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WHEN DOES A BOUNDED DOMAIN COVER A PROJECTIVE MANIFOLD? (SURVEY)

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Abstract. The present survey introduces in some classical properties of the universal coverings of the projective algebraic manifolds. All the results are non-original. A forthcoming note is intended to discuss the corresponding fundamental groups.

In complex dimension 1, all the bounded, simply connected domains are biholomorphic to the unit disk, according to Riemann Mapping Theorem (cf. [31]). As a consequence, they admit projective discrete quotients with ample canonical bundles. Conversely, Riemann Uniformization Theorem (cf. [1] or [34]) asserts that all the complex projective curves with ample canonical bundle are covered by the disk.

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Actually, for a bounded domain D and a discrete fixed point free subgroup Γ of biholomorphic automorphisms of D, the quotient $X = \Gamma \setminus D$ is a projective algebraic manifold if and only if it is compact. The projective manifolds are, certainly, compact. Conversely, the canonical bundle $\mathcal{K}_X$ of a compact $X = \Gamma \setminus D$ is ample, as far as its first Chern class is represented by the Kähler form of the Bergman metric on D (cf. [36]). Thus, Kodaira Embedding Theorem implies the projectiveness of X, i.e., the existence of holomorphic sections of a sufficiently high power $\mathcal{K}_X^{\otimes n}$ of $\mathcal{K}_X$, separating the points and the tangent directions on X. Explicit lower bounds on the number of the linearly independent holomorphic sections of $\mathcal{K}_X^{\otimes n}$ for comparatively small $n \in \mathbb{N}$, can be found in Kollár's [24], [25]. The existence of projective embeddings of the compact quotients of the bounded domains can be justified directly by construction of automorphic forms (cf. [12], [25]).

A sort of higher dimensional generalizations of the disk are the bounded symmetric domains, i.e., the bounded homogeneous domains whose origin (and therefore, any point) is an isolated fixed point of an involutive biholomorphism. Recall that the Riemannian (Hermitian) globally symmetric spaces of noncompact type, consisting of the isolated fixed points of (holomorphic) involutive isometries, are quotients G/K of noncompact semisimple Lie groups G by maximal compact subgroups $K \subset G$. The bounded symmetric domains are exactly the Hermitian globally symmetric spaces of noncompact type (cf. [17]).

In [3] Borel has constructed compact discrete quotients of the Riemannian globally symmetric spaces G/K of noncompact type, and called them compact Clifford-Klein forms. In order to formulate precisely, let us recall few definitions.

Definition 1. (i) *A lattice Γ of a locally compact group G is a discrete subgroup $\Gamma \subset G$, whose quotient $\Gamma \setminus G$ admits a finite invariant measure. The lattices with compact $\Gamma \setminus G$ are called uniform.*

(ii) *An algebraic group $\mathbf{G}$ is a subgroup of some $GL(n, \mathbb{C})$, defined as a zero set of polynomials in $\{X_{i,j}\}_{i,j=1}^n$, $det X^{-1}$ $(X \in GL(n, \mathbb{C}))$ with complex coefficients. Whenever the defining polynomials of $\mathbf{G}$ have rational coefficients, $\mathbf{G}$ is said to be defined over $\mathbb{Q}$. For a $\mathbb{C}$-vector space V and a subring $S \subset \mathbb{C}$, let $V_S \subset V$ be the S-submodule, generated by a basis of V, and $\mathbf{G}_S := \{g \in \mathbf{G} | gV_S = V_S\}$. An arithmetic subgroup Γ of an algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$, is any subgroup $\Gamma \subset \mathbf{G}_{\mathbb{Q}}$ commensurable with $\mathbf{G}_{\mathbb{Z}}$, i.e., having an intersection $\Gamma \cap \mathbf{G}_{\mathbb{Z}}$ of finite index in Γ and $\mathbf{G}_Z$.*

The following are classical results for lattices in semisimple Lie groups:

Theorem 2. (*i*) (Borel and Harish-Chandra [5], [6], or [4]) *Let* $\mathbf{G}$ *be a connected semisimple algebraic group defined over* $\mathbb{Q}$. *Then any arithmetic subgroup* $\mathbf{\Gamma} \subset \mathbf{G}_{\mathbb{R}}$ *is a lattice of* $\mathbf{G}_{\mathbb{R}}$.

