h_o \mapsto h_o^{-1}$. Then by Proposition 22 one has $h_o = e^{\frac{2\pi i}{3}} I_2$ or $\lambda_1(h_o) = e^{\frac{4\pi i}{3}}$, $\lambda_2(h_o) = 1$. Assume that $\lambda_1(h_o) = e^{\frac{4\pi i}{3}}$, $\lambda_2(h_o) = 1$ and note that the commuting g and h_o can be simultaneously diagonalized by an appropriate $S \in GL(2, \mathbb{C})$. Consequently,

$$D = S^{-1} g S = \left(\begin{array}{cc} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{array} \right) \quad \text{and} \quad D_o = S^{-1} h_o S = \left(\begin{array}{cc} e^{\frac{4\pi i}{3}} & 0 \\ 0 & 1 \end{array} \right)$$

are subject to $D^2 D_o = e^{\frac{2\pi i}{3}} I_2$. As a result,

$$g^2 h_o = (SDS^{-1})^{-1} (SD_o S^{-1}) = S(D^2 D_o) S^{-1} = e^{\frac{2\pi i}{3}} I_2$$

and $H = \langle g, h_o \rangle = \langle g, g^2 h_o \rangle \simeq H_{C3}(3)$.

Finally, for $r = s = 6$, let us assume that $\det(h_o) = e^{-\frac{\pi i}{3}}$. Then

$$\{\lambda_1(h_o), \lambda_2(h_o)\} = \left\{ e^{\frac{\pi i}{3}}, e^{-\frac{2\pi i}{3}} \right\}, \quad \left\{ e^{-\frac{\pi i}{3}}, 1 \right\} \quad \text{or} \quad \left\{ e^{\frac{2\pi i}{3}}, -1 \right\}.$$

Similarly to the case of $r = s = 3$, the commuting g and h_o admit a simultaneous diagonalization

$$D = S^{-1} g S = \left(\begin{array}{cc} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{array} \right), \quad D_o = S^{-1} h_o S = \left(\begin{array}{cc} \lambda_1(h_o) & 0 \\ 0 & \lambda_2(h_o) \end{array} \right).$$

If $\lambda_1(h_o) = e^{-\frac{\pi i}{3}}$, $\lambda_2(h_o) = 1$ then

$$DD_o = \left(\begin{array}{cc} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{array} \right) \quad \text{and} \quad H \simeq \langle D, D_o \rangle = \langle D, DD_o \rangle \simeq H_{C3}(4).$$

For $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$ and $\lambda_2(h_o) = -1$ note that

$$DD_o = \begin{pmatrix} e^{-\frac{2\pi i}{3}} & 0 \\ 0 & e^{\frac{\pi i}{3}} \end{pmatrix}, \quad \text{so that again} \quad H \simeq \langle D, D_o \rangle = \langle D, DD_o \rangle \simeq H_{C3}(4).$$

Note that

$$g = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix}, \quad h_o = \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix} \in GL(2, \mathcal{O}_{-3})$$

generate a group, isomorphic to $H_{C3}(4)$.

□

Corollary 36. *Let H be a finite subgroup of $GL(2, R)$,*

$$H \cap SL(2, R) = \langle g \rangle \simeq \mathbb{C}_4$$

and $h_o \in H$ be an element of order r with $\det(H) = \langle \det(h_o) \rangle \simeq \mathbb{C}_s$ and eigenvalues $\lambda_1(h_o), \lambda_2(h_o)$. Then H is isomorphic to some $H_{C4}(i)$, $1 \leq i \leq 9$, where

$$H_{C4}(1) = \langle h_o \rangle \simeq \mathbb{C}_8$$

with $R = \mathcal{O}_{-2}$, $\lambda_1(h_o) = e^{\frac{\pi i}{4}}$, $\lambda_2(h_o) = e^{\frac{3\pi i}{3}}$,

$$H_{C4}(2) = \langle g \rangle \times \langle h_o \rangle \simeq \mathbb{C}_4 \times \mathbb{C}_2$$

with $R = R_{-1,f}$, $\lambda_1(h_o) = -1$, $\lambda_2(h_o) = 1$,

$$H_{C4}(3) = \langle g, h_o \mid g^2 = -I_2, \quad h_o^2 = I_2, \quad h_o g h_o^{-1} = g^{-1} \rangle \simeq \mathcal{D}_4$$

is the dihedral group of order 8 with $\lambda_1(h_o) = -1$, $\lambda_2(h_o) = 1$,

$$H_{C4}(4) = \langle h_o \rangle \simeq \mathbb{C}_{12}$$

with $R = \mathcal{O}_{-3}$, $\lambda_1(h_o) = e^{\frac{5\pi i}{6}}$, $\lambda_2(h_o) = e^{-\frac{\pi i}{6}}$,

$$H_{C4}(5) = \langle g \rangle \times \langle e^{\frac{2\pi i}{3}} I_2 \rangle \simeq \mathbb{C}_4 \times \mathbb{C}_3$$

for $R = \mathcal{O}_{-3}$ and $\forall g \in SL(2, \mathcal{O}_{-3})$ with $\mathrm{tr}(g) = 0$,

$$H_{C4}(6) = \langle g \rangle \times \langle h_o \rangle \simeq \mathbb{C}_4 \times \mathbb{C}_4$$

with $R = \mathbb{Z}[i]$, $\lambda_1(h_o) = i$, $\lambda_2(h_o) = 1$,

$$H_{C4}(7) = \langle ig \rangle \times \langle h_o \rangle \simeq \mathbb{C}_2 \times \mathbb{C}_8$$

with $R=\mathbb{Z}[i]$, $\lambda_1(h_o)=e^{\frac{3\pi i}{4}}$, $\lambda_2(h_o)=e^{-\frac{\pi i}{4}}$,

$$H_{C4}(8)=\langle g,h_o \mid g^2=h_o^4=-I_2,\quad h_ogh_o^{-1}=g^{-1}\rangle$$

of order 16 *with* $R=\mathbb{Z}[i]$, $\lambda_1(h_o)=e^{\frac{3\pi i}{4}}$, $\lambda_2(h_o)=e^{-\frac{\pi i}{4}}$,

$$H_{C4}(9)=\langle g,h_o \mid g^2=-I_2,\quad h_o^6=I_2,\quad h_ogh_o^{-1}=g^{-1}\rangle$$

of order 24 *with* $R=\mathcal{O}_{-3}$, $\lambda_1(h_o)=e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o)=e^{-\frac{\pi i}{3}}$.

There exist subgroups

$$H_{C4}(1)<GL(2,\mathcal{O}_{-2}),\quad H_{C4}(4),H_{C4}(5)<GL(2,\mathcal{O}_{-3}),$$

$$H_{C4}(2),H_{C4}(6)<GL(2,\mathbb{Z}[i]),$$

as well as subgroups

$$H^o_{C4}(7),H^o_{C4}(8)<GL(2,\mathbb{Q}(\sqrt{2},i)),\quad H^o_{C4}(3)<GL(2,\mathbb{Q}(\sqrt{-d})),$$

$$H^o_{C4}(9)<GL(2,\mathbb{Q}(\sqrt{-3})),$$

with $H^o_{C4}(j)\simeq H_{C4}(j)$ *for* $j\in\{3,7,8,9\}$.

Proof. If $\frac{r}{s}=4$ then either $(s,r)=(2,8)$ and $H\simeq H_{C4}(1)$ or $(s,r)=(3,12)$ and $H\simeq H_{C4}(4)$. By Proposition 16 there exists an element $h_o\in GL(2,\mathcal{O}_{-2})$ of order 8 with $\det(h_o)=-1$. Proposition 21 provides an example of $h_o\in GL(2,\mathcal{O}_{-3})$ of order 12 with $\det(h_o)=e^{\frac{2\pi i}{3}}$. There remain to be considered the cases with $\frac{r}{s}\in\{1,2\}$. According to Lemma 33, the non-abelian H under consideration are isomorphic to $H_{C4}(3)$, $H_{C4}(8)$ or $H_{C4}(9)$. By Lemma 34 (i) there is a subgroup $H^o_{C4}(3)<GL(2,\mathbb{Q}(\sqrt{-d}))$, conjugate to $H_{C4}(3)$. Lemma 34 (iii) provides an example of $S^{-1}H_{C4}(8)S=H^o_{C4}(8)<GL(2,\mathbb{Q}(\sqrt{2},i))$, while Lemma 34(ii) justifies the existence of $S^{-1}H_{C4}(9)S=H^o_{C4}(9)<GL(2,\mathbb{Q}(\sqrt{-3}))$.

There remain to be classified the non-cyclic abelian groups $H=[H\cap SL(2,R)]\langle h_o\rangle$ with $H\cap SL(2,R)\simeq\mathbb{C}_4$, $\langle h_o\rangle\simeq\mathbb{C}_r$, $\det(h_o)=e^{\frac{2\pi i}{3}}$ for $s\in\{2,3,4,6\}$, $r\in\{s,2s\}$.

If $r=s=2$ then by Proposition 16, the eigenvalues of h_o are $\lambda_1(h_o)=-1$ and $\lambda_2(h_o)=1$. There exists a matrix $S\in GL(2,\mathbb{Q}(\sqrt{-d})$, such that

$$D_o=S^{-1}h_oS=\begin{pmatrix}-1&0\\0&1\end{pmatrix}.$$

Proposition 15 establishes that $g\in SL(2,R)$ is of order 4 exactly when $\mathrm{tr}(g)=0$. The trace and the determinant are invariant under conjugation, so that

$$D=S^{-1}gS=\begin{pmatrix}a&b\\c&-a\end{pmatrix}\in SL(2,\mathbb{Q}(\sqrt{-d})).$$

The commutation

$$DD_o = \begin{pmatrix} -a & b \\ -c & -a \end{pmatrix} = \begin{pmatrix} -a & -b \\ c & -a \end{pmatrix} = D_o D$$

holds only when $b = c = 0$ and

$$D = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Bearing in mind that $D \in SL(2, \mathbb{Q}(\sqrt{-d}))$, one concludes that $i \in \mathbb{Q}(\sqrt{-d})$, whereas $d = 1$ and $R = R_{-1,f}$. The matirces

$$g = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad h_o = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{Z}[i])$$

generate a subgroup of $GL(2, \mathbb{Z}[i])$, isomorphic to $H_{C4}(2)$.

For $s = 2$ and $r = 4$ one has $R = \mathbb{Z}[i]$ and $h_o = \pm I_2$. Bearing in mind that $g \in SL(2, R)$ is of order 4 if and only if $\mathrm{tr}(g) = 0$, let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}[o]).$$

Then

$$gh_o = \pm \begin{pmatrix} ai & bi \\ ci & -ai \end{pmatrix} \in \mathbb{Z}[i]_{2\times 2}$$

has determinant $\det(gh_o) = \det(g)\det(h_o) = \det(h_o) = -1$ and trace $\mathrm{tr}(gh_o) = 0$. By Proposition 16, gh_o has eigenvalues $\lambda_1(gh_o) = -1$, $\lambda_2(gh_o) = 1$ and $H \simeq H_{C4}(2)$.

If $s = r = 3$ then $R = \mathcal{O}_{-3}$ and either $h_o = e^{-\frac{2\pi i}{3}}$ or $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o) = 1$. Replacing $e^{-\frac{2\pi i}{3}} I_2$ by its inverse, one observes that $H_{C4}(5) = \langle g, e^{-\frac{2\pi i}{3}} I_2 \rangle < GL(2, \mathcal{O}_{-3})$. If $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o) = 1$, then there exists $S \in GL(2, \mathbb{Q}(\sqrt{-3}))$, such that

$$D_o = S^{-1} h_o S = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & 1 \end{pmatrix}.$$

The determinant and the trace are invariant under conjugation, so that

$$D = S^{-1} g S = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{-3})).$$

Note that

$$DD_o = \begin{pmatrix} e^{\frac{2\pi i}{3}} a & b \\ e^{\frac{2\pi i}{3}} c & -a \end{pmatrix} = \begin{pmatrix} e^{\frac{2\pi i}{3}} a & e^{\frac{2\pi i}{3}} b \\ c & -a \end{pmatrix} = D_o D$$

is equivalent to $b = c = 0$ and $1 = \det(g) = \det(D) = -a^2$ specifies that

$$D = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

That contradicts $F \in SL(2, \mathbb{Q}(\sqrt{-3}))$ and justifies the non-existence of H with $s = r = 3$.

Let $s = 3$, $r = 6$. According to Proposition 21, there follows $R = \mathcal{O}_{-3}$ with $h_o = e^{\frac{\pi i}{3}} I_2$ or $\lambda_1(h_o) = e^{-\frac{\pi i}{3}}$, $\lambda_2(h_o) = 1$. If $h_o = e^{\frac{\pi i}{3}}$ then $H = \langle g, h_o \rangle = \langle g, g^2 h_o = -h_o = e^{-\frac{2\pi i}{3}} I_2 \rangle \simeq H_{C4}(5)$. In the case of $\lambda_1(h_o) = e^{-\frac{\pi i}{3}}$, $\lambda_2(h_o) = 1$ let us choose $S \in GL(2, \mathbb{Q}(\sqrt{-3}))$ with

$$D_o = S^{-1} h_o S = \begin{pmatrix} e^{-\frac{\pi i}{3}} & 0 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{Q}(\sqrt{-3})) \quad \text{and}$$

$$D = S^{-1} g S = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{-3})).$$

Then

$$DD_o = \begin{pmatrix} e^{-\frac{\pi i}{3}} a & b \\ e^{-\frac{\pi i}{3}} c & -a \end{pmatrix} = \begin{pmatrix} e^{-\frac{\pi i}{3}} a & e^{-\frac{\pi i}{3}} b \\ c & -a \end{pmatrix} = D_o D$$

if and only if

$$D = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{-3})),$$

which is an absurd.

Let us suppose that $s = r = 4$. The Proposition 17 specifies that $R = \mathbb{Z}[i]$ and $\lambda_1(h_o) = \varepsilon i$, $\lambda_2(h_o) = \varepsilon$ for some $\varepsilon \in \{\pm 1\}$. As far as $g^2 = -I_2 \in H$, there is no loss of generality in assuming that $\lambda_1(h_o) = i$, $\lambda_2(h_o) = 1$ and $H \simeq H_{C4}(6)$. Note that

$$g = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad h_o \in \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{Z}[i])$$

generate a subgroup, isomorphic to $H_{C4}(6)$.

For $s = 4$, $r = 8$, Proposition 17 implies that $R = \mathbb{Z}[i]$ and $\lambda_1(h_o) = e^{\frac{3\pi i}{4}}$, $\lambda_2(h_o) = e^{-\frac{\pi i}{4}}$. Note that $(ig)^2 = -g^2 = I_2$, so that $ig \in H = \langle g, h_o \rangle$ is of order 2 and $h_o^6 = i I_2$, according to $\lambda_1(h_o^6) = \lambda_1(h_o)^6 = i$, $\lambda_2(h_o^6) = \lambda_2(h_o)^6 = i$. Consequently,

$$H = \langle g, h_o \rangle = \langle h_o^6 g = ig, h_o \rangle = \langle ig \rangle \times \langle h_o \rangle \simeq \mathbb{C}_2 \times \mathbb{C}_8,$$

as far as $\langle ig \rangle \cap \langle h_o \rangle = \{I_2\}$. More precisely, if $ig = h_o^m$, then the second eigenvalue

$$1 = -i^2 = \lambda_2(ig) = \lambda_2(h_o^m) = e^{-\frac{\pi i m}{4}},$$

whereas $m \in 8\mathbb{Z}$ and the first eigenvalue

$$-1 = \lambda_1(ig) = \lambda_1(h_o^m) = e^{\frac{3\pi i m}{4}} = 1,$$

which is an absurd. Thus, $H \simeq H_{C4}(7)$ and there exists a subgroup

$$H_{C4}^o(7) = \langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} e^{\frac{3\pi i}{4}} & 0 \\ 0 & e^{-\frac{\pi i}{4}} \end{pmatrix} \rangle < GL(2, \mathbb{Q}(\sqrt{2}, i)),$$

conjugate to $H_{C4}(7)$.

Let us assume that $s = r = 6$. Then Proposition 19 applies to provide $R = \mathcal{O}_{-3}$ and

$$\{\lambda_1(h_o), \lambda_2(h_o)\} = \left\{ e^{\frac{2\pi i}{3}}, e^{-\frac{\pi i}{3}} \right\}, \quad \left\{ e^{\frac{\pi i}{3}}, 1 \right\}, \quad \left\{ e^{-\frac{2\pi i}{3}}, -1 \right\}.$$

Choose a matrix $S \in GL(2, \mathbb{Q}(\sqrt{-3}))$ with

$$D_o = S^{-1} h_o S = \begin{pmatrix} \lambda_1(h_o) & 0 \\ 0 & \lambda_2(h_o) \end{pmatrix} \in GL(2, \mathbb{Q}(\sqrt{-3})),$$

$$D = S^{-1} g S = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{-3})).$$

If $\lambda_1(h_o) \neq \lambda_2(h_o)$ then

$$DD_o = \begin{pmatrix} \lambda_1(h_o)a & \lambda_2(h_o)b \\ \lambda_1(h_o)c & -\lambda_2(h_o)a \end{pmatrix} = \begin{pmatrix} \lambda_1(h_o)a & \lambda_1(h_o)b \\ \lambda_2(h_o)c & -\lambda_2(h_o)a \end{pmatrix} = D_o D$$

is tantamount to $b = c = 0$, $a = \pm i$ and

$$D = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{-3})$$

is an absurd.

Similarly, in the case of $s = 6$, $r = 12$, Proposition 19 derives that $R = \mathcal{O}_{-3}$ and

$$\{\lambda_1(h_o), \lambda_2(h_o)\} = \left\{ e^{\frac{2\pi i}{3}}, e^{-\frac{\pi i}{3}} \right\}, \quad \left\{ e^{\frac{\pi i}{3}}, 1 \right\}, \quad \left\{ e^{-\frac{2\pi i}{3}}, -1 \right\}.$$

Note that $\lambda_1(h_o) \neq \lambda_2(h_o)$ for all the possibilities and apply the considerations for $s = r = 6$, in order to exclude the case $s = 6$, $r = 12$.

□

Corollary 37. *Let H be a finite subgroup of $GL(2, R)$,*

$$H \cap SL(2, R) = \langle g \rangle \simeq \mathbb{C}_6$$

and $h_o \in H$ be an element of order r with $\det(H) = \langle \det(h_o) \rangle \simeq \mathbb{C}_s$ and eigenvalues $\lambda_1(h_o)$, $\lambda_2(h_o)$. Then H is isomorphic to some $H_{C6}(i)$, $1 \leq i \leq 7$, where

$$H_{C6}(1) = \langle h_o \rangle \simeq \mathbb{C}_{12}$$

with $R = \mathbb{Z}[i]$, $\lambda_1(h_o) = e^{\frac{\pi i}{6}}$, $\lambda_2(h_o) = e^{\frac{5\pi i}{6}}$,

$$H_{C6}(2) = \langle g \rangle \times \langle h_o \rangle \simeq \mathbb{C}_6 \times \mathbb{C}_{12}$$

with $R = \mathcal{O}_{-3}$ or $R = R_{-3,2}$, $\lambda_1(h_o) = -1$, $\lambda_2(h_o) = 1$,

$$H_{C6}(3) = \langle g, h_o \mid g^3 = -I_2, \quad h_o^2 = I_2, \quad h_o g h_o^{-1} = g^{-1} \rangle \simeq \mathcal{D}_6$$

is the dihedral group of order 12, $\lambda_1(h_o) = -1$, $\lambda_2(h_o) = 1$,

$$H_{C6}(4) = \langle g \rangle \times \langle e^{\frac{2\pi i}{3}} I_2 \rangle \simeq \mathbb{C}_6 \times \mathbb{C}_3$$

with $R = \mathcal{O}_{-3}$ *and* $\forall g \in SL(2, \mathcal{O}_{-3})$ *of* $\mathrm{tr}(g) = 1$,

$$H_{C6}(5) = \langle g, h_o \ \mid \ g^3 = h_o^4 = -I_2, \quad h_o g h_o^{-1} = g^{-1} \rangle$$

of order 24 *with* $R = \mathbb{Z}[i]$, $\lambda_1(h_o) = e^{\frac{3\pi i}{4}}$, $\lambda_2(h_o) = e^{-\frac{\pi i}{4}}$,

$$H_{C6}(6) = \langle g, h_o \ \mid \ g^3 = -I_2, \quad h_o^6 = I_2, \quad h_o g h_o^{-1} = g^{-1} \rangle$$

of order 36 *with* $R = \mathcal{O}_{-3}$, $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o) = e^{-\frac{\pi i}{3}}$,

$$H_{C6}(7) = \langle g \rangle \times \langle h_o \rangle \simeq \mathbb{C}_6 \times \mathbb{C}_6$$

of order 36 *with* $R = \mathcal{O}_{-3}$, $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o) = e^{-\frac{\pi i}{3}}$.

There exist subgroups

$$H_{C6}(1) < GL(2, \mathbb{Z}[i]), \quad H_{C6}(2), H_{C6}(4), H_{C6}(7) < GL(2, \mathcal{O}_{-3}),$$

as well as subgroups

$$H^o_{C6}(3) < GL(2, \mathbb{Q}(\sqrt{-d})), \quad H^o_{C6}(5) < GL(2, \mathbb{Q}(\sqrt{2}, i)),$$

$$H^o_{C6}(6) < GL(2, \mathbb{Q}(\sqrt{-3}))$$

with $H^o_{C6}(j) \simeq H_{C6}(j)$ *for* $j \in \{3, 5, 6\}$.

Proof. According to Lemma 33(i), the ratio $\frac{r}{s} \in \{1, 2, 3, 6\}$ is a divisor of $t = 6$. If $r = 6s$ then $s = 2$ and $H = \langle h_o \rangle \simeq \mathbb{C}_{12} \simeq H_{C6}(1)$ by Lemma 33 (i), (ii). According to Proposition 16, the existence of $h_o \in GL(2, R)$ of order 12 with $\det(h_o) = -1$ requires $R = \mathbb{Z}[i]$ and there exist $h_o \in GL(2, \mathbb{Z}[i])$ of order 12 with $\det(h_o) = -1$.

For $r = 3s$ Lemma 33(i) specifies that $s = 2$. Combining with Lemma 33(iv), one concludes that

$$H = \langle g, h_o \ \mid \ g^3 = -I_2, \quad h_o^6 = I_2, \quad h_o g = g h_o \rangle$$

is a non-cyclic abelian group of order $st = 12$. By Proposition 16, $R = \mathcal{O}_{-3}$ or $R = R_{-3,2}$ and h_o has eigenvalues $\lambda_1(h_o) = e^{\frac{\varepsilon \pi i}{3}}$, $\lambda_2(h_o) = e^{\frac{\varepsilon 2\pi i}{3}}$ for some $\varepsilon \in \{\pm 1\}$. Due to $\langle g, h_o \rangle = \langle g, h_o^{-1} = h_o^5 \rangle$ by $h_o = (h_o^5)^5$, one can assume that $\lambda_1(h_o) = e^{\frac{\pi i}{3}}$, $\lambda_2(h_o) = e^{\frac{2\pi i}{3}}$. The commuting matrices g and h_o admit a simultaneous diagonalization

$$D = S^{-1} g S = \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{pmatrix}, \quad D_o = S^{-1} h_o S = \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{\frac{2\pi i}{3}} \end{pmatrix}$$

by an appropriate $S \in GL(2, \mathbb{Q}(\sqrt{-3}))$. Then

$$D^2 D_o = \left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right)$$

implies that $\lambda_1(g^2 h_o) = -1$, $\lambda_2(g^2 h_o) = 1$. As a result, $H = \langle g, h_o \rangle = \langle g, g^2 h_o \rangle$ is a subgroup of $GL(2, \mathcal{O}_{-3})$, isomorphic to $H_{C6}(2)$.

Form now on, $\frac{r}{s} \in \{1, 2\}$. In particular, $\frac{r}{s} < t = 6$ and the non-abelian

$$H = \langle g, h_o \mid g^6 = h_o^r = I_2, \quad h_o g h_o^{-1} = g^{-1} \rangle$$

occurs for $(r, s) \in \{(2, 2), (8, 4), (6, 6)\}$, according to Lemma 33(iv). Namely, for $r = s = 2$ one has a dihedral group $H \simeq \mathcal{D}_6 \simeq H_{C6}(3)$ of order 12, which is realized as a subgroup of $GL(2, \mathbb{Q}(\sqrt{-d}))$ by Lemma 34(i). In the case of $s = 4$ and $r = 8$ the group $H \simeq H_{C6}(5)$ of order 24 is embedded in $GL(2, \mathbb{Q}(\sqrt{2}, i))$ by Lemma 34(iii). In the case of $r = s = 6$ one has $H \simeq H_{C6}(6)$ of order 36, realized as a subgroup of $GL(2, \mathbb{Q}(\sqrt{-3}))$ by Lemma 34(ii).

There remain to be considered the non-cyclic abelian H with $r = 2s$, $s \in \{2, 3, 4\}$ or $r = s \in \{2, 3, 4, 6\}$. If $s = 2$, $r = 4$ then Proposition 16 requires $R = \mathbb{Z}[i]$ and $h_o = \pm i I_2$. Up to an inversion of h_o, one can assume that $h_o = i I_2$. Then $H = \langle g, i I_2 \rangle = \langle -g = (i I_2)^2 g, i I_2 \rangle$ is generated by the element $-g$ of order 3 and the scalar matrix $i I_2 \in H$ of order 4, so that $-ig = (i I_2)(-g) \in H$ of order 12 generates H, $H \simeq H_{C6}(1) \simeq \mathbb{C}_{12}$. (Note that for $g \in SL(2, \mathbb{Z}[i])$ of order 6 one has $g^3 = -I_2$, whereas $(-g)^3 = -g^3 = I_2$. The assumptions $-g = I_2$ and $(-g)^2 = g^2 = I_2$ lead to an absurd.)

