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DUURSMA'S REDUCED POLYNOMIAL

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Abstract. The weight distribution $\{\mathcal{W}_C^{(w)}\}_{w=0}^n$ of a linear code $C \subset \mathbb{F}_q^n$ is put in an explicit bijective correspondence with Duursma's reduced polynomial $D_C(t) \in \mathbb{Q}[t]$ of C. We prove that the Riemann Hypothesis Analogue for a linear code C requires the formal self-duality of C. Duursma's reduced polynomial $D_F(t) \in \mathbb{Z}[t]$ of the function field $F = \mathbb{F}_q(X)$ of a curve X of genus g over $\mathbb{F}_q$ is shown to provide a generating function $\frac{D_F(t)}{(1-t)(1-qt)} = \sum_{i=0}^{\infty} \mathcal{B}_i t^i$ for the numbers $\mathcal{B}_i$ of the effective divisors of degree $i \geq 0$ of a virtual function field of a curve of genus $g-1$ over $\mathbb{F}_q$.

Let $\overline{\mathbb{F}_q} = \cup_{m=1}^{\infty} \mathbb{F}_{q^m}$ be the algebraic closure of a finite field $\mathbb{F}_q$ and $X/\mathbb{F}_q \subset \mathbb{P}^N(\overline{\mathbb{F}_q})$ be a smooth irreducible projective curve of genus g, defined over $\mathbb{F}_q$. Denote by $F = \mathbb{F}_q(X)$ the function field of X over $\mathbb{F}_q$ and choose n different $\mathbb{F}_q$-rational points $P_1, \ldots, P_n \in X(\mathbb{F}_q) := X \cap \mathbb{P}^N(\mathbb{F}_q)$. Suppose that G is an effective divisor of F of degree $2g-2 < \deg G = m < n$, whose support is disjoint from the support of $D = P_1 + \ldots + P_n$. The space $L(G) := H^0(X, \mathcal{O}_X(G))$ of the global holomorphic sections of the line bundle, associated with G will be referred to as to the Riemann-Roch space of G. We put $l(G) := \dim_{\mathbb{F}_q} L(G)$ and observe that the evaluation map

$$\mathcal{E}_D : L(G) \longrightarrow \mathbb{F}_q^n,$$

$$\mathcal{E}_D(f) = (f(P_1), \ldots, f(P_n)) \quad \text{for} \quad \forall f \in L(G)$$

is an $\mathbb{F}_q$-linear embedding. Its image $C := \mathrm{im}(\mathcal{E}_D) = \mathcal{E}_D L(G)$ is known as an algebraic geometry code or Goppa code. The minimum distance of C is $d(C) \geq n-m$. The equality $d(C) = n-m$ holds if and only if there exists a rational function $f_o \in L(G)$, vanishing at exactly m of the points $P_1, \ldots, P_n$. For an arbitrary $s \in \mathbb{N}$

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let $N_s(F) := |X(\mathbb{F}_{q^s})|$ be the number of the $\mathbb{F}_{q^s}$-rational points of X. Then the formal power series

$$Z_F(t) := \exp\left(\sum_{s=1}^{\infty} \frac{N_s(F)}{s} t^s\right)$$

is called the Hasse-Weil zeta function of F. It is well known (cf. Theorem 4.1.11 from [6]) that

$$Z_F(t) = \frac{L_F(t)}{(1-t)(1-qt)}$$

for a polynomial $L_F(t) \in \mathbb{Z}[t]$ of degree $2g$. We refer to $L_F(t)$ as to the Hasse-Weil polynomial of F.

In [2], [3] Duursma introduces the genus of a linear code $C \subset \mathbb{F}_q^n$ as the deviation $g := n+1-k-d$ of its dimension $k := \dim_{\mathbb{F}_q} C$ and minimum distance d from the equality in Singleton bound. Let $\mathcal{W}_C^{(w)}$ be the number of the codewords $c \in C$ of weight $d \leq w \leq n$. Then

$$\mathcal{W}_C(x,y) := x^n + \sum_{w=d(C)}^{n} \mathcal{W}_C^{(w)} x^{n-w} y^w$$

is called the homogeneous weight enumerator of C. Denote by $\mathcal{M}_{n,s}(x,y)$ the MDS-weight enumerator of length n and minimum distance s. Put $g^{\perp}$ for the genus of the dual code $C^{\perp}$ of C and $r := g + g^{\perp}$. In [2], [3] Duursma proves that the homogeneous weight enumerator

$$\mathcal{W}_C(x,y) = a_0\mathcal{M}_{n,d}(x,y) + a_1\mathcal{M}_{n,d+1}(x,y) + \ldots + a_r\mathcal{M}_{n,d+r}(x,y). \quad (1)$$

of an arbitrary linear code $C \subset \mathbb{F}_q^n$ has uniquely determined coordinates $a_0, \ldots, a_r \in \mathbb{Q}$ with respect to the MDS-weight enumerators $\mathcal{M}_{n,d+i}(x,y)$, $0 \leq i \leq r$. He refers to $P_C(t) := \sum_{i=0}^{r} a_i t^i \in \mathbb{Q}[t]$ as to the ζ-polynomial of C. The present note establishes that the difference

$$\mathcal{W}_C(x,y) - \mathcal{M}_{n,n+1-k}(x,y) = (q-1)\sum_{i=0}^{r-2} c_i \binom{n}{d+i} (x-y)^{n-d-i} y^{d+i}$$