(*ii*) (Borel [3]) *Any connected noncompact simple Lie group* G_1 *has a uniform lattice.*

For the proof of (*i*), it suffices to construct a subset $U \subset \mathbf{G}_{\mathbb{R}}$ with a finite invariant measure, such that $\mathbf{G}_{\mathbb{R}} = U\mathbf{G}_{\mathbb{Z}}$. Let $\mathcal{A}_t \subset SL(n,\mathbb{R})$ be the set of the diagonal matrices with $0 \leq a_{i,i} \leq ta_{i+1,i+1}$, $1 \leq i < n$ for some $t \geq \frac{2}{\sqrt{3}}$, and $\mathcal{N}_u \subset SL(n,\mathbb{R})$ be the set of the upper triangular unipotent matrices with $|n_{i,j}| \leq u$, $1 \leq i < j \leq n$ for some $u \geq \frac{1}{2}$. It is well known that the Siegel domains $\mathcal{S}_{t,u} = SO(n)\mathcal{A}_t\mathcal{N}_u$ of $SL(n,\mathbb{R})$ have finite invariant measures and $SL(n,\mathbb{R}) = \mathcal{S}_{t,u}SL(n,\mathbb{Z})$, i.e., $SL(n,\mathbb{Z})$ is a lattice of $SL(n,\mathbb{R})$. Regarding $\mathbf{G}$ as a subgroup of $SL(n,\mathbb{C})$, Borel and Harish-Chandra establish the existence of $a \in SL(n,\mathbb{R})$ and $b_1,\ldots,b_m \in SL(n,\mathbb{Z})$ such that $\mathbf{G}_{\mathbb{R}} = \mathit{Interior}\left(\bigcup_{i=1}^{m} a^{-1}\mathcal{S}_{t,u}b_i \cap a^{-1}\mathbf{G}_{\mathbb{R}}a\right)\mathbf{G}_{\mathbb{Z}}$. The argument continues by showing that a Siegel domain $\mathcal{S}(\mathbf{G}_{\mathbb{R}})$ of a semisimple algebraic group $\mathbf{G}_{\mathbb{R}}$ has a finite invariant measure, and the intersections $\mathcal{S}_{t,u}b_ia^{-1} \cap \mathbf{G}_{\mathbb{R}}$ are contained in finite unions of right $\mathbf{G}_{\mathbb{R}}$-translates of $\mathcal{S}(\mathbf{G}_{\mathbb{R}})$.

Concerning (*ii*), the crucial step is to establish the existence of a Lie subalgebra $\mathfrak{g}_{\mathbb{Q}} \subset \mathfrak{g} = \mathit{Lie}G_1$ over $\mathbb{Q}$ and a $\mathbb{Q}$-linear involution $\theta_{\mathbb{Q}} : \mathfrak{g}_{\mathbb{Q}} \rightarrow \mathfrak{g}_{\mathbb{Q}}$ such that $\mathfrak{g} = \mathfrak{g}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ and $\theta_{\mathbb{Q}}$ extends to a Cartan involution of $\mathfrak{g}$. Let $\mathfrak{g}_u$ be the compact real form of $\mathfrak{g}$ and $\mathcal{G} = \mathfrak{g} \oplus \mathfrak{g}_u$. By the means of an appropriately chosen basis of $\mathcal{G}$, Borel identifies $\mathcal{G}_{\mathbb{C}} = \mathcal{G} \otimes_{\mathbb{R}} \mathbb{C}$ with $\mathbb{C}^n$ and considers the algebraic group $\mathbf{G} := \mathit{Aut}(\mathcal{G}_{\mathbb{C}}) \subset GL(n,\mathbb{C})$, defined over $\mathbb{Q}$ and consisting of the Lie algebra automorphisms of $\mathcal{G}_{\mathbb{C}}$. Whenever the identity component $\mathbf{G}^0$ of $\mathbf{G}$ does not admit a nontrivial rational character $\mathbf{G}^0 \rightarrow \mathbb{C}^*$ and $\mathbf{G}_{\mathbb{Q}}$ consists entirely of semisimple elements, the arithmetic lattices $\mathbf{\Gamma} \subset \mathbf{G}_{\mathbb{R}}$ are uniform. The aforementioned compactness criterion for $\mathbf{\Gamma} \setminus \mathbf{G}_{\mathbb{R}}$ with a reductive algebraic $\mathbf{G}$ defined over $\mathbb{Q}$, is due independently to Borel and Harish-Chandra [5], [6], as well as to Mostow and Tamagawa [27]. Since the identity component $\mathbf{G}^0_{\mathbb{R}}$ of $\mathbf{G}_{\mathbb{R}}$ has a surjective homomorphism $\varphi : \mathbf{G}^0_{\mathbb{R}} \rightarrow G_1$ with a compact kernel G_u, $\mathit{Lie}G_u = \mathfrak{g}_u$, the uniform arithmetic lattices $\mathbf{\Gamma} \subset \mathbf{G}^0_{\mathbb{R}}$ supply uniform lattices $\Gamma_1 := \varphi(\mathbf{\Gamma}) \subset G_1$.