Let us assume that $s = 3$ and $r = 6$. Then Proposition 21 implies that $R = \mathcal{O}_{-3}$ with $h_o = E^{\frac{\pi i}{3}} I_2$ or $\lambda_1(h_o) = e^{-\frac{\pi i}{3}}$, $\lambda_2(h_o) = -1$. Note that $H = \langle g, e^{\frac{\pi i}{3}} I_2 \rangle = \langle g, e^{-\frac{\pi i}{3}} I_2 \rangle$ by $e^{-\frac{\pi i}{3}} = \left(e^{\frac{\pi i}{3}} \right)^5$, $e^{\frac{\pi i}{3}} = \left(e^{-\frac{\pi i}{3}} \right)^5$. Further,

$$g^3 \left(e^{-\frac{\pi i}{3}} I_2 \right) = \left(e^{\pi i} I_2 \right) \left(e^{-\frac{\pi i}{3}} I_2 \right) = e^{\frac{2\pi i}{3}} I_2$$

implies that

$$H = \langle g, e^{-\frac{\pi i}{3}} I_2 \rangle = \langle g, g^3 \left(e^{-\frac{\pi i}{3}} I_2 \right) = e^{\frac{2\pi i}{3}} I_2 \rangle = \langle g \rangle \times \langle e^{\frac{2\pi i}{3}} \rangle \simeq \mathbb{C}_6 \times \mathbb{C}_3 \simeq H_{C6}(4).$$

For any $g \in SL(2, \mathcal{O}_{-3})$ of order 6, there is a subgroup $H_{C6}(4) = \langle g, e^{\frac{2\pi i}{3}} I_2 \rangle < GL(2, \mathcal{O}_{-3})$.

For $s = 4$, $r = 8$ there follow $R = \mathbb{Z}[i]$ and $\lambda_1(h_o) = e^{\frac{3\pi i}{4}}$, $\lambda_2(h_o) = e^{-\frac{\pi i}{4}}$, according to Proposition 17. Suppose that $S \in GL(2, \mathbb{Q}(\sqrt{2}, i))$ diagonalizes h_o,

$$D_o = S^{-1} h_o S = \left(\begin{array}{cc} e^{\frac{3\pi i}{4}} & 0 \\ 0 & e^{-\frac{\pi i}{4}} \end{array} \right) \in GL(2, \mathbb{Q}(\sqrt{2}, i)).$$

By Proposition 15, $g \in SL(2, \mathbb{Z}[i])$ is of order 6 exactly when $\mathrm{tr}(g) = 1$. Since the determinant and the trace are invariant under conjugation, one has

$$D = S^{-1}gS = \left(\begin{array}{cc} a & b \\ c & 1-a \end{array} \right) \in SL(2, \mathbb{Q}(\sqrt{2}, i).$$

However,

$$DD_o = \left(\begin{array}{cc} e^{\frac{3\pi i}{4}} a & e^{-\frac{\pi i}{4}} b \\ e^{\frac{3\pi i}{4}} c & e^{-\frac{\pi i}{4}}(1-a) \end{array} \right) = \left(\begin{array}{cc} e^{\frac{3\pi i}{4}} a & e^{\frac{3\pi i}{4}} b \\ e^{-\frac{\pi i}{4}} c & e^{-\frac{\pi i}{4}}(1-a) \end{array} \right) = D_o D$$

if and only if $b = c = 0$ and $a = e^{\frac{\varepsilon \pi i}{3}}$ for some $\varepsilon \in \{\pm 1\}$. Now,

$$D = \left(\begin{array}{cc} e^{\frac{\varepsilon \pi i}{3}} & 0 \\ 0 & 1 - e^{\frac{\varepsilon \pi i}{3}} \end{array} \right) \in SL(2, \mathbb{Q}(\sqrt{2}, i))$$

is an absurd, justifying the non-existence of H with $s = 4$ and $r = 8$.

In the case of $r = s = 2$ Proposition 16 implies that $\lambda_1(h_o) = -1$, $\lambda_2(h_o) = 1$, so that $H \simeq H_{C6}(2) \simeq \mathbb{C}_6 \times \mathbb{C}_2$.

For $r = s = 3$ Proposition 21 reveals that $R = \mathcal{O}_{-3}$ with $h_o = e^{-\frac{2\pi i}{3}} I_2$ or $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o) = 1$. It is clear that

$$H = \langle g, e^{-\frac{2\pi i}{3}} I_2 = \left(e^{\frac{2\pi i}{3}} I_2 \right)^2 \rangle = \langle g, e^{\frac{2\pi i}{3}} I_2 = \left(e^{-\frac{2\pi i}{3}} I_2 \right)^2 \rangle \simeq H_{C6}(4) \simeq \mathbb{C}_3 \times \mathbb{C}_3.$$

If $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o) = 1$ then the commuting matrices g and h_o admit a simultaneous diagonalization

$$D = S^{-1}gS = \left(\begin{array}{cc} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{array} \right), \quad D_o = S^{-1}h_oS = \left(\begin{array}{cc} e^{\frac{2\pi i}{3}} & 0 \\ 0 & 1 \end{array} \right) \in GL(2, \mathbb{Q}(\sqrt{-3}))$$

by an appropriate $S \in GL(2, \mathbb{Q}(\sqrt{-3}))$. Then $D^2 D_o = e^{-\frac{2\pi i}{3}} I_2$, whereas $g^2 h_o = S \left(e^{-\frac{2\pi i}{3}} I_2 \right) S^{-1} = e^{-\frac{2\pi i}{3}} I_2$ and

$$H = \langle g, h_o \rangle = \langle g, g^2 h_o = e^{-\frac{2\pi i}{3}} I_2 \rangle \simeq H_{C6}(4) \simeq \mathbb{C}_6 \times \mathbb{C}_3.$$

The assumption $r = s = 4$ implies that $R = \mathbb{Z}[i]$ and $\lambda_1(h_o) = \varepsilon i$, $\lambda_2(h_o) = \varepsilon$ for some $\varepsilon \in \{\pm 1\}$, according to Proposition 17. Due to $g^3 = -I_2$, one has $\langle g, h_o \rangle = \langle g, -h_o = g^3 h_o \rangle$, so that there is no loss of generality in assuming $\varepsilon = 1$. If $S \in GL(2, \mathbb{Q}(i))$ conjugates h_o to its diagonal form

$$D_o = S^{-1} h_o S = \left(\begin{array}{cc} i & 0 \\ 0 & 1 \end{array} \right) \in GL(2, \mathbb{Q}9i)),$$

then

$$D = S^{-1}gS = \left(\begin{array}{cc} a & b \\ c & 1-a \end{array} \right) \in SL(2,\mathbb{Q}(i)).$$

The relation

$$DD_o = \left(\begin{array}{cc} ia & b \\ ic & 1-a \end{array} \right) = \left(\begin{array}{cc} ia & ib \\ c & 1-a \end{array} \right) = D_o D$$

implies that

$$D = \left(\begin{array}{cc} e^{\frac{\varepsilon \pi i}{3}} & 0 \\ 0 & e^{-\frac{\varepsilon \pi i}{3}} \end{array} \right) \in SL(2,\mathbb{Q}(i)) \quad \text{for some} \quad \varepsilon \in \{\pm\}.$$

The contradiction proves the non-existence of H with $r = s = 4$.

Finally, for $r = s = 6$ Proposition 19 specifies that $R = \mathcal{O}_{-3}$ and

$$\{\lambda_1(h_o), \lambda_2(h_o)\} = \left\{ e^{\frac{2\pi i}{3}}, e^{-\frac{\pi i}{3}} \right\}, \quad \left\{ 1, e^{\frac{\pi i}{3}} \right\} \quad \text{or} \quad \left\{ e^{-\frac{2\pi i}{3}}, -1 \right\}.$$

The commuting matrices g and h_o admit simultaneous diagonalization

$$D = S^{-1}gS = \left(\begin{array}{cc} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{array} \right),$$

$$D_o = S^{-1}h_oS = \left(\begin{array}{cc} \lambda_1(h_o) & 0 \\ 0 & \lambda_2(h_o) \end{array} \right) \in GL(2,\mathbb{Q}(\sqrt{-3})$$

by an appropriate $S \in GL(2,\mathbb{Q}(\sqrt{-3}))$. Let us denote

$$D_o := \left(\begin{array}{cc} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{array} \right), \quad D'_o := \left(\begin{array}{cc} 1 & 0 \\ 0 & e^{\frac{\pi i}{3}} \end{array} \right), \quad D''_o := \left(\begin{array}{cc} e^{-\frac{2\pi i}{3}} & 0 \\ 0 & -1 \end{array} \right) \in GL(2,\mathcal{O}_{-3})$$

and observe that

$$D^2 D_o = D''_o, \quad D62D''_o = D'_o.$$

By its very definition,

$$H = \langle D, D_o \rangle < GL(2,\mathcal{O}_{-3})$$

is isomorphic to $H_{C6}(7)$. The equalities $\langle D, D'_o = D^2 D''_o \rangle = \langle D, D''_o \rangle$ and $\langle D, D''_o = D^2 D_o \rangle = \langle D, D_o \rangle$ conclude the proof of the proposition.

□

Proposition 38. *Let H be a finite subgroup of $GL(2,R)$,*

$$H \cap SL(2,R) = \langle g_1, g_2 \mid g_1^2 = g_2^2 = -I_2, \quad g_2 g_1 = -g_1 g_2 \rangle \simeq \mathbb{Q}_8,$$

and $h_o \in H$ be an element of order r with $\det(H) = \langle \det(h_o) \rangle \simeq \mathbb{C}_s$ *and eigenvalues $\lambda_1(h_o), \lambda_2(h_o)$. Then H is isomorphic to some $H_{\mathbb{Q}8}(i)$, $1 \leq i \leq 9$, where*

$$H_{\mathbb{Q}8}(1) = \langle g_1, g_2, iI_2 \mid g_1^2 = g_2^2 = -I_2, \quad g_2 g_1 = -g_1 g_2 \rangle$$

is of order 16 *with* $R = \mathbb{Z}[i]$,

$$H_{\mathbb{Q}8}(2) = \langle g_1, g_2, h_o \ \mid \ g_1^2 = g_2^2 = -I_2, \quad h_o^2 = I_2, \quad g_2 g_1 = -g_1 g_2, \\ h_o g_1 h_o^{-1} = -g_1, \quad h_o g_2 h_o^{-1} = -g_2 \rangle$$

is of order 16 *with* $R = \mathbb{Z}[i]$, $\lambda_1(h_o) = -1$, $\lambda_2(h_o) = 1$,

$$H_{\mathbb{Q}8}(3) = \langle g_1, g_2, h_o \ \mid \ g_1^2 = g_2^2 = h_o^4 = -I_2, \quad g_2 g_1 = -g_1 g_2, \\ h_o g_1 h_o^{-1} = g_2, \quad h_o g_2 h_o^{-1} = -g_1 \rangle$$

is of order 16 *with* $R = \mathcal{O}_{-2}$, $\lambda_1(h_o) = e^{\frac{\pi i}{4}}$, $\lambda_2(h_o) = e^{\frac{3\pi i}{4}}$, $h_o^2 = \pm g_1 g_2$,

$$H_{\mathbb{Q}8}(4) = \langle g_1, g_2, h_o \ \mid \ g_1^2 = g_2^2 = -I_2, \quad h_o^2 = I_2, \quad g_2 g_1 = -g_1 g_2, \\ h_o g_1 h_o^{-1} = g_2, \quad h_o g_2 h_o^{-1} = g_1 \rangle$$

is of order 16 *with* $R = R_{-2,f}$, $\lambda_1(h_o) = -1$, $\lambda_2(h_o) = 1$,

$$H_{\mathbb{Q}8}(5) = \langle g_1, g_2 \rangle \times \langle e^{\frac{2\pi i}{3}} \rangle \simeq \mathbb{Q}_8 \times \mathbb{C}_3$$

is of order 24 *with* $R = \mathcal{O}_3$,

$$H_{\mathbb{Q}8}(6) = \langle g_1, g_2, h_o \ \mid \ g_1^2 = g_2^2 = -I_2, \quad h_o^3 = I_2, \quad g_2 g_1 = -g_1 g_2, \\ h_o g_1 h_o^{-1} = g_2, \quad h_o g_2 h_o^{-1} = g_1 g_2 \rangle$$

is of order 24 *with* $R = \mathcal{O}_{-3}$, $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o) = 1$,

$$H_{\mathbb{Q}8}(7) = \langle g_1, g_2, h_o \ \mid \ g_1^2 = g_2^2 = h_o^4 = -I_2, \quad g_2 g_1 = -g_1 g_2, \\ h_o g_1 h_o^{-1} = -g_1, \quad h_o g_2 h_o^{-1} = -g_2 \rangle$$

is of order 32 *with* $R = \mathbb{Z}[i]$, $\lambda_1(h_o) = e^{\frac{3\pi i}{4}}$, $\lambda_2(h_o) = e^{-\frac{\pi i}{4}}$,

$$H_{\mathbb{Q}8}(8) = \langle g_1, g_2, h_o \ \mid \ g_1^2 = g_2^2 = h_o^4 = -I_2, \quad g_2 g_1 = -g_1 g_2, \\ h_o g_1 h_o^{-1} = g_2, \quad h_o g_2 h_o^{-1} = g_1 \rangle$$

is of order 32 *with* $R = \mathbb{Z}[i]$, $\lambda_1(h_o) = e^{\frac{3\pi i}{4}}$, $\lambda_2(h_o) = e^{-\frac{pii}{4}}$,

$$H_{\mathbb{Q}8}(9) = \langle g_1, g_2, h_o \ \mid \ g_1^2 = g_2^2 = -I_2, \quad h_o^4 = I_2, \quad g_2 g_1 = -g_1 g_2, \\ h_o g_1 h_o^{-1} = g_2, \quad h_o g_1 h_0^{-1} = g_2 \rangle$$

is of order 32 *with* $R = \mathbb{Z}[i]$, $\lambda_1(h_o) = i$, $\lambda_2(h_o) = 1$.

There exist subgroups

$$H_{\mathbb{Q}8}(1), \quad H_{\mathbb{Q}8}(2), \quad H_{\mathbb{Q}8}(9) < GL(2, \mathbb{Z}[i]), \quad {}_{\mathbb{Q}8}(5) < GL(2, \mathcal{O}_{-3}),$$

as well as subgroups

$$H_{\mathbb{Q}8}^o(4) < GL(2, \mathbb{Q}(\sqrt{-2})), \quad H_{\mathbb{Q}8}^o(6) < GL(2, \mathbb{Q}(\sqrt{-3})),$$

$$H_{\mathbb{Q}8}^o(3), \quad H_{\mathbb{Q}8}^o(7), \quad H_{\mathbb{Q}8}^o(8) < GL(2, \mathbb{Q}(\sqrt{2}, i)),$$

such that $H_{\mathbb{Q}8}^o(j) \simeq H_{\mathbb{Q}8}(j)$ *for* $j \in \{3, 4, 6, 7, 8\}$.

Proof. According to Lemmas 26 and 27, the group $H = \langle g_1, g_2 \rangle \langle h_o \rangle$ with $\det(H) = \langle \det(h_o) \rangle \simeq \mathbb{C}_s$ is completely determined by the order r of h_o and the elements $x_j = h_o g_j h_o^{-1} \in \langle g_1, g_2 \rangle$, $1 \leq j \leq 2$ of order 4. Bearing in mind that $\langle g_1, g_2 \rangle^{(4)} = \{\pm g_1, \pm g_2, \pm g_1 g_2\}$, let us split the considerations into Case A with $x_j \in \{\pm g_j\}$ for $1 \leq j \leq 2$, Case B with $h_o g_1 h_o^{-1} = g_2$, $h_o g_2 h_o^{-1} = \varepsilon g_1$ for some $\varepsilon = \pm 1$ and Case C with $h_o g_1 h_o^{-1} = g_2$, $h_o g_2 h_o^{-1} = \varepsilon g_1 g_2$ for some $\varepsilon = \pm 1$.

In the case A, let us distinguish between Case A1 with $x_j = h_o g_j h_o^{-1} = g_j$ for $\forall 1 \leq j \leq 2$ and Case A2 with $x_k = h_o g_k h_o^{-1} = -g_k$ for some $k \in \{1, 2\}$. Note that if $h_o g_j = g_j h_o$ for $\forall 1 \leq j \leq 2$ then $h_o \in H$ is a scalar matrix. Indeed, if h_o has diagonal form

$$D_o = S^{-1} h_o S = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

for some $S \in GL(2, \mathbb{Q}(\sqrt{-d}, \lambda_1))$ and

$$D_j = S^{-1} g_j S = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{-d}, \lambda_1)) \quad \text{for} \quad 1 \leq j \leq 2 \quad \text{then}$$

$$D_o D_j D_o^{-1} = \begin{pmatrix} a_j & \frac{\lambda_1}{\lambda_2} b_j \\ \frac{\lambda_2}{\lambda_1} c_j & -a_j \end{pmatrix} \tag{14}$$

coincides with D_j if and only if

$$\left| \begin{array}{l} \left(\frac{\lambda_1}{\lambda_2} - 1 \right) b_j = 0 \\ \left(\frac{\lambda_2}{\lambda_1} - 1 \right) c_j = 0 \end{array} \right. .$$

The assumption $\lambda_1(h_o) = \lambda_1 \neq \lambda_2 = \lambda_2(h_o)$ implies $b_j = c_j = 0$ for $\forall 1 \leq j \leq 2$, so that

$$D_1 = \pm i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad D_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are diagonal. In particular, D_1 commutes with D_2, contrary to $D_2 D_1 = -D_1 D_2$. Thus, in the Case A1 with $h_o g_j = g_j h_o$ for $\forall 1 \leq j \leq 2$ the matrix $h_o \in H$ is to be scalar. By Propositions 16, 17, 18, 19, 20, 21, 22, the scalar matrices $h_o \in GL(2, R) \setminus SL(2, R)$ are

$$h_o = i I_2 \in GL(2, \mathbb{Z}[i]) \quad \text{of order} \quad 4,$$

$$h_o = e^{\pm \frac{2\pi i}{3}} I_2 \in GL(2, \mathbb{Z}[i]) \quad \text{of order} \quad 3 \quad \text{and}$$

$$h_o = e^{\pm \frac{\pi i}{3}} I_2 \in GL(2, \mathbb{Z}[i]) \quad \text{of order} \quad 6.$$

For any subgroup

$$\mathbb{Q}_8 \simeq \langle g_1, g_2 \mid g_1^2 = g_2^2 = -I_2, \quad g_2 g_1 = -g_1 g_2 \rangle < SL(2, \mathbb{Z}[i])$$

one obtains a group

$$H_{Q8}(1) = \langle g_1, g_2, iI_2 \ \mid \ g_1^2 = g_2^2 = -I_2, \quad g_2 g_1 = -g_1 g_2 \rangle < GL(2, \mathbb{Z}[i])$$

of order 16. As far as $-I_2 \in H \cap SL(2,R)$, the group H contains $e^{\frac{2\pi i}{3}} I_2$ if and only if it contains $-e^{\frac{2\pi i}{3}} I_2 = e^{-\frac{\pi i}{3}} I_2$. Since $\langle g_1, g_2 \rangle \cap \langle e^{\frac{2\pi i}{3}} I_2 \rangle = \{I_2\}$, any finite group H with $e^{\frac{2\pi i}{3}} I_2 \in H$ is a subgroup of $GL(, \mathcal{O}_{-3})$ of the form

$$H_{Q8}(5) = \langle g_1, g_2 \rangle \times \langle e^{\frac{2\pi i}{2}} I_2 \rangle \simeq \mathbb{Q}_8 \times \mathbb{C}_3.$$

These deplete $H = [H \cap SL(2,R)] \langle h_o \rangle = \langle g_1, g_2 \rangle \langle h_o \rangle \simeq \mathbb{Q}_8 \mathbb{C}_s$ of Case A1.

In the Case A2, one can assume that $h_o g_1 h_o^{-1} = -g_1$. If $h_o g_2 h_o = g_2$ then $h_o(g_1 g_2) h_o^{-1} = -g_1 g_2$, so that there is no loss of generality in supposing $h_o g_2 h_o^{-1} = -g_2$. By Lemma 33(iv), $h_o g_1 h_o^{-1} = -g_1$ requires $\lambda_1(h_o) = i e^{\frac{\pi i}{s}}$, $\lambda(h_o) = -i e^{\frac{\pi i}{s}}$, whereas $\frac{\lambda_1(h_o)}{\lambda_2(h_o)} + 1 = \frac{\lambda_2(h_o)}{\lambda_1(h_o)} + 1 = 0$. If

$$D_o = S^{-1} h_o S = \left(\begin{array}{cc} i e^{\frac{\pi i}{s}} & 0 \\ 0 & -i e^{\frac{\pi i}{s}} \end{array} \right) \in GL(2, \mathbb{Q}(\sqrt{-d}, i e^{\frac{\pi i}{s}}))$$

is a diagonal form of $h_o \in H$ and

$$D_j = S^{-1} g_j S = \left(\begin{array}{cc} a_j & b_j \\ c_j & -a_j \end{array} \right) \in GL(2, \mathbb{Q}(\sqrt{-d}, i e^{\frac{\pi i}{s}})) \quad \text{for} \quad 1 \leq j \leq 2,$$

then $D_o D_j D_o^{-1} = -D_j$ for $1 \leq j \leq 2$ is equivalent to $a_1 = a_2 = 0$. As a result, $b_j \neq 0$ and $c_j = -\frac{1}{b_j}$. Straightforwardly, $D_2 D_1 = -D_1 D_2$ amounts to $2a_1 a_2 + b_1 c_2 + b_2 c_1 = 0$, whereas $\frac{b_2}{b_1} + \frac{b_1}{b_2} = 0$. Denoting $\beta := \frac{b_2}{b_1} \in \mathbb{Q}(\sqrt{-d}, i e^{\frac{\pi i}{s}})$, one computes that $\beta = \pm i \in \mathbb{Q}(\sqrt{-d}, i e^{\frac{\pi i}{s}})$. Then by Lemma 28 there follows $s = 2$ with $d = 1$ or $s = 4$. For $s = 2$ one has $\lambda_1(h_o) = -1$, $\lambda_2(h_o) = 1$, so that $h_o \in H$ is of order 2 and

$$H = H_{Q8}(2) = \langle g_1, g_2, h_o \ \mid \ g_1^2 = g_2^2 = -I_2, \quad h_o^2 = I_2,$$

$$g_2 g_1 = -g_1 g_2, \quad h_o g_1 h_o^{-1} = -g_1, \quad h_o g_2 h_o^{-1} = -g_2 \rangle$$

is a subgroup of $GL(2, R_{-1,f})$ of order 16. Note that

$$h_o = \left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right), \quad g_1 = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \quad \text{and} \quad g_2 = \left(\begin{array}{cc} 0 & i \\ i & 0 \end{array} \right)$$

generate a subgroup of $GL(2, \mathbb{Z}[i])$, isomorphic to $H_{Q8}(2)$. In the case of $s = 4$, the element $h_o \in H$ with eigenvalues $\lambda_1(h_o) = e^{\frac{3\pi i}{4}}$, $\lambda_2(h_o) = e^{-\frac{\pi i}{4}}$ is of order 8 and

$$H = H_{Q8}(7) = \langle g_1, g_2, h_o \ \mid \ g_1^2 = g_2^2 = h_o^4 = -I_2, \quad g_2 g_1 = -g_1 g_2$$

$$h_o g_1 h_o^{-1} = -g_1, \quad h_o g_2 h_o^{-1} = -g_2 \rangle$$

is a subgroup of $GL(2, \mathbb{Z}[]i)$ of order 32. The matrices

$$D_o = \begin{pmatrix} e^{\frac{3\pi i}{4}} & 0 \\ 0 & e^{-\frac{\pi i}{4}} \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

generate a subgroup $H^o_{Q8}(7)$ of $GL(2, \mathbb{Q}(\sqrt{2}, i))$, isomorphic to $H_{Q8}(7)$. That concludes the Case A.