of the homogeneous weight enumerator $\mathcal{W}_C(x,y)$ of C and the MDS-weight enumerator $\mathcal{M}_{n,n+1-k}(x,y)$ of the same length n and dimension k as C has uniquely determined coordinates $c_0, \ldots, c_{r-2} \in \mathbb{Q}$ with respect to $(x-y)^{n-d-i}y^{d+i}$, $0 \leq i \leq r-2$ (cf.Proposition 1). The polynomial $D_C(t) = \sum_{i=0}^{r-2} c_i t^i \in \mathbb{Q}[t]$ is in a bijective correspondence with $P_C(t) = (1-t)(1-qt)D_C(t) + t^g$. Theorem 11.1 from Duursma's [4] expresses the generating function $\zeta_{C,j}(t) = D_{C,j}(t) + ht^{g+j-1}Z_F(t)$ for the j-th support weights of C by a polynomial $D_{C,j}(t)$ and the Hasse-Weil ζ-function $Z_F(t)$ of the function field $F = \mathbb{F}_q(\mathbb{P}^j(\overline{\mathbb{F}_q}))$ of the projective space $\mathbb{P}^j(\overline{\mathbb{F}_q})$. In the case of $j = 1$, Duursma's $D_{C,1}(t)$ coincides with our $D_C(t)$ and that is why we call $D_C(t)$ Duursma's reduced polynomial of C.

The classical Hasse-Weil Theorem establishes that all the roots of the Hasse-Weil polynomial $L_F(t) \in \mathbb{Z}[t]$ of the function field $\mathbb{F}_q(X)$ of a curve X of genus g over $\mathbb{F}_q$ are on the circle $S\left(\frac{1}{\sqrt{q}}\right) : \left\{ z \in \mathbb{C} \;\middle|\; |z| = \frac{1}{\sqrt{q}} \right\}$ (cf. Theorem 4.2.3 form [6]). Suppose that there is a complete set of representatives $G_1, \ldots, G_h$ of the linear equivalence classes of the divisors of $\mathbb{F}_q(X)$ of degree $2g-2 < \deg G_i < n$ with

$\mathrm{Supp}(G_i) \cap \mathrm{Supp}(D) = \emptyset$ for $\forall 1 \leq i \leq n$, $D = P_1 + \ldots + P_n$. If $C_i = \mathcal{E}_D L(G_i)$ are the algebro-geometric Goppa codes, associated with these divisors, then according to Theorem 12.1 from Duursma's [4], the ζ-polynomials of C_i are related by the equality

$$\sum t^{g-g(C_i)} P_{C_i}(t) = L_F(t).$$

to the Hasse-Weil polynomial $L_F(t)$ of F. Baring in mind this fact, Duursma says that a linear code $C \subset \mathbb{F}_q^n$ satisfies the Riemann Hypothesis Analogue if all the roots of its zeta polynomial $P_C(t) = \sum\limits_{i=0}^{r} a_i t^i \in \mathbb{Q}[t]$ are on the circle $S\left(\frac{1}{\sqrt{q}}\right)$. Let C be an $\mathbb{F}_q$-linear code of dimension k and minimum distance d, which satisfies the Riemann Hypothesis Analogue. Proposition 2 shows that C is formally self-dual. Let us recall that C is formally self-dual if it has the same weight distribution $\mathcal{W}_C^{(w)} = \mathcal{W}_{C^\perp}^{(w)}$, $\forall 0 \leq w \leq n$ as its dual code $C^\perp \subset \mathbb{F}_q^n$. In the light of Duursma's results and our Proposition 1, the formal self-duality of C turns to be equivalent to the functional equation $P_C(t) = P_C\left(\frac{1}{qt}\right) q^g t^{2g}$ for $P_C(t)$ and to the functional equation $D_C(t) = D_C\left(\frac{1}{qt}\right) q^{g-1} t^{2g-2}$ for $D_C(t)$. Proposition 3 from the present note expresses explicitly the homogeneous weight enumerator $\mathcal{W}_C(x,y)$ of a formally self-dual code $C \subset \mathbb{F}_q^n$ by the lowest half of the coefficients of $D_C(t)$ or by the numbers $\mathcal{W}_C^{(d)}, \ldots, \mathcal{W}_C^{(k)}$ of the codewords $c \in C$, whose weights are between the minimum distance d of C and the dimension k.

In [1] Dodunekov and Landgev introduce the near-MDS code $C \subset \mathbb{F}_q^n$ as the ones with quadratic zeta polynomial $P_C(t)$. Kim and Hyun's article [5] provides a necessary and sufficient condition for a near-MDS code to satisfy the Riemann Hypothesis Analogue. By Theorem 3 from Duursma's [3], the zeta polynomial $P_C(t)$ of a formally self-dual code $C \subset \mathbb{F}_q^n$ is of even degree. Our Proposition 4 is a necessary and sufficient condition for a formally self-dual code $C \subset \mathbb{F}_q^n$ with zeta polynomial $P_C(T)$ of $\deg P_C(t) = 4$ to be subject to the Riemann Hypothesis Analogue. In analogy with the classical Hasse-Weil Theorem, we intend to express the Riemann Hypothesis Analogue for a linear code $C \subset \mathbb{F}_q^n$ in terms of the coefficients of the power series expansion of $\log\left[\frac{P_C(t)}{(1-t)(1-qt)}\right]$.