Borel's Existence Theorem 2.(*ii*) implies, in particular, that all the bounded symmetric domains G/K cover projective algebraic manifolds $\Gamma \setminus G/K$ after, eventually, replacing Γ by a normal torsion-free subgroup $\Gamma' \subset \Gamma$ of finite index.

Let X be a compact complex algebraic manifold X of an arbitrary $\dim_{\mathbb{C}} X = n$. Locally, Griffiths [16] has established that an arbitrary smooth point $x \in X$ has a Zariski open neighborhood $U \subset X$, covered by a bounded contractible pseudoconvex domain in $\mathbb{C}^n$ (cf. Definition 7). In the case of an algebraic surface X, by further removing of divisors from U, Shabat [32] proves the existence of a Zariski open neighborhood, whose universal covering has a discrete biholomorphism group.

Globally, quite a lot of projective algebraic manifolds X are not covered by bounded domains. All compact complex manifolds X with finite fundamental groups $\pi_1(X)$ have compact universal coverings which, obviously, cannot be biholomorphic to domains in $\mathbb{C}^n$. Campana [8] shows that if a compact Kähler manifold X with $\chi(\mathcal{O}_X) := \sum_{i=0}^{n}(-1)^i h^{0,i}(X) \neq 0$, $\dim_{\mathbb{C}} X = n \geq 2$, does not coincide with the union of its irreducible compact complex analytic subspaces Y of $0 < \dim_{\mathbb{C}} Y < n$, then the fundamental group $\pi_1(X)$ is finite and of cardinality at most 2^{n-1}.

Prominent achievements in the study of the compact complex manifolds with ample canonical bundle are Frankel-Nadel's Uniformization results :

Theorem 3. *Let $\widetilde{M}$ be the universal covering of the compact complex manifold M with ample canonical bundle and $Aut(\widetilde{M})$ be the biholomorphism group of $\widetilde{M}$.*

(i) (Nadel [28]) *The identity component $G = Aut(\widetilde{M})^o$ of $Aut(\widetilde{M})$ is a real semisimple Lie group without compact factors.*

(ii) (Frankel [15], [14]) *There is a splitting $\widetilde{M} = \mathcal{M}_1 \times \mathcal{M}_2$ into a product of a Hermitian globally symmetric space $\mathcal{M}_1 = G/K$ (K - maximal compact subgroup of G) and a simply connected complex manifold $\mathcal{M}_2$ with a discrete biholomorphism group $Aut(\mathcal{M}_2)$. The fundamental group $\pi_1(M)$ has a finite index subgroup $\Gamma_1 \times \Gamma_2$ where Γ_1 is a uniform lattice of G and Γ_2 is a finite index subgroup of $Aut(\mathcal{M}_2)$.*