In the Case B, let us observe that $h_o g_1 h_o^{-1} = g_2$ and $h_o g_2 h_o^{-1} = \varepsilon g_1$ imply $h_o^2 g_1 h_o^{-2} = \varepsilon g_1$ and $h_o^2 g_2 h_o^{-2} = \varepsilon g_2$. If $h_o^2 \in H \cap SL(2, R)$ then $\det(h_o) = \lambda_1(h_o)\lambda_2(h_o) = -1$. The matrices

$$D_o = S^{-1} h_o S = \begin{pmatrix} \lambda_1(h_o) & 0 \\ 0 & \lambda_2(h_o) \end{pmatrix} \quad \text{and} \quad D_j = S^{-1} g_j S = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix}$$

with $a_j^2 + b_j c_j = -1$, $2a_1 a_2 + b_1 c_2 + b_2 c_1 = 0$ satisfy $D_o D_1 D_o^{-1} = D_2$ if and only if

$$D_2 = \begin{pmatrix} a_1 & -\lambda_1^2(h_o) b_1 \\ -\frac{c_1}{\lambda_1^2(h_o)} & -a_1 \end{pmatrix}.$$

Then $D_o D_2 D_o^{-1} = \varepsilon D_1$ is equivalent to

$$\left| \begin{array}{r} (\varepsilon - 1) a_1 = 0 \\ (\varepsilon - \lambda_1^4(h_o)) b_1 = 0 \\ \left(\varepsilon - \frac{1}{\lambda_1^4(h_o)} \right) c_1 = 0 \end{array} \right. .$$

According to $\det(D_1) = 1 \neq 0$, there follows $(\varepsilon - 1)(\varepsilon - \lambda_1^4(h_o)) = 0$. In the case of $-1 = \varepsilon = \lambda_1^4(h_o)$, Proposition 16 implies that $R = \mathcal{O}_{-2}$, h_o is of order 8 and

$$D_o = S^{-1} h_o S = \begin{pmatrix} e^{\frac{\pi i}{4}} & 0 \\ 0 & e^{\frac{3\pi i}{4}} \end{pmatrix} \in GL(2, \mathbb{Q}(\sqrt{2}, i)).$$

Moreover,

$$D_1 = \begin{pmatrix} 0 & b_1 \\ -\frac{1}{b_1} & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & -ib_1 \\ -\frac{i}{b_1} & 0 \end{pmatrix},$$

so that the subgroup

$$H_{Q8}(3) = \langle g_1, g_2, h_o \mid g_1^2 = g_2^2 = h_o^4 = -I_2, \quad g_2 g_1 = -g_1 g_2,$$

$$h_o g_1 h_o^{-1} = g_2, \quad h_o g_2 h_o^{-1} = -g_1 \rangle < GL(2, \mathcal{O}_{-2})$$

of order 16 is conjugate to the subgroup

$$H^o_{Q8}(3) = \langle D_o = \begin{pmatrix} e^{\frac{\pi i}{4}} & 0 \\ 0 & e^{\frac{3\pi i}{4}} \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$D_2 = \left(\begin{array}{cc} 0 & -i \\ -i & 0 \end{array} \right) \rangle < GL(2, \mathbb{Q}(\sqrt{2}, i)).$$

For $\varepsilon = 1$ and $\lambda_1^4(h_o) \neq 1$ there follows

$$D_2 = D_1 = \pm \left(\begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right),$$

which contradicts $D_2 D_1 = -D_1 D_2$. Therefore $\varepsilon = 1$ implies $\lambda_1^4(h_o) = 1$ and

$$D_o = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$$

is of order 2, since all $h_o \in H$ of order 4 with $\det(h_o) = -1$ are scalar matrices and commute with g_1, g_2. In such a way, one obtains the group

$$H_{Q8}(4) = \langle g_1, g_2, h_o \mid g_1^2 = g_2^2 = -I_2, \quad h_o^2 = I_2, \quad g_2 g_1 = -g_1 g_2,$$

$$h_o g_1 h_o^{-1} = g_2, \quad h_o g_2 h_o^{-1} = g_1 \rangle$$

of order 16. The matrices

$$D_1 = \left(\begin{array}{cc} a_1 & b_1 \\ c_1 & -a_1 \end{array} \right) \quad \text{and} \quad D_2 = \left(\begin{array}{cc} a_1 & -b_1 \\ -c_1 & -a_1 \end{array} \right)$$

generate a subgroup of $GL(2, \mathbb{Q}(\sqrt{-d}))$, isomorphic to $\mathbb{Q}_8$ exactly when $a_1 = \pm \frac{\sqrt{-2}}{2} \in \mathbb{Q}(\sqrt{-d})$ and $c_1 = -\frac{1}{b_1}$ for some $b_1 \in \mathbb{Q}(\sqrt{-d})^*$. Therefore $H_{Q8}(4)$ occurs only as a subgroup of $GL(2, R_{-2,f})$ and

$$D_o = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad D_1 = \left(\begin{array}{cc} \frac{\sqrt{-2}}{2} & 1 \\ -\frac{1}{2} & -\frac{\sqrt{-2}}{2} \end{array} \right), \quad D_2 = \left(\begin{array}{cc} \frac{\sqrt{-2}}{2} & -1 \\ \frac{1}{2} & -\frac{\sqrt{-2}}{2} \end{array} \right)$$

generate a subgroup $H_{Q8}^o(4)$ of $GL(2, \mathbb{Q}(\sqrt{-2}))$, isomorphic to $H_{Q8}(4)$. That concludes the Case B with $h_o^2 \in H \cap SL(2, R)$.

Let us suppose that $h_o g_1 h_o^{-1} = g_2$, $h_o g_2 h_o^{-1} = \varepsilon g_1$ with $\det(h_o) \in R^*$ of order $s > 2$. Note that $h_o^s \in H \cap SL(2, R) = \langle g_1, g_2 \rangle$ implies $h_o^s g_j h_o^{-s} \in \{\pm g_j\}$ for $\forall 1 \leq j \leq 2$, so that $s \in \{4, 6\}$ has to be an even natural number. The group

$$H' = \langle g_1, g_2, h_o^2 \mid g_1^2 = g_2^2 = -I_2, \quad h_o^r = I_2, g_2 g_1 = -g_1 g_2,$$

$$h_o^2 g_1 h_o^{-2} = \varepsilon g_1, \quad h_o^2 g_2 h_o^{-2} = \varepsilon g_2 \rangle$$

with $h_o^2 \in GL(2, R) \setminus SL(2, R)$, $H' \cap SL(2, R) = \langle g_1, g_2 \rangle \simeq \mathbb{Q}_8$ is of order $8\frac{s}{2} \in \{16, 24\}$ and satisfies the assumptions of Case A. Thus, for $\varepsilon = 1$ one has $h_o^2 = iI_2$ or $h_o^2 = e^{\frac{2\pi i}{3}} I_2$. If $h_o^2 = iI_2$ then $h_o \in H$ is of order 8 with $\det(h_o) = \pm i$. Therefore $R = \mathbb{Z}[i]$

and h_o has eigenvalues $\lambda_1(h_o) = e^{\frac{3\pi i}{4}}$, $\lambda_2(h_o) = e^{-\frac{\pi i}{4}}$ with $\frac{\lambda_1(h_o)}{\lambda_2(h_o)} = \frac{\lambda_2(h_o)}{\lambda_1(h_o)} = -1$. The relations $D_o D_1 D_o^{-1} = D_2$, $D_o D_2 D_o^{-1} = D_1$ on the diagonal form D_o of h_o hold for

$$D_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} a_1 & -b_1 \\ -c_1 & -a_1 \end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{2}, i)).$$

The group $\langle D_1, D_2 \rangle$ is isomorphic to $\mathbb{Q}_8$ if and only if $a_1 = \pm \frac{\sqrt{-2}}{2}$ and $c_1 = -\frac{1}{b_1}$ for some $b_1 \in \mathbb{Q}(\sqrt{2}, i)$. In such a way, one obtains the group

$$H_{Q8}(8) = \langle g_1, g_2, h_o \ \ | g_1^2 = g_2^2 = h_o^4 = -I_2, \quad g_2 g_1 = -g_1 g_2,$$

$$h_o g_1 h_o^{-1} = g_2, \quad h_o g_2 h_o^{-1} = g_1 \rangle$$

for $R = \mathbb{Z}[i]$. Note that $H_{Q8}(8)$ is of order 32 and has a conjugate $H_{Q8}^o(8) = \langle D_1, D_2, D_o \rangle < GL(2, \mathbb{Q}(\sqrt{2}, i))$. If $h_o^2 = e^{\frac{2\pi i}{3}} I_2$ then $R = \mathcal{O}_{-3}$ and $h_o \in H$ is of order 6 with $\det(h_o) = e^{\pm \frac{2\pi i}{3}}$. According to $h_o g_1 h_o^{-1} = g_2 \neq g_1$, h_o is not a scalar matrix, so that $\lambda_1(h_o) = e^{-\frac{\pi i}{3}}$, $\lambda_2(h_o) = -1$ for $\det(h_o) = e^{\frac{2\pi i}{3}}$. Now, $D_o D_1 D_o^{-1} = D_2$ is tantamount to

$$D_2 = \begin{pmatrix} a_1 & e^{\frac{2\pi i}{3}} b_1 \\ & \\ e^{-\frac{2\pi i}{3}} c_1 & -a_1 \end{pmatrix}$$

and $D_o D_2 D_o^{-1} = D_1$ reduces to

$$\left| \begin{array}{l} \left(1 - e^{-\frac{2\pi i}{3}}\right) b_1 = 0 \\ \left(1 - e^{\frac{2\pi i}{3}}\right) c_1 = 0 \end{array} \right. .$$

As a result, $b_1 = c_1$ and

$$D_1 = D_2 = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

commute with each other. Thus, there is no group H of Case B with $h_o^2 = e^{\frac{2\pi i}{3}} I_2$. If $h_o g_1 h_o^{-1} = g_2$, $h_o g_2 h_o^{-1} = -g_1$ and $h_o^2 \notin \langle g_1, g_2 \rangle$ then

$$H' = \langle g_1, g_2, h_o^2 \ \ | \ \ g_1^2 = g_2^2 = -I_2, \quad h_o^r = I_2, \quad g_2 g_1 = -g_1 g_2,$$

$$h_o^2 g_1 h_o^{-2} = -g_1, \quad h_o^2 g_2 h_o^{-2} = -g_2 \rangle$$

is isomorphic to $H_{Q8}(2)$ or $H_{Q8}(7)$, according to the considerations for Case A. More precisely, if $H' \simeq H_{Q8}(2)$ then h_o of order 4 has $\det(h_o) = \pm i$ and $R = \mathbb{Z}[i]$. Due to $-I_2 \in \langle g_1, g_2 \rangle$, one can assume that

$$D_o = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $D_o D_1 D_o^{-1} = D_2$ requires

$$D_2 = \begin{pmatrix} a_1 & ib_1 \\ -ic_1 & -a_1 \end{pmatrix},$$

so that $D_o D_2 D_o^{-1} = -D_1$ results in $a_1 = 0$. Bearing in mind that $\det(D_1) = \det(D_2) = 1$, one concludes that

$$D_1 = \begin{pmatrix} 0 & b_1 \\ -\frac{1}{b_1} & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & ib_1 \\ \frac{i}{b_1} & 0 \end{pmatrix}.$$

For $b_1 = 1$, one obtains a subgroup $\langle D_1, D_2, D_o \rangle$ of $GL(2, \mathbb{Z}[i])$, isomorphic to

$$H_{Q8}(9) = \langle g_1, g_2, h_o \mid g_1^2 = g_2^2 = -I_2, \quad h_o^4 = I_2, \quad g_2 g_1 = -g_1 g_2,$$

$$h_o g_1 h_o^{-1} = g_2, \quad h_o g_2 h_o^{-1} = -g_1 \rangle < GL(2, \mathbb{Z}[i]).$$

Since $\det(h_o) = i$ is of order $s = 4$, the group $H_{Q8}(9)$ is of order 32. If $H' = \langle g_1, g_2, h_o^2 \rangle \simeq H_{Q8}(7)$ then $h_o \in H$ is to be of order 16, since h_o^2 is of order 8. The lack of $h_o \in GL(2, R)$ of order 16 reveals that the groups $H_{Q8}(3)$, $H_{Q8}(4)$, $H_{Q8}(8)$, $H_{Q8}(9)$ deplete Case B.

There remains to be considered Case C with $h_o g_1 h_o^{-1} = g_2$, $h_o g_2 h_o^{-1} = \varepsilon g_1 g_2$, $h_o (g_1 g_2) h_o^{-1} = \varepsilon g_1$ for some $\varepsilon = \pm 1$. Note that $h_o^2 g_1 h_o^{-2} = \varepsilon g_1 g_2$, $h_o^2 g_2 h_o^{-2} = g_1$, $h_o^3 g_1 h_o^{-3} = g_1$, $h_o^3 g_2 h_o^{-3} = g_2$ require the divisibility of s by 3, as far as $\langle g_j \rangle$ are normal subgroups of $\langle g_1, g_2 \rangle$ and $h_o^s \in \langle g_1, g_2 \rangle$. In other words, $s \in \{3, 6\}$ and $R = \mathcal{O}_{-3}$. The non-scalar matrices $h_o \in GL(2, \mathcal{O}_{-3})$ with $\det(h_o) = e^{\frac{2\pi i}{3}}$ have eigenvalues $\{\lambda_1(h_o), \lambda_2(h_o)\} = \left\{e^{\frac{2\pi i}{3}}, 1\right\}$, $\left\{e^{-\frac{\pi i}{3}}, -1\right\}$ or $\left\{e^{\frac{5\pi i}{6}}, e^{-\frac{\pi i}{6}}\right\}$. If h_o is of order 3 or 6 then $\frac{\lambda_1(h_o)}{\lambda_2(h_o)} = e^{\frac{2\pi i}{3}}$ and $D_o D_1 D_o^{-1} = D_2$ specifies that

$$D_2 = \begin{pmatrix} a_1 & e^{\frac{2\pi i}{3}} b_1 \\ e^{-\frac{2\pi i}{3}} c_1 & -a_1 \end{pmatrix}.$$

Now, $2a_1 a_2 + b_1 c_2 + b_2 c_1 = 0$ reduces to $2a_1^2 = b_1 c_1$ and $a_1^2 + b_1 c_1 = -1$ requires $a_1 = \pm \frac{-3}{3}$, $c_1 = -\frac{2}{3b_1}$ for some $b_1 \in \mathbb{Q}(\sqrt{-3})^*$. Replacing, eventually, D_j by D_j^3, one has

$$D_1 = \begin{pmatrix} \frac{\sqrt{-3}}{3} & b_1 \\ -\frac{2}{3b_1} & -\frac{\sqrt{-3}}{3} \end{pmatrix}, \quad D_2 = \begin{pmatrix} \frac{\sqrt{-3}}{3} & e^{\frac{2\pi i}{3}} b_1 \\ -\frac{2e^{-\frac{2\pi i}{3}}}{3b_1} & -\frac{\sqrt{-3}}{3} \end{pmatrix}.$$

Now,

$$D_1 D_2 = \begin{pmatrix} \frac{\sqrt{-3}}{3} & e^{-\frac{2\pi i}{3}} b_1 \\ -\frac{2e^{\frac{2\pi i}{3}}}{3b_1} & -\frac{\sqrt{-3}}{3} \end{pmatrix}$$

and $D_oD_2D_o^{-1} = \varepsilon D_1D_2$ holds for $\varepsilon = 1$. Thus,

$$H^o_{Q8}(6) = \langle D_1, D_2, D_o\rangle < GL(2, \mathbb{Q}(\sqrt{-3}))$$

is conjugate to

$$H_{Q8}(6) = \langle g_1, g_2, h_o \ | \ g_1^2 = g_2^2 = -I_2, \ \ h_o^3 = I_2, \ \ g_2g_1 = -g_1g_2$$
$$h_og_1h_o^{-1} = g_2, \ \ h_og_2h_o^{-1} = g_1g_2\rangle < GL(2, \mathcal{O}_{-3})$$

of order 24 with $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o) = 1$ or to

$$H = \langle g_1, g_2, h_o \ | \ g_1^2 = g_2^2 = -I_2, \ \ h_o^3 = -I_2, g_2g_1 = -g_1g_2, \tag{15}$$
$$h_og_1h_o^{-1} = g_2, \ \ h_og_2h_o^{-1} = g_1g_2\rangle < GL(2, \mathcal{O}_{-3})$$

of order 24 with $\lambda_1(h_o) = e^{-\frac{\pi i}{3}}$, $\lambda_2(h_o) = -1$. Due to $-I_2 \in \langle g_1, g_2\rangle$, the presence of $h_o \in H$ of order 6 with $\det(H) = \langle \det(h_o)\rangle \simeq \mathbb{C}_3$ is equivalent to the existence of $-h_o \in H$ of order 3 with $\det(H) = \langle\det(-h_o)\rangle \simeq \mathbb{C}_3$ and H from (15) is isomorphic to $H_{Q8}(6)$. If h_o has diagonal form

$$D_o = \begin{pmatrix} e^{\frac{5\pi i}{6}} & 0 \\ 0 & e^{-\frac{\pi i}{6}} \end{pmatrix} \in GL(2, \mathbb{Q}(\sqrt{-3}))$$

of order 12 with $\det(D_o) = e^{\frac{2\pi i}{3}}$, $\frac{\lambda_1(h_o)}{\lambda_2(h_o)} = \frac{\lambda_2(h_o)}{\lambda_1(h_o)} = -1$, then $D_oD_1D_o^{-1} = D_2$ implies that

$$D_2 = \begin{pmatrix} a_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix}$$

with $a_1^2 = b_1c_1 = -\frac{1}{2}$. Therefore, $a_1 = \pm\frac{\sqrt{-2}}{2} \in GL(2, \mathbb{Q}(\sqrt{-3}))$, which is an absurd. If $h_og_1h_o^{-1} = g_2$, $h_og_2h_o^{-1} = \varepsilon g_1g_2$ and $s = 6$ then $h_o \in H$ is of order 6, according to Proposition 19. Now $H'' = \langle g_1, g_2, h_o^3\rangle < GL(2, R)$ with $h_o^3 \not\in \langle g_1, g_2\rangle$ is subject to Case A with a scalar matrix $h_o \in H$, according to $h_o^3g_1h_o^{-3} = g_1$, $h_o^3g_2h_o^{-3} = g_2$. If $h_o^3 = iI_2$ then h_o is of order $r = 12$. The assumption $h_o^3 = e^{\frac{2\pi i}{3}}I_2$ holds for h_o of order $r = 9$. Both contradict to $r = 6$ and establish that any subgroup $H < GL(2, R)$ with $H \cap SL(2, R) \simeq \mathbb{Q}_8$ is isomorphic to $H_{Q8}(i)$ for some $1 \leq i \leq 9$.

□

Proposition 39. *Let H be a finite subgroup of $GL(2, R)$,*

$$H \cap SL(2, R) = K_7 = \langle g_1, g_4, \ \ g_1^2 = g_4^3 = -I_2, \ \ g_1g_4g_1^{-1} = g_4^{-1}\rangle \simeq \mathbb{Q}_{12}$$

and $h_o \in H$ be an element of order r with $\det(H) = \langle\det(h_o)\rangle \simeq \mathbb{C}_s$ and eigenvalues $\lambda_1(h_o)$, $\lambda_2(h_o)$. Then H is isomorphic to $H_{Q12}(i)$ for some $1 \leq i \leq 10$, where

$$H_{Q12}(1) = \langle g_1, g_4, h_o = iI_2 \ \ | \ \ g_1^2 = g_4^3 = -I_2, \ \ g_1g_4g_1^{-1} = g_4^{-1}\rangle$$

is of order 24 *with* $R - \mathbb{Z}[i]$,

$$H_{Q12}(2) = \langle g_1, g_4, h_o \;\mid\; g_1^2 = g_4^3 = -I_2, \quad h_o^6 = I_2, \quad g_1 g_4 g_1^{-1} = g_4^{-1},$$

$$h_o g_1 h_o^{-1} = g_1 g_4, \quad h_o g_4 h_o^{-1} = g_4 \rangle$$

of order 24, *with* $R = \mathcal{O}_{-3}$, $\lambda_1(h_o) = e^{\frac{\pi i}{3}}$, $\lambda_2(h_o) = e^{\frac{2\pi i}{3}}$,

$$H_{Q12}(3) = \langle g_1, g_4, h_o \;\mid\; g_1^2 = g_4^3 = h_o^6 = -I_2, g_1 g_4 g_1^{-1} = g_4^{-1},$$

$$h_o g_1 h_o^{-1} = g_1 g_4^2, \quad h_o g_4 h_o^{-1} = g_4 \rangle$$

is of order 24 *with* $R = \mathcal{O}_{-3}$, $\lambda_1(h_o) = e^{\frac{\pi i}{6}}$, $\lambda_2(h_o) = e^{\frac{5\pi i}{6}}$,

$$H_{Q12}(4) = \langle g_1, g_4, h_o \;\mid\; g_1^2 = g_4^3 = -I_2, \quad h_o^2 = I_2, \quad g_1 g_4 g_1^{-1} = g_4^{-1},$$

$$h_o g_1 h_o^{-1} = -g_1, \quad h_o g_4 h_o^{-1} = g_4 \rangle$$

is of order 24 *with* $\lambda_1(h_o) = -1$, $\lambda_2(h_o) = 1$,

$$H_{Q12}(5) = \langle g_1, g_4, h_o = e^{\frac{2\pi i}{3}} I_2 \;\mid\; g_1^2 = g_4^3 = -I_2, \quad g_1 g_4 g_1^{-1} = g_4^{-1} \rangle$$

is of order 36 *with* $R = \mathcal{O}_{-3}$,

$$H_{Q12}(6) = \langle g_1, g_4, h_o \;\mid\; g_1^2 = g_4^3 = -I_2, \quad h_o^3 = I_2, \quad g_1 g_4 g_1^{-1} = g_4^{-1},$$

$$h_o g_1 h_o^{-1} g_1 g_4^2, \quad h_o g_4 h_o^{-1} = g_4 \rangle$$

is of order 36 *with* $R = \mathcal{O}_{-3}$, $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o) = 1$,

$$H_{Q12}(7) = \langle g_1, g_4, h_o \;\mid\; g_1^2 = g_4^3 = h_o^6 = -I_2, \quad g_1 g_4 g_1^{-1} = g_4^{-1},$$

$$h_o g_1 h_o^{-1} = -g_1, \quad h_o g_4 h_o^{-1} = g_4 \rangle$$

is of order 36 *with* $R = \mathcal{O}_{-3}$, $\lambda_1(h_o) = e^{-\frac{\pi i}{6}}$, $\lambda_2(h_o) = e^{\frac{5\pi i}{6}}$,

$$H_{Q12}(8) = \langle g_1, g_4, h_o \;\mid\; g_1^2 = g_4^3 = h_o^4 = -I_2, \quad g_1 g_4 g_1^{-1} = g_4^{-1},$$

$$h_o g_1 h_o^{-1} = -g_1, \quad h_o g_4 h_o^{-1} = g_4 \rangle$$

is of order 48 *with* $R = \mathbb{Z}[i]$, $\lambda_1(h_o) = e^{\frac{3pii}{4}}$, $\lambda_2(h_o) = e^{-\frac{\pi i}{4}}$,

$$H_{Q12}(9) = \langle g_1, g_4, h_o \;\mid\; g_1^2 = g_4^3 = -I_2, \quad h_o^6 = I_2, \quad g_1 g_4 g_1^{-1} = g_4^{-1},$$

$$h_o g_1 h_o^{-1} = g_1 g_4, \quad h_o g_4 h_o^{-1} = g_4 \rangle$$

is of order 72 *with* $R = \mathcal{O}_{-3}$, $\lambda_1(h_o) = 1$, $\lambda_2(h_o) = e^{\frac{\pi i}{3}}$,

$$H_{Q12}(10) = \langle g_1, g_4, h_o \;\mid\; g_1^2 = g_4^3 = -I_2, \quad h_o^6 = I_2, \quad g_1 g_4 g_1^{-1} = g_4^{-1},$$

$$h_o g_1 h_o^{-1} = -g_1, \quad h_o g_4 h_o^{-1} = g_4 \rangle$$

is of order 72 *with* $R = \mathcal{O}_{-3}$, $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o) = e^{-\frac{\pi i}{3}}$.

There exist subgroups

$$H_{Q12}(2), H_{Q12}(4), H_{Q12}(5), H_{Q12}(6), H_{Q12}(9), H_{Q12}(10) < GL(2, \mathcal{O}_{-3}),$$

as well as subgroups

$$H^o_{Q12}(1), H^o_{Q12}(3), H^o_{Q12}(7) < GL(2, \mathbb{Q}(\sqrt{3}, i)), \quad H^o_{Q12}(8) < GL(2, \mathbb{Q}(\sqrt{2}, \sqrt{3}, i))$$

with $H^o_{Q12}(j) \simeq H_{Q12}(j)$ *for* $j \in \{1, 3, 7, 8\}$.

Proof. According to Lemma 27, the groups $H = K_7\langle h_o\rangle$ with $\det(H) = \langle \det(h_o)\rangle \simeq \mathbb{C}_s$ are determined up to an isomorphism by the order r of h_o, the element $h_o g_1 h_o^{-1} \in K_7$ of order 4 and the element $h_o g_4 h_o^{-1} \in K_7$ of order 6. Let us denote by $K_7^{(m)}$ the set of the elements of K_7 of order m. Straightforwardly,

$$K_7^{(6)} = \{g_4, g_4^{-1}\}, \quad K_7^{(4)} = \{\pm g_1 g_4 \ \mid \ 0 \leq i \leq 3\}.$$

Inverting $g_1 g_4 g_1^{-1} = g_4^{-1}$, one obtains $g_1 g_4^{-1} g_1^{-1} = g_4$. If $h_o g_4 h_o^{-1} = g_4^{-1}$ then

$$(g_1 h_o) g_4 (g_1 h_o^{-1} = g_1 (h_o g_4 h_o^{-1}) g_1^{-1} = g_1 g_4^{-1} g_1^{-1} = g_4.$$

As far as $K_7 = \langle g_1, g_4, h_o\rangle = \langle g_1, g_4, g_1 h_o\rangle$, there is no loss of generality in assuming $h_o g_4 h_o^{-1} = g_4$.

