

COMPRESSED PRODUCT OF BALLS AND LOWER BOUNDARY ESTIMATES ON BERGMAN KERNELS

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Abstract. The image $B_{p^\sigma, q}$ of a product of balls $B_p \times B_q$ under a compression $c_\sigma(X, V) = (X, V(1 - {}^t\bar{X}X)^{\frac{\sigma}{2}})$ is called a compressed product of balls of exponent $\sigma \in \mathbb{R}$. The present note obtains the group $\text{Aut}(B_{p^\sigma, q})$ of the holomorphic automorphisms and the $\text{Aut}(B_{p^\sigma, q})$ -orbit structure of $B_{p^\sigma, q}$ and its boundary $\partial B_{p^\sigma, q}$ for $\sigma > 1$. The Bergman completeness of $B_{p^\sigma, q}$ is verified by an explicit calculation of the Bergman kernel. As a consequence, local lower boundary estimates on the Bergman kernels of the bounded pseudoconvex domains are obtained, which are locally inscribed in $B_{p^\sigma, q}$ at a common boundary point.

For a strictly pseudoconvex domain $\mathcal{D} = \{z \in \mathbb{C}^n; \rho(z) < 0\}$ with a smooth boundary, Fefferman [6] and Boutet de Monvel–Sjöstrand [2] have expanded the diagonal values $k_{\mathcal{D}}(z) := k_{\mathcal{D}}(z, z)$ of the Bergman kernel in the form $k_{\mathcal{D}}(z) = \varphi_{\mathcal{D}}(\rho)\rho^{-n-1} + \psi_{\mathcal{D}}(\rho)\log(-\rho)$, where $\varphi_{\mathcal{D}}(\rho)$ and $\psi_{\mathcal{D}}(\rho)$ are power series in the defining function $\rho = \rho(z)$.

Only few results are known for the boundary behavior of the Bergman kernel of a weakly pseudoconvex domain. For arbitrary $m = (m_1, \dots, m_n) \in \mathbb{N}^n$ let $\mathcal{E}_m := \{z \in \mathbb{C}^n; \rho_m(z) = \sum_{j=1}^n |z_j|^{2m_j} - 1 < 0\}$, $z^o \in \partial\mathcal{E}_m$,

$$P_m := \{j \in \mathbb{N}; z_j^o = 0 \text{ and } m_j > 1\}, \quad Q_m := \{j \in \mathbb{N}; z_j^o \neq 0 \text{ or } m_j = 1\}.$$

Kamimoto has established in [11] the existence of an open subset $U \subset \mathbb{C}^n$ with $z^o \in \partial U$ and a real analytic function $\Phi_m: U \rightarrow \{r \in \mathbb{R}; r > 0\}$, such that $k_{\mathcal{E}_m}(z) = \Phi_m(z)\rho_m(z)^{-\sum_{j \in P_m} m_j^{-1} - \text{card } Q_m - 1}$ on U . If $P_m = \emptyset$ then $\Phi_m(z)$ is bounded around z^o , while $\lim_{z \rightarrow z^o} \Phi_m(z) = \infty$ for $P_m \neq \emptyset$.

In the spirit of Kamimoto's result, the present note provides local lower boundary estimates on the Bergman kernels of the bounded pseudoconvex domains, which are locally inscribed in a compressed product of balls. More precisely, on a product of balls

$$B_p \times B_q = \{X \in \text{Mat}_{p,1}(\mathbb{C}), V \in \text{Mat}_{q,1}(\mathbb{C}); {}^t\bar{X}X < 1, {}^t\bar{V}V < 1\} \quad (1)$$

let us consider a compression map

$$c_\sigma : B_p \times B_q \rightarrow B_p \times B_q, \quad c_\sigma(X, V) := \left(X, V(1 - {}^t\bar{X}X)^{\frac{\sigma}{2}} \right) \quad (2)$$

of a positive real exponent σ . Its image

$$\begin{aligned} B_{p^\sigma, q} &:= c_\sigma(B_p \times B_q) = \{X \in B_p, Y \in B_q; \rho_{p^\sigma, q}(X, Y) \\ &= {}^t\bar{Y}Y - (1 - {}^t\bar{X}X)^\sigma < 0\} \end{aligned} \quad (3)$$

will be referred to as a **compressed product of balls**.

In Section 1 an explicit holomorphic action of the group $G = SU(p, 1) \times U_q$ on $B_{p^\sigma, q}$ (cf. (5) from Lemma 1(ii)) is recognized. After describing the G -orbit decompositions of $B_{p^\sigma, q}$, $\partial B_{p^\sigma, q}$ and the geometry of the corresponding orbits, Proposition 1 establishes that the entire group $\text{Aut}(B_{p^\sigma, q})$ of the holomorphic automorphisms of $B_{p^\sigma, q}$ is depleted by G .