The opposite of statement (i) is equivalent to the presence of a nontrivial solvable radical $\mathcal{R} \neq 0$ of $LieG$. Let $V_1, \dots, V_r$ be an $\mathbb{R}$-basis of the last term $D^k\mathcal{R} \neq 0$ in the derived series of $\mathcal{R}$. Making use of the semistability of the holomorphic tangent bundle of M, Nadel shows the linear dependence of $V_1, \dots, V_r$ over the field of the meromorphic functions on $\widetilde{M}$. Then bearing in mind that $V_1, \dots, V_r$ are commuting vector fields on $\widetilde{M}$, whose real parts are infinitesimal isometries for the complete Kähler-Einstein metric $\widetilde{g}$ on $\widetilde{M}$, he derives the indefiniteness of the Ricci form $Ricci(\widetilde{g})$ of $\widetilde{g}$. On the other hand, $Ricci(\widetilde{g})$ has to be a negative constant multiple of the Kähler form of $\widetilde{g}$, so that the contradiction implies $D^k\mathcal{R} = 0$, i.e., the semisimplicity of $LieG$.

Towards the uniformization splitting of $\widetilde{M}$, announced in (ii), Frankel establishes the existence of a G-equivariant harmonic map $f : M \to \pi_1(M) \cap G \setminus G/K$. The proof is based on the non-increasing of the energy density under averaging. Moreover, the results of [9] imply that the harmonic submersion f with a locally symmetric target has to be a holomorphic map onto a Hermitian locally symmetric space, and the isotropy subgroups $I_x \subset G$ of all points $x \in \widetilde{M}$ are maximal compact, i.e., $I_x \simeq K$. For the product of an orbit $Orb(x) = G(x) \simeq G/I_x$ and the fixed point set $Fix(I_x)$ of the associated stabiliser, Frankel shows in [14] that the map $\Phi : Orb(x) \times Fix(I_x) \to \widetilde{M}$, given by $\Phi(gx, y) = gy$, is a biholomorphism onto $\widetilde{M}$. The assertion of (ii) on the fundamental group $\pi_1(M)$, is equivalent to the finiteness of the image of the extension homomorphism $Ext : \pi_1(M) \cap G \setminus \pi_1(M) \to Out(\pi_1(M) \cap G)$ in the outer automorphisms of $\pi_1(M) \cap G$. As far as both $Image(Ext) \subset Ad\pi_1(M)$ act on G, normalizing $\pi_1(M) \cap G$, and can be embedded in G, the image of Ext appears to be a subgroup of $Aut(\pi_1(M) \cap G \setminus G/K)$, which is finite by a theorem of Bochner-Yano.

Let us make a brief overview of Nadel's [28] and Frankel's [15] articles. In [10], [11] H. Cartan proves that the group of the biholomorphic automorphisms of a bounded domain D is a locally compact real Lie group (cf. also [29]). As a consequence, $Aut(D)$ has at most countably many connected components and the orbits of each component are closed subsets of D. These two observations are exploited by Shabat in his Ph.D. Thesis [32]. Nadel's Special Uniformization Theorem in dimension two is a generalization of Shabat's results, obtained independently of them.

Definition 4. (i) *Let D be a bounded domain and $\Gamma \subset Aut(D)$, be a discrete subgroup of biholomorphic automorphisms of D. A Γ-fundamental domain F on D is a connected subset $F \subset D$, containing a single point from each Γ-orbit on D. With respect to an arbitrary $Aut(D)$-invariant metric ρ on D, one can construct a Dirichlett fundamental domain*

$$F(z_0) := \{z \in D \mid \rho(z, z_0) < \rho(z, \gamma z_0), \quad \forall 1 \neq \gamma \in \Gamma\},$$

centered at $z_0 \in D$.

(ii) *A discrete group Γ acts properly discontinuously on the locally compact space Y if any point $y \in Y$ has a neighborhood U_y such that $\{\gamma \in \Gamma | \gamma(U_y) \cap U_y \neq \emptyset\}$ is finite.*

Any quotient $\Gamma \setminus Y$ of a complex analytic space Y by a properly discontinuously acting group Γ, inherits the complex analytic structure of Y by announcing $Y \to \Gamma \setminus Y$ to be holomorphic. If D is a bounded domain then any discrete subgroup of $Aut(D)$ is known to act properly discontinuously.