We start the study of H by a realization of K_7 as a subgroup of the special linear group $SL(2, \mathbb{Q}(\sqrt{-d}, \sqrt{-3}))$. Let

$$D_4 = S^{-1} g_4 S = \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{pmatrix}$$

be a diagonal form of g_4 for some $S \in GL(2, \mathbb{Q}(\sqrt{-d}, \sqrt{-3}))$ and

$$D_1 = S^{-1} g_1 S = \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix} \quad \text{with} \quad a_1^2 + b_1 c_1 = -1.$$

Then

$$D_1 D_4 D_1^{-1} = \begin{pmatrix} -\sqrt{-3}a_1^2 + e^{-\frac{\pi i}{3}} & -\sqrt{-3}a_1 b_1 \\ -\sqrt{-3}a_1 c_1 & \sqrt{-3}a_1^2 + E^{\frac{\pi i}{3}} \end{pmatrix} \in SL(2, \mathbb{Q}(\sqrt{-d}, \sqrt{-3}))$$

coincides with D_4^{-1} if and only if

$$D_1 = \begin{pmatrix} 0 & b_1 \\ -b_1^{-1} & 0 \end{pmatrix} \quad \text{for some} \quad b_1 \in \mathbb{Q}(\sqrt{-d}, \sqrt{-3})^*.$$

That allows to compute explicitly

$$K_7^{(4)} = \left\{ \pm D_1 = \pm \left(\begin{array}{cc} 0 & b_1 \\ -b_1^{-1} & 0 \end{array} \right), \ \ \pm D_1 D_4 = \pm \left(\begin{array}{cc} 0 & e^{-\frac{\pi i}{3}} b_1 \\ -\left(e^{-\frac{\pi i}{3}} b_1 \right)^{-1} & 0 \end{array} \right), \right.$$

$$\left. \pm D_1 D_4^2 = \pm \left(\begin{array}{cc} 0 & e^{-\frac{2\pi i}{3}} b_1 \\ -\left(e^{-\frac{2\pi i}{3}} b_1 \right)^{-1} & 0 \end{array} \right) \right\},$$

$$K_7^{(4)} = \left\{ D_1 D_4^j = \left(\begin{array}{cc} 0 & e^{-\frac{j\pi i}{3}} b_1 \\ -\left(e^{-\frac{j\pi i}{3}} b_1 \right)^{-1} & 0 \end{array} \right) \ \Bigg| \ 0 \leq j \leq 5 \right\}.$$

Now, $D_o D_4 D_o^{-1} = D_4$ amounts to

$$D_o = \left(\begin{array}{cc} \lambda_1(h_o) & 0 \\ 0 & \lambda_2(h_o) \end{array} \right) \quad \text{and}$$

$$D_o D_1 D_o^{-1} = \left(\begin{array}{cc} 0 & \frac{\lambda_1(h_o)}{\lambda_2(h_o)} b_1 \\ -\left[\frac{\lambda_1(h_o)}{\lambda_2(h_o)} b_1 \right]^{-1} & 0 \end{array} \right) = \left(\begin{array}{cc} 0 & e^{-\frac{j\pi i}{3}} b_1 \\ -\left(e^{-\frac{j\pi i}{3}} b_1 \right)^{-1} & 0 \end{array} \right) = D_1 D_4^j$$

if and only if $\frac{\lambda_1(h_o)}{\lambda_2(h_o)} = e^{-\frac{j\pi i}{3}}$. Note that the ratio $\frac{\lambda_1(h_o)}{\lambda_2(h_o)}$ of the eigenvalues of h_o is determined up to an inversion and

$$\left\{ e^{-\frac{j\pi i}{3}} \ \mid \ 0 \leq j \leq 5 \right\} = \left\{ 1 = e^0, \ \ e^{\mp \frac{j\pi i}{3}}, \ \ -1 = e^{\pi i} \ \mid \ 1 \leq j \leq 2 \right\}.$$

For any $h_o \in H$ with $\frac{\lambda_1(h_o)}{\lambda_2(h_o)} = e^{\mp \frac{j\pi i}{3}}$, $0 \leq j \leq 3$ the group

$$H = \langle g_1, g_4, h_o \ \mid \ g_1^2 = g_4^3 = -I_2, \ \ h_o^r = I_2, \ \ g_1 g_4 g_1^{-1} = g_4^{-1},$$
$$h_o g_1 h_o^{-1} = g_1 g_4^j, \ \ h_o g_4 h_o^{-1} = g_4 \rangle.$$

Note that $\frac{\lambda_1(h_o)}{\lambda_2(h_o)} = 1$ exactly when $h_o \in H \setminus SL(2,R)$ is a scalar matrix. According to Propositions 16, 17, 18, 19, 20, 21, 22, the only scalar matrices $h_o \in GL(2,R) \setminus SL(2,R)$ are $h_o = \pm i I_2$ for $R = \mathbb{Z}[i]$ and $h_o = e^{\pm \frac{2\pi i}{3}} I_2$ or $e^{\pm \frac{\pi i}{3}} I_2$ with $R = \mathcal{O}_{-3}$. Replacing, eventually, $h_o = -iI_2$ by $h_o^{-1} = iI_2$, one obtains the group $H_{Q12}(1) = \langle g_1, g_4, iI_2 \rangle$ with $R = \mathbb{Z}[i]$. Note that $H_{Q12}^o(1) = \langle D_1, D_4, h_o = iI_2 \rangle$ is a realization of $H_{Q12}(1)$ as a subgroup of $GL(2, \mathbb{Q}(\sqrt{3}, i))$. Bearing in mind that $-I_2 \in K_7$, one observes that $e^{-\frac{\pi i}{3}} I_2 \in H$ if and only if $-e^{-\frac{\pi i}{3}} I_2 = e^{\frac{2\pi i}{3}} I_2 \in H$. Replacing, eventually, $e^{\frac{\pi i}{3}} I_2$ and $e^{-\frac{2\pi i}{3}} I_2$ by their inverse matrices, one observes that $h_o = e^{\frac{2\pi i}{3}} I_2 \in H$ whenever H contains a scalar matrix of order 3 or 6. That provides the group $H_{Q12}(5) = \langle g_1, g_4, e^{\frac{2\pi i}{3}} I_2 \rangle$. Note that

$$\langle D_1 = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \ \ D_4 = \left(\begin{array}{cc} e^{\frac{\pi i}{3}} & 0 \\ 0 & E^{-\frac{\pi i}{3}} \end{array} \right), \ \ D_o = e^{\frac{2\pi i}{3}} I_2 \rangle < GL(2, \mathcal{O}_{-3})$$

is a realization of $H_{Q12}(5)$ as a subgroup of $GL(2,\mathcal{O}_{-3})$.

For $\frac{\lambda_1(h_o)}{\lambda_2(h_o)} = e^{\mp\frac{\pi i}{3}}$, Corollary 29 specifies that either $R=\mathcal{O}_{-3}$, $s=2$, $r=6$, $\lambda_1(h_o)=e^{\frac{\pi i}{3}}$, $\lambda_2(h_o)=e^{\frac{2\pi i}{3}}$ and $H\simeq H_{Q12}(2)$ or $R=\mathcal{O}_{-3}$, $s=6$, $r=6$, $\lambda_1(h_o)=\varepsilon e^{\frac{\eta\pi i}{3}}$, $\lambda_2(h_o)=\varepsilon$. In the second case, one can restrict to $\varepsilon=1$, due to $-I_2\in K_7\subset H$. The corresponding group $H\simeq H_{Q12}(9)$. Both, $H_{Q12}(2)$ and $H_{Q12}(9)$ can be realized as subgroups of $GL(2,\mathcal{O}_{-3})$, setting

$$g_1=\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right),\quad g_4=\left(\begin{array}{cc} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{array}\right),$$

$$h_o=\left(\begin{array}{cc} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{\frac{2\pi i}{3}} \end{array}\right) \quad \text{or, respectively,} \quad h_o=\left(\begin{array}{cc} e^{-\frac{\pi i}{3}} & 0 \\ 0 & 1 \end{array}\right).$$

If $\frac{\lambda_1(h_o)}{\lambda_2(h_o)} = e^{\mp\frac{2\pi i}{3}}$ then, eventually, replacing h_o by h_o^{-1}, one has $\lambda_1(h_o)=e^{\frac{\pi i}{6}}$, $\lambda_2(h_o)=e^{\frac{5\pi i}{6}}$, $s=2$, $r=12$, $R=\mathbb{Z}[i]$ and $H\simeq H_{Q12}(3)$ or $\lambda_1(h_o)=\varepsilon$, $\lambda_2(h_o)=\varepsilon e^{\frac{2\pi i}{3}}$, $s=3$, $R=\mathcal{O}_{-3}$, by Corollary 29. Note that $-I_2\in K_7\subset H$ reduces the second case to $\lambda_1(h_o)=1$, $\lambda_2(h_o)=e^{\frac{2\pi i}{3}}$, $s=3$, $r=3$, $R=\mathcal{O}_{-3}$ and $H\simeq H_{Q12}(6)$. Note that

$$g_1=\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right),\quad g_4=\left(\begin{array}{cc} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{array}\right),\quad h_o=\left(\begin{array}{cc} 1 & 0 \\ 0 & e^{\frac{2\pi i}{3}} \end{array}\right)$$

generate a subgroup of $GL(2,\mathcal{O}_{-3})$, isomorphic to $H_{Q12}(6)$. In the case of $H\simeq H_{Q12}(3)$ the eigenvalues of h_o are primitive twelfth roots of unity, so that

$$D_1=\left(\begin{array}{cc} 0 & b_1 \\ -b_1^{-1} & 0 \end{array}\right),\quad D_4=\left(\begin{array}{cc} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{array}\right),\quad D_o=\left(\begin{array}{cc} e^{\frac{\pi i}{6}} & 0 \\ 0 & e^{\frac{5\pi i}{6}} \end{array}\right)$$

generate a subgroup $H^o_{Q12}(3)<GL(2,\mathbb{Q}(\sqrt{3},i))$, isomorphic to $H_{Q12}(3)$.

For $\frac{\lambda_1(h_o)}{\lambda_2(h_o)}=-1$ there are four non-equivalent possibilities for the eigenvalues $\lambda_1(h_o)$, $\lambda_2(h_o)$ of h_o. The first one is $\lambda_1(h_o)=1$, $\lambda_2(h_o)=-1$ with $s=2$, $r=2$ for any $R=R_{-d,f}$ and $H\simeq H_{Q12}(4)$ of order 24. Note that

$$D_1=\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right),\quad D_4=\left(\begin{array}{cc} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{array}\right),\quad h_o=\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right)$$

realizes $H_{Q12}(4)$ as a subgroup of $GL(2,\mathcal{O}_{-3})$. The second one is $\lambda_1(h_o)=e^{\frac{3\pi i}{4}}$, $\lambda_2(h_o)=E^{-\frac{\pi i}{4}}$ with $s=4$, $r=8$, $R=\mathbb{Z}[i]$ and $H\simeq H_{Q12}(8)$ of order 48. Observe that

$$D_1=\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right),\quad D_4=\left(\begin{array}{cc} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{array}\right),\quad D_o=\left(\begin{array}{cc} e^{\frac{3\pi i}{4}} & 0 \\ 0 & e^{-\frac{\pi i}{4}} \end{array}\right)$$

generate a subgroup of $GL(2,\mathbb{Q}(\sqrt{2},\sqrt{3},i))$, isomorphic to $H_{Q12}(8)$. In the third case, $\lambda_1(h_o)=e^{-\frac{\pi i}{6}}$, $\lambda_2(h_o)=e^{\frac{5\pi i}{6}}$ with $s=3$, $r=12$, $R=\mathcal{O}_{-3}$ and $H\simeq H_{Q12}(7)$ of order 36, realized by

$$D_1=\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right),\quad D_4=\left(\begin{array}{cc} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{array}\right),\quad D_o=\left(\begin{array}{cc} e^{-\frac{\pi i}{6}} & 0 \\ 0 & e^{\frac{5\pi i}{6}} \end{array}\right)$$

as a subgroup of $GL(2,\mathbb{Q}(\sqrt{3},i))$. In the fourth case, $\lambda_1(h_o)=e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o)=e^{-\frac{\pi i}{3}}$ with $s=6$, $r=6$, $R=\mathcal{O}_{-3}$ and $H\simeq H_{Q12}(10)$ of order 72. The matrices

$$g_1=\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right),\quad g_4=\left(\begin{array}{cc} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{array}\right),\quad h_o=\left(\begin{array}{cc} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{array}\right)$$

generate a subgroup of $GL(2,\mathcal{O}_{-3})$, isomorphic to $H_{Q12}(10)$. The groups $H_{Q12}(4)$, $H_{Q12}(7)$, $H_{Q12}(8)$, $H_{Q12}(10)$ with $\frac{\lambda_1(h_o)}{\lambda_2(h_o)}=-1$ are non-isomorphic, as far as they are of different orders.

$\square$

Proposition 40. *Let H be a finite subgroup of $GL(2,R)$,*

$$H\cap SL(2,R)=K_8=\langle g_1,g_2,g_3 \ \mid \ g_1^2=g_2^2=-I_2,\quad g_3^3=I_2,\quad g_2g_1=-g_1g_2,$$

$$g_3g_1g_3^{-1}=g_2,\quad g_3g_2g_3^{-1}=g_1g_2\rangle\simeq SL(2,\mathbb{F}_3)$$

and $h_o\in H$ be an element of order r with $\det(H)=\langle\det(h_o)\rangle\simeq\mathbb{C}_s$ *and eigenvalues $\lambda_1(h_o)$, $\lambda_2(h_o)$. Then H is isomorphic to $H_{SL(2,3)}(i)$ for some $1\leq i\leq 9$, where*

$$H_{SL(2,3)}(1)=\langle g_1,g_2,g_3,iI_2 \ \mid \ g_1^2=g_2^2=-I_2,\quad g_3^3=I_2,\quad g_2g_1=-g_1g_2,$$

$$g_3g_1g_3^{-1}=g_2,\quad g_3g_2g_3^{-1}=g_1g_2,\rangle$$

of order 48 with $R=\mathbb{Z}[i]$,

$$H_{SL(2,3)}(2)=\langle g_1,g_2,g_3,h_o \ \mid \ g_1^2=g_2^2=-I_2,\quad g_3^3=I_2,\quad h_o^2=I_2,\quad g_2g_1=-g_1g_2$$

$$g_3g_1g_3^{-1}=g_2,\quad g_3g_2g_3^{-1}=g_1g_2,\quad h_og_1h_o^{-1}=-g_1,\quad h_og_2h_o^{-1}=-g_2,\quad h_og_3h_o^{-1}=-g_2g_3\rangle$$

of order 48 with $R=\mathbb{Z}[i]$, $\lambda_1(h_o)=-1$, $\lambda_2(h_o)=1$,

$$H_{SL(2,3)}(3)=\langle g_1,g_2,g_3,h_o \ \mid \ g_1^2=g_2^2=h_o^4=-I_2,\quad g_3^3=I_2,\quad g_2g_1=-g_1g_2,$$

$$g_3g_1g_3^{-1}=g_2,\quad g_3g_2g_3^{-1}=g_1g_2,\quad h_og_1h_o^{-1}=g_2,\quad h_og_2h_o^{-1}=-g_1,\quad h_og_3h_o^{-1}=g_2g_3^2\rangle$$

of order 48 with $R=\mathcal{O}_{-2}$, $\lambda_1(h_o)=e^{\frac{\pi i}{4}}$, $\lambda_2(h_o)=e^{\frac{3\pi i}{4}}$,

$$H_{SL(2,3)}(4)=\langle g_1,g_2,g_3,h_o \ \mid \ g_1^2=g_2^2=-I_2,\quad g_3^3=I_2,\quad h_o^2=I_2,\quad g_2g_1=-g_1g_2$$

$$g_3g_1g_3^{-1} = g_2, \quad g_3g_2g_3^{-1} = g_1g_2, \quad h_og_1h_o^{-1} = g_2, \quad h_og_2h_o^{-1} = g_1, \quad h_og_3h_o^{-1} = g_1g_3^2\rangle$$

of order 48 *with* $R = R_{-2,f}$, $\lambda_1(h_o) = -1$, $\lambda_2(h_o) = 1$,

$$H_{SL(2,3)}(5) = K_8 \times \langle e^{\frac{2\pi i}{3}} I_2 \rangle \simeq SL(2, \mathbb{F}_3) \times \mathbb{C}_3$$

of order 72 *with* $R = \mathcal{O}_{-3}$,

$$H_{SL(2,3)}(6) = \langle g_1, g_2, g_3, h_o \ \mid \ g_1^2 = g_2^2 = -I_2, \quad g_3^3 = I_2, \quad h_o^3 = I_2, \quad g_2g_1 = -g_1g_2,$$

$$g_3g_1g_3^{-1} = g_2, \quad g_3g_2g_3^{-1} = g_1g_2, \quad h_og_1h_o^{-1} = g_2, \quad h_og_2h_o^{-1} = g_1g_2, \quad h_og_3h_o^{-1} = g_3\rangle$$

of order 72 *with* $R = \mathcal{O}_{-3}$, $\lambda_1(h_o) = e^{\frac{2\pi i}{3}}$, $\lambda_2(h_o) = 1$,

$$H_{SL(2,3)}(7) = \langle g_1, g_2, g_3, h_o \ \mid \ g_1^2 = g_2^2 = h_o^4 = -I_2, \quad g_3^3 = I_2, \quad g_2g_1 = -g_1g_2$$

$$g_3g_1g_3^{-1} = g_2, \quad g_3g_2g_3^{-1} = g_1g_2, \quad h_og_1h_o^{-1} = -g_1, \quad h_og_2h_o^{-1} = -g_2, \quad h_og_3h_o^{-1} = -g_2g_3\rangle$$

of order 96 *with* $R = \mathbb{Z}[i]$, $\lambda_1(h_o) = e^{\frac{3\pi i}{4}}$, $\lambda_2(h_o) = e^{-\frac{\pi i}{4}}$,

$$H_{SL(2,3)}(8) = \langle g_1, g_2, g_3, h_o \ \mid \ g_1^2 = g_2^2 = h_o^4 = -I_2, \quad g_3^3 = I_2, \quad g_2g_1 = -g_1g_2'$$

$$g_3g_1g_3^{-1} = g_2, \quad g_3g_2g_3^{-1} = g_1g_2, \quad h_og_1h_o^{-1} = g_2, \quad h_og_2h_o^{-1} = g_1, \quad h_og_3h_o^{-1} = g_1g_3^2\rangle$$

of order 96 *with* $R = \mathbb{Z}[i]$, $\lambda_1(h_o) = e^{\frac{3\pi i}{4}}$, $\lambda_2(h_o) = e^{-\frac{\pi i}{4}}$,

$$H_{SL(2,3)}(9) = \langle g_1, g_2, g_3, h_o \ \mid \ g_1^2 = g_2^2 = -I_2, \quad g_3^3 = I_2, \quad h_o^4 = I_2, \quad g_2g_1 = -g_1g_2,$$

$$g_3g_1g_3^{-1} = g_2, \quad g_3g_2g_3^{-1} = g_1g_2, \quad h_og_1h_o^{-1} = g_2, \quad h_og_2h_o^{-1} = -g_1, \quad h_og_3h_o^{-1} = g_2g_3^2\rangle$$

of order 96 *with* $R = \mathbb{Z}[i]$, $\lambda_1(h_o) = i$, $\lambda_2(h_o) = 1$.

There exists a subgroup

$$H_{SL(2,3)}(5) < GL(2, \mathcal{O}_{-3}),$$

as well as subgroups

$$H^o_{SL(2,3)}(1), H^o_{SL(2,3)}(2), H^o_{SL(2,3)}(9) < GL(2, \mathbb{Q}(\sqrt{3}, i)),$$

$$H^o_{SL(2,3)}(4) < GL(2, \mathbb{Q}(\sqrt{-2}, \sqrt{-3})),$$

$$H^o_{SL(2,3)}(3), H^o_{SL(2,3)}(7), H^o_{SL(2,3)}(8) < GL(2, \mathbb{Q}(\sqrt{2}, \sqrt{3}, i))$$

with $H^o_{SL(2,3)}(j) \simeq H_{SL(2,3)}(j)$ *for* $1' \leq j \leq 4$ *or* $6 \leq j \leq 9$.

Proof. According to Lemma 27, the groups H under consideration are uniquely determined up to an isomorphism by the order r of h_o and by the elements $h_o g_j h_o^{-1} \in K_8^{(4)}$, $1 \leq j \leq 2$, $x_3 := h_o g_3 h_o^{-1} \in K_8^{(3)}$. (Throughout, $G^{(\nu)}$ denotes the set of the elements of order ν from a group G.) Recall by Proposition 24 the realization of $K_8 \simeq SL(2, \mathbb{F}_3)$ as a subgroup $\mathcal{K}_8$ of $GL(2, \mathbb{Q}(\sqrt{-d}, \sqrt{-3}))$, generated by the matrices

$$D_1 = \begin{pmatrix} -\frac{\sqrt{-3}}{3} & b_1 \\ -\frac{2}{3b_1} & \frac{\sqrt{-3}}{3} \end{pmatrix}, \quad D_2 = \begin{pmatrix} -\frac{\sqrt{-3}}{3} & e^{-\frac{2\pi i}{3}} b_1 \\ -\frac{2e^{\frac{2\pi i}{3}}}{3b_1} & \frac{\sqrt{-3}}{3} \end{pmatrix}, \quad D_3 = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix}$$

with some $b_1 \in \mathbb{Q}(\sqrt{-d}, \sqrt{-3})^*$. After computing

$$D_1 D_2 = \begin{pmatrix} -\frac{\sqrt{-3}}{3} & e^{-\frac{4\pi i}{3}} b_1 \\ -\frac{2e^{\frac{4\pi i}{3}}}{3b_1} & \frac{\sqrt{-3}}{3} \end{pmatrix},$$

one puts

$$\delta_j := \begin{pmatrix} -\frac{\sqrt{-3}}{3} & e^{-\frac{2j\pi i}{3}} b_1 \\ -\frac{2e^{\frac{2j\pi i}{3}}}{3b_1} & \frac{\sqrt{-3}}{3} \end{pmatrix} \quad \text{for} \quad 0 \leq j \leq 2$$

and observes that $\delta_0 = D_1$, $\delta_1 = D_2$, $\delta_2 = D_1 D_2$. The elements of $\mathcal{K}_8$ of order 4 constitute the subset

$$\mathcal{K}_8^{(4)} = \{\pm \delta_j \mid 0 \leq j \leq 2\}.$$

In order to list the elements of $\mathcal{K}_8$ of order 3, let us note that $D_3 D_1 D_3^{-1} = D_2$ and $D_3 D_2 D_3^{-1} = D_1 D_2$ imply $D_3 (D_1 D_2) D_3^{-1} = D_1$. Thus, for any even permutation j, l, m of $0, 1, 2$, one has

$$\left| \begin{array}{l} D_3 \delta_j D_3^{-1} = \delta_l \\ D_3 \delta_l D_3^{-1} = \delta_m \\ D_3 \delta_m D_3^{-1} = \delta_j \end{array} \right. \quad \text{or, equivalently,} \quad \left| \begin{array}{l} D_3 \delta_j = \delta_l D_3 \\ D_3 \delta_l = \delta_m D_3 \\ D_3 \delta_m = \delta_j D_3 \end{array} \right. . \tag{16}$$

Making use of (16, one computes that

$$(-\delta_j D_3)^2 = \delta_m D_3^2, \quad (-\delta_j D_3)^3 = (-\delta_j D_3)(-\delta_j D_3)^2 = I_2 \quad \text{for all} \quad 0 \leq j \leq 2,$$

so that $-\delta_j D_3 \in \mathcal{K}_8^{(3)}$. As a result, $\delta_j D_3^2 = (-\delta_l D_m)^2 \in \mathcal{K}_8^{(3)}$ for all $0 \leq j \leq 2$ and

$$\mathcal{K}_8^{(3)} = \{D_3, \ \ -\delta_j D_3, \ \ D_3^2, \ \ \delta_j D_3^2 \mid 0 \leq j \leq 2\}.$$

Proposition 24 has established that $\mathcal{K}_8$ has a unique Sylow 2-subgroup

$$\mathcal{H}_8 = \langle \delta_0, \delta_1 \mid \delta_0^2 = \delta_1^2 = -I_2, \ \ \delta_1 \delta_0 = -\delta_0 \delta_1 \rangle = \{\pm I_2, \pm \delta_j \mid 0 \leq j \leq 2\},$$

so that the set $\mathcal{K}_8^{(4)} = \mathcal{H}_8^{(4)}$ of the elements of $\mathcal{K}_8$ of order 4 are contained in $\mathcal{H}_8 \simeq \mathbb{Q}_8$. In other words, $x_j := h_o \delta_j h_o^{-1} \in \mathcal{H}_8$ and $H' = \langle g_1, g_2, h_o \rangle \simeq \mathcal{H}' = \langle \delta_0, \delta_1, D_o \rangle$ is a subgroup of H with $H \cap SL(2,R) \simeq \mathbb{Q}_8$. Proposition 38 establishes that any such H' is isomorphic to $H_{Q8}(i)$ for some $1 \leq i \leq 9$.