The last, third section is devoted to Duursma's reduced polynomial $D_F(t)$ of the function field $F = \mathbb{F}_q(X)$ of a curve $X/\mathbb{F}_q \subset \mathbb{P}^N(\overline{\mathbb{F}_q})$ of genus g over $\mathbb{F}_q$. Corollary 5.2 from Duursma's [2] shows the existence of $D_F(t)$. Explaining formula (10.1) from [4], he mentions that $D_F(t)$ accounts for the contribution of the special divisors of F to the zeta function $Z_F(t)$. The present article establishes that $D_F(t) \in \mathbb{Z}[t]$ is determined uniquely by its lowest g coefficients, which equal the numbers $\mathcal{A}_i$ of the effective divisors of F of degree $0 \leq i \leq g-1$. Our Proposition 5 reveals that the zeta function

$$\frac{D_F(t)}{(1-t)(1-qt)} = \sum_{i=0}^{\infty} \mathcal{B}_i t^i,$$

associated with $D_F(t)$ has the properties of a generating function for the numbers $\mathcal{B}_i$ of the effective divisors of degree $i \geq 0$ of a virtual function field of genus $g-1$ over $\mathbb{F}_q$. There arises the following

Open Problem: To characterize the function fields $F = \mathbb{F}_q(X)$ of curves $X/\mathbb{F}_q \subset \mathbb{P}^N(\overline{\mathbb{F}_q})$ of genus g over $\mathbb{F}_q$, for which there are curves $Y/\mathbb{F}_q \subset \mathbb{P}^M(\overline{\mathbb{F}_q})$

of genus $g-1$, defined over $\mathbb{F}_q$ with Hasse-Weil zeta function

$$Z_{\mathbb{F}_q(Y)}(t) = \frac{D_F(t)}{(1-t)(1-qt)}.$$

1. **The homogeneous weight enumerator of an arbitrary code.**

Proposition 1. *Let $C \subset \mathbb{F}_q^n$ be a linear code of dimension $k = \dim_{\mathbb{F}_q} C$, minimum distance d and genus $g = n+1-k-d \geq 1$, whose dual $C^{\perp} \subset \mathbb{F}_q^n$ is of minimum distance $d^{\perp}$ and genus $g^{\perp} = k+1-d^{\perp} \geq 1$. If*

$$D_C(t) = \sum_{i=0}^{g+g^{\perp}-2} c_i t^i \in \mathbb{Q}[t]$$

is Duursma's reduced polynomial of C and $\mathcal{M}_{n,n+1-k}(x,y)$ is MDS-weight enumerator of length n, dimension k and minimum distance $n+1-k$, then the homogeneous weight enumerator of C is

$$\mathcal{W}_C(x,y) = \mathcal{M}_{n,n+1-k}(x,y) + (q-1)\sum_{i=0}^{g+g^{\perp}-2} c_i \binom{n}{d+i}(x-y)^{n-d-i}y^{d+i}. \tag{2}$$

More precisely, Duursma's reduced polynomial $D_C(t) = \sum_{i=0}^{g+g^{\perp}-2} c_i t^i$ determines uniquely the weight distribution of C, according to

$$\mathcal{W}_C^{(w)} = (q-1)\binom{n}{w}\sum_{i=0}^{w-d}(-1)^{w-d-i}\binom{w}{d+i}c_i \quad \text{for} \quad d \leq w \leq d+g-1, \tag{3}$$

$$\begin{aligned} \mathcal{W}_C^{(w)} =& (q-1)\binom{n}{w}\sum_{i=0}^{\min(w-d,n-d-d^{\perp})}(-1)^{w-d-i}\binom{w}{d+i}c_i \\ &+ \binom{n}{w}\sum_{j=0}^{w-n-1+k}(-1)^j\binom{w}{j}(q^{w-n+k-j}-1) \quad \text{for} \quad d+g \leq w \leq n. \end{aligned} \tag{4}$$

Conversely, for $\forall 0 \leq i \leq g+g^{\perp}-2$ the numbers $\mathcal{W}_C^{(d)}, \ldots, \mathcal{W}_C^{(d+i)}$ determine uniquely the coefficient c_i of Duursma's reduced polynomial $D_C(t) = \sum_{i=0}^{g+g^{\perp}-2} c_i t^i$ by

$$c_i = (q-1)^{-1}\binom{n}{d+i}^{-1}\sum_{w=d}^{d+i}\binom{n-w}{n-d-i}\mathcal{W}_C^{(w)} \tag{5}$$

for $0 \leq i \leq g-1$,

$$\begin{aligned} c_i =& (q-1)^{-1}\binom{n}{d+i}^{-1}\left\{\sum_{w=d}^{d+g-1}\binom{n-w}{n-d-i}\mathcal{W}_C^{(w)}\right. \\ &\left.+ \sum_{w=d+g}^{d+i}\binom{n-w}{n-d-i}\left[\mathcal{W}_C^{(w)} - \binom{n}{w}\sum_{j=0}^{w-n-1+k}(-1)^j\binom{w}{j}(q^{w-n+k-j}-1)\right]\right\} \end{aligned} \tag{6}$$

for $g \leq i \leq g+g^{\perp}-2$.

In particular,

$$(q-1)\binom{n}{d+i}c_i \in \mathbb{Z}$$

are integers for all $0 \leq i \leq g + g^{\perp} - 2$.

The aforementioned formulae imply that $\mathcal{W}_C^{(d)}, \ldots, \mathcal{W}_C^{(d+g+g^{\perp}-2)}$ *determine uniquely the homogeneous weight enumerator* $\mathcal{W}_C(x,y)$ *of* C *by the formula*

$$\mathcal{W}_C(x,y) = \sum_{w=d}^{d+g+g^{\perp}-2} \mathcal{W}_C^{(w)} \lambda_w(x,y) + \Lambda(x,y), \tag{7}$$

with explicit polynomials

$$\lambda_w(x,y) := \sum_{s=w}^{d+g+g^{\perp}-2} \binom{n-w}{n-s}(x-y)^{n-s}y^s \quad \textit{for} \quad d \leq w \leq d+g+g^{\perp}-2 \tag{8}$$

and

$$\Lambda(x,y) := \mathcal{M}_{n,n+1-k}(x,y) - \sum_{w=d+g}^{d+g+g^{\perp}-2} \mathcal{M}_{n,n+1-k}^{(w)} \lambda_w(x,y). \tag{9}$$