Theorem 5. (i) (Shabat [32]) *The universal covering of a family $S \to R$ of (eventually open) Riemann surfaces, covered by the disk, over a (not necessarily closed) Riemann surface R, covered by the disk, is a bounded contractible domain D. If D is not symmetric then its biholomorphism group $Aut(D)$ is discrete and the index of $\pi_1(S)$ in $Aut(D)$ is bounded by the ratio*

$$\frac{vol(F)}{vol(B(\frac{r}{2}))},$$

where r is the minimal distance between a pair of points from an $Aut(D)$-orbit, $B(\frac{r}{2})$ is a ball of radius $\frac{r}{2}$, F is a $\pi_1(S)$-fundamental domain on D and all the distances and volumes are calculated with respect to the Bergman metric of D.

(ii) (Nadel [28]) *Let S be a compact complex surface with ample canonical bundle. Then either the universal covering $\widetilde{S}$ is biholomorphic to the 2-ball or the bi-disk, or its biholomorphism group $Aut(\widetilde{S})$ is discrete, acts properly discontinuously on $\widetilde{S}$, and contains the group of deck transformations as a subgroup of finite index.*

The compact Clifford-Klein forms of the Hermitian symmetric spaces realize the bounded symmetric domains as universal coverings of projective algebraic manifolds. The presence of non-symmetric bounded domains with compact discrete quotients is illustrated by the next

Example 6. (Kodaira [21], Atiyah [2], Shabat [32]) There exist compact complex surfaces, namely, the Kodaira surfaces $M_{n,m}$, whose universal coverings are bounded contractible domains with discrete biholomorphism groups.

The Kodaira surfaces $M_{n,m}$ are constructed independently by Kodaira [21] and Atiyah [2]. Let R_0 be a Riemann surface of genus $n \geq 2$, and R be an unramified double covering of R_0 with genus $g = 2n-1$ and involution $\sigma : R \to R$, interchanging the sheets of $R \to R_0$. For an arbitrary $m \in \mathbb{N}$, let us consider the group homomorphism $\Phi : \pi_1(R) \to (\mathbb{Z}_m)^{2g}$, transforming the standard a- and b-cycles on R to generators of the $\mathbb{Z}_m$-factors.The Riemann surface S with $\pi_1(S) = Kernel(\Phi)$, is an m^{2g}-sheeted covering $\pi : S \to R$ of $genus(S) = m^{2g}(g-1)+1$. From the product $W = S \times R$ one removes the graphs of π and $\sigma\pi$ to obtain W'. The homologies $H_1^c(W', \mathbb{Z})$ with compact support are proved to decompose into a direct sum

$$H_1^c(W', \mathbb{Z}) = H_1^c(R, \mathbb{Z}) + H_1^c(S, \mathbb{Z}) + \mathbb{Z}_m,$$

where $\mathbb{Z}_m$ is generated by the bounding circle of a small disk on $s_0 \times R$, for a fixed $s_0 \in S$. Now, the epimorphism $\Psi : \pi_1(W') \to \mathbb{Z}_m$ determines an m-fold covering $M' \to W'$. The completion $M = M_{n,m} \to W$ is a branched covering of W, whose ramification locus consists of the graphs of π and $\sigma\pi$. Explicit calculation of the Chern numbers reveals that

$$2 < \frac{c_1^2(M_{n,m})}{c_2(M_{n,m})} = 2 + \frac{m^2 - 1}{m^2(2n-1) - m} < 3 \quad \text{for } n \geq 2, m \geq 1.$$

According to Hirzebruch's Proportionality Principle [18], the compact quotients of the bi-disk have $c_1^2 = 2c_2$, while the compact quotients of the 2-ball are characterized by $c_1^2 = 3c_2$ (cf. also [38]). Thus, for any $n \geq 2$ and $m \geq 1$ the Kodaira surface $M_{n,m}$ is not covered by a bounded symmetric domain in $\mathbb{C}^2$. Shabat's Thesis [32] implies that the universal coverings of $M_{n,m}$ are bounded contractible domains with discrete biholomorphism groups. Kas [19] has shown that the deformations of the complex structure of the Kodaira surfaces, are unobstructed. That is one more way of justifying that $M_{n,m}$ are not covered by the 2-ball.