We claim that for any $1 \leq i \leq 9$ there is (at most) a unique finite subgroup $H = \langle g_1, g_2, g_3, h_o \rangle$ of $GL(2,R)$ with $\langle g_1, g_2, h_o \rangle \simeq H_{Q8}(i)$, $H \cap SL(2,R) = \langle g_1, g_2, g_3 \rangle \simeq SL(2, \mathbb{F}_3)$ and $\det(H) = \langle \det(h_o) \rangle$. To this end, let us consider the adjoint representation

$$\mathrm{Ad} : \mathcal{K}_8 \longrightarrow S(\mathcal{K}_8^{(4)}) \simeq S_6$$

$$\mathrm{Ad}_x(y) = xyx^{-1} \quad \text{for} \quad \forall x \in \mathcal{K}_8, \quad \forall y \in \mathcal{K}_8^{(4)}$$

and its restriction

$$\mathrm{Ad} : \mathcal{K}_8^{(3)} \longrightarrow S(\mathcal{K}_8^{(4)}) \simeq S_6$$

to the elements of $\mathcal{K}_8$ of order 3. Note that

$$\langle x_0, x_1 \rangle = h_o \langle \delta_0, \delta_1 \rangle h_o^{-1} = h_o \mathcal{H}_8 h_o^{-1} = \mathcal{H}_8,$$

as far as $\mathcal{H}_8 \simeq \mathbb{Q}_8$ is normal subgroup of $\mathcal{H}' = \mathcal{H}_8 \langle h_o \rangle$. The adjoint action

$$\mathrm{Ad}_{h_o} : \mathcal{K}_8 \longrightarrow \mathcal{K}_8$$

$$\mathrm{Ad}_{h_o}(x) = h_o x h_o^{-1} \quad \text{for} \quad \forall x \in \mathcal{K}_8$$

of h_o is a group homomorphism and transforms the relations $D_3 \delta_s D_3^{-1} = \delta_{s+1}$ for $0 \leq s \leq 1$ into the relations $x_3 x_s x_3^{-1} = x_{s+1}$ for $0 \leq s \leq 1$. For any $1 \leq i \leq 9$ the subgroup $\mathcal{H}' \simeq H_{Q8}(i)$ of $\mathcal{H}$ determines uniquely $x_0, x_1 \in \mathcal{H}_8$. We claim that for any such x_0, x_1 there is a unique $x_3 \in \mathcal{K}_8^{(3)}$ with

$$\mathrm{Ad}_{x_3}(x_0) = x_1, \quad \mathrm{Ad}_{x_3}(x_1) = x_0 x_1. \tag{17}$$

Indeed, Proposition 38 specifies the following five possibilities:

$$\text{Case 1} \quad x_0 = \delta_0, \quad x_1 = \delta_1;$$

$$\text{Case 2} \quad x_0 = -\delta_0, \quad x_1 = -\delta_1;$$

$$\text{Case 3} \quad x_0 = \delta_1, \quad x_1 = -\delta_0;$$

$$\text{Case 4} \quad x_0 = \delta_1, \quad x_1 = \delta_0;$$

$$\text{Case 5} \quad x_0 = \delta_1, \quad x_1 = \delta_2.$$

For any $0 \leq s \neq t \leq 2$ and $\varepsilon, \eta \in \{\pm 1\}$ note that

$$\mathrm{Ad}_{\varepsilon \delta_s}(\eta \delta_s) = \eta \delta_s, \quad \mathrm{Ad}_{\varepsilon \delta_s}(\eta \delta_t) = -\eta \delta_t.$$

Combining with (14), one concludes that

$$\mathrm{Ad}_{D_3}(\langle \delta_j \rangle) = \mathrm{Ad}_{(-\delta_s D_3)}(\langle \delta_j \rangle) = \langle \delta_l \rangle,$$

$$\mathrm{Ad}_{D_3}(\langle \delta_l \rangle) = \mathrm{Ad}_{(-\delta_s D_3)}(\langle \delta_l \rangle) = \langle \delta_m \rangle,$$

$$\mathrm{Ad}_{D_3}(\langle \delta_m \rangle) = \mathrm{Ad}_{(-\delta_s D_3)}(\langle \delta_m \rangle) = \langle \delta_j \rangle$$

for any $0 \leq s \leq 2$ and any even permutation j, l, m of $0, 1, 2$. Similarly,

$$\mathrm{Ad}_{D_3^2}(\langle \delta_j \rangle) = \mathrm{Ad}_{\delta_s D_3^2}(\langle \delta_j \rangle) = \langle \delta_m \rangle,$$

$$\mathrm{Ad}_{D_3^2}(\langle \delta_l \rangle) = \mathrm{Ad}_{\delta_s D_3^2}(\langle \delta_l \rangle) = \langle \delta_j \rangle,$$

$$\mathrm{Ad}_{D_3^2}(\langle \delta_m \rangle) = \mathrm{Ad}_{\delta_s D_3^2}(\langle \delta_m \rangle) = \langle \delta_l \rangle$$

for any $0 \leq s \leq 2$ and any even permutation j, l, m of $0, 1, 2$. In the case 1, (17) read as $\mathrm{Ad}_{x_3}(\delta_0) = \delta_1$, $\mathrm{Ad}_{x_3}(\delta_1) = \delta_2$ and imply that $x_3 = D_3$, according to (16) and $\mathrm{Ad}_{(-\delta_s)} \not\equiv Id_{\mathcal{K}_8}$ for all $0 \leq s \leq 2$. In the Case 2, $\mathrm{Ad}_{x_3}(\delta_0) = \delta_1$ and $\mathrm{Ad}_{x_3}(\delta_1) = -\delta_2$ specify that $x_3 = -\delta_1 D_3 = -D_2 D_3$. In the next Case 3, the relations $\mathrm{Ad}_{x_3}(\delta_1) = -\delta_0$, $\mathrm{Ad}_{x_3}(\delta_0 = \delta_2$ hold if and only if $x_3 = \delta_1 D_3^2 = D_2 D_3^2$. Further, $\mathrm{Ad}_{x_3}(\delta_1) = \delta_0$, $\mathrm{Ad}_{x_3}(\delta_0) = -\delta_2$ in Case 4 are satisfied by $x_3 = \delta_0 D_3^2 = D_1 D_3^2$ and $\mathrm{Ad}_{x_3}(\delta_1) = \delta_2$, $\mathrm{Ad}_{x_3}(\delta_2) = \delta_0$ in Case 5 are valid for $x_3 = D_3$. Given a presentation of $H' \simeq H_{Q8}(i)$ with generators g_1, g_2, h_o, one adjoins a generator $g_3 \in SL(2, R)$ of order 3 and the relation $h_o g_3 h_o^{-1} = x_3$, in order to obtain a presentation of $H \simeq H_{SL(2,3)}(i)$, $1 \leq i \leq 9$. □

4 Explicit Galois groups for A/H of fixed Kodaira-Enriques type

In order to classify the finite subgroups H of $Aut(A)$, for which A/H is of a fixed Kodaira-Enriques classification type, one needs to describe the finite subgroups H of $Aut(A)$ for $A = E \times E$. Making use of the classification of the finite subgroups $\mathcal{L}(H)$ of $GL(2, R)$, done in section 3, let $\det \mathcal{L}(H) = \langle \det \mathcal{L}(h_o) = e^{\frac{2\pi i}{s}} \rangle \simeq \mathbb{C}_s$ for some $s \in \{1, 2, 3, 4, 6\}$, $h_o \in H$. (In the case of $s = 1$, we choose $h_o = Id_A$.) By Proposition 24 one has $\mathcal{L}(H) \cap SL(2, R) = \langle \mathcal{L}(h_1), \ldots, \mathcal{L}(h_t) \rangle$ for some $0 \leq t \leq 3$. (Assume $\mathcal{L}(H) \cap SL(2, R) = \{I_2\}$ for $t = 0$.) The linear part

$$\mathcal{L}(H) = [\mathcal{L}(h) \cap SL(2, R)] \langle \mathcal{L}(h_o) \rangle = \langle \mathcal{L}(h_1), \ldots, \mathcal{L}(h_t) \rangle \langle \mathcal{L}(h_o) \rangle$$

of H is a product of its normal subgroup $\langle \mathcal{L}(h_1), \ldots, \mathcal{L}(h_t) \rangle$ and the cyclic group $\langle \mathcal{L}(h_o) \rangle$. The translation part $\mathcal{T}(H) = \ker(\mathcal{L}|_H)$ of H is a finite subgroup of $(\mathcal{T}_A, +) \simeq (A, +)$. The lifting $\left(\widetilde{\mathcal{T}_A}, +\right) < \left(\widetilde{A} = \mathbb{C}^2, +\right)$ of $\mathcal{T}(H)$ is a free $\mathbb{Z}$-module of rank 4. Therefore $\left(\widetilde{\mathcal{T}(H)}, +\right)$ has at most four generators and

$$\mathcal{T}(H) = \langle \tau_{(P_i, Q_i)} \ | \ 1 \leq i \leq m \rangle \quad \text{for some} \quad 0 \leq m \leq 4.$$

(In the case of $m=0$ one has $\mathcal{T}(H)=\{Id_A\}$.) We claim that

$$H=\mathcal{T}(H)\langle h_1,\dots,h_t,h_o\rangle=\langle \tau_{(P_i,Q_i)},h_j,h_o \ \mid\ 1\leq i\leq m,\ \ 1\leq j\leq t\rangle$$

for some $0\leq m\leq 4$, $0\leq t\leq 3$. The choice of $\tau_{(P_i,Q_i)},h_j,h_o\in H$ justifies the inclusion $\langle \tau_{(P_i,Q_i)},h_j,h_o\ \mid\ 1\leq i\leq m, 1\leq j\leq t\rangle\subseteq H$. For the opposite inclusion, an arbitrary element $h\in H$ with $\mathcal{L}(h)=\mathcal{L}(h_1)^{k_1}\dots\mathcal{L}(h_t)^{k_t}\mathcal{L}(h_o)^{k_o}$ for some $k_j\in\mathbb{Z}$ produces a translation $\tau_{(U,V)}:=hh_o^{-k_o}h_t^{-k_t}\dots h_1^{-k_1}\in\ker(\mathcal{L}|_H)=\mathcal{T}(H)=\langle\tau_{(P_i,Q_i)}\ \mid\ 1\leq i\leq m\rangle$, so that $h=\tau_{(U,V)}h_1^{k_1}\dots h_t^{k_t}h_o^{k_o}\in\langle\tau_{(P_i,Q_i)},h_j,h_o\ \mid\ 1\leq i\leq m,\ \ 1\leq j\leq t\rangle$ and $H\subseteq\langle\tau_{(P_i,Q_i)},h_j,h_o\ \mid\ 1\leq i\leq m,\ \ 1\leq j\leq t\rangle$. In such a way, we have derived the following

Lemma 41. *If H is a finite subgroup of $Aut(A)$, $A=E\times E$ with*

$$\det\mathcal{L}(H)=\langle\det\mathcal{L}(h_o)=e^{\frac{2\pi i}{s}}\rangle\simeq\mathbb{C}_s\quad \textit{and}$$

$$\mathcal{L}(H)\cap SL(2,R)=\langle\mathcal{L}(h_1),\dots,\mathcal{L}(h_t)\rangle\quad \textit{for some}\quad 0\leq t\leq 3\quad\textit{then}$$

$$H=\langle\tau_{(P_i,Q_i)},h_jh_o\ \mid\ 1\leq i\leq m,\ \ 1\leq j\leq t\rangle$$

is generated by $0\leq m\leq 3$ translations and at most four non-translation elements.

Bearing in mind that A/H is birational to a K3 surface exactly when $\mathcal{L}(H)$ is a subgroup of $SL(2,R)$, one obtains the following

Corollary 42. *The quotient A/H by a finite subgroup H of $Aut(A)$ has a smooth K3 model if and only if H is isomorphic to some $H^{K3}(j,m)$ with $1\leq j\leq 8$, $0\leq m\leq 3$, where*

$$H^{K3}(1.m)=\langle\tau_{(P_i,Q_i)},\quad \tau_{(U_1,V_1)}(-I_2)\ \mid\ 1\leq i\leq m\rangle$$

$$H^{K3}(2,m)=\langle\tau_{(P_i,Q_i)},\quad h_1\ \mid\ 1\leq i\leq m\rangle$$

for $\mathcal{L}(h_1)\in SL(2,R)$, $\mathrm{tr}\mathcal{L}(h_1)=0$,

$$H^{K3}(3,m)=\langle\tau_{(P_i,Q_i)},\quad h_1,\quad h_2\ \mid\ 1\leq i\leq m\rangle$$

for $\mathcal{L}(h_1),\mathcal{L}(h_2)\in SL(2,R)$, $\mathrm{tr}\mathcal{L}(h_1)=\mathrm{tr}\mathcal{L}(h_2)=0$, $\mathcal{L}(h_2)\mathcal{L}(h_1)=-\mathcal{L}(h_1)\mathcal{L}(h_2)$,

$$H^{K3}(4,m)=\langle\tau_{(P_i,Q_i)},\quad h_3\ \mid\ 1\leq i\leq m\rangle$$

for $\mathcal{L}(h_3)\in SL(2,R)$, $\mathrm{tr}\mathcal{L}(h_3)=-1$,

$$H^{K3}(5,m)=\langle\tau_{(P_i,Q_i)},\quad h_4\ \mid\ 1\leq i\leq m\rangle$$

for $\mathcal{L}(h_4)\in SL(2,R)$, $\mathrm{tr}\mathcal{L}(h_4)=1$,

$$H^{K3}(6,m)=\langle\tau_{(P_i,Q_i)},\quad h_1,\quad h_4\ \mid\ 1\leq i\leq m\rangle$$

for $\mathcal{L}(h_1), \mathcal{L}(h_4) \in SL(2,R)$, $\mathrm{tr}\mathcal{L}(h_1) = 0$, $\mathrm{tr}\mathcal{L}(h_4) = 1$, $\mathcal{L}(h_1)\mathcal{L}(h_4)[\mathcal{L}(h_1)]^{-1} = [\mathcal{L}(h_4)]^{-1}$,

$$H^{K3}(7,m) = \langle \tau_{(P_i,Q_i)}, \quad h_1, h_2, h_3 \ \mid \ 1 \leq i \leq m \rangle$$

for $\mathcal{L}(h_1), \mathcal{L}(h_2), \mathcal{L}(h_3) \in SL(2,R)$, $\mathrm{tr}\mathcal{L}(h_1) = \mathrm{tr}\mathcal{L}(h_2) = 0$, $\mathrm{tr}\mathcal{L}(h_3) = -1$,

$$\mathcal{L}(h_2)\mathcal{L}(h_1) = -\mathcal{L}(h_1)\mathcal{L}(h_2),$$
$$\mathcal{L}(h_3)\mathcal{L}(h_1)[\mathcal{L}(h_3)]^{-1} = \mathcal{L}(h_2) \quad \mathcal{L}(h_3)\mathcal{L}(h_2)[\mathcal{L}(h_3)]^{-1} = \mathcal{L}(h_1)\mathcal{L}(h_2),$$
$$H^{K3}(8,m) = \langle \tau_{(P_i,Q_i)}, \quad h_1, h_2, h_3 \ \mid \ 1 \leq i \leq m \rangle$$

for $\mathcal{L}(h_1), \mathcal{L}(h_2), \mathcal{L}(h_3) \in SL(2,R)$, $\mathrm{tr}\mathcal{L}(h_1) = \mathrm{tr}\mathcal{L}(h_2) = 0$, $\mathrm{tr}\mathcal{L}(h_3) = -1$,

$$\mathcal{L}(h_2)\mathcal{L}(h_1) = -\mathcal{L}(h_1)\mathcal{L}(h_2),$$
$$\mathcal{L}(h_3)\mathcal{L}(h_1)[\mathcal{L}(h_3)]^{-1} = \mathcal{L}(h_2), \quad \mathcal{L}(h_3)\mathcal{L}(h_2)[\mathcal{L}(h_3)]^{-1} = \mathcal{L}(h_1)\mathcal{L}(h_2).$$

We are going to show that for an arbitrary finite subgroup $H < Aut(A)$ with an abelian linear part $\mathcal{L}(H) < GL(2,R)$, there exist an isomorphic model $F_1 \times F_2$ of A and a normal subgroup N_1 of H, embedded in $Aut(F_1)$, such that the quotient group H/N_1 is an automorphism group of F_2. This result can be viewed as a generalization of Bombieri-Mumford's classification [3] of the hyper-elliptic surfaces. More precisely, if $H = \mathcal{T}(H)\langle h_o \rangle$ for some $h_o \in H$ with eigenvalues $\lambda_1 \mathcal{L}(h_o) = 1$, $\lambda_2 \mathcal{L}(h_o) = \det \mathcal{L}(h_o) = e^{\frac{2\pi i}{s}}$, $s \in \{2,3,4,6\}$, then there is a translation subgroup N_1 of $Aut(F_1)$, such that $G \simeq H/N_1$ is a non-translation group, acting on the split abelian surface $F_1' \times F_2 = (F_1/N_1) \times F_2$. According to Proposition 5, the quotient A/H is hyper-elliptic (respectively, ruled with elliptic base) exactly when the finite Galois covering $A \to A/H$ is unramified (respectively, ramified). Since $F_1 \to F_1/N_1 = F_1'$ is unramified for a translation subgroup $N_1 \mathcal{T}_{F_1} < Aut(F_1)$, the covering $A \to A/H$ is unramified is and only if the covering $F_1' \times F_2 \to (F_1' \times F_2)/G$ is unramified for $G = H/N_1$. In particular, the first canonical projection $\mathrm{pr}_1 : G \to Aut(F_1')$ is a group monomorphism and G is an abelian group with at most two generators, according to the classification of the finite translation groups of F_1'. Thus, Bombieri-Mumford's classification of the hyper-elliptic surfaces $(F_1' \times F_2)/G$ reduces to the classification of the split, fixed point free abelian subgroups $G < Aut(F_1' \times F_2)$ with at most two generators, for which the canonical projections $\mathrm{pr}_1 : G \to Aut(F_1')$ and $\mathrm{pr}_2 : G \to Aut(F_2)$ are injective group homomorphisms.

Towards the classification of the finite subgroups of $Aut(E)$, let us recall that the semi-direct products $\langle a \rangle \rtimes \langle b \rangle \simeq \mathbb{C}_m \rtimes \mathbb{C}_s$ of cyclic groups are completely determined by the adjoint action of b on a. Namely, $\mathrm{Ad}_b(a) = bab^{-1} = a^j$ for some residue $j \in \mathbb{Z}_m^*$ modulo m, relatively prime to m. Now $\mathrm{Ad}_{b^s}(a) = a^{j^s} = a$ requires $j^s \equiv 1 (\mathrm{mod} m)$. In other words, $j \in \mathbb{Z}_m^*$ is of order r, dividing s and $\langle a \rangle \leftthreetimes \langle b \rangle$ is isomorphic to

$$G_s^{(j)}(m) := \mathbb{C}_m \rtimes_j \mathbb{C}_s = \langle a, \quad b \ \mid \ a^m = 1, \quad b^s = 1, \quad bab^{-1} = a^j \rangle \tag{18}$$

for some $j \in \mathbb{Z}_m^*$ of order r, dividing s. Form now on, we use the notation (18) without further reference. Note that the only $j \in \mathbb{Z}_m^*$ of order 1 is $j \equiv 1(\mathrm{mod} m)$ and $G_s^{(1)}(m) = \langle a \rangle \times \langle b \rangle \simeq \mathbb{C}_m \times \mathbb{C}_s$ is the direct product of $\langle a \rangle = \mathbb{C}_m$ and $\langle b \rangle = \mathbb{C}_s$.

Lemma 43. *Let G be a finite subgroup of the automorphism group $Aut(E)$ of an elliptic curve E with endomorphism ring $End(E) = R$. Then G is isomorphic to some of the groups $G_1(m,n)$, $G_2^{(-1,-1)}(m,n)$, $G_s^{(j)}(m)$, $s \in \{3,4,6\}$, where*

$$G_1(m,n) = \langle \tau_{P_1}, \quad \tau_{P_2} \rangle \simeq \mathbb{C}_m \times \mathbb{C}_n, \quad m,n \in \mathbb{N}$$

is a translation group with at most two generators,

$$G_2^{(-1,-1)}(m,n) = \langle \tau_{P_1}, \quad \tau_{P_2} \rangle \rtimes \langle -1 \rangle \simeq (\mathbb{C}_m \times \mathbb{C}_n) \rtimes_{(-1,-1)} \mathbb{C}_2 = (\langle a \rangle \times \langle b \rangle) \rtimes_{(-1,-1)} \langle c \rangle =$$

$$= \langle a, \quad b, \quad c \quad | \quad a^m = 1, \quad b^n = 1, \quad c^2 = 1, \quad cac^{-1} = a, \quad cbc^{-1} = b^{-1} \rangle$$

$$G_3^{(j)}(m) = \langle \tau_{P_1} \rangle \rtimes_j \langle e^{\frac{2\pi i}{3}} \rangle \simeq \mathbb{C}_m \rtimes_j \mathbb{C}_3 = \langle a \rangle \rtimes_j \langle c \rangle =$$

$$= \langle a, \quad c \quad | \quad a^m = 1, \quad c^3 = 1, \quad cac^{-1} = a^j \rangle$$

for some $j \in \mathbb{Z}_m^*$ of order 1 or 3, $R = \mathcal{O}_{-3}$,

$$G_4^{(j)}(m) = \langle \tau_{P_1} \rangle \rtimes_j \langle i \rangle \simeq \mathbb{C}_m \rtimes_j \mathbb{C}_4 = \langle a \rangle \rtimes_j \langle c \rangle =$$

$$= \langle a, \quad c \quad | \quad a^m = 1, \quad c^4 = 1, \quad cac^{-1} = a^j \rangle$$

for some $j \in \mathbb{Z}_m^*$ of order $1, 2$ or 4, $R = \mathbb{Z}[i]$,

$$G_6^{(j)}(m) = \langle \tau_{P_1} \rangle \rtimes_j \langle e^{\frac{\pi i}{3}} \rangle \simeq \mathbb{C}_m \rtimes_j \mathbb{C}_6 = \langle a \rangle \rtimes_j \langle c \rangle =$$

$$= \langle a, \quad c \quad | \quad a^m = 1, \quad c^6 = 1, \quad cac^{-1} = a^j \rangle$$

for some $j \in \mathbb{Z}_m^*$ of order $1, 2, 3$ or 6.

Proof. Any finite translation group $G < (\mathcal{L}_E, +)$ lifts to a lattice $\widetilde{G} < (\widetilde{E} = \mathbb{C}, +)$ of rank 2, containing $\pi_1(E)$. By the Structure Theorem for finitely generated modules over the principal ideal domain $\mathbb{Z}$, there exists a $\mathbb{Z}$-basis λ_1, λ_2 of $\widetilde{G}$ and natural numbers $m, n \in \mathbb{N}$, such that

$$\widetilde{G} = \lambda_1 \mathbb{Z} + \lambda_2 \mathbb{Z}, \quad \pi_1(E) = m\lambda_1 \mathbb{Z} + mn\lambda_2 \mathbb{Z}.$$

As a result, $P_1 = \lambda_1 + \pi_1(E) \in (E, +)$ of order m and $P_2 = \lambda_2 + \pi_1(E) \in (E, +)$ of order mn generate the finite translation group $G = \widetilde{G}/\pi_1(E) \simeq \mathbb{C}_m \times \mathbb{C}_{mn}$.

If G is a finite non-translation subgroup of $Aut(E)$ then the linear part $\mathcal{L}(G)$ of G is a non-trivial subgroup of the units group R^*. Bearing in mind that

$$R^* = \begin{cases} \langle -1 \rangle \simeq \mathbb{C}_2 & \text{for } R \neq \mathbb{Z}[i], \mathcal{O}_{-3}, \\ \langle i \rangle \simeq \mathbb{C}_4 & \text{for } R = \mathbb{Z}[i], \\ \langle e^{\frac{\pi i}{3}} \rangle & \text{for } R = \mathcal{O}_{-3}, \end{cases}$$

one concludes that $\mathcal{G} = \langle e^{\frac{2\pi i}{s}} \rangle \simeq \mathbb{C}_s$ for some $s \in \{2,3,4,6\}$. Any lifting $g_0 = \tau_U e^{\frac{2\pi i}{s}} \in G$ of $\mathcal{L}(g_0) = e^{\frac{2\pi i}{s}}$ has a fixed point $P_0 \in E$. After moving the origin of E at P_0, one can assume that $g_0 = e^{\frac{2\pi i}{s}}$. Bearing in mind that the translation part $\mathcal{T}(G) = \ker(|_G)$, one observes that $G = \mathcal{T}(G)\langle e^{\frac{2\pi i}{s}} \rangle$. The inclusion $\mathcal{T}(G)\langle e^{\frac{2\pi i}{s}} \rangle \subseteq G$ is clear. For any $g \in G$ with $\mathcal{L}(g) = e^{\frac{2\pi ij}{s}}$ for some $0 \leq j \leq s-1$, one has $g \left(e^{\frac{2\pi i}{s}} \right)^{-j} \in \ker(\mathcal{L}|_G) = \mathcal{T}(G)$, so that $G \subseteq \mathcal{T}(G)\langle e^{\frac{2\pi i}{s}} \rangle$ and $G = \mathcal{T}(G)\langle e^{\frac{2\pi i}{s}} \rangle$. Note that $\mathcal{T}(G)$ is a normal subgroup of G with $\mathcal{T}(G) \cap \langle e^{\frac{2\pi i}{s}} \rangle = \{Id_E\}$, so that

$$G = \mathcal{T}(G) \rtimes \langle e^{\frac{2\pi i}{s}} \rangle$$

is a semi-direct product. As a result, there is an adjoint action

$$\mathrm{Ad} : \langle e^{\frac{2\pi i}{s}} \rangle \longrightarrow Aut(\mathcal{T}(G)),$$

$$\mathrm{Ad}_{e^{\frac{2\pi ij}{s}}}(\tau_{P_1}) = e^{\frac{2\pi ij}{s}} \tau_{P_1} e^{-\frac{2\pi ij}{s}} = \tau_{{}_s \frac{2\pi ij}{s} P_1}$$

of $\langle e^{\frac{2\pi i}{s}} \rangle$ on $\mathcal{T}(G)$, which is equivalent to the invariance of $\mathcal{T}(G)$ under a multiplication by $e^{\frac{2\pi i}{s}} \in R^*$. The translation group $'\mathcal{T}(G) = \langle \tau_{P_1}, \quad \tau_{P_2} \rangle$ has at most two generators, so that

$$G = \langle \tau_{P_1}, \quad \tau_{P_2} \rangle \rtimes \langle e^{\frac{2\pi i}{s}} \rangle$$

for some $s \in \{2,3,4,6\}$. If $s = 2$ and $\langle \tau_{P_1}, \tau_{P_2} \rangle \simeq \mathbb{C}_m \times \mathbb{C}_n = \langle \tau_{Q_1} \rangle \times \langle \tau_{Q_2} \rangle$, then $\mathrm{Ad}_{-1}(\tau_{Q_1}) = \tau_{-Q_k}$ for $1 \leq k \leq 2$. The residue classes $-1(\mathrm{mod} m) \in \mathbb{Z}_m^*$ and $-1(\mathrm{mod} n) \in \mathbb{Z}_n^*$ are order 1 or 2.