Proof. In the case of $g = 0$, note that C is an MDS-code and $\mathcal{W}_C(x,y) = \mathcal{M}_{n,n+1-k}(x,y)$. Form now on, we assume that $g > 0$ and put $r := g + g^{\perp}$. According to Proposition 9.2 from Duursma's [2], the ζ-polynomials of C and $C^{\perp}$ satisfy the functional equation

$$P_{C^{\perp}}(t) = P_C\left(\frac{1}{qt}\right) q^g t^{g+g^{\perp}} \tag{10}$$

and $P_C(1) = P_{C^{\perp}}(1) = 1$. Therefore $P_C\left(\frac{1}{q}\right) = P_{C^{\perp}}(1)q^{-g} = \left(\frac{1}{q}\right)^g$ and the polynomial $P_C(t) - t^g \in \mathbb{Q}[t]$ vanishes at $t = 1$ and $t = \frac{1}{q}$. As a result, there is a polynomial

$$D_c(t) := \frac{P_C(t) - t^g}{(1-t)(1-qt)} = \sum_{i=0}^{r-2} c_i t^i \in \mathbb{Q}[t]. \tag{11}$$

Making use of (1), let us express

$$\mathcal{W}_C(x,y) = \mathcal{M}_{n,d+g}(x,y) + \sum_{i=0}^{r} b_i \mathcal{M}_{n,d+i}(x,y)$$

by the coefficients of $P_C(t) - t^g = \sum_{i=0}^{r} b_i t^i$. The comparison of the coefficients of

$$P_C(t) - t^g = (1-t)(1-qt)D_C(t). \tag{12}$$

yields

$$b_i = c_i - (q+1)c_{i-1} + qc_{i-2} \quad \text{for} \quad \forall 0 \leq i \leq r$$

with $c_{-2} = c_{-1} = c_{r-1} = c_r = 0$. Therefore

$$\mathcal{W}_C(x,y) = \mathcal{M}_{n,d+g}(x,y) + \sum_{i=0}^{r} c_i \mathcal{M}_{n,d+i}(x,y)$$

$$-(q+1)\sum_{i=0}^{r} c_{i-1}\mathcal{M}_{n,d+i}(x,y) + q\sum_{i=0}^{r} c_{i-2}\mathcal{M}_{n,d+i}(x,y).$$

Setting $j=i-1$, respectively, $j=i-2$ in the last two sums, one obtains

$$\mathcal{W}_C(x,y)=\mathcal{M}_{n,d+g}(x,y)+\sum_{i=0}^{r}c_i\mathcal{M}_{n,d+i}(x,y)$$
$$-(q+1)\sum_{j=-1}^{r-1}c_j\mathcal{M}_{n,d+j+1}(x,y)+q\sum_{j=-2}^{r-2}c_j\mathcal{M}_{n,d+j+2}(x,y),$$

whereas

$$\mathcal{W}_C(x,y)=\mathcal{M}_{n,d+g}(x,y)$$
$$+\sum_{j=0}^{r-2}c_j[\mathcal{M}_{n,d+j}(x,y)-(q+1)\mathcal{M}_{n,d+j+1}(x,y)+q\mathcal{M}_{n,d+j+2}(x,y)]. \tag{13}$$

Let us put

$$\mathcal{W}_{n,d+j}(x,y):=\mathcal{M}_{n,d+j}(x,y)-(q+1)\mathcal{M}_{n,d+j+1}(x,y)+q\mathcal{M}_{n,d+j+2}(x,y)$$

and recall that the MDS-weight enumerator of length n and minimum distance $d+j$ equals

$$\mathcal{M}_{n,d+j}(x,y)=x^n+\sum_{w=d+j}^{n}\mathcal{M}_{n,d+j}^{(w)}x^{n-w}y^w$$

with

$$\mathcal{M}_{n,d+j}^{(w)}=\binom{n}{w}\sum_{i=0}^{w-d-j}(-1)^i\binom{w}{i}(q^{w+1-d-j-i}-1). \tag{14}$$

Therefore

$$\mathcal{W}_{n,d+j}(x,y)=\mathcal{M}_{n,d+j}^{(d+j)}x^{n-d-j}y^{d+j}+[\mathcal{M}_{n,d+j}^{(d+j+1)}-(q+1)\mathcal{M}_{n,d+j+1}^{(d+j+1)}]x^{n-d-j-1}y^{d+j+1}$$
$$+\sum_{w=d+j+2}^{n}[\mathcal{M}_{n,d+j}^{(w)}-(q+1)\mathcal{M}_{n,d+j+1}^{(w)}+q\mathcal{M}_{n,d+j+2}^{(w)}]x^{n-w}y^w.$$

Making use of the MDS-weight distribution (14) and introducing

$$\mathcal{W}_{n,d+j}^{(w)}:=\mathcal{M}_{n,d+j}^{(w)}-(q+1)\mathcal{M}_{n,d+j+1}^{(w)}+q\mathcal{M}_{n,d+j+2}^{(w)}\quad\text{for}\quad d+j+2\leq w\leq n,$$

one expresses

$$\mathcal{W}_{n,d+j}(x,y)=\binom{n}{d+j}(q-1)x^{n-d-j}y^{d+j}$$
$$-\binom{n}{d+j+1}(q-1)(d+j+1)x^{n-d-j-1}y^{d+j+1}+\sum_{w=d+j+2}^{n}\mathcal{W}_{n,d+j}^{(w)}x^{n-w}y^w.$$