Let us return to the unit disk $\Delta \subset \mathbb{C}$ and observe that it is geometrically convex, i.e., with any pair of points, it contains the entire real line segment between them.

Definition 7. (i) *If the domain $D = \{z \in \mathbb{C}^n | \rho(z) < 0\}$ has a defining function $\rho : \mathbb{C}^n \to \mathbb{R}$ of class C^2, then D is geometrically convex exactly when*

$$Re\left(\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z^j \partial z^k}(z) w_j w_k\right) + \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z^j \partial \overline{z}^k}(z) w_j \overline{w}_k \geq 0$$

for all boundary points $z \in \partial D$ and real tangent vectors

$$w \in T_z^{\mathbb{R}}(\partial D) := \left\{ w \in \mathbb{C}^n \;\mid\; \sum_{j=1}^{n} \frac{\partial \rho}{\partial z^j}(z) w_j + \sum_{j=1}^{n} \frac{\partial \rho}{\partial \overline{z}^j}(z) \overline{w}_j = 0 \right\}.$$

(ii) *A domain $D = \{z \in \mathbb{C}^n | \rho(z) < 0\}$ with a C^2 boundary is called pseudoconvex (resp., strictly pseudoconvex) if*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z^j \partial \overline{z}^k}(z) w_j \overline{w}_k \geq 0(\text{ resp., } > 0)$$

for all boundary points and complex tangent vectors

$$w \in T_z^{\mathbb{C}}(\partial D) := \left\{ w \in \mathbb{C}^n \;\mid\; \sum_{j=1}^{n} \frac{\partial \rho}{\partial z^j}(z) w_j = 0 \right\}.$$

It is straightforward that the geometric convexity is not invariant under biholomorphisms. Any geometrically convex domain of class C^2 is pseudoconvex. The pseudoconvexity is a biholomorphic invariant, equivalent to the so called *Kontinuitätssatz*: If $\varphi_\alpha : \Delta \to \mathbb{C}^n$, $\alpha \in A$ is a family of nonconstant holomorphic maps of the unit disk, extending continuously to the closure $\overline{\Delta}$, and the union of the boundaries $\cup\varphi_\alpha(\partial\Delta)$ is compactly embedded in D, then the union of the closed analytic disks $\cup\varphi_\alpha(\overline{\Delta})$ is also compactly embedded in D (cf. [26]).

The pseudoconvexity characterizes the domains of holomorphy, without involving the notion of a holomorphic function. By definition, a domain $D \subset \mathbb{C}^n$ is called a domain of holomorphy if there is a holomorphic function on D, which cannot be analytically continued to a strictly larger domain $\widehat{D}$. It is well known that any domain of holomorphy with a C^2 boundary is pseudoconvex and any pseudoconvex domain is a domain of holomorphy (Levi problem). Our interest in the pseudoconvex domains is based on a classical result of Siegel [33] that the bounded domains which admit compact discrete quotients are domains of holomorphy.

Definition 8. *For a domain $D_o = \{z \in \mathbb{C}^n | \rho_o < 0\}$ with a smooth boundary (i.e., $grad\rho_o|_{\partial D_o} \neq 0$), let $\varepsilon > 0$ be sufficiently small such that any smooth function $\rho : \mathbb{C}^n \to \mathbb{R}$ with $\|\rho - \rho_o\|_{C^\infty} < \varepsilon$ defines a domain $D_\rho := \{z \in \mathbb{C}^n | \rho(z) < 0\}$ with a smooth boundary. The set of domains $\mathcal{U}_\varepsilon(D_o) = \{D_\rho \subset \mathbb{C}^n | \|\rho - \rho_o\|_{C^\infty} < \varepsilon\}$ is called an ε-neighborhood of D_o in the C^∞ topology.*

In a vast distinction with the case of one complex variable, there exist bounded domains $D \subset \mathbb{C}^n$, $n \geq 2$ which do not admit compact discrete quotients.