The second section provides the diagonal values $k_{p^\sigma, q}(X, Y)$ of the Bergman kernel of $B_{p^\sigma, q}$. This is done by means of the $\text{Aut}(B_{p^\sigma, q})$ -transformation law of $k_{p^\sigma, q}$ and an explicit calculation of $k_{p^\sigma, q}$ at the canonical $\text{Aut}(B_{p^\sigma, q})$ -orbit representatives on $B_{p^\sigma, q}$ (cf. (8) from Lemma 1(iv)). If the diagonal values of the Bergman kernel of a bounded domain \mathcal{D} tend to infinity, while approaching each of the boundary points, then \mathcal{D} is called Bergman complete (cf. [1, 16]). Bremermann has shown in [4] that any Bergman complete domain is pseudoconvex and any bounded domain with a strictly plurisubharmonic exhaustion function is Bergman complete. However, there exist bounded pseudoconvex domains, which are not Bergman complete, e. g., the punctured disc $\Delta^* := \Delta \setminus \{0\}$, $\Delta := \{z \in \mathbb{C}; |z| < 1\}$. This raises the problem of obtaining sufficient conditions for the Bergman completeness of bounded pseudoconvex domains. Blocki and Pflug [1] and, independently, Herbort [8] have shown that the existence of a plurisubharmonic exhaustion function of a bounded domain \mathcal{D} is sufficient for its Bergman completeness. In [16] Zwonek proves that if a bounded pseudoconvex Reinhardt domain \mathcal{D} does not admit an embedding $\varepsilon: \Delta^* \rightarrow \mathcal{D}$, which extends to $\varepsilon: \Delta \rightarrow \mathcal{D}$ with $\varepsilon(0) \in \partial\mathcal{D}$, then \mathcal{D} is Bergman complete. The present note derives the Bergman completeness of the domain $B_{p^\sigma, q}$ from the explicit formula for $k_{p^\sigma, q}$.

A domain $\mathcal{D} \subset \mathbb{C}^n$ is locally geometrically approximated by a family \mathcal{F} of domains in \mathbb{C}^n , if for any $Z \in \partial\mathcal{D}$ there exist a domain $\Omega_Z \in \mathcal{F}$ and a

neighborhood $U(\Omega_Z)$ of Z on \mathbb{C}^n , such that $\mathcal{D} \cap U(\Omega_Z)$ is a domain in Ω_Z . A result of Diederich, Fornæss and Herbort (cf. [5] or Lemma 4) implies that the local geometric approximation by a family of bounded Bergman complete domains suffices for the Bergman completeness of a bounded pseudoconvex domain. The third section exposes the ultimate results of the present note in the form of local lower boundary estimates on the Bergman kernels of bounded pseudoconvex domains \mathcal{D} , which are locally inscribed in some $B_{p^\sigma, q}$ with $\sigma > 1$ at their common boundary point (cf. Corollary 2).

1. The Geometry and the Group of Holomorphic Automorphisms of a Compressed Product of Balls

Lemma 1. *Let $B_{p^\sigma, q}$ be a compressed product of balls of exponent $\sigma > 0$ (cf. (3)). Then:*

i) $B_{p^\sigma, q}$ fibers by q -balls of variable radii $(1 - {}^t\bar{X}X)^{\frac{\sigma}{2}}$ over the p -ball B_p

$$B_{p^\sigma, q} = \sqcup_{X \in B_p} B_q \left((1 - {}^t\bar{X}X)^{\frac{\sigma}{2}} \right);$$

ii) the group

$$G := SU(p, 1) \times U_q \tag{4}$$

acts by holomorphic automorphisms on $B_{p^\sigma, q}$, according to the rule

$$g(X, Y) = ((AX + B)(CX + D)^{-1}, LY(CX + D)^{-\sigma}) \tag{5}$$

where

$$g = (N, L) \in G, N = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(p, 1), L \in U_q$$

and

$$(X, Y) \in B_{p^\sigma, q};$$

iii) for any $(X_o, Y_o) \in B_{p^\sigma, q}$ the $(p + 1) \times (p + 1)$ -matrix

$$\xi^{X_o} := \begin{pmatrix} (I_p - X_o {}^t\bar{X}_o)^{-\frac{1}{2}} & -(I_p - X_o {}^t\bar{X}_o)^{-\frac{1}{2}} X_o \\ -(1 - {}^t\bar{X}_o X_o)^{-\frac{1}{2}} ({}^t\bar{X}_o) & (1 - {}^t\bar{X}_o X_o)^{-\frac{1}{2}} \end{pmatrix} \tag{6}$$

belongs to $SU(p, 1)$ and

$$\xi^{X_o}(X_o, Y_o) = \left(0, Y_o(1 - {}^t\bar{X}_o X_o)^{-\frac{\sigma}{2}} \right) \in \{0\} \times B_q; \tag{7}$$

iv) $B_{p^\sigma, q} = \sqcup_{s \in [0,1]} GM(s)$ splits into a real one-parameter family of G -orbits with canonical representatives

$$M(s) = (X = 0, Y = {}^t(s, 0, \dots, 0)); \tag{8}$$

v) for $s > 0$ the orbits $GM(s) = \sqcup_{X \in B_p} S^{2q-1}(s(1 - {}^t\bar{X}X)^{\frac{\sigma}{2}})$ fiber by $(2q-1)$ -spheres of variable radii over the p -ball, while $GM(0) = B_p \times \{0\}$ is an ordinary p -ball.

Proof: Claim (i) is immediate from the definition (3) of a compressed product of balls.