We claim that $G = \langle \tau_{P_1} \rangle \rtimes \langle e^{\frac{2\pi i}{s}} \rangle$ has at most two generators for $s \in \{3,4,6\}$. Indeed, $\tau_{P_1} \in \mathcal{T}(G)$ implies that $\mathrm{Ad}_{e^{\frac{2\pi i}{s}}}(\tau_{P_1}) = \tau_{e^{\frac{2\pi i}{s}} P_1} \in \mathcal{T}(G)$. For $s \in \{3,4,6\}$ the points P_1, $e^{\frac{2\pi i}{s}} P_1$ have $\mathbb{Z}$-linearly independent liftings from $\widetilde{\mathcal{T}(G)}$, so that $\mathcal{T}(G) = \langle \tau_{P_1}, \quad \tau_{P_2} \rangle = \langle \tau_{P_1}, \quad \tau_{e^{\frac{2\pi i}{s}} P_1} \rangle$. As a result,

$$G == \langle \tau_{P_1}, \quad e^{\frac{2\pi i}{s}} \tau_{P_1} e^{-\frac{2\pi i}{s}} \rangle \rtimes \langle e^{\frac{2\pi i}{s}} \rangle = \langle \tau_{P_1} \rangle \rtimes \langle e^{\frac{2\pi i}{s}} \rangle \simeq {}_m \rtimes_j \mathbb{C}_s = \langle a \rangle \rtimes_j \langle c \rangle =$$

$$\langle a, \quad c \quad | \quad a^m = 1, \quad c^s = 1, \quad cac^{-1} = a^j \rangle$$

for some $j \in \mathbb{Z}_m^*$ of order r, dividing $s \in \{3,4,6\}$.

□

Let us put $G_1^{(1,1)}(m,n) := G_1(m,n)$, in order to list the finite subgroups of $Aut(E)$ as $G_s^{(j_1,j_2)}(m,n)$ with $s \in \{1,2\}$ and $G_s^{(j)}(m)$ with $s \in \{3,4,6\}$.

Lemma 44. *Let H be a finite subgroup of $Aut(A)$ with abelian linear part $\mathcal{L}(H)$. Then:*

(i) there exists $S \in GL(2,\mathbb{C})$, such that all the elements of

$$S^{-1}HS = \{S^{-1}hS = (\tau_{U_1}\lambda_1\mathcal{L}(h), \tau_{U_2}\lambda_2\mathcal{L}) \quad | \quad h \in H\} < Aut(S^{-1}A)$$

have diagonal linear parts;

(ii) if $F_1 = S^{-1}(E \times \check{o}_E)$, $F_2 = S^{-1}(\check{o}_E \times E)$ *then* $S^{-1}A = F_1 \times F_2$ *and the canonical projections*

$$\mathrm{pr}_k : S^{-1}HS \longrightarrow Aut(F_k),$$

$$\mathrm{pr}_k(\tau_{U_1}\lambda_1\mathcal{L}(h), \tau_{U_2}\lambda_2\mathcal{L}(h)) = \tau_{U_k}\lambda_k\mathcal{L}(h),$$

are group homomorphisms with $\mathrm{pr}_k(S^{-1}HS) \simeq G_s^{(j_1,j_2)}(m,n)$, $s \in \{1,2\}$ *or* $G_s^{(j)}$, $s \in \{3,4,6\}$*;*

(iii) $S^{-1}HS = \ker(\mathrm{pr}_2)\langle h_1, \ldots, h_t\rangle$ *for any liftings* $h_j = (\alpha_j, \beta_j) \in S^{-1}HS$ *of the generators* $\beta_1, \ldots, \beta_t$ *of* $\mathrm{pr}_2(S^{-1}HS)$, $1 \leq t \leq 3$*;*

(iv) $S^{-1}A/\ker(\mathrm{pr}_2) = C_1 \times F_2$, *where* C_1 *is an elliptic curve for a translation subgroup* $\ker(\mathrm{pr}_2) < (\mathcal{T}_{F_1}, +) < Aut(F_1)$ *or a rational curve for a non-translation subgroup* $\ker(\mathrm{pr}_2) < Aut(F_1)$, $\ker(\mathrm{pr}_2) \setminus (\mathcal{T}_{F_1}, +) \neq \emptyset$*;*

(v) $A/H \simeq (C_1 \times F_2)/G$ *for '*

$$G := \langle h_1, \ldots, h_t\rangle/(\langle h_1, \ldots, h_t\rangle \cap \ker(\mathrm{pr}_2))$$

with isomorphic second projection

$$\overline{\mathrm{pr}_2} : G \longrightarrow \mathrm{pr}_2(S^{-1}HS)$$

and first projection

$$\overline{\mathrm{pr}_1} : G \to \overline{\mathrm{pr}_1}(G) < Aut(C_1)$$

with kernel $\ker(\overline{\mathrm{pr}_1}|_G) \simeq \ker(\mathrm{pr}_1|_{S^{-1}HS})$.

Proof. (i) It is well known that for any finite set $\{\mathcal{L}(h) \mid h \in H\}$ of commuting matrices, there exists $S \in GL(2,\mathbb{C})$, such that

$$S^{-1}\mathcal{L}(h)S = \mathcal{L}(S^{-1}hS) = \begin{pmatrix} \lambda_1\mathcal{L}(h) & 0 \\ 0 & \lambda_2\mathcal{L}(h) \end{pmatrix}$$

are diagonal for all $h \in H$. Namely, if there is $h_o \in H$, whose linear part $\mathcal{L}(h_o)$ has two different eigenvalues $\lambda_1\mathcal{L}(h_o) \neq \lambda_2\mathcal{L}(h_o)$, then one takes the j-th column of $S \in \mathbb{Q}(\sqrt{-1})_{2\times 2}$ to be an eigenvector, associated with $\lambda_j\mathcal{L}(h_o)$, $1 \leq j \leq 2$. The conjugate $S^{-1}\mathcal{L}(h_o)S$ is a diagonal matrix. It suffices to show that v_j are eigenvectors of all $\mathcal{L}(h)$, in order to conclude that $S^{-1}\mathcal{L}(h)S$ are diagonal, as the matrices of $\mathcal{L}(h)$ with respect to the basis v_1, v_2 of $\mathbb{C}^2$. Indeed, for any $h \in H$ the relation $\mathcal{L}(h)\mathcal{L}(h_o) = \mathcal{L}(h_o)\mathcal{L}(h)$ implies that

$$\lambda_j\mathcal{L}(h_o)[\mathcal{L}(h)v_j] = \mathcal{L}(h)\mathcal{L}(h_o)v_j = \mathcal{L}(h_o)[\mathcal{L}(h)v_j.$$

Therefore $\mathcal{L}(h)v_j$ is an eigenvector of $\mathcal{L}(h_o)$ with associated eigenvalue $\lambda_j\mathcal{L}(h_o$, so that $\mathcal{L}(h)v_j$ is proportional to v_j, i.e., $\mathcal{L}(h)v_j = c_hv_j$ for some $c_h \in \mathbb{C}$, which turns to be an eigenvalue $c_h = \lambda_j\mathcal{L}(h)$ of $\mathcal{L}(h)$. If $\lambda_1\mathcal{L}(h) = \lambda_2\mathcal{L}(h)$ for $\forall h \in H$ then all $\mathcal{L}(h)$ are scalar matrices. In particular, $\mathcal{L}(h)$ are diagonal.

(ii) Note that the direct product $A = E \times E$ of elliptic curves coincides with their direct sum. If

$$S^{-1}A := S^{-1}\widetilde{A}/S^{-1}\pi_1(A) = \mathbb{C}^2/S^{-1}\pi_1(A),$$

then $S^{-1}A \to S^{-1}A$ is an isomorphism of abelian surfaces and

$$S^{-1}(A) = S^{-1}(E \times E) = S^{-1}[(E \times \check{o}_E) \times (\check{o}_E \times E)] =$$

$$= S^{-1}(E \times \check{o}_E) \times S^{-1}(\check{o}_E \times E) = F_1 \times F_2.$$

The canonical projections $\mathrm{pr}_k : S^{-1}HS \to Aut(F_k)$ are group homomorphisms, according to

$$\mathrm{pr}_k((\tau_{V_1}\lambda_1\mathcal{L}(g), \tau_{V_2}\lambda_2\mathcal{L}(g))(\tau_{U_1}\lambda_1\mathcal{L}(h), \tau_{U_2}\lambda_2\mathcal{L}(h)) =$$
$$= \mathrm{pr}_k(\tau_{V_1+\lambda_1\mathcal{L}(g)U_1}(\lambda\mathcal{L}(g).\lambda_1\mathcal{L}(h)), \tau_{V_2+\lambda_2\mathcal{L}(g)U_2}(\lambda_2\mathcal{L}(g).\lambda_2\mathcal{L}(h))) =$$
$$= \tau_{V_k\lambda_k\mathcal{L}(g)U_k}(\lambda_k\mathcal{L}(g).\lambda_k\mathcal{L}(h)) = (\tau_{V_k}\lambda_k\mathcal{L}(g))(\tau_{U_k}\lambda_j\mathcal{L}(h)) =$$
$$= \mathrm{pr}_k(\tau_{V_1}\lambda_1\mathcal{L}(g), \tau_{V_2}\lambda_2\mathcal{L}(h)).(\mathrm{pr}_k(\tau_{U_1}\lambda_1\mathcal{L}(h), \tau_{U_2}\lambda_2\mathcal{L}(h))$$

for $\forall g, h \in H$ with $S^{-1}gS = \tau_{(V_1,V_2)}\mathcal{L}(S^{-1}gS)$, $S^{-1}hS = \tau_{(U_1,U_2)}\mathcal{L}(S^{-1}hS)$. The image $\mathrm{pr}_k(S^{-1}HS)$ of $S^{-1}HS$ is a finite subgroup of $Aut(F_k)$ for $1 \leq k \leq 2$.

(iii) If $h_j = (\alpha_j, \beta_j) \in S^{-1}HS$ are liftings of the generators β_j of $\mathrm{pr}_2(S^{-1}HS)$, then $\ker(\mathrm{pr}_2)\langle h_1, \ldots, h_t\rangle$ is a subgroup of $S^{-1}HS$, as far as $\ker(\mathrm{pr}_2)$ is a normal subgroup of $S^{-1}HS$. For any $\mathrm{pr}_2(S^{-1}hS) = \beta_1^{m_1} \ldots \beta_t^{m_t}$ for some $m_i \in \mathbb{Z}$, one has $(S^{-1}HS)(h_1^{m_1} \ldots h_t^{m_t}) \in \ker(\mathrm{pr}_2)$, so that $S^{-1}hS \in \ker(\mathrm{pr}_2)\langle h_1, \ldots, h_t\rangle$and $S^{-1}HS = \ker(\mathrm{pr}_2)\langle h_1, \ldots, h_t\rangle$.

(iv) The subgroup $\ker(\mathrm{pr}_2)$ of $S^{-1}HS$ acts identically on F_2 and can be thought of as a subgroup of $Aut(F_1)$, $\mathrm{pr}_1(\ker(\mathrm{pr}_2)) \simeq \ker(\mathrm{pr}_2)$. Thus,

$$S^{-1}A/\ker(\mathrm{pr}_2) \simeq [F_1/\mathrm{pr}_1(\ker(\mathrm{pr}_2)] \times F_2 = C_1 \times F_2$$

with an elliptic curve C_1 exactly when $\mathrm{pr}_1(\ker(\mathrm{pr}_2))$ is a translation subgroup of $Aut(F_1)$ or a rational curve C_1 for a non-translation subgroup $\mathrm{pr}_1(\ker(\mathrm{pr}_2))$ of the automorphism group $Aut(F_1)$ of F_1.

(v) Since $\ker(\mathrm{pr}_2)$ is a normal subgroup of $S^{-1}HS$ with quotient

$$S^{-1}HS/\ker(\mathrm{pr}_2) = [\ker(\mathrm{pr}_2)\langle h_1, \ldots, h_t\rangle]/\ker(\mathrm{pr}_2) =$$

$$= \langle h_1, \ldots, h_t\rangle/(\langle h_1, \ldots, h_t\rangle \cap \ker(\mathrm{pr}_2)) = G,$$

one has

$$A/H \simeq (S^{-1}A)/(S^{-1}HS) \simeq [S^{-1}A/\ker(\mathrm{pr}_2)]/[S^{-1}HS/\ker(\mathrm{pr}_2)] = (C_1 \times F_2)/G.$$

By the First Isomorphism Theorem, the epimorphism $\mathrm{pr}_2 : S^{-1}HS \to \mathrm{pr}_2(S^{-1}HS)$ gives rise to an isomorphism

$$\overline{\mathrm{pr}_2} : S^{-1}HS/\ker(\mathrm{pr}_2) = G \longrightarrow \mathrm{pr}_2(S^{-1}HS).$$

The homomorphism $\mathrm{pr}_1 : S^{-1}HS \to Aut(F_1)$ induces a homomorphism

$$\overline{\mathrm{pr}_1} : S^{-1}HS/\ker(\mathrm{pr}_2) = G \longrightarrow Aut(F_1)/\mathrm{pr}_1(\ker(\mathrm{pr}_2)) \simeq Aut(C_1).$$

in the automorphism group of $C_1 = F_1/\mathrm{pr}_1(\ker(\mathrm{pr}_2))$. It suffices to show that the kernel

$$\ker(\overline{\mathrm{pr}_1}) = \{S^{-1}hS\ker(\mathrm{pr}_2) \mid \mathrm{pr}_1(S^{-1}hS) \in \mathrm{pr}_1\ker(\mathrm{pr}_2)\} =$$
$$[\ker(\mathrm{pr}_2)\ker(\mathrm{pr}_1)]/\ker(\mathrm{pr}_2),$$

since

$$[\ker(\mathrm{pr}_2)\ker(\mathrm{pr}_1)]/\ker(\mathrm{pr}_2) \simeq \ker(\mathrm{pr}_1)/[\ker(\mathrm{pi}_2) \cap \ker(\mathrm{pr}_1)] = \ker(\mathrm{pr}_1).$$

Indeed, if there exists $S^{-1}h_1S(\mathrm{pr}_1(S^{-1}hS), Id_{F_2}) \in \ker(\mathrm{pr}_2)$ then

$$S^{-1}(h_1^{-1}h)S = (Id_{F_1}, \mathrm{pr}_2(S^{-1}hS)) \in S^{-1}HS \cap \ker(\mathrm{pr}_1),$$

so that $S^{-1}hS \in S^{-1}h_1S\ker(\mathrm{pr}_1) \subset \ker(\mathrm{pr}_2)\ker(\mathrm{pr}_1)$ for $\forall S^{-1}hS\ker(\mathrm{pr}_2) \in \ker(\overline{\mathrm{pr}_1})$. Conversely, any element of $[\ker(\mathrm{pr}_2)\ker(\mathrm{pr}_1)]/\ker(\mathrm{pr}_2)$ is of the form

$$(g_1, Id_{F_2})(Id_{F_1}, g_2)\ker(\mathrm{pr}_2) = (g_1, g_2)\ker(\mathrm{pr}_2)$$

for some $(g_1, Id_{F_2}), (Id_{F_1}, g_2) \in S^{-1}HS \cap [Aut(F_1) \times Aut(F_2)]$, so that

$$\mathrm{pr}_1(g_1, g_2) = g_1 = \mathrm{pr}_1((g_1, Id_{F_2})) \in \mathrm{pr}_1\ker(\mathrm{pr}_2)$$

reveals that $(g_1, g_2)\ker(\mathrm{pr}_2) \in \ker(\overline{\mathrm{pr}_1})$.

□

According to Lemma 43, the finite automorphism groups of elliptic curves have at most three generators. Combining with Lemma 44(iii), one concludes that the finite subgroups H of $Aut(E \times E)$ with abelian linear part $\mathcal{L}(H)$ have at most six generators. Their linear parts $\mathcal{L}(H)$ have at most two generators.

Lemma 45. *Let $h = \tau_{(U,V)}\mathcal{L}(h)$ be an automorphism of $A = E \times E$ and $w = (u, v) \in \mathbb{C}^2 = \widetilde{A}$ be a lifting of $(u, v) + \pi_1(A) = (U, V) \in A$. Then h has no fixed points on A if and only if for any $\mu = (\mu_1, \mu_2) \in \pi_1(A)$ the affine-linear transformation*

$$\widetilde{h}(w, \mu) = \tau_{w+\mu}\mathcal{L}(h) \in Aff(\mathbb{C}^2, R) := (\mathbb{C}^2, +) \rtimes GL(2, R)$$

has no fixed points on $\mathbb{C}^2$.

Proof. The statement of the lemma is equivalent to the fact that $Fix_A(h) \neq q\emptyset$ exactly when $Fix_{\mathbb{C}^2}(\widetilde{h}(w, \mu)) \neq \emptyset$ for some $\mu \in \pi_1(A)$. Indeed, if $(p, q) \in Fix_{\mathbb{C}^2}(\widetilde{h}(w, \mu))$ then $(P, Q) = (p + \pi_1(E), q + \pi_1(E)) \in A$ is a fixed point of h, according to

$$h(P, Q) = \mathcal{L}(h)\begin{pmatrix} P \\ Q \end{pmatrix} + \begin{pmatrix} U \\ V \end{pmatrix} = \mathcal{L}(h)\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} \pi_1(E) \\ \pi_1(E) \end{pmatrix} =$$

$$= \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} \pi_1(E) \\ \pi_1(E) \end{pmatrix} = \begin{pmatrix} P \\ Q \end{pmatrix}.$$

Conversely, if

$$\mathcal{L}(h) \begin{pmatrix} P \\ Q \end{pmatrix} + \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} P \\ Q \end{pmatrix},$$

then for any lifting $(p,q) \in \mathbb{C}^2$ of $(P,Q) = (p + \pi_1(E), q + \pi_1(E))$, one has

$$\mathcal{L}(h) \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} \pi_1(E) \\ \pi_1(E) \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} \pi_1(E) \\ \pi_1(E) \end{pmatrix}.$$

In other words,

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} := \mathcal{L}(h) \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} p \\ q \end{pmatrix} \in \begin{pmatrix} \pi_1(E) \\ \pi_1(E) \end{pmatrix}$$

and $(p,q) \in Fix_{\mathbb{C}^2}(\widetilde{h}(w, -\mu))$.

□

Now we are ready to characterize the automorphisms $h \in Aut(A)$ without fixed points

Lemma 46. *An automorphism $h = \tau_{(U,V)}\mathcal{L}(h) \in Aut(A) \setminus (\mathcal{T}_A, +)$ acts without fixed points on $A = E \times E$ if and only if its linear part $\mathcal{L}(h)$ has eigenvalues $\lambda_1 \mathcal{L}(h) = 1$, $\lambda_2 \mathcal{L}(h) \neq 1$ and*

$$\mathcal{L}(h) \begin{pmatrix} u \\ v \end{pmatrix} \neq \lambda_2 \begin{pmatrix} u \\ v \end{pmatrix}$$

for any lifting $(u,v) \in \mathbb{C}^2$ of $(u + \pi_1(E), v + \pi_1(E)) = (U,V)$.

Proof. The fixed points $(P,Q) \in A$ of $h = \tau_{(U,V)}\mathcal{L}(h)$ are described by the equality

$$(\mathcal{L}(h) - I_2) \begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} -U \\ -V \end{pmatrix}. \tag{19}$$

If $\det(\mathcal{L}(h) - I_2) \neq 0$ or $1 \in \mathbb{C}$ is not an eigenvalues of $\mathcal{L}(h)$, then consider the adjoint matrix

$$(\mathcal{L}(h) - I_2)^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in R_{2\times 2} \quad \text{of}$$

$$\mathcal{L}(h) - I_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R_{2\times 2}.$$

According to $(\mathcal{L}(h) - I_2)^*(\mathcal{L}(h) - I_2) = \det(\mathcal{L}(h) - I_2)I_2 = (\mathcal{L}(h) - I_2)(\mathcal{L}(h) - I_2)^*$, one obtains

$$\det(\mathcal{L}(h) - I_2) \begin{pmatrix} P \\ Q \end{pmatrix} = (\mathcal{L}(h) - I_2)^*(\mathcal{L}(h) - I_2) \begin{pmatrix} u \\ v \end{pmatrix} = -(\mathcal{L}(h) - I_2)^* \begin{pmatrix} U \\ V \end{pmatrix}. \tag{20}$$

Then for an arbitrary lifting $(u_1, v_1) \in \mathbb{C}^2$ of

$$\left(\begin{array}{c} u_1 + \pi_1(E) \\ v_1 + \pi_1(E) \end{array} \right) = \left(\begin{array}{c} U_1 \\ V_1 \end{array} \right) := -(\mathcal{L}(h) - I_2)^* \left(\begin{array}{c} U \\ V \end{array} \right),$$

the point

$$(p,q) = \left(\frac{u_1}{\det(\mathcal{L}(h) - I_2)}, \frac{v_1}{\det(\mathcal{L}(h) - I_2)} \right) \in {}^2$$

descends to $(P,Q) = (p + \pi_1(E), q + \pi_1(E))$, subject to (20). As a result,

$$(\mathcal{L}(h) - I_2) \left(\begin{array}{c} P \\ Q \end{array} \right) = \frac{1}{\det(\mathcal{L}(h) - I_2)} (\mathcal{L}(h) - I_2) \left(\begin{array}{c} u_1 \\ v_1 \end{array} \right) + \left(\begin{array}{c} \pi_1(E) \\ \pi_1(E) \end{array} \right) =$$

$$= \left(\begin{array}{c} u \\ v \end{array} \right) + \left(\begin{array}{c} \pi_1(E) \\ \pi_1(E) \end{array} \right)$$

and $(P,Q) \in Fix_A(h)$.

From now on, let us suppose that the linear part $\mathcal{L}(h) \in GL(2,R)$ of $h \in Aut(A) \setminus (\mathcal{T}_A, +)$ has eigenvalues $\lambda_1 \mathcal{L}(h) = 1$ and $\lambda_2 \mathcal{L}(h) = \det \mathcal{L}(h) \in R^* \setminus \{1\}$. We claim that a lifting $(u,v) \in {}^2$ of $(u + \pi_1(E), v + \pi_1(E)) = (U,V) \in A$ satisfies

$$\mathcal{L}(h) \left(\begin{array}{c} u \\ v \end{array} \right) = \lambda_2 \mathcal{L}(h) \left(\begin{array}{c} u \\ v \end{array} \right)$$

if and only if there exists $(p,q) \in \mathbb{C}^2$ with

$$(\mathcal{L}(h) - I_2) \left(\begin{array}{c} p \\ q \end{array} \right) = \left(\begin{array}{c} -u \\ -v \end{array} \right),$$

which amounts to $(p,q) \in Fix_{\mathbb{C}^2}(\tau_{(u,v)} \mathcal{L}(h))$. To this end, let us view $\mathcal{L}(h) : \mathbb{C}^2 \to \mathbb{C}^2$ as a linear operator in $\mathbb{C}^2$ and reduce the claim to the equivalence of $(-u,-v) \in \ker(\mathcal{L}(h) - \lambda_2 \mathcal{L}(h) I_2)$ with $(-u,-v) \in Im(\mathcal{L}(h) - I_2)$. In other word, the statement of the lemma reads as $\ker(\mathcal{L}(h) - \lambda_2 \mathcal{L}(h) I_2) = Im(\mathcal{L}(h) - I_2)$ for the linear operators $\mathcal{L}(h) - \lambda_2 \mathcal{L}(h) I_2$ and $\mathcal{L}(h) - I_2$ in $\mathbb{C}^2$. By Hamilton -Cayley Theorem, $\mathcal{L}(h) \in \mathbb{C}_{2\times 2}$ is a root of its characteristic polynomial

$$\mathcal{X}_{\mathcal{L}(h)}(\lambda) = (\lambda - \lambda_1 \mathcal{L}(h))(\lambda - 1).$$

Thus,

$$(\mathcal{L}(h) - \lambda_2 \mathcal{L}(h) I_2) Im(\mathcal{L}(h) - I_2) = \{(0,0)\}$$

is the zero subspace of $\mathbb{C}^2$ and $Im(\mathcal{L}(h) - I_2) \subseteq \ker(\mathcal{L}(h) - \lambda_2 \mathcal{L}(h) I_2)$. However, $\dim Im(\mathcal{L}(h) - I_2) = \mathrm{rk}(\mathcal{L}(h) - I_2) = 1$ and

$$\dim \ker(\mathcal{L}(h) - \lambda_2 \mathcal{L}(h)) = 2 - \mathrm{rk}(\mathcal{L}(h) - \lambda_2 \mathcal{L}(h) I_2) = 2 - 1 = 1,$$

so that $Im(\mathcal{L}(h) - I_2) = \ker(\mathcal{L}(h) - \lambda_2 \mathcal{L}(h) I_2)$.