For any $d+j+2\leq w\leq n$ one has

$$\mathcal{W}_{n,d+j}^{(w)}=\binom{n}{w}\binom{w}{d+j}(q-1)(-1)^{w-d-j}.$$

Baring in mind that

$$\binom{n}{w}\binom{w}{d+j}=\binom{n-d-j}{w-d-j}\binom{n}{d+j},$$

one obtains

$$\mathcal{W}_{n,d+j}(x,y) = \binom{n}{d+j}(q-1)x^{n-d-j}y^{d+j} - \binom{n}{d+j+1}(q-1)(d+j+1)x^{n-d-j-1}y^{d+j+1} + \\ + \sum_{w=d+j+2}^{n} \binom{n}{d+j}\binom{n-d-j}{w-d-j}(q-1)(-1)^{w-d-j}x^{n-w}y^{w}.$$

Then by the means of

$$(d+j+1)\binom{n}{d+j+1} = (n-d-j)\binom{n}{d+j},$$

one derives that

$$\mathcal{W}_{n,d+j}(x,y) = \binom{n}{d+j}(q-1)\left[x^{n-d-j}y^{d+j} - (n-d-j)x^{n-d-j-1}y^{d+j+1} + \right. \\ \left. + \sum_{w=d+j+2}^{n} (-1)^{w-d-j}\binom{n-d-j}{w-d-j}x^{n-w}y^{w}\right].$$

Introducing $s := w-d-j$, one expresses

$$\sum_{w=d+j+2}^{n} (-1)^{w-d-j}\binom{n-d-j}{w-d-j}x^{n-w}y^{w} = \sum_{s=2}^{n-d-j}(-1)^{s}\binom{n-d-j}{s}x^{n-d-j-s}y^{d+j+s}$$

and concludes that

$$\mathcal{W}_{n,d+j}(x,y) = \binom{n}{d+j}(q-1)(x-y)^{n-d-j}y^{d+j}. \tag{15}$$

The equality $\mathcal{W}_{n,n-k}(x,y) = \binom{n}{k}(q-1)(x-y)^{k}y^{n-k}$ is exactly the claim (c) of Lemma 1 from Kim and Nyun's work [5]. Plugging in (15) in (13) and bearing in mind that $d+g = n+1-k$, one obtains (2).

In order to prove (3) and (4), let us put

$$\mathcal{V}_C(x,y) := \mathcal{W}_C(x,y) - \mathcal{M}_{n,n+1-k}(x,y)$$

and note that $\mathcal{V}_C(x,y) = \sum\limits_{w=d}^{n}\mathcal{V}_C^{(w)}x^{n-w}y^{w}$ with $\mathcal{V}_C^{(w)} = \mathcal{W}_C^{(w)}$ for $d \leq w \leq n-k$,

$$\mathcal{V}_C^{(w)} = \mathcal{W}_C^{(w)} - \mathcal{M}_{n,n+1-k}^{(w)} = \mathcal{W}_C^{(w)} - \binom{n}{w}\sum_{i=0}^{w-n-1+k}(-1)^{i}\binom{w}{i}(q^{w-n+k-i}-1)$$

for $d+g = n+1-k \leq w \leq n$. Making use of (2), one expresses

$$\mathcal{V}_C(x,y) = (q-1)\sum_{i=0}^{g+g^{\perp}-2} c_i\binom{n}{d+i}\sum_{s=0}^{n-d-i}\binom{n-d-i}{s}(-1)^{n-d-i-s}x^{s}y^{n-s}$$
$$= (q-1)\sum_{s=0}^{n-d}\left[\sum_{i=0}^{\min(n-d-s,g+g^{\perp}-2)} c_i\binom{n}{d+i}\binom{n-d-i}{s}(-1)^{n-d-i-s}\right]x^{s}y^{n-s},$$

after changing the summation order. Setting $w := n-s$, one obtains

$$\mathcal{V}_C(x,y) = (q-1)\sum_{w=d}^{n}\left[\sum_{i=0}^{\min(w-d,n-d-d^{\perp})} c_i\binom{n}{d+i}\binom{n-d-i}{n-w}(-1)^{w-d-i}\right]x^{n-w}y^{w}.$$

Then

$$\binom{n}{d+i}\binom{n-d-i}{n-w} = \binom{n}{w}\binom{w}{d+i},$$

allows to concludes that

$$\mathcal{V}_C^{(w)} = (q-1)\binom{n}{w}\sum_{i=0}^{\min(w-d,n-d-d^{\perp})} c_i \binom{w}{d+i}(-1)^{w-d-i} \quad \text{for} \quad \forall d \leq w \leq n,$$

which proves (3), (4).

Towards (5), (6), let us introduce $z := x - y$ and express (2) in the form

$$\mathcal{V}_C(y+z, y) = (q-1)\sum_{i=0}^{g+g^{\perp}-2} c_i \binom{n}{d+i} z^{n-d-i} y^{d+i}. \tag{16}$$

On the other hand,

$$\mathcal{V}_C(y+z,y) = \sum_{w=d}^{n} \mathcal{V}_C^{(w)} (y+z)^{n-w} y^w$$

$$= \sum_{w=d}^{n}\sum_{s=0}^{n-w}\binom{n-w}{s}\mathcal{V}_C^{(w)} y^{n-s} z^s = \sum_{s=0}^{n-d}\left[\sum_{w=d}^{n-s}\binom{n-w}{s}\mathcal{V}_C^{(w)}\right] y^{n-s} z^s,$$

after changing the summation order. Comparing the coefficients of $y^{d+i}z^{n-d-i}$ in the left and right hand side of (16), one obtains

$$\sum_{w=d}^{d+i}\binom{n-w}{n-d-i}\mathcal{V}_C^{(w)} = (q-1)c_i\binom{n}{d+i},$$

whereas

$$c_i = (q-1)^{-1}\binom{n}{d+i}^{-1}\sum_{w=d}^{d+i}\binom{n-w}{n-d-i}\mathcal{V}_C^{(w)}.$$

Combining with (14), one justifies (5) and (6). These formulae imply also that $(q-1)\binom{n}{d+i}c_i \in \mathbb{Z}$ are integers for all $0 \leq i \leq g + g^{\perp} - 2$.