The indefinite special unitary group

$$SU(p, 1) := \left\{ N = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL_{p+1}(\mathbb{C}); {}^t\bar{N} \begin{pmatrix} I_p & 0 \\ 0 & -1 \end{pmatrix} N = \begin{pmatrix} I_p & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

has defining equations

$$\begin{aligned} {}^t\bar{A}A - {}^t\bar{C}C &= I_p, & {}^t\bar{A}B - {}^t\bar{C}D &= 0, \\ {}^t\bar{B}B - {}^t\bar{D}D &= -1, & \det(N) &= 1 \end{aligned} \tag{9}$$

in terms of the matrices $A \in \text{Mat}_{p,p}(\mathbb{C})$, $B \in \text{Mat}_{p,1}(\mathbb{C})$, $C \in \text{Mat}_{1,p}(\mathbb{C})$, $D \in \mathbb{C}$. The action of $N \in SU(p, 1)$ on $X \in B_p$ is $N(X) := (AX + B)(CX + D)^{-1}$. Bearing in mind (9), one observes that $1 - {}^t\bar{N}(X)N(X) = (1 - {}^t\bar{X}X)|CX + D|^{-2}$ and verifies that g from (5) is a holomorphic bijection of $B_{p^\sigma, q}$. Straightforward verification of $N_2(N_1(X, Y)) = (N_2N_1)(X, Y)$ and $I_{p+1}(X, Y) = (X, Y)$ concludes the proof of (ii).

Concerning (iii), let us choose $T \in U_p$, rotating X_o to ${}^t\bar{T}X_o = {}^t(w_o, 0, \dots, 0)$ for some $w_o \in [0, 1)$. Then

$$(I_p - X_o {}^t\bar{X}_o)^{-\frac{1}{2}} = T \begin{pmatrix} (1 - w_o^2)^{-\frac{1}{2}} & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix} {}^t\bar{T} \in \text{Mat}_{p,p}(\mathbb{C})$$

is well defined positively definite Hermitian matrix. Expressing ξ^{X_o} from (6) by T and w_o , one verifies immediately the defining equations (9) of $SU(p, 1)$. For the proof of (iv) one multiplies the second component of (7) by an appropriate $L \in U_q$, in order to obtain $L\xi^{X_o}(X_o, Y_o) = (0, s_o)$ for some $s_o \in [0, 1)$. Towards the description of the G -orbits on $B_{p^\sigma, q}$, we note that the projection $\Pi: B_{p^\sigma, q} \rightarrow B_p$, $\Pi(X, Y) = X$ commutes with the $SU(p, 1)$ -actions on $B_{p^\sigma, q}$ and B_p . Thus, for an arbitrary $X \in B_p$ there exists $N \in SU(p, 1)$, such that $X = N(0) = N\Pi M(s) = \Pi N M(s) \in \Pi GM(s)$ for all $s \in [0, 1)$. In other words, any $\Pi_s: GM(s) \rightarrow B_p$ is surjective and the fibers

$\Pi_s^{-1}(X) = \Pi_s^{-1}(N(0)) = \Pi_s^{-1}(BD^{-1}) = \{(BD^{-1}, L^t(s, 0, \dots, 0)D^{-\sigma}); L \in U_q\} \simeq \{Y \in \text{Mat}_{q,1}(\mathbb{C}); {}^t\bar{Y}Y = (s|D|^{-\sigma})^2\}$ are $(2q - 1)$ -spheres, which degenerate to the origin $0 \in \mathbb{C}^q$ for $s = 0$. \square

The next lemma describes some geometric and analytic properties of the boundary $\partial B_{p^\sigma, q}$. The case of an ordinary ball $B_{p^1, q} \equiv B_{p+q}$ is well known.

Lemma 2. *Let $\partial B_{p^\sigma, q} = \{X \in \bar{B}_p, Y \in \bar{B}_q; \rho_{p^\sigma, q}(X, Y) = {}^t\bar{Y}Y - (1 - {}^t\bar{X}X)^\sigma = 0\}$ be the boundary of a compressed product of balls $B_{p^\sigma, q}$. Then:*

- i) *in the notations of part (ii) of Lemma 1, any $g \in G$ extends to a real analytic diffeomorphism of $\partial B_{p^\sigma, q}$;*
- ii) *for $\sigma > 1$ the boundary $\partial B_{p^\sigma, q}$ is of class C^1 and the singular boundary locus is the orbit*

$$\partial B_{p^\sigma, q} \cap \{Y = 0\} = S^{2p-1} \times \{0\} = U_p(N)$$

of

$$N = (X = {}^t(1, 0, \dots, 0), Y = 0) \in \partial B_{p^\sigma, q};$$

- iii) *the smooth boundary locus*

$$(\partial B_{p^\sigma, q})^\circ = \sqcup_{X \in B_p} S^{2q-1} \left((1 - {}^t\bar{X}X)^{\frac{\sigma}{2}} \right) \tag{10}$$

fibers by $(2q - 1)$ -spheres of variable radii over the p -ball and constitutes the orbit