Theorem 9. (Burns, Shneider and Wells [7]) *For any bounded strictly pseudoconvex domain $D_o \subset \mathbb{C}^n$, $n \geq 2$ with a smooth boundary and a sufficiently small $\varepsilon > 0$, there exists an infinite dimensional family of non-biholomorphic to each other domains $D \in \mathcal{U}_\varepsilon(D_o)$ with $Aut(D) = 1$ for $D \neq D_o$. In particular, such D do not cover compact complex analytic varieties.*

The biholomorphism classes of $D \in \mathcal{U}_\varepsilon(D_o)$ are distinguished by the means of the diffeomorphism classes of their boundaries, according to the following

Theorem 10. (Fefferman [13]) *Any biholomorphism $\Phi : D_1 \to D_2$ of bounded strictly pseudoconvex domains $D_1, D_2 \subset \mathbb{C}^n$ with smooth boundaries, extends to a smooth diffeomorphism $\Phi : \partial D_1 \to \partial D_2$ of their boundaries.*

The rough idea of the proof of Theorem 9 is that the defining functions of generic $D_{\rho_1}, D_{\rho_2} \in \mathcal{U}_\varepsilon(D_o)$ cannot be matched by a smooth extension Φ of a biholomorphism, since the nontrivial terms from the Taylor expansions of ρ_1 and ρ_2 are considerably more than the ones from the expansion of Φ.

In certain classes of bounded domains, the only members which admit compact discrete quotients are the bounded symmetric ones. Wong [37] shows that if a bounded strictly pseudoconvex domain $D \subset \mathbb{C}^n$ with a smooth boundary covers a compact complex analytic space, then D is biholomorphic to the ball $B_{1,n} \subset \mathbb{C}^n$. Similarly, the bounded convex domains which admit compact discrete quotients are bounded symmetric, according to Frankel [14].

Theorem 11. (i) (Vey [35]) *Let $D \subset \mathbb{C}^n \times \mathbb{C}^m$ be an S-domain or a Siegel domain of exponent $c \in \mathbb{R}$ (cf.* [20]*). Namely, D is biholomorphic to a bounded domain, contains a point $(z_0, 0)$, and is invariant under the following holomorphic transformations:*

$$\begin{array}{ll} (i) & (z,w) \mapsto (z+a, w) \quad \forall a \in \mathbb{R}^n; \\ (ii) & (z,w) \mapsto (z, e^{it}w) \quad \forall t \in \mathbb{R}; \\ (iii) & (z,w) \mapsto (e^t z, e^{ct}w) \quad \forall t \in \mathbb{R}. \end{array}$$

Then D covers a projective manifold if and only if it is bounded symmetric. In particular, the only bounded circular domains with compact discrete quotients are the bounded symmetric ones.

(ii) (Kodama [22], [23]) *Let $D \subset \mathbb{C}^n \times \mathbb{C}^{m_1} \times \ldots \times \mathbb{C}^{m_k}$ be a generalized S-domain of exponent $(c_1, \ldots, c_k) \in \mathbb{R}^k$ around $(z_0, 0, \ldots, 0)$, i.e., D is biholomorphic to a bounded domain and invariant under the following holomorphic transformations:*

$$\begin{array}{ll} (i) & (z, w_1, \ldots, w_k) \mapsto (z+a, w_1, \ldots, w_k) \quad \forall a \in \mathbb{R}^n; \\ (ii) & (z, w_1, \ldots, w_k) \mapsto (z, \ldots, w_{l-1}, e^{it}w_l, w_{l+1}, \ldots) \quad \forall t \in \mathbb{R}, 1 \leq l \leq k; \\ (iii) & (z, w_1, \ldots, w_k) \mapsto (e^t z, e^{c_1 t}w_1, \ldots, e^{c_k t}w_k) \quad \forall t \in \mathbb{R}. \end{array}$$

Then D admits a compact discrete quotient $\Gamma \setminus D$, $\Gamma \subset Aut(D)$, if and only if it is bounded symmetric.

Let us compare with Wong [37] and Rosay's [30] result that a bounded strictly pseudoconvex domain $D_0 \subset \mathbb{C}^n$ with a smooth boundary covers a compact complex analytic variety if and only if its biholomorphism group $Aut(D_0)$ is noncompact. Theorem 11, illustrates that the non-compactness of $Aut(D)$ is, in general, insufficient for the existence of projective or quasiprojective quotients.

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