The substitution by (5), (6), (14) in (2) yields

$$\mathcal{W}_C(x,y) = \mathcal{M}_{n,n+1-k}(x,y) + \sum_{i=0}^{g+g^{\perp}-2}\sum_{w=d}^{d+i}\binom{n-w}{n-d-i}\mathcal{W}_C^{(w)}(x-y)^{n-d-i}y^{d+i}$$

$$- \sum_{i=g}^{g+g^{\perp}-2}\sum_{w=d+g}^{d+i}\binom{n-w}{n-d-i}\mathcal{M}_{n,n+1-k}^{(w)}(x-y)^{n-d-i}y^{d+i}.$$

One exchanges the summation order in the double sums towards

$$\mathcal{W}_C(x,y) = \mathcal{M}_{n,n+1-k}(x,y) + \sum_{w=d}^{d+g+g^{\perp}-2}\mathcal{W}_C^{(w)}\sum_{i=w-d}^{g+g^{\perp}-2}\binom{n-w}{n-d-i}(x-y)^{n-d-i}y^{d+i}$$

$$- \sum_{w=d+g}^{d+g+g^{\perp}-2}\mathcal{M}_{n,n+1-k}^{(w)}\sum_{i=w-d}^{g+g^{\perp}-2}\binom{n-w}{n-d-i}(x-y)^{n-d-i}y^{d+i}.$$

Introducing $s := d + i$, one obtains (7) with (8) and (9).

□

Comparing the coefficients of $x^{n-d}y^d$ in the left and right hand sides of (2), one obtains $\mathcal{W}_C^{(d)} = (q-1)\binom{n}{d}c_0$ for a linear code C of genus $g \geq 1$. We claim that $c_0 < 1$. To this end, note that for any d-tuple $\{i_1, \ldots, i_d\} \subset \{1, \ldots, n\}$, supporting a word $c \in C$ of weight d there are exactly $q-1$ words $c' \in C$ with $\mathrm{Supp}(c') = \mathrm{Supp}(c) = \{i_1, \ldots, i_d\}$. That is due to the fact that the columns $H_{i_1}, \ldots, H_{i_d}$ of an arbitrary parity check matrix H of C are of rank $d-1$ and there are no words of weight $\leq d-1$ in the right null space of the matrix $(H_{i_1} \ldots H_{i_d})$. If ν is the number of the supports of the words of C of weight d then $\nu(q-1) = \mathcal{W}_C^{(d)}$, whereas

$$c_0 = \frac{\mathcal{W}_C^{(d)}}{(q-1)\binom{n}{d}} = \frac{\nu}{\binom{n}{d}} \leq 1.$$

If we assume that $c_0 = 1$ then any d-tuple of columns of H is linearly dependent. Bearing in mind that $\mathrm{rk}H = n-k$, one concludes that $d > n-k$. Combining with Singleton Bound $d \leq n-k+1$, one obtains $d = n-k+1$. That contradicts the assumption that C is not an MDS-code and proves that $c_0 < 1$ for any $\mathbb{F}_q$-linear code $C \subset \mathbb{F}_q^n$ of genus $g \geq 1$. Note that c_0 can be interpreted as the probability for a d-tuple to support a word of weight d from C.

2. **The Riemann Hypothesis Analogue and the formal self-duality of a linear code.** Recall that a linear code $C \subset \mathbb{F}_q^n$ with dual $C^\perp \subset \mathbb{F}_q^n$ is formally self-dual if C and $C^\perp$ have one and a same number $\mathcal{W}_C^{(w)} = \mathcal{W}_{C^\perp}^{(w)}$ of codewords of weight $0 \leq w \leq n$. Let us mention some trivial consequences of the formal self-duality of C. First of all, C and $C^\perp$ have one and a same minimum distance $d = d(C) = d(C^\perp) = d^\perp$. Further, C and $C^\perp$ have one and a same cardinality

$$q^{\dim C} = \sum_{w=0}^{n} \mathcal{W}_C^{(w)} = \sum_{w=0}^{n} \mathcal{W}_C^{(w)} = q^{\dim C^\perp},$$

so that $k = \dim C = \dim C^\perp = k^\perp$ and the length $n = k + k^\perp = 2k$ is an even integer. The genera $g = k+1-d = g^\perp$ also coincide. Let $P_C(t) = \sum\limits_{i=0}^{2g} a_i t^i$ and $P_{C^\perp} = \sum\limits_{i=0}^{2g} a_i^\perp t^i$ be the zeta polynomials of C, respectively, of $C^\perp$. The consecutive comparison of the coefficients of $x^{n-d}y^d, x^{n-d-1}y^{d+1}, \ldots, x^{n-d-2g}y^{d+2g}$ from the homogeneous polynomial

$$a_0\mathcal{M}_{2k,d}(x,y) + a_1\mathcal{M}_{2k,d+1}(x,y) + \ldots + a_{2g}\mathcal{M}_{2k,d+2g}(x,y) = \mathcal{W}_C(x,y)$$
$$= \mathcal{W}_{C^\perp}(x,y) = a_0^\perp\mathcal{M}_{2k,d}(x,y) + a_1^\perp\mathcal{M}_{2k,d+1}(x,y) + \ldots + a_{2g}^\perp\mathcal{M}_{2k,d+2g}(x,y)$$