$$(\partial B_{p^\sigma, q})^\circ = (SU(p, 1) \times U_q)M \tag{11}$$

of

$$M = (X = 0, Y = {}^t(1, 0, \dots, 0)) \in (\partial B_{p^\sigma, q})^\circ .$$

Proof: We claim that ξ^{X_o} from (6), Lemma 1 (iii) extends in a neighborhood of $\partial B_{p^\sigma, q}$. Otherwise, there is $X_1 \in \text{Mat}_{p,1}(\mathbb{C})$ with ${}^t\bar{X}_o X_1 = 1$ and ${}^t\bar{X}_1 X_1 = 1$. Cauchy-Schwarz inequality for the standard Hermitian inner product in $\text{Mat}_{p,1}(\mathbb{C})$ implies that $1 = {}^t\bar{X}_o X_1 \leq \sqrt{{}^t\bar{X}_o X_o} \sqrt{{}^t\bar{X}_1 X_1} = \sqrt{{}^t\bar{X}_o X_o} < 1$, which is an absurd. This justifies the extendability of ξ^{X_o} over $\partial B_{p^\sigma, q}$. For an arbitrary $g = (N, L)$ with $N \in SU(p, 1)$, let $X_o := N^{-1}(0)$ where $0 \in \mathbb{C}^p$ stands for the origin of B_p . Then $\xi^{X_o} N^{-1} \in \text{Aut}_o(B_p) = U_p$ acts linearly on B_p and extends to $\text{Mat}_{p,1}(\mathbb{C}) \supset \bar{B}_p$. Similarly, $L \in U_q$ induces a diffeomorphism of $\partial B_{p^\sigma, q}$.

Towards the verification of (ii), one calculates that the real gradient

$$\begin{aligned} & \left(\frac{\partial \rho_{p^\sigma, q}}{\partial (rx_1)}, \frac{\partial \rho_{p^\sigma, q}}{\partial (ix_1)}, \dots, \frac{\partial \rho_{p^\sigma, q}}{\partial (rx_p)}, \frac{\partial \rho_{p^\sigma, q}}{\partial (ix_p)}, \dots, \right. \\ & \left. \frac{\partial \rho_{p^\sigma, q}}{\partial (ry_1)}, \frac{\partial \rho_{p^\sigma, q}}{\partial (iy_1)}, \dots, \frac{\partial \rho_{p^\sigma, q}}{\partial (ry_q)}, \frac{\partial \rho_{p^\sigma, q}}{\partial (iy_q)} \right) \\ & = (2\sigma(rx_1)(1 - {}^t\bar{X}X)^{\sigma-1}, 2\sigma(ix_1)(1 - {}^t\bar{X}X)^{\sigma-1}, \dots, 2ry_q, 2iy_q) \end{aligned}$$

vanishes exactly when $X \in S^{2p-1} = U_p(N)$ and $Y = 0 \in \mathbb{C}^q$.

As an immediate consequence of (ii) and the defining equation (10) of $\partial B_{p^\sigma, q}$, follows. The argument of Lemma 1 (ii) reveals the G -invariance of $(B_{p^\sigma, q})^\circ$, whereas $GM \subseteq (B_{p^\sigma, q})^\circ$. Conversely, for any $(X_o, Y_o) \in (B_{p^\sigma, q})^\circ$ the proof of Lemma 1 (iii) provides $\xi^{X_o} \in SU(p, 1)$ with $\xi^{X_o}(X_o, Y_o) = (0, Y_1)$, $Y_1 = Y_o(1 - {}^t\bar{X}_o X_o)^{-\frac{\sigma}{2}}$. Then an appropriate $L \in U_q$ brings $Y_1 \in S^{2q-1}$ to ${}^t(1, 0, \dots, 0)$. \square

We are now in the position to describe the reduction of the identity component $\text{Aut}(B_p \times B_q)_1 = SU(p, 1) \times SU(q, 1)$, caused by a compression map c_σ (cf. (2)).

Proposition 1.

i) *The holomorphic automorphisms, fixing the origin $\delta = (X = 0, Y = 0)$ of a compressed product of balls $B_{p^\sigma, q}$ of exponent $\sigma > 1$ form the group*

$$\text{Aut}_\delta(B_{p^\sigma, q}) = \text{Aut}_\delta(B_{p^\sigma, q}) \cap G = S(U_p \times U_1) \times U_q \simeq U_p \times U_q. \quad (12)$$

ii) *The group of the holomorphic automorphisms of a compressed product of balls $B_{p^\sigma, q}$ of exponent $\sigma > 1$ is*

$$\text{Aut}(B_{p^\sigma, q}) = SU(p, 1) \times U_q. \quad (13)$$

Proof:

i) By definition (3), the domain $B_{p^\sigma, q}$ is circular, i. e., invariant under the action $(X, Y) \mapsto (e^{i\alpha}X, e^{i\alpha}Y)$ of $U_1 = \{e^{i\alpha}; \alpha \in [0, 2\pi)\}$. The holomorphic automorphisms, fixing the origin of a bounded circular domain act by unitary linear transformations on the ambient Euclidean space (cf. [10, 13]). Therefore, any $\varphi \in \text{Aut}_\delta(B_{p^\sigma, q}) \subseteq U_{p+q}$ extends to a real analytic diffeomorphism of the boundary $\partial B_{p^\sigma, q}$, preserving the smooth and the singular boundary loci. More precisely, if $\varphi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U_{p+q}$ with $A \in \text{Mat}_{p,p}(\mathbb{C}), B \in \text{Mat}_{p,q}(\mathbb{C}), C \in \text{Mat}_{q,p}(\mathbb{C}), D \in \text{Mat}_{q,q}(\mathbb{C})$ transforms any $(X, Y = 0) \in S^{2p-1} \times \{0\}$ in $(AX, CX = 0) \in S^{2p-1} \times \{0\}$, then $C = 0$. Further, ${}^t\bar{\varphi}\varphi = I_{p+q}$ implies that $B = 0, A \in U_p$

and $D \in U_q$, whereas $\text{Aut}_{\delta}(B_{p^\sigma, q}) \subseteq U_p \times U_q$. The obvious inclusion $U_p \times U_q = \text{Aut}_{\delta}(B_{p^\sigma, q}) \cap G \subseteq \text{Aut}_{\delta}(B_{p^\sigma, q})$ concludes the proof of (i).