□

Corollary 47. *Let $H = \mathcal{T}(h)\langle h_o \rangle$ be a finite subgroup of $Aut(A)$ for some $h_o \in H$ with*

$$\lambda_1 \mathcal{L}(h_o) = 1, \quad \lambda_2 \mathcal{L}(h_o) = e^{\frac{2\pi i}{s}}, \quad s \in \{2,3,4,6\},$$

$S \in GL(2, \mathbb{Q}(\sqrt{-d}))$ be a diagonalizing matrix for h_o and

$$S^{-1} h_o S = \left(\tau_W, e^{\frac{2\pi i}{s}} \right)$$

after appropriate choice of an origin of $S^{-1}A = F_1 \times F_2$, $F_1 = S^{-1}(E \times \check{o}_E)$, $F_2 = S^{-1}(\check{o}_E \times E)$. Then A/H is a hyper-elliptic surface if and only if the kernel $\ker(\mathrm{pr}_1)$ of the first canonical projection $\mathrm{pr}_1 : S^{-1}HS \to Aut(F_1)$ is a translation subgroup of $Aut(F_2)$. If so, then

$$S^{-1}A/[\ker(\mathrm{pr}_2)\ker(\mathrm{pr}_1)] \simeq C_1 \times C_2$$

for some elliptic curves C_1, C_2 and

$$A/H \simeq (C_1 \times C_2)/G,$$

where the group G is isomorphic to some of the groups

$$G_2^{HE} = \langle (\tau_{U_1}, -1) \rangle \simeq \mathbb{C}_2$$

with $U_1 \in C_1^{2-\mathrm{tor}} \setminus \{\check{o}_{C_1}\}$,

$$G_{2,2}^{HE} = \langle \tau_{(P_1,Q_1)} \rangle \times \langle (\tau_{U_1}, -1) \rangle \simeq \mathbb{C}_2 \times \mathbb{C}_2$$

with $P_1, U_1 \in C_1^{2-\mathrm{tor}} \setminus \{\check{o}_{C_1}\}$, $Q_1 \in C_2^{2-\mathrm{tor}}$,

$$G_3^{HE} = \langle (\tau_{U_1}, e^{\frac{2\pi i}{3}}) \rangle \simeq \mathbb{C}_3$$

with $R = \mathcal{O}_{-3}$, $U_1 \in C_1^{3-\mathrm{tor}} \setminus C_1^{2-\mathrm{tor}}$,

$$G_{3,3}^{HE} = \langle \tau_{(P_1,Q_1)} \rangle \times \langle \left(\tau_{U_1}, e^{\frac{2\pi i}{3}} \right) \rangle \simeq \mathbb{C}_3 \times \mathbb{C}_3$$

with $R = \mathcal{O}_{-3}$, $P_1, U_1 \in C_1^{3-\mathrm{tor}} \setminus C_1^{2-\mathrm{tor}}$, $Q \in C_2^{3-\mathrm{tor}} \setminus \{\check{o}_{C_2}\}$,

$$G_4^{HE} = \langle (\tau_{U_1}, i) \rangle \simeq \mathbb{C}_4$$

with $R = \mathbb{Z}[i]$, $U_1 \in C_1^{4-\mathrm{tor}} \setminus (C_1^{2-\mathrm{tor}} \cup C_1^{3-\mathrm{tor}})$,

$$G_{4,4}^{HE} = \langle \tau_{(P_1,Q_1)} \rangle \times \langle (\tau_{U_1}, i) \rangle \simeq \mathbb{C}_2 \times \mathbb{C}_4$$

with $R = \mathbb{Z}[i]$, $P_1 \in C_1^{2-\mathrm{tor}} \setminus \{\check{o}_{C_1}\}$, $Q_1 \in C_2^{(1_i)-\mathrm{tor}} \setminus \{\check{o}_{C_2}\}$, $U_1 \in C_1^{4-\mathrm{tor}} \setminus (C_1^{2-\mathrm{tor}} \cup C_1^{3-\mathrm{tor}})$,

$$G_6^{HE} = \langle \left(\tau_{U_1}, e^{\frac{\pi i}{3}} \right) \rangle \simeq \mathbb{C}_6$$

with $R = \mathcal{O}_{-3}$, $U_1 \in C_1^{6-\mathrm{tor}} \setminus (C_1^{3-\mathrm{tor}} \cup C_1^{4-\mathrm{tor}} \cup C_1^{5-\mathrm{tor}})$.

In the notations from Proposition 30, A/H is a hyper-elliptic surface exactly when $H \simeq S^{-1}HS$ is isomorphic to some of the groups:

$$H_2^{HE}(m,n) = \langle (\tau_{M_j}, Id_{F_2}), (Id_{F_1}, \tau_{N_k}), (\tau_W, -1) \mid 1 \leq j \leq m, \quad 1 \leq k \leq n \rangle$$

with $W \not\in \ker(\mathrm{pr}_2)$, $2W \in \ker(\mathrm{pr}_2)$, $\mathcal{L}(H_2^{HE}(m,n)) \simeq H_{C1}(1) \simeq \mathbb{C}_2$,

$$H_{2,2}^{HE}(m,n) = \langle (\tau_{M_j}, Id_{F_2}), \quad (Id_{F_1}, \tau_{N_k}), \quad \tau_{(X,Y)}, \quad (\tau_W, -1) \mid 1 \leq j \leq m, \quad 1 \leq k \leq n \rangle$$

with $2X.2W \in \ker(\mathrm{pr}_2)$, $X, W \not\in \ker(\mathrm{pr}_2)$, $2Y \in \ker(\mathrm{pr}_1)$, $Y \not\in \ker(\mathrm{pr}_1)$, $\mathcal{L}(H_{2,2}^{HE}(m,n)) \simeq H_{C1}(1) \simeq \mathbb{C}_2$

$$H_3^{HE}(m,n) = \langle (\tau_{M_j}, If_{F_2}), \quad (Id_{F_1}, \tau_{N_k}), \quad \left(\tau_W, e^{\frac{2\pi i}{3}}\right) \quad \mid \quad 1 \leq j \leq m, \quad 1 \leq k \leq n \rangle$$

with $R = \mathcal{O}_{-3}$, $3W \in \ker(\mathrm{pr}_2)$, $2W \not\in \ker(\mathrm{pr}_2)$, $\mathcal{L}(H_3^{HE}(m,n)) \simeq H_{C1}(2) \simeq \mathbb{C}_3$,

$$H_{3,3}^{HE}(m,n) = \langle (\tau_{M_j}, Id_{F_2}), (Id_{F_1}, \tau_{N_k}), \tau_{(X,Y)}, \left(\tau_W, e^{\frac{2\pi i}{3}}\right) \mid 1 \leq j \leq m, 1 \leq k \leq n \rangle$$

with $R = \mathcal{O}_{-3}$, $3X, 3W \in \ker(\mathrm{pr}_2)$, $2X, 2W \not\in \ker(\mathrm{pr}_2)$, $3Y \in \ker(\mathrm{pr}_1)$, $Y \not\in \ker(\mathrm{pr}_1)$, $\mathcal{L}(H_{3,3}^{HE}(m,n)) \simeq H_{C1}(2) \simeq \mathbb{C}_3$,

$$H_4^{HE}(m,n) = \langle (\tau_{M_j}, Id_{F_2}), \quad (Id_{F_1}, \tau_{N_k}), \quad (\tau_W, i) \mid 1 \leq j \leq m, \quad 1 \leq k \leq n \rangle$$

with $R = \mathbb{Z}[i]$, $4W \in \ker(\mathrm{pr}_2)$, $2W, 3W \not\in \ker(\mathrm{pr}_2)$, $\mathcal{L}(H_4^{HE}(m,n)) \simeq H_{C1}(e) \simeq \mathbb{C}_4$,

$$H_{4,4}^{HE}(m,n) = \langle (\tau_{M_j}, Id_{F_2}), \quad (Id_{F_1}, \tau_{N_k}), \quad \tau_{(X,Y)}, \quad (\tau_W, i) \mid 1 \leq j \leq m, \quad 1 \leq k \leq n \rangle$$

with $R = \mathbb{Z}[i]$, $2X \in \ker(\mathrm{pr}_2)$, $X \not\in \ker(\mathrm{pr}_2)$, $(1_i)Y \in \ker(\mathrm{pr}_1)$, $Y \not\in \ker(\mathrm{pr}_1)$, $4W \in \ker(\mathrm{pr}_2)$, $2W, 3W \not\in \ker(\mathrm{pr}_2)$, $\mathcal{L}(H_{4,4}^{HE}(m,n) \simeq H_{C1}(3) \simeq \mathbb{C}_4$,

$$H_6^{HE}(m,n) = \langle (\tau_{M_j}, Id_{F_2}), \quad (Id_{F_1}, \tau_{N_k}), \left(\tau_W, e^{\frac{\pi i}{3}}\right) \quad \mid \quad 1 \leq j \leq m, \quad 1 \leq k \leq n \rangle$$

with $R = \mathcal{O}_{-3}$, $6W \in \ker(\mathrm{pr}_2)$, $3W, 4W, 5W \not\in \ker(\mathrm{pr}_2)$, *where* $m, n \in \{0, 1, 2\}$.

Proof. In the notations from Lemma 44, the kernel $\ker(\mathrm{pr}_2)$ of the second canonical projection $\mathrm{pr}_2 : S^{-1}HS \to Aut(F_2)$ is a translation group, so that

$$S^{-1}A \to S^{-1}A/\ker(\mathrm{pr}_2) = C_1 \times F_2$$

is unramified and C_1 is an elliptic curve. Thus, the covering $A \to A/H$ is unramified if and only if $C_1 \times F_2 \to (C_1 \times F_2)/G \simeq A/H$ is unramified. In other words, A/H is a hyper-elliptic surface exactly when the group G has no fixed point on $C_1 \times F_2$. For any $g \in G$ with $\mathcal{L}(g) \neq I_2$ the second component $\overline{\mathrm{pr}_2}(g) = \tau_{V_2} e^{\frac{2\pi ij}{s}}$ for some $1 \leq j \leq s-1$, $V_2 \in F_2$ has a fixed point on F_2. Towards $Fix_{C_1 \times F_2}(g) = \emptyset$ one has to have

$\overline{\mathrm{pr}_1}(g) \neq Id_{C_1}$, so that $\ker(\overline{\mathrm{pr}_1}) \subseteq \mathcal{T}(G) = G \cap \ker(\mathcal{L})$ and $\ker(\mathrm{pr}_1) \subseteq \mathcal{H} = H \cap \ker(\mathcal{L})$ are translation groups. The covering $C_1 \times F_2 \to (C_1 \times F_2)/\ker(\overline{\mathrm{pr}_1}) = C_1 \times C_2$ is unramified, C_2 is an elliptic curve and A/H is a hyper-elliptic surface exactly when $G_o = G/\ker(\overline{\mathrm{pr}_1})$ has no fixed points on $(C_1 \times F_2)/\ker(\overline{\mathrm{pr}_1})$. The canonical projections

$$\overline{\mathrm{pr}_1} : G_o \longrightarrow Aut(C_1) \quad \text{and} \quad \overline{\mathrm{pr}_2} : G_o \longrightarrow Aut(C_2)$$

are injective. Since $\overline{\mathrm{pr}_1}(G_o)$ is a translation subgroup of $Aut(C_1)$, the group $G_o \simeq \overline{\mathrm{pr}_1}$ is abelian and has at most two generators. As a result, $\overline{\mathrm{pr}_2}(G_o) \simeq G_o$ is an abelian subgroup of $Aut(C_2)$ with at most two generators and non-trivial linear part $\mathcal{L}\left(\overline{\mathrm{pr}_2}(G_o)\right) = \langle e^{\frac{2\pi i}{s}} \rangle \simeq \mathbb{C}_s$ for some $s \in \{2,3,4,6\}$. According to Lemma 43,

$$\overline{\mathrm{pr}_2}(G_o) \simeq \langle \tau_{Q_1} \rangle \times \langle e^{\frac{2\pi i}{s}} \rangle \simeq \mathbb{C}_m \times \mathbb{C}_s$$

for some $Q_1 \in C_2$ with $\tau_{Q_1} = \mathrm{Ad}_{e^{\frac{2\pi i}{s}}}(\tau_{Q_1}) = \tau_{e^{\frac{2\pi i}{s}} Q_1}$. In other words, the point $Q_1 \in C_2^{\left(e^{\frac{2\pi i}{s}}-1\right)-\mathrm{tor}} \setminus \{\check{o}_{C_2}\}$. If $s=2$ then any $Q_1 \in C_2^{2-\mathrm{tor}}$ works out and the order of $Q_1 \in (C_2,+)$ is $m=2$.

For $s=3$ note that the endomorphism ring of C_2 is $End(C_2) = \mathcal{O}_{-3}$. Therefore the fundamental group $\pi_1(C_2) = c(\mathbb{Z} + \tau\mathbb{Z})$ for some $\tau \in \mathbb{Q}(\sqrt{-3})$ and $c \in \mathbb{C}^*$. By $c \in \pi_1(C_2)$ and $e^{\frac{\pi i}{3}} \in End(C_2)$ one has $e^{\frac{\pi i}{3}} c \in \pi_1(C_2)$. Due to the linear independence of c and $e^{\frac{\pi i}{3}}$ over $\mathbb{Z}$, one has $\pi_1(C_2) = c\mathbb{Z} + e^{\frac{\pi i}{3}} c\mathbb{Z} = c\mathcal{O}_{-3}$. For $\alpha = e^{\frac{2\pi i}{3}} - 1 = -\frac{3}{2} + \frac{\sqrt{3}}{2} i$ the equation

$$\alpha \left(x + e^{\frac{\pi i}{3}} y\right) = \left(a + e^{\frac{\pi i}{3}} b\right) c \quad \text{for some} \quad a,b \in \mathbb{Z}$$

has a solution $x = \frac{-a+b}{3}$, $y = \frac{-a-2b}{3}$. Note that $x(\mathrm{mod}\mathbb{Z}) \equiv y(\mathrm{mod}\mathbb{Z})$ and

$$\left(x + e^{\frac{\pi i}{3}} y\right) c \left(\mathrm{mod}\mathbb{Z} + e^{\frac{\pi i}{3}}\mathbb{Z}\right) = \left(x + e^{\frac{\pi i}{3}}\right) (\mathrm{mod}\pi_1(C_2)) \in$$

$$\left\{\check{o}_{C_2}, \quad \pm\left(1 + e^{\frac{\pi i}{3}}\right)(\mathrm{mod}\pi_1(C_2))\right\} = C_2^{3-\mathrm{tor}},$$

whereas $C_2^{\alpha-\mathrm{tor}} = C_2^{3-\mathrm{tor}}$ and $m=3$. Thus, $Q_1 \in C_2^{3-\mathrm{tor}} \setminus \{\check{o}_{C_2}\}$ in the case of $s=3$.

If $s=4$ then $End(C_2) = \mathbb{Z}[i]$ and $\pi_1(C_2) = c\mathbb{Z}[i]$ for some $c \in \mathbb{C}^*$. The equation $(i-1)(x+iy)c = (a+bi)c$ for some $a,b \in \mathbb{Z}$ has a solution $x = \frac{-a+b}{2}$, $y = \frac{-a-b}{2}$ with

$$(x+iy)c(\mathrm{mod}\mathbb{Z}[i]) = x + iy(\mathrm{mod}\pi_1(C_2)) \in$$

$$\left\{\check{o}_{C_2}, \left(\frac{1+i}{2}\right) c(\mathrm{mod}\pi_1(C_2))\right\} = C_2^{(i+1)-\mathrm{tor}},$$

so that $m=4$ and $Q_1 \in C_2^{(i+1)-\mathrm{tor}} \setminus \{\check{o}_{C_2}\}$.

For $s=6$ one has $e^{\frac{\pi i}{3}} - 1 = e^{\frac{2\pi i}{3}}$ and $C_2^{e^{\frac{2\pi i}{3}}-\mathrm{tor}} = \{\check{o}_{C_2}\}$, Therefore $\overline{\mathrm{pr}_2}(G_o) = \langle e^{\frac{\pi i}{3}} \rangle \simeq \mathbb{C}_6$ in this case.

The restrictions on $P_1, U_1 \in C_1$ arise from the isomorphism $G_o \simeq \overline{\text{pr}_1}(G_o) \simeq \overline{\text{pr}_2}(G_o)$. Namely, $\left(\tau_{U_1}, e^{\frac{2\pi i}{s}}\right) \in G_o$ with $\overline{\text{pr}_2}\left(\tau_{U_1}, e^{\frac{2\pi i}{s}}\right) = E^{\frac{2\pi i}{s}}$ of order $s \in \{2,,34,6\}$ has to have $\tau_{U_1} = \overline{\text{pr}_1}\left(\tau_{U_1}, e^{\frac{2\pi i}{s}}\right) \in (C_1,+)$ of order s. That amounts to $U_1 \in C_1^{s-\text{tor}}$ and $U_1 \notin C_1^{t-\text{tor}}$ for all $1 \leq t < s$. If $\overline{\text{pr}_2}(G_o) = \langle \tau_{Q_1} \rangle \times \langle e^{\frac{2\pi i}{s}} \rangle$ with $Q_1 \neq \check{o}_{C_2}$ then the order m of $Q_1 \in C_2$ has to coincide with the order of $P_1 \in C_1$.

In order to relate the classification G_s^{HE}, $G_{m,s}^{HE}$ of G_o with the classification of the groups $H_s^{HE}(m,n)$, $H_{s,s}^{HE}(m,n)$ of $H \simeq S^{-1}HS$, note that $P_1, U_1 \in C_1^{p-\text{tor}} \setminus C_1^{q-\text{tor}}$ for some natural numbers $p > q$ exactly when the corresponding liftings $X, W \in F_1$ are subject to $pX, pQ \in \ker(\text{pr}_2)$, $qX, qW \notin \ker(\text{pr}_2)$. Similarly, $Q_1 \in C_2^{p-\text{tor}} \setminus C_2^{q-\text{tor}}$ for $p,q \in \mathbb{N}$, $P > q$ if and only if an arbitrary lifting $Y \in F_2$ satisfies $pY \in \ker(\text{pr}_1)$, $qY \notin \ker(\text{pr}_1)$.

□

Bearing in mind that A/H with $H = \mathcal{T}(H)\langle h_o \rangle$, $\lambda_1\mathcal{L}(h_o) = 1$, $\lambda_2\mathcal{L}(h_o) \in R^* \setminus \{1\}$ is either hyper-elliptic or a ruled surface with an elliptic base, one obtains the following

Corollary 48. *Let $H = \mathcal{T}(H)\langle h_o \rangle$ be a finite subgroup of $Aut(A)$ for some $h_o \in H$ with $\lambda_1\mathcal{L}(h_o) = 1$, $\lambda_2\mathcal{L}(h_o) = e^{\frac{2\pi i}{s}}$, $s \in \{2,3,4,6\}$, $S \in GL(2,\mathbb{Q}(\sqrt{-d}))$ be a diagonalizing matrix for h_o and*

$$S^{-1}h_oS = \left(\tau_{U_1}, e^{\frac{2\pi i}{s}}\right)$$

after an appropriate choice of an origin of $S^{-1}(A) = F_1 \times F_2$, $F_1 = S^{-1}(E \times \check{o}_E)$, $F_2 = S^{-1}(\check{o}_E \times E)$. Then A/H is a ruled surface with an elliptic base if and only if the kernel $\ker(\text{pr}_1)$ of the first canonical projection $\text{pr}_1 : S^{-1}HS \to Aut(F_1)$ contains a non-translation element $S^{-1}hS = \left(Id_{F_1}, \tau_{V_2}e^{\frac{2\pi ik}{s}}\right)$ for some $1 \leq k \leq s-1$, $V_2 \in F_2$.

In the notations from Lemma 44, the quotient $A/H \simeq (C_1 \times F_2)/G$ of the split abelian surface $C_1 \times F_2 = S^{-1}A/\ker(\text{pr}_2)$ by its finite automorphism group $G = S^{-1}HS/\ker(\text{pr}_2)$ is a ruled surface with an elliptic base exactly when G is isomorphic to some of the groups

$$G_2^{RE}(m,n) = \langle \tau_{(P_1,Q_1)}, \quad \tau_{(P_2,Q_2)}, \quad \rangle \rtimes \langle (\tau_{U_1}, -1) \rangle \simeq (\mathbb{C}_m \times \mathbb{C}_n) \rtimes_{(-1,-1)} \mathbb{C}_2 =$$

$$= (\langle a \rangle \times \langle b \rangle) \rtimes_{(-1,-1)} \langle c \rangle = \langle a, \quad b, \quad c \quad | \quad a^m = 1, \quad b^n = 1, \quad cac^{-1} = a^{-1}, \quad cbc^{-1} = b^{-1} \rangle$$

with $\tau_{U_1} \in (\langle \tau_{P_1}, \tau_{P_2} \rangle, +) \simeq \mathbb{C}_m \times \mathbb{C}_n$ for some $m,n \in \mathbb{N}$,

$$G_3^{RE}(m,j) = \langle \tau_{(P_1,Q_1)} \rangle \rtimes \langle \left(\tau_{U_1}, e^{\frac{2\pi i}{3}}\right) \rangle \simeq \mathbb{C}_m \rtimes_j \mathbb{C}_3 =$$

$$= \langle a \rangle \rtimes_j \langle c \rangle = \langle a, \quad c \quad | \quad a^m = 1, \quad c^3 = 1, \quad cac^{-1} = a^j \rangle$$

with $R = \mathcal{O}_{-3}$, $2U_1 \in (\langle \tau_{P_1} \rangle, +) \simeq \mathbb{C}_m$ for some $j \in \mathbb{Z}_m^*$ of order 1 or 3,

$$G_4^{RE}(m,j) = \langle \tau_{(P_1,Q_1)} \rangle \rtimes \langle (\tau_{U_1}, i) \rangle \simeq \mathbb{C}_m \rtimes_j \mathbb{C}_4 =$$

$$= \langle a \rangle \rtimes_j \langle c \rangle = \langle a, \quad c \mid a^m = 1, \quad c^4 = 1, \quad cac^{-1} = a^j \rangle$$

with $R = \mathbb{Z}[i]$ *for some* $j \in \mathbb{Z}_m^*$ *or order* $1, 2$ *or* 4,

$$G_6^{RE}(m, j) = \langle \tau_{(P_1, Q_1)} \rangle \rtimes \langle \left(\tau_{U_1}, e^{\frac{\pi i}{3}} \right) \rangle \simeq \mathbb{C}_m \rtimes_j \mathbb{C}_6 =$$

$$= \langle a \rangle \rtimes_j \langle c \rangle = \langle a, \quad c \mid a^m = 1, \quad c^6 = 1, \quad cac^{-1} = a^j \rangle$$

with $R = \mathcal{O}_{-3}$ *and at least one of* $3U_1, 4U_1$ *or* $5U_1$ *from* $(\langle \tau_{P_1} \rangle, +)$ *for some* $j \in \mathbb{Z}_m^*$ *of order* $1, 2, 3$ *or* 6.

The classification of G is an immediate application of the group isomorphism $\overline{\mathrm{pr}_2} : G \to \mathrm{pr}_2(S^{-1}HS)$ from Lemma 44 (v) and the classification of $Aut(F_2)$, given in Lemma 43.

Lemma 49. *Let* G *be a finite subgroup of* $GL(2, R)$ *with* $G \cap SL(2, R) \neq \{I_2\}$, *such that any* $g \in G \setminus SL(2, R) \neq \emptyset$ *has an eigenvalue* $\lambda_1(g) = 1$. *Then:*

(i) $G = G_s = \langle g_s, g_o \rangle$ *is generated by* $g_s \in SL(2, R)$ *of order* $s \in \{2, 3, 4, 6\}$ *and* $g_o \in GL(2, R)$ *with* $\det(g_o) = -1$, $\mathrm{tr}(g_o) = 0$, *subject to* $g_o g_s g_o^{-1} = g_s^{-1}$;

(ii) and $g \in G \setminus SL(2, R)$ *has eigenvalues* $\lambda_1(g) = 1$ *and* $\lambda_2(g) = -1$;

(iii) the group

$$G_s = \langle g_s, \quad g_o \mid g_s^s = I_2, \quad g_o^2 = I_2, \quad g_o g_s g_o^{-1} = g_s^{-1} \rangle \simeq \mathcal{D}_s$$

is dihedral of order $2s$ *for* $s \in \{3, 4, 6\}$ *or the Klein group* $G_2 \simeq \mathbb{C}_2 \times \mathbb{C}_2$ *for* $s = 2$.