in x, y yields $a_i = a_i^\perp$ for $\forall 0 \leq i \leq 2g$. It is clear that $a_i = a_i^\perp$ for $\forall 0 \leq i \leq 2g$ suffices for $\mathcal{W}_C(x,y) = \mathcal{W}_{C^\perp}(x,y)$, so that the formal self-duality of C is tantamount to the coincidence $P_C(t) = P_{C^\perp}(t)$ of the zeta polynomials of C and $C^\perp$. Duursma has shown in Proposition 9.2 from [2] that Mac Williams identities for $\mathcal{W}_C^{(w)}$ and $\mathcal{W}_{C^\perp}^{(w)}$ are equivalent to the functional equation (10) for the zeta polynomials $P_C(t)$, $P_{C^\perp}(t)$ of $C, C^\perp \subset \mathbb{F}_q^n$ with genera $g, g^\perp$. Thus, an $\mathbb{F}_q$-linear code $C \subset \mathbb{F}_q^n$ is formally self-dual if and only if its zeta polynomial $P_C(t)$ satisfies the functional equation

$$P_C(t) = P_C\left(\frac{1}{qt}\right) q^g t^{2g} \tag{17}$$

of the Hasse-Weil polynomial of the function field of a curve of genus g over $\mathbb{F}_q$.

Proposition 2. *If a linear code $C \subset \mathbb{F}_q^n$ satisfies the Riemann Hypothesis Analogue then C is formally self-dual, i.e., the zeta polynomial $P_C(t)$ of C is subject to the functional equation (17) of the Hasse-Weil polynomial of the function field of a curve of genus g over $\mathbb{F}_q$.*

Proof. Let us assume that $P_C(t)$ of degree $r := g + g^{\perp}$ satisfies the Riemann Hypothesis Analogue, i.e.,

$$P_C(t) = a_r \prod_{j=1}^{r} (t - \alpha_j) \in \mathbb{Q}[t]$$

for some $\alpha_j \in \mathbb{C}$ with $|\alpha_j| = \frac{1}{\sqrt{q}}$ for all $1 \leq j \leq r$. If α_j is a real root of $P_C(t)$ then $\alpha_j = \frac{\varepsilon}{\sqrt{q}}$ with $\varepsilon = \pm 1$. We claim that in the case of an even degree $r = 2m$, the zeta polynomial $P_C(t)$ is of the form

$$P_C(t) = a_{2m} \prod_{i=1}^{m} (t - \alpha_i)(t - \overline{\alpha_i}) \tag{18}$$

or of the form

$$P_C(t) = a_{2m} \left(t^2 - \frac{1}{q} \right) \prod_{i=1}^{m-1} (t - \alpha_i)(t - \overline{\alpha_i}), \tag{19}$$

while for an odd degree $r = 2m + 1$ one has

$$P_C(t) = a_{2m+1} \left(t - \frac{\varepsilon}{\sqrt{q}} \right) \prod_{i=1}^{m} (t - \alpha_i)(t - \overline{\alpha_i}) \tag{20}$$

for some $\varepsilon \in \{\pm 1\}$. Indeed, if $\alpha_i \in \mathbb{C} \backslash \mathbb{R}$ is a complex, non-real root of $P_C(t) \in \mathbb{Q}[t] \subset \mathbb{R}[t]$ then $\overline{\alpha_i} \neq \alpha_i$ is also a root of $P_C(t)$ and $P_C(t)$ is divisible by $(t - \alpha_i)(t - \overline{\alpha_i})$. If $P_C(t) = 0$ has three real roots $\alpha_1, \alpha_2, \alpha_3 \in \left\{ \frac{1}{\sqrt{q}}, -\frac{1}{\sqrt{q}} \right\}$, then at least two of them coincide. For $\alpha_1 = \alpha_2 = \frac{\varepsilon}{\sqrt{q}}$ one has $(t - \alpha_1)(t - \alpha_2) = (t - \alpha_1)(t - \overline{\alpha_1})$. Thus, $P_C(t)$ has at most two real roots, which are not complex conjugate (or, equivalently, equal) to each other and $P_C(t)$ is of the form (18), (19) or (20).

If $P_C(t)$ is of the form (18), then $P_C(t) = a_{2m} \prod_{i=1}^{m} \left(t^2 - 2\mathrm{Re}(\alpha_i) + \frac{1}{q} \right)$ and (10) reads as

$$P_{C^{\perp}}(t) = a_{2m} \left[\prod_{i=1}^{m} \left(\frac{1}{q} - 2\mathrm{Re}(\alpha_i) t + t^2 \right) \right] q^{g-m} = P_C(t) q^{g-m}, \tag{21}$$

after multiplying each of the factors $\frac{1}{q^2 t^2} - \frac{2\mathrm{Re}(\alpha_i)}{qt} + \frac{1}{q}$ by qt^2. If $D_C(t)$ is Duursma's reduced polynomial of C and $D_{C^{\perp}}(t)$ is Duursma's reduced polynomial of $C^{\perp}$, then

$$(1-t)(1-qt)D_{C^{\perp}}(t) + t^{g^{\perp}} = P_{C^{\perp}}(t) = P_C(t) q^{g-m} = (1-t)(1-qt) q^{g-m} D_C(t) + q^{g-m} t^g$$

implies that

$$(1 - t)(1 - qt)[D_{C^{\perp}}(t) - q^{g-m} D_C(t)] = q^{g-m} t^g - t^{g^{\perp}}.$$

Plugging in $t = 1$, one concludes that $q^{g-m} = 1$, whereas $g = m$. As a result, $g + g^{\perp} = 2m = 2g$ specifies that $g = g^{\perp}$ and (21) yields $P_C(t) = P_{C^{\perp}}(t)$, which is equivalent to the formal self-duality of C.