ii) Let us suppose that $\varphi \in \text{Aut}(B_{p^\sigma, q})$ maps $\delta = M(0)$ in $(X_o, Y_o) \in B_{p^\sigma, q}$. According to Lemma 1 (iv), there exist $g \in G$ and $s \in [0, 1)$, such that $g(X_o, Y_o) = M(s)$ (cf. (8)). If $s = 0$ then $g\varphi \in \text{Aut}_{\delta}(B_{p^\sigma, q}) \subset G$, so that $\varphi \in G$ and $\text{Aut}(B_{p^\sigma, q}) = G$.

The opposite assumption $s \neq 0$ implies that $GM(s) \subseteq \text{Aut}(B_{p^\sigma, q})M(0)$, whereas $2p + 2q - 1 = \dim_{\mathbb{R}} GM(s) \leq \dim_{\mathbb{R}} \text{Aut}(B_{p^\sigma, q})M(0)$. According to Braun, Kaup and Upmeyer's article [3], the $\text{Aut}(B_{p^\sigma, q})$ -orbit of the origin $M(0)$ of the bounded circular domain $B_{p^\sigma, q}$ is a bounded symmetric domain, embedded as a closed complex analytic submanifold of $B_{p^\sigma, q}$. Therefore, $\text{Aut}(B_{p^\sigma, q})M(0) = B_{p^\sigma, q}$ is a Hermitian symmetric space of noncompact type. In particular, the boundary $\partial B_{p^\sigma, q}$ is smooth. This contradicts Lemma 2 (ii), thus justifying that $s = 0$.

□

2. The Bergman Kernel of a Compressed Product of Balls

Lemma 3. *At the canonical $\text{Aut}(B_{p^\sigma, q})$ -orbit representatives $M(s)$ of $B_{p^\sigma, q}$ with $\sigma > 0$ (cf. (8) from Lemma 1), the diagonal values of the Bergman kernel are*

$$k_{p^\sigma, q}(M(s)) = C_o \left[\sum_{j=0}^p F_j(\sigma) s^{2j} (1 - s^2)^{p-j} \right] \times [(1 - s^2)^{q+1} - s^{2(q+1)}](1 - s^2)^{-p-q-1} (1 - 2s^2)^{-1} \tag{14}$$

where

$$C_o := (-1)^{[\frac{p+1}{2}] + [\frac{q+1}{2}]} \pi^{-p-q}, \quad F_p(\sigma) := (p + q)! \sigma^p \tag{15}$$

$$F_j(\sigma) := \left[\prod_{l=1}^q (j + l) \right] \left\{ \sum_{i=j}^{p-1} \left[\prod_{l=1}^{p-i} (\sigma q + i + l) \right] a_{i,j}(\sigma) + a_{p,j}(\sigma) \right\} \sigma^j, \tag{16}$$

for all $0 \leq j \leq p - 1$,

are expressed by the recursively defined $a_{i,j}$,

$$a_{i,0}(\sigma) := 0, \text{ for all } i \in \mathbb{N}$$

$$a_{i,1}(\sigma) := \prod_{l=1}^{i-1} (\sigma - l), \text{ for all } i \in \mathbb{N} \setminus \{1\} \tag{17}$$

$$a_{i,i}(\sigma) := i!, \text{ for all } i \in \mathbb{N} \cup \{0\}$$

$$a_{i,j}(\sigma) := (\sigma j - i + 1)a_{i-1,j}(\sigma) + ja_{i-1,j-1}(\sigma), \quad (18)$$

for all $i \geq 3, 2 \leq j \leq i - 1$.

Proof: The compressed product of balls $B_{p,\sigma,q}$ is a bounded Reinhardt domain, i. e., $B_{p,\sigma,q}$ is invariant under the action ${}^t(x_1, \dots, x_p) \times {}^t(y_1, \dots, y_q) \mapsto {}^t(e^{i\alpha_1}x_1, \dots, e^{i\alpha_p}x_p) \times {}^t(e^{i\beta_1}y_1, \dots, e^{i\beta_q}y_q)$ of the torus $T^{p+q} := \underbrace{U_1 \times \dots \times U_1}_{p+q}$, $e^{i\alpha_j}, e^{i\beta_k} \in U_1$. Therefore the monomials $x_1^{i_1} \dots x_p^{i_p} y_1^{j_1} \dots y_q^{j_q}$ with $i_l, j_l \in \mathbb{N} \cup \{0\}$ form an orthogonal basis of the Hilbert space of the holomorphic square integrable functions $\mathcal{O}_{B_{p,\sigma,q}} \cap L^2_{B_{p,\sigma,q}}$ and

$$k_{p,\sigma,q}(M(s)) = \sum_{j=0}^{\infty} \frac{s^{2j}}{\|y_1^j\|_{B_{p,\sigma,q}}^2}.$$