Proof. Note that $g \in G \setminus SL(2, R)$ has an eigenvalue 1 exactly when the characteristic polynomial $\mathcal{X}_g(\lambda) = \lambda^2 - \mathrm{tr}(g)\lambda + \det(g) \in R[\lambda]$ of g vanishes at $\lambda = 1$. This is equivalent to

$$\mathrm{tr}(g) = \det(g) + 1.$$

If $-I_2 \notin G$, then Proposition 24 specifies that $G \cap SL(2, R) = \langle g_3 \rangle \simeq \mathbb{C}_3$. In the notations from Proposition 35, all the finite subgroups $H_{C3}(i) = [H_{C3}(i) \cap SL(2, R)] \langle g_o \rangle$ of $GL(2, R)$ with $H_{C3}(i) \cap SL(2, R) \simeq \mathbb{C}_3$, such that g_o has an eigenvalue $\lambda_1(g_o) = 1$ are isomorphic to

$$H_{C3}(4) = \langle g, \quad g_o \quad g^3 = g_o^3 = I_2, \quad g_o g g_o^{-1} = g^{-1} \rangle \simeq S_3 \simeq \mathcal{D}_3$$

for some $g \in SL(2, R)$ with $\mathrm{tr}(g) = -1$ and $\lambda_1(g_o) = 1$, $\lambda_2(g_o) = -1$. Since g_o is of order 2, the complement

$$H_{C3}(4) \setminus SL(2, R) = \langle g \rangle g_o = \{ g^j g_o \mid 0 \leq j \leq 2 \}$$

consists of matrices $g^j g_o$ of determinant $\det(g^j g_o) = \det(g_o) = -1$ and $g \in H_{C3}(4) \setminus SL(2, R)$ has as eigenvalue 1 exactly when $\mathrm{tr}(g^j g_o) = 0$. Bearing in mind the invariance of the trace under conjugation, one can consider

$$g = \left(\begin{array}{cc} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{array} \right) \quad \text{and} \quad g_o = \left(\begin{array}{cc} a_o & b_o \\ c_o & -a_o \end{array} \right)$$

with $a_o^2 + b_o c_o = 1$. Then

$$g_o g g_o^{-1} = g_o g g_o = \left(\begin{array}{cc} e^{-\frac{2\pi i}{3}} + \sqrt{-3} a_o^2 & \sqrt{-3} a_o b_o \\ \sqrt{-3} a_o c_o & e^{\frac{2\pi i}{3}} + \sqrt{-3} a_o^2 \end{array} \right) = \left(\begin{array}{cc} e^{-\frac{2\pi i}{3}} & 0 \\ 0 & e^{\frac{2\pi i}{3}} \end{array} \right) = g^{-1}$$

is equivalent to $a_o = 0$ and

$$g^j g_o = \left(\begin{array}{cc} e^{\frac{2\pi i j}{3}} & 0 \\ 0 & e^{-\frac{2\pi i j}{3}} \end{array} \right) \left(\begin{array}{cc} 0 & b_o \\ \frac{1}{b_o} & 0 \end{array} \right) = \left(\begin{array}{cc} 0 & e^{\frac{2\pi i j}{3}} b_o \\ \frac{e^{-\frac{2\pi i j}{3}}}{b_o} & 0 \end{array} \right)$$

have $\mathrm{tr}(g^j g_o) = 0$ for all $0 \leq j \leq 2$. Thus, any $g \in H_{C3}(4) \backslash SL(2, R)$ has an eigenvalue $\lambda_1(g) = 1$.

If $-I_2 \in G$, then for any $g \in G \setminus SL(2, R)$ with $\lambda_1(g) = 1$, $\lambda_2(g) = \det(g) \in R^* \setminus \{1\}$, one has $-g \in G \setminus SL(2, R)$ with $\lambda_1(-g) = -1$, $\lambda_2(-g) = -\det(g)$. Thus, $-g$ has an eigenvalue 1 exactly when $\lambda_2(-g) = -\det(g) = 1$ or $\lambda_2(g) = \det(g) = -1$. In particular,

$$G = [G \cap SL(2, R)] \langle g_o \rangle$$

for some $g_o \in G$ with $\det(g_o) = -1$, $\mathrm{tr}(g_o) = 0$ and $G \setminus SL(2, R) = [G \cap SL(2, R)] g_o$. Thus, for any $g \in G \setminus SL(2, R)$ has $\det(g) = -1$ and g has an eigenvalue $\lambda_1(g) = 1$ exactly when $\mathrm{tr}(g) = 0$.

We claim that $\mathrm{tr}(g_1 g_o) = 0$ for all $g_1 \in G \cap SL(2, R)$ and some $g_o \in G$ with $\det(g_o) = -1$, $\mathrm{tr}(g_o) = -1$ requires $G \cap SL(2, R)$ to be a cyclic group. Assume the opposite. Then by Proposition 24, either $G \cap SL(2, R)$ contains a subgroup

$$K_4 = \langle g_1, g_2 \mid g_1^2 = g_2^2 = -I_2, \quad g_1 g_2 g_1^{-1} = g_2^{-1} \rangle \simeq \mathbb{Q}_8$$

isomorphic to the quaternion group $\mathbb{Q}_8$ of order 8, or

$$G \cap SL(2, R) = K_7 = \langle g_1, \quad g_4 \mid g_1^2 = g_4^3 = -I_2, \quad g_1 g_4 g_1^{-1} = g_4^{-1} \rangle \simeq \mathbb{Q}_{12}$$

is isomorphic to the dicyclic group $\mathbb{Q}_{12}$ of order 12. In either case, one has $h_1, h_2 \in SL(2, R)$ with $\mathrm{tr}(h_1) = 0$ and h_2 of order $s \in \{4, 6\}$, such that $h_1 h_2 h_1^{-1} = h_2^{-1}$. Let us consider

$$D_1 = S^{-1} h_1 S = \left(\begin{array}{cc} a_1 & b_1 \\ c_1 & -a_1 \end{array} \right) \in SL\left(2, \mathbb{Q}\left(\sqrt{-d}, E^{\frac{2\pi i}{s}}\right)\right),$$

$$D_2 = S^{-1} h_2 S = \left(\begin{array}{cc} e^{\frac{2\pi i}{s}} & 0 \\ 0 & e^{-\frac{2\pi i}{s}} \end{array} \right) \quad \text{and}$$

$$D_o = S^{-1} g_o S = \left(\begin{array}{cc} a_o & b_o \\ c_o & -a_o \end{array} \right) \in GL\left(2, \mathbb{Q}\left(\sqrt{-d}, e^{\frac{2\pi i}{s}}\right)\right)$$

with $a_o^2 + b_o c_o = 1$. The relation

$$D_1 D_2 D_1^{-1} = -D_1 D_2 D_1 = \left(\begin{array}{cc} e^{-\frac{2\pi i}{s}} - 2iIm\left(e^{\frac{2\pi i}{s}}\right) a_1^2 & -2iIm\left(e^{\frac{2\pi i}{s}}\right) a_1 b_1 \\ -2iIm\left(e^{\frac{2\pi i}{s}}\right) a_1 c_1 & e^{\frac{2\pi i}{s}} + 2iIm\left(e^{\frac{2\pi i}{s}}\right) a_1^2 \end{array} \right) =$$

$$= \left(\begin{array}{cc} e^{-\frac{2\pi i}{s}} & 0 \\ 0 & e^{\frac{2\pi i}{s}} \end{array} \right) = D_2^{-1}$$

requires $a_1 = 0$ and

$$D_1 = \left(\begin{array}{cc} 0 & b_1 \\ -\frac{1}{b_1} & 0 \end{array} \right) \quad \text{for some} \quad b_1 \in \mathbb{Q}\left(\sqrt{-d}, e^{\frac{2\pi i}{s}}\right).$$

Now,

$$\text{tr}(D_2 D_o) = \text{tr} \left(\begin{array}{cc} e^{\frac{2\pi i}{s}} a_o & e^{\frac{2\pi i}{s}} b_o \\ e^{-\frac{2\pi i}{s}} c_o & -e^{-\frac{2\pi i}{s}} a_o \end{array} \right) = 2iIm\left(e^{\frac{2\pi i}{s}}\right) a_o = 0$$

specifies the vanishing of a_o, whereas

$$D_o = \left(\begin{array}{cc} 0 & b_o \\ \frac{1}{b_o} & 0 \end{array} \right) \quad \text{for some} \quad b_o \in \mathbb{Q}\left(\sqrt{-d} e^{\frac{2\pi i}{s}}\right).$$

The condition

$$\text{tr}(D_1 D_o) = \text{tr} \left(\begin{array}{cc} \frac{b_1}{b_o} & 0 \\ 0 & -\frac{b_o}{b_1} \end{array} \right) = \frac{b_1}{b_o} - \frac{b_o}{b_1} = 0$$

requires $b_1 = \varepsilon b_o$ for some $\varepsilon \in \{\pm\}$ and

$$\text{tr}(D_1 D_2 D_o) = \text{tr} \left(\begin{array}{cc} \varepsilon e^{-\frac{2\pi i}{s}} & 0 \\ mbox & \\ 0 & -\varepsilon e^{\frac{2\pi i}{s}} \end{array} \right) = -\varepsilon \left(e^{\frac{2\pi i}{s}} - e^{-\frac{2\pi i}{s}} \right) = -2iIm\left(e^{\frac{2\pi i}{s}}\right) \varepsilon \neq 0$$

contradicts the assumption. Therefore $G \cap SL(2, R) = \langle g \rangle \simeq \mathbb{C}_s$ is cyclic group of order $s \in \{2, 4, 6\}$. If $G = [G \cap SL(2, R)]\langle g_o \rangle$ has a normal subgroup $G \cap SL(2, R) = \langle g \rangle \simeq \mathbb{C}_2$ then $g = -I_2$ and $g_o(-I_2) = (-I_2)g_o$, as far as $-I_2$ is a scalar matrix. As a result, $G = \langle g \rangle \times \langle g_o \rangle \simeq \mathbb{C}_2 \times \mathbb{C}_2$. For $G = [G \cap SL(2, R)]\langle g_o \rangle$ with a normal subgroup $G \cap SL(2, R) = \langle g \rangle \simeq \mathbb{C}_s$ of order $\{4, 6\}$ note that the element $g_o g g_o^{-1}$ of $\langle g \rangle$ is of order s, so that either $g_o g g_o^{-1} = g$ or $g_o g g_o^{-1} = g^{-1}$, according to $\mathbb{Z}_4^* = \{\pm 1 (\text{mod} 4)\}$, $\mathbb{Z}_6^* = \{\pm 1 (\text{mod} 6)\}$. If $g_o g = g g_o$ then there exists a matrix $S \in GL\left(2, \mathbb{Q}\left(\sqrt{-d}, e^{\frac{2\pi i}{s}}\right)\right)$, such that

$$D = S^{-1} g S = \left(\begin{array}{cc} e^{\frac{2\pi i}{s}} & 0 \\ 0 & e^{-\frac{2\pi i}{s}} \end{array} \right) \quad \text{and} \quad D_o = S^{-1} g_o S = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$$

are diagonal. Then $\mathrm{tr}(gg_o) = \mathrm{tr}(DD_o) = e^{\frac{2\pi i}{s}} - e^{-\frac{2\pi i}{s}} = 2iIm\left(e^{\frac{2\pi i}{s}}\right) \neq 0$ and 1 is not an eigenvalue of gg_o. Therefore $g_o g g_o^{-1} = g^{-1}$. If

$$D = S^{-1}gS = \begin{pmatrix} e^{\frac{2\pi i}{s}} & 0 \\ 0 & e^{-\frac{2\pi i}{s}} \end{pmatrix} \quad \text{and}$$

$$D_o = S^{-1}g_oS = \begin{pmatrix} a_o & b_o \\ C_o & -a_o \end{pmatrix} \in GL\left(2, \mathbb{Q}\left(\sqrt{-d}, e^{\frac{2\pi i}{s}}\right)\right) \quad \text{with} \quad a_o^2 + b_oc_o = 1,$$

then the relation

$$D_oDD_o^{-1} = D_oDD_o = \begin{pmatrix} e^{-\frac{2\pi i}{s}} + 2iIm\left(e^{\frac{2\pi i}{s}}\right)a_o^2 & 2iIm\left(e^{\frac{2\pi i}{s}}\right)a_ob_o \\ 2iIm\left(e^{\frac{2\pi i}{s}}\right)a_oc_o & e^{\frac{2\pi i}{s}} - 2iIm\left(e^{\frac{2\pi i}{s}}\right)a_o^2 \end{pmatrix} =$$

$$= \begin{pmatrix} e^{-\frac{2\pi i}{s}} & 0 \\ 0 & e^{\frac{2\pi i}{s}} \end{pmatrix} = D^{-1}$$

specifies that $a_o = 0$ and

$$D_o = \begin{pmatrix} 0 & b_o \\ \frac{1}{b_o} & 0 \end{pmatrix} \quad \text{for some} \quad b_o \in \mathbb{Q}\left(\sqrt{-d}, e^{\frac{2\pi i}{s}}\right).$$

The non-trivial coset

$$S^{-1}GS \setminus SL\left(2, \mathbb{Q}\left(\sqrt{-d}, e^{\frac{2\pi i}{s}}\right)\right) = \langle D\rangle D_o = \{D^jD_o \mid 0 \leq j \leq s-1\}$$

consists of elements of trace

$$\mathrm{tr}(D^jD_o) = \mathrm{tr}\begin{pmatrix} 0 & e^{\frac{2\pi ij}{s}}b_o \\ \frac{e^{-\frac{2\pi ij}{s}}}{b_o} & 0 \end{pmatrix} = 0,$$

so that any $\Delta \in S^{-1}GS \setminus SL\left(2, \mathbb{Q}\left(\sqrt{-d}, e^{\frac{2\pi i}{s}}\right)\right)$ has an eigenvalue 1 and any $g = S\Delta S^{-1} \in G \setminus SL(2, R)$ has an eigenvalue 1.

□

Proposition 50. *The quotient A/H of $A = E \times E$ is an Enriques surface if and only if H is generated by $h \in H$ of order $s \in \{2, 3, 4, 6\}$ with $\mathcal{L}(h) \in SL(2, R)$ and $h_o \in H$ with $\lambda_1\mathcal{L}(h_o) = 1$, $\lambda_2\mathcal{L}(h_o) = -1$, $\tau(h_o) = h_o\mathcal{L}(h_o)^{-1} = \tau_{(U_o,V_o)}$, subject to $h_ohh_o^{-1} = h_ohh_o = h^{-1}$ and*

$$\mathcal{L}(h_o)\begin{pmatrix} U_o \\ V_o \end{pmatrix} \neq -\begin{pmatrix} U_o \\ V_o \end{pmatrix}. \tag{21}$$

In particular, for $s=2$ the group

$$H \simeq \mathcal{L}(H) \simeq \mathbb{C}_2 \times \mathbb{C}_2$$

is isomorphic to the Klein group of order 4, while for $s \in \{3,4,6\}$ one has a dihedral group

$$H \simeq \mathcal{L}(H) \simeq \mathcal{D}_s = \langle a, \quad b \quad | \quad a^s = 1, \quad b^2 = 1, \quad bab^{-1} = a^{-1} \rangle$$

of order $2s$.

Proof. According to Lemmas 41 and 49, the finite subgroups H of $Aut(E \times E)$ with Enriques quotient A/H are of the form

$$H = \langle \tau_{(P_i,Q_i)}, \quad h, \quad h_o \quad | \quad 1 \leq i \leq m \rangle$$

with $0 \leq m \leq 3$ and

$$\mathcal{L}(H) = \langle \mathcal{L}(h), \quad \mathcal{L}(h_o) \quad \mathcal{L}(h)^s = I_2, \quad \mathcal{L}(h_o)^2 = I_2, \quad \mathcal{L}(h_o)\mathcal{L}(h)\mathcal{L}(h_o)^{-1} = \mathcal{L}(h_)^{-1} \simeq \mathcal{D}_s$$

for some $\mathcal{L}(h) \in SL(2,R)$, $\mathcal{L}(h_o) \in GL(2,R)$, $\lambda_1\mathcal{L}(h_o) = 1$, $\lambda_2\mathcal{L}(h_o) = -1$. Note that

$$K := \mathcal{L}^{-1}(\mathcal{L}(H) \cap SL(2,R)) = \langle \tau_{(P_i,Q_i)} \quad | \quad 1 \leq i \leq m \rangle \langle h \rangle$$

is a normal subgroup of H with a single non-trivial coset

$$H \setminus K = Kh_o = \left\{ \tau_{h(z,j) = \sum\limits_{i=1}^{m} z_i(P_i,Q_i)} h^j h_o \quad | \quad z_i \in \mathbb{Z}, \quad 0 \leq j \leq s-1 \right\}.$$

The automorphism h, whose linear part $\mathcal{L}(h)$ has eigenvalues $\lambda_1\mathcal{L}(h) = e^{\frac{2\pi i}{s}}$, $\lambda_2\mathcal{L}(h) = e^{-\frac{2\pi i}{s}}$, different from 1 has always a fixed point on A. Without loss of generality, one can assume that $h = \mathcal{L}(h) \in GL(2,R)$, after moving the origin of A at a fixed point of h. If $h_o = \tau_{(U_o,V_o)}\mathcal{L}(h_o)$ for some $(U_o,V_o) \in A$ then the translation parts

$$\tau(h(z,j)) = h(z,j)\mathcal{L}(h(z,j))^{-1} = \tau_{\sum\limits_{i=1}^{m} z_i(P_i,Q_i) + h^j(U_o,V_o)} \quad \text{for} \quad \forall z = (z_1,\ldots,z_m) \in \mathbb{Z}^m$$

and $0 \leq j \leq s-1$. The linear parts $\mathcal{L}(h(z,j)) = \mathcal{L}(h^j h_o) = h^j\mathcal{L}(h_o)$ have eigenvalues $\lambda_1(h^j\mathcal{L}(h_o)) = 1$, $\lambda_2(h^j\mathcal{L}(h_o)) = -1$ for all $0 \leq j \leq s-1$. Applying Lemma 46, one concludes that $Fix_A(h(z,j)) = \emptyset$ if and only if no one lifting $(x(z,j), y(z,j)) \in \mathbb{C}^2$ of $\tau(h(z,j))$ is in the kernel of the linear operator $\psi_j = h^j\mathcal{L}(h_o) + I_2 : \mathbb{C}^2 \rightarrow \mathbb{C}^2$. For any fixed $0 \leq j \leq s-1$, note that $(x(z,j), y(z,j)) \notin \ker(\phi_j)$ for all $z = (z_1,\ldots,z_m) \in \mathbb{Z}^m$ implies that the lifting of the $\mathbb{R}$-span of $\langle \tau_{(P_i,Q_i)} \quad | \quad 1 \leq i \leq m \rangle$ to $\mathbb{C}^2$ is parallel to $\ker(\psi_j)$. It suffices to establish that $\ker(\psi_0) \cap \ker(\psi_1) = \{(0,0)\}$, in order to conclude that $m=0$ and $H = \langle h, \quad h_o \rangle = \langle h_o, \quad h \rangle$. Since the claim $\ker(\psi_0) \cap \ker(\psi_1) = \{(0,0)\}$

is independent on the choice of a coordinate system on $\mathbb{C}^2$, one can use Lemma 49 to assume that

$$\mathcal{L}(h_o) = D_o = \left(\begin{array}{cc} 0 & b_o \\ \frac{1}{b_o} & 0 \end{array} \right) \quad \text{and} \quad h = \mathcal{L}(h) = \left(\begin{array}{cc} e^{\frac{2\pi i}{s}} & 0 \\ 0 & e^{-\frac{2\pi i}{s}} \end{array} \right)$$

for some $s \in \{2,3,4,6\}$. Then $\psi_0 = \mathcal{L}(h_o) + I_2$ has kernel $\ker(\psi_0) = \mathrm{Span}_{\mathbb{C}}(b_o, -1)$, while

$$\psi_1 = h\mathcal{L}(h_o) + I_2 = \left(\begin{array}{cc} 1 & e^{\frac{2\pi i}{s}} b_o \\ e^{-\frac{2\pi i}{s}} b_o^{-1} & 1 \end{array} \right)$$

has kernel $\ker(\psi_1) = \mathrm{Span}_{\mathbb{C}} \left(e^{\frac{2\pi i}{s}} b_o, -1 \right)$. For $s \in \{2,,34,6\}$ the vectors $(b_o, -1)$ and $\left(e^{\frac{2\pi i}{s}} b_o, -1 \right)$ are linearly independent over $\mathbb{C}$, so that $\ker(\psi_0) \cap \ker(\psi_1) = \{(0,0)\}$. Now, $\mathcal{L}(h^j h_o) = h^j \mathcal{L}(h_o) \neq I_2$ for any $0 \leq j \leq s-1$, as far as $\mathcal{L}(h_o) \notin \langle h \rangle < SL(2,R)$. On the other hand, the subgroup $\langle h = \mathcal{L}(h) \rangle$ of H is contained in $SL(2,R)$, so that the translation part $\mathcal{T}(H) = \ker(\mathcal{L}|_H) = Id_A$ is trivial. As a result, $\mathcal{L} : H \rightarrow \mathcal{L}(H)$ is a group isomorphism and the relation $\mathcal{L}(h_o) h \mathcal{L}(h_o)^{-1} = h^{-1}$ implies that

$$h_o h h_o^{-1} = \left(\tau_{(U_o,V_o)} \mathcal{L}(h_o) \right) h \left(\tau_{-\mathcal{L}(h_o)^{-1}(U_o,V_o)} \mathcal{L}(h_o)^{-1} \right) =$$

$$= \tau_{(U_o,V_o) - \mathcal{L}(h_o) h \mathcal{L}(h_o)^{-1}(U_o,V_o)} [\mathcal{L}(H_o) h \mathcal{L}(h_o)^{-1}] = \tau_{(U_o,V_o) - h^{-1}(U_o,V_o)} h^{-1} = h^{-1}.$$

After acting by h on $(U_o,V_o) = h^{-1}(U_o,V_o)$, one obtains that $h(U_o,V_o) = (U_o,V_o)$, or $(U_o,V_o) \in A$ is a fixed point of h. Bearing in mind that $K = \langle h \rangle \simeq \langle \mathcal{L}(h) \rangle = \mathcal{L}(H) \cap SL(2,R)$ is a normal subgroup of $H \simeq \mathcal{L}(H) = [\mathcal{L}(H) \cap SL(2,R)] \langle \mathcal{L}(h_o) \rangle$, let us represent the complement $H \setminus K$ as the set of the entries of the left coset

$$H \setminus K = h_o K = \{ h_o h^j \ \mid \ 0 \leq j \leq s-1 \}.$$

Then $h_o h^j = \tau_{(U_o,V_o)}(\mathcal{L}(h_o) h^j)$ have translation parts

$$\tau(h_o h^j) = h_o h^j \mathcal{L}(h_o h^j)^{-1} = h_o \mathcal{L}(h_o)^{-1} = \tau(h_o) = \tau_{(U_o,V_o)}$$

and linear parts $\mathcal{L}(h_o) h^j$ with eigenvalues $\lambda_1(\mathcal{L}(h_o) h^j) = 1$, $\lambda_2(\mathcal{L}(h_o) h^j) = -1$. According to Lemma 46, the automorphism $h_o h^j \in Aut(A)$ has no fixed point on A if and only if no one lifting $(u_o, v_o) \in \mathbb{C}^2$ of $(u_o + \pi_1(E), v_o + \pi_1(E)) = (U_o, V_o)$ is in the kernel of $\varphi_j = \mathcal{L}(h_o) h^j + I_2$. We claim that if

$$h \left(\begin{array}{c} u_o \\ v_o \end{array} \right) = \left(\begin{array}{c} u_o \\ v_o \end{array} \right) + \left(\begin{array}{c} \mu_1 \\ \mu_2 \end{array} \right) \quad \text{for some} \quad (\mu_1, \mu_2) \in \pi_1(A),$$

then $\varphi_j(u_o, v_o) - \varphi_0(u_o, v_o) \in \pi_1(A)$. Indeed, by an induction on j, one has

$$h^j \left(\begin{array}{c} u_o \\ v_o \end{array} \right) - \left(\begin{array}{c} u_o \\ v_o \end{array} \right) \in \pi_1(A),$$

whereas

$$\varphi_j(u_o, v_o) - \varphi_0(u_o, v_o) = \mathcal{L}(h_o)h^j \begin{pmatrix} u_o \\ v_o \end{pmatrix} - \mathcal{L}(h_o) \begin{pmatrix} u_o \\ v_o \end{pmatrix} \in \pi_1(A).$$

Thus, the assumption $(u_o, v_o) \in \ker(\varphi_j)$ implies that

$$\varphi_0(u_o, v_o) = \mathcal{L}(h_o)(u_o, v_o) + (u_o, v_o) = (\mu_1', \mu_2') \in \pi_1(A),$$

whereas

$$\mathcal{L}(h_o) \begin{pmatrix} U_o \\ V_o \end{pmatrix} = - \begin{pmatrix} U_o \\ V_o \end{pmatrix},$$

contrary to the assumption (21). Note that (21) is equivalent to $\varphi_0(u_o, v_o) \notin \pi_1(A)$ for all liftings $(u_o, v_o) \in \mathbb{C}^2$ of $(u_o + \pi_1(E), v_o + \pi_1(E)) = (U_o, V_o)$ and is slightly stronger than $Fix_A(h_o) = \emptyset$, which amounts to $\varphi_0(u_o, v_o) \neq 0$ for $\forall (u_o, v_o) \in \mathbb{C}^2$ with $(u_o + \pi_1(E), v_o + \pi_1(E)) = (U_o, V_o)$.

□

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