Here $\|y_1^j\|_{B_{p,\sigma,q}}^2 := \left(\frac{i}{2}\right) \int_{B_{p,\sigma,q}} |y_1|^{2j} dx_1 \wedge d\bar{x}_1 \wedge \dots \wedge dx_p \wedge d\bar{x}_p \wedge dy_1 \wedge d\bar{y}_1 \wedge \dots \wedge dy_q \wedge d\bar{y}_q$. The real analytic diffeomorphism $B_{p,\sigma,q} \ni (X, Y) \mapsto (X, Z := Y(1 - {}^t\bar{X}X)^{\frac{1-\sigma}{2}}) \in B_{p+q}$, followed by polar coordinate changes $x_l := u_l e^{i\theta_l}$, $z_m := v_m e^{i\tau_m}$ where $\theta_l, \tau_m \in [0, 2\pi)$, $u_l, v_m \in \mathbb{R}$, $r := \sum_{l=1}^p u_l^2 + \sum_{m=1}^q v_m^2 < 1$, yields

$$\begin{aligned} & \|y_1^j\|_{B_{p,\sigma,q}}^2 \\ &= C_o^{-1} \int_{r < 1} \left(1 - \sum_{l=1}^p u_l^2\right)^{(\sigma-1)(j+q)} v_1^{2j} dv_1^2 \wedge dv_2^2 \wedge \dots \wedge dv_q^2 \wedge du_1^2 \wedge \dots \wedge du_p^2 \\ &= C_o^{-1} \prod_{m=1}^q (j+m)^{-1} \int_{\sum_{l=1}^p u_l^2 < 1} \left(1 - \sum_{l=1}^p u_l^2\right)^{\sigma(j+q)} du_1^2 \wedge \dots \wedge du_p^2 \\ &= C_o^{-1} \prod_{m=1}^q (j+m)^{-1} \left\{ \prod_{l=1}^p [\sigma(j+q) + l]^{-1} \right\} \end{aligned}$$

bearing in mind the fact that

$$\int_{\sum_{l=k}^m w_l^2 < \varepsilon} \left(\varepsilon - \sum_{l=k}^m w_l^2\right)^{\delta(n+k-1)} dw_k^2 \wedge \dots \wedge dw_m^2 =$$

$$[\delta(n+k-1) + 1]^{-1} \int_{\sum_{l=k+1}^m w_l^2 < \varepsilon} \left(\varepsilon - \sum_{l=k+1}^m w_l^2\right)^{\delta(n+k-1)+1} dw_{k+1}^2 \wedge \dots \wedge dw_m^2$$

for arbitrary $k, m, n \in \mathbb{N}$, $k < m$, real positive ε, δ and real variables

w_k, w_{k+1}, \dots, w_m . Then

$$\begin{aligned}
 k_{p^\sigma, q}(M(s)) &= C_o \sum_{j=0}^{\infty} \left\{ \prod_{m=1}^q (j+m) \prod_{l=1}^p [\sigma(j+q)+l] \right\} s^{2j} \\
 &= C_o \frac{d^q}{d\theta^q} \left[\frac{d^p}{dt^p} \left(\frac{t^{\sigma q+p}}{1-t^\sigma} \right) \Big|_{t=\sqrt[q]{\theta}} \right] \Big|_{\theta=s^2}
 \end{aligned}$$

according to $\frac{t^{\sigma q+p}}{1-t^\sigma} = \sum_{j=0}^{\infty} t^{\sigma(j+q)+p}$. By an induction on i one verifies that

$$\frac{d^i}{dt^i} (1-t^\sigma)^{-1} = \sum_{j=0}^i a_{i,j}(\sigma) \sigma^j t^{\sigma j-i} (1-t^\sigma)^{-j-1} \quad \text{for all } i \in \mathbb{N} \cup \{0\}$$

where $a_{i,j}(\sigma)$ are defined by (17) and (18). Consequently,

$$\begin{aligned}
 &\frac{d^p}{dt^p} [(1-t^\sigma)^{-1} t^{\sigma q+p}] \Big|_{t=\sqrt[q]{\theta}} \\
 &= \sum_{j=0}^{p-1} \left\{ \sum_{i=j}^{p-1} \left[\prod_{l=1}^{p-i} (\sigma q + i + l) \right] a_{i,j}(\sigma) + a_{p,j}(\sigma) \right\} \\
 &\quad \times \sigma^j \theta^{j+q} (1-\theta)^{-j-1} + p! \sigma^p \theta^{p+q} (1-\theta)^{-p-1}.
 \end{aligned}$$

Making use of

$$\frac{d^q}{d\theta^q} [\theta^{j+q} (1-\theta)^{-j-1}] \Big|_{\theta=s^2} = \left[\prod_{l=1}^q (j+l) \right] s^{2j} (1-s^2)^{-j-1} \left[\sum_{m=0}^q \left(\frac{s^2}{1-s^2} \right)^m \right]$$

(15) and (16), one obtains (14). \square

Proposition 2. *At an arbitrary point (X, Y) of $B_{p^\sigma, q}$ with $\sigma > 1$, the diagonal values of the Bergman kernel*

$$\begin{aligned}
 k_{p^\sigma, q}(X, Y) &= C_o \left\{ \sum_{j=0}^p F_j(\sigma) \left[\frac{t\bar{Y}Y}{(1-t\bar{X}X)^\sigma} \right]^j \left[1 - \frac{t\bar{Y}Y}{(1-t\bar{X}X)^\sigma} \right]^{p-j} \right\} \\
 &\quad \times \left\{ \left[1 - \frac{t\bar{Y}Y}{(1-t\bar{X}X)^\sigma} \right]^{q+1} - \left[\frac{t\bar{Y}Y}{(1-t\bar{X}X)^\sigma} \right]^{q+1} \right\} \\
 &\quad \times \left[1 - \frac{t\bar{Y}Y}{(1-t\bar{X}X)^\sigma} \right]^{-p-q-1} \left[1 - 2 \frac{t\bar{Y}Y}{(1-t\bar{X}X)^\sigma} \right]^{-1} (1-t\bar{X}X)^{-q\sigma-p-1}
 \end{aligned}$$

in the notations from (15), (16), (17), (18), Lemma 3.

Proof: The $\text{Aut}(B_{p^\sigma, q})$ -transformation law of the Bergman kernel states that

$$k_{p^\sigma, q}(Z_o) = k_{p^\sigma, q}(g(Z_o)) |\det J^{\mathbb{C}} g|_{Z_o}|^2$$

for arbitrary $Z_o = (X_o, Y_o) \in B_{p^\sigma, q}$ and $g \in \text{Aut}(B_{p^\sigma, q})$ with holomorphic Jacobian matrix $J^{\mathbb{C}} g$ (cf. [12]). According to Lemma 1(iii) and the proof of Lemma 1(iv), there exist $\xi^{X_o} \in SU(p, 1)$, given by (6) and $L \in U_q$, such that $(\xi^{X_o}, L)(X_o, Y_o) = M(s_o)$ for $s_o = \left[{}^t \bar{Y}_o Y_o (1 - {}^t \bar{X}_o X_o)^{-\sigma} \right]^{\frac{1}{2}} \in [0, 1)$. It is straightforward that $g^{Z_o} := (\xi^{X_o}, L)$ has

$$J^{\mathbb{C}} g^{Z_o} |_{Z_o} = \begin{pmatrix} (I_p - X_o {}^t \bar{X}_o)^{-\frac{1}{2}} (1 - {}^t \bar{X}_o X_o)^{-\frac{1}{2}} & 0_{p, q} \\ * & L(1 - {}^t \bar{X}_o X_o)^{-\frac{\sigma}{2}} \end{pmatrix} \in \text{Mat}_{p+q, p+q}(\mathbb{C}).$$

Plugging in $X_o = T^t(w_o, 0, \dots, 0)$ for appropriate $T \in U_p$, $w_o \in [0, 1)$, one observes that $\det(I_p - X_o {}^t \bar{X}_o) = 1 - w_o^2 = 1 - {}^t \bar{X}_o X_o$. Consequently, $|\det J^{\mathbb{C}} g^{Z_o}|^2 = (1 - {}^t \bar{X}_o X_o)^{-q\sigma - p - 1}$. Putting together with Lemma 3, one obtains the announced $k_{p^\sigma, q}(X_o, Y_o)$. \square

A bounded domain $\mathcal{D} \subset \mathbb{C}^n$ is Bergman complete if for any $z^o \in \partial \mathcal{D}$ and any $z_n \in \mathcal{D}$ with $\lim_{n \rightarrow \infty} z_n = z^o$, the diagonal values of the Bergman kernel $\lim_{z_n \rightarrow z^o} k_{\mathcal{D}}(z_n) = \infty$.

Corollary 1. *Let $B_{p^\sigma, q}$ be a compressed product of balls of exponent $\sigma > 1$. Then:*

- i) *for any smooth $(X_o, Y_o) \in \partial B_{p^\sigma, q}$ there exist a neighborhood U of (X_o, Y_o) , a positive real constant c and a real analytic function $\Phi_{p^\sigma, q} : U \rightarrow (0, c)$, such that*

$$k_{p^\sigma, q}(X, Y) = \Phi_{p^\sigma, q}(X, Y) \left[1 - \frac{{}^t \bar{Y} Y}{(1 - {}^t \bar{X} X)^\sigma} \right]^{-p-q-1} \tag{19}$$

for all $(X, Y) \in B_{p^\sigma, q} \cap U$;

- ii) *for any singular $(X_1, 0) \in \partial B_{p^\sigma, q}$ there exist a neighborhood U of $(X_1, 0)$, a positive real constant c and a real analytic function $\Psi_{p^\sigma, q} : U \rightarrow (0, c)$, such that*

$$k_{p^\sigma, q}(X, Y) = \Psi_{p^\sigma, q}(X, Y) \left[1 - \frac{{}^t \bar{Y} Y}{(1 - {}^t \bar{X} X)^\sigma} \right]^{-p-q-1} (1 - {}^t \bar{X} X)^{-\sigma q - p - 1} \tag{20}$$

for all $(X, Y) \in B_{p^\sigma, q} \cap U$. In particular, $B_{p^\sigma, q}$ is Bergman complete.

Proof: Let $B_{p^\sigma, q} = \left\{ X \in B_p, Y \in B_q; \tau(X, Y) = \frac{{}^t Y Y}{(1 - {}^t X X)^\sigma} < 1 \right\}$, so that

$$k_{p^\sigma, q} = \frac{C_o \left[\sum_{j=0}^p F_j(\sigma) \tau^j (1 - \tau)^{p-j} \right] \left[\sum_{j=0}^q \tau^j (1 - \tau)^{q-j} \right]}{(1 - \tau)^{p+q+1} (1 - {}^t X X)^{\sigma q + p + 1}}.$$

Then $\Psi_{p^\sigma, q} := k_{p^\sigma, q} (1 - \tau)^{p+q+1} (1 - {}^t X X)^{\sigma q + p + 1}$ is a bounded real analytic positively valued function on the closure $\bar{B}_{p^\sigma, q}$. In particular, if (X_o, Y_o) is a smooth boundary point of $B_{p^\sigma, q}$ then ${}^t X_o X_o < 1$ (cf. Lemma 2(iii)), so that the function $\Phi_{p^\sigma, q} := (1 - {}^t X X)^{-\sigma q - p - 1} \Psi_{p^\sigma, q}$ satisfies Corollary 1 (i). \square

3. Lower Boundary Estimates on Bergman Kernels of Bounded Pseudoconvex Domains

A domain $\mathcal{D} \subset \mathbb{C}^n$ is locally geometrically approximated by a family \mathcal{F} of domains in \mathbb{C}^n if for any $z \in \partial \mathcal{D}$ there exist a domain $\Omega_z \in \mathcal{F}$ and a neighborhood $U(\Omega_z)$ of Z such that $\mathcal{D} \cap \Omega_z$ is a domain in Ω_z . We need a result of Diederich, Fornaess and Herbort (cf. [5]) in the following slightly modified form:

Lemma 4. (Diederich, Fornaess, and Herbort [5]) *Let $\mathcal{D} \subset \mathbb{C}^n$ be a bounded pseudoconvex domain and U be a neighborhood of $Z_o \in \partial \mathcal{D}$, such that $\mathcal{D} \cap U$ is a subdomain in a bounded domain $\mathcal{D}_o \subset \mathbb{C}^n$. Then for any neighborhood $V \subset \subset U$ of Z_o there exists a constant $c \in (0, 1]$ such that*

$$k_{\mathcal{D}}(z) \geq c k_{\mathcal{D}_o}(z)$$

for all $z \in \mathcal{D} \cap V$. In particular, if a bounded pseudoconvex domain \mathcal{D} admits a local geometric approximation by a family \mathcal{F} of bounded Bergman complete domains, then \mathcal{D} is Bergman complete.

Proof: The aforementioned theorem from [5] asserts that for any $V \subset \subset U$, and $Z_o \in V$, there exists a constant $c \in (0, 1]$, such that $k_{\mathcal{D}}(z) \geq c k_{\mathcal{D} \cap U}(z)$ for arbitrary $z \in \mathcal{D} \cap V$. As far as $\mathcal{D} \cap U$ is a domain in \mathcal{D}_o , a result of Kobayashi from [9] applies to provide $k_{\mathcal{D} \cap U}(z) \geq k_{\mathcal{D}_o}(z)$ for $z \in \mathcal{D} \cap U$. \square

Putting together Corollary 1 and Lemma 4, one obtains the following

Corollary 2. *Let $\mathcal{D} \subset \text{Mat}_{p,1}(\mathbb{C}) \times \text{Mat}_{q,1}(\mathbb{C}) \simeq \mathbb{C}^{p+q}$ be a bounded pseudoconvex domain, $B_{p^\sigma, q}$ be a compressed product of balls of exponent $\sigma > 1$ and $Z_o = (X_o, Y_o) \in \partial \mathcal{D} \cap \partial B_{p^\sigma, q}$ be a common boundary point. Suppose that there is a neighborhood U of Z_o with connected $\mathcal{D} \cap U \subset B_{p^\sigma, q}$.*

(i) If Z_o is a smooth boundary point of $B_{p^\sigma, q}$ then there exists a neighborhood $V \subset\subset U$ and a real analytic function $\Phi_{p^\sigma, q}: V \rightarrow (0, \infty)$, such that

$$k_{\mathcal{D}}(Z) = \Phi_{p^\sigma, q}(X, Y) \left[1 - \frac{{}^t\bar{Y}Y}{(1 - {}^t\bar{X}X)^\sigma} \right]^{-p-q-1} \quad (21)$$

for all $(X, Y) \in \mathcal{D} \cap V$.

(ii) If Z_o is a singular boundary point of $B_{p^\sigma, q}$ then there exist $V \subset\subset U$ and a real analytic function $\Psi_{p^\sigma, q}: V \rightarrow (0, \infty)$, such that

$$k_{\mathcal{D}}(Z) = \Psi_{p^\sigma, q}(X, Y) \left[1 - \frac{{}^t\bar{Y}Y}{(1 - {}^t\bar{X}X)^\sigma} \right]^{-p-q-1} (1 - {}^t\bar{X}X)^{-\sigma q - p - 1} \quad (22)$$

for all $(X, Y) \in \mathcal{D} \cap V$.

In particular, if \mathcal{D} admits a local geometric approximation by the family of the compressed products of balls $B_{p^\sigma, q}$ of exponent $\sigma > 1$, then \mathcal{D} is Bergman complete.

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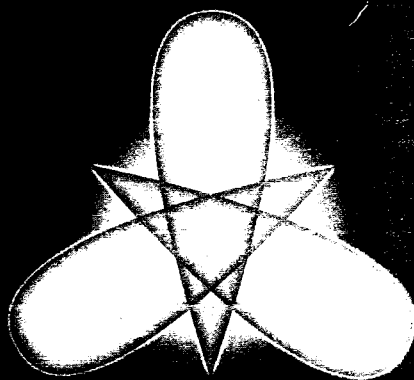
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