If $P_C(t)$ is of the form (19) then (10) provides

$$P_{C^\perp}(t) = a_{2m}\left(\frac{1}{q} - t^2\right)\left[\prod_{i=1}^{m-1}\left(\frac{1}{q} - 2\mathrm{Re}(\alpha_i)t + t^2\right)\right] q^{g-m} = -P_C(t)q^{g-m}. \quad (22)$$

Expressing by Duursma's reduced polynomials $D_C(t), D_{C^\perp}(t)$, one obtains

$$(1-t)(1-qt)D_{C^\perp}(t) + t^{g^\perp} = P_{C^\perp}(t) =$$
$$-P_C(t)q^{g-m} = -(1-t)(1-qt)q^{g-m}D_C(t) - q^{g-m}t^g,$$

whereas

$$(1-t)(1-qt)[D_{C^\perp}(t) + q^{g-m}D_C(t)] = -t^{g^\perp} - q^{g-m}t^g.$$

The substitution $t = 1$ in the last equality of polynomials yields $-1 - q^{g-m} = 0$, which is an absurd, justifying that a zeta polynomial $P_C(t)$, subject to the Riemann Hypothesis Analogue cannot be of the form (19).

If $P_C(t)$ is of odd degree $2m+1$, then (20) and (10) yield

$$P_{C^\perp}(t) = -\varepsilon\sqrt{q}a_{2m+1}\left(t - \frac{\varepsilon}{\sqrt{q}}\right)\left[\prod_{i=1}^{m}\left(\frac{1}{q} - 2\mathrm{Re}(\alpha_i)t + t^2\right)\right] q^{g-m-1}$$
$$= -\varepsilon\sqrt{q}P_C(t)q^{g-m-1}$$

after multiplying $\frac{1}{qt} - \frac{\varepsilon}{\sqrt{q}}$ by $-\frac{\varepsilon}{\sqrt{q}}qt$ and each $\frac{1}{q^2t^2} - \frac{2\mathrm{Re}(\alpha_i)}{qt} + \frac{1}{q}$ by qt^2. Expressing by Duursma's reduced polynomials

$$(1-t)(1-qt)D_{C^\perp}(t) + t^{g^\perp} = P_{C^\perp}(t) = -\varepsilon q^{g-m-\frac{1}{2}}P_C(t)$$
$$= -\varepsilon q^{g-m-\frac{1}{2}}(1-t)(1-qt)D_C(t) - \varepsilon q^{g-m-\frac{1}{2}}t^g,$$

one obtains

$$(1-t)(1-qt)\left[D_{C^\perp}(t) + \varepsilon q^{g-m-\frac{1}{2}}D_C(t)\right] = -t^{g^\perp} - \varepsilon q^{g-m-\frac{1}{2}}t^g.$$

The substitution $t = 1$ implies $-1 - \varepsilon q^{g-m-\frac{1}{2}} = 0$, which is an absurd, as far as $q^x = 1$ if and only if $x = 0$, while $g - m - \frac{1}{2}$ cannot vanish for integers g, m. Thus, none zeta polynomial of odd degree satisfies the Riemann Hypothesis Analogue. □

Proposition 3. *The following conditions are equivalent for a linear code* $C \subset \mathbb{F}_q^n$*:*

(i) C *is formally self-dual, i.e., the zeta polynomial* $P_C(t)$ *of* C *satisfies the functional equation*

$$P_C(t) = P_C\left(\frac{1}{qt}\right)q^g t^{2g}$$

of the Hasse-Weil polynomial of the function field of a curve of genus g *over* $\mathbb{F}_q$*;*

(ii) Duursma's reduced polynomial $D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i$ *satisfies the functional equation*

$$D_C(t) = D_C\left(\frac{1}{qt}\right)q^{g-1}t^{2g-2} \quad (23)$$

of the Hasse-Weil polynomial of the function field of a curve of genus $g-1$ *over* $\mathbb{F}_q$*;*

(iii) the coefficients of Duursma's reduced polynomial $D_C(t) = \sum\limits_{i=0}^{g+g^{\perp}-2} c_i t^i$ *of* C *satisfy the equalities*

$$c_{g-1+i} = q^i c_{g-1-i} \quad \text{for} \quad \forall 1 \leq i \leq g-1; \tag{24}$$

(iv) the dual code $C^{\perp} \subset \mathbb{F}_q^n$ *of* C *has dimension* $\dim_{\mathbb{F}_q} C^{\perp} = \dim_{\mathbb{F}_q} C = k$*, genus* $g(C^{\perp}) = g(C) = g$ *and the homogeneous weight enumerator of* C *is*

$$\mathcal{W}_C(x,y) = \mathcal{M}_{2k,k+1}(x,y) + \sum_{j=0}^{g-1} c_{g-1-j} w_j(x,y), \tag{25}$$

where

$$w_j(x,y) := (q-1)\binom{2k}{k+j}\left[(x-y)^{k+j}y^{k-j} + q^j (x-y)^{k-j} y^{k+j}\right] \tag{26}$$

for $1 \leq j \leq g-1$.

$$w_0(x,y) := (q-1)\binom{2k}{k}(x-y)^k y^k. \tag{27}$$

(v) the dual code $C^{\perp} \subset \mathbb{F}_q^n$ *of* C *has dimension* $\dim_{\mathbb{F}_q} C^{\perp} = \dim_{\mathbb{F}_q} C = k$*, genus* $g(C^{\perp}) = g(C) = g$ *and the homogeneous weight enumerator*

$$\mathcal{W}_C(x,y) = \mathcal{M}_{2k,k+1}(x,y) + \sum_{w=d}^{k-1} \mathcal{W}_C^{(w)} \varphi_w(x,y) + \mathcal{W}_C^{(k)} (x-y)^k y^k \tag{28}$$

with

$$\varphi_w(x,y) := \sum_{s=w}^{k-1} \binom{2k-w}{s-w}\left[(x-y)^{2k-s}y^s + q^{k-s}(x-y)^s y^{2k-s}\right] + \binom{2k-w}{k}(x-y)^k y^k \tag{29}$$

for $d \leq w \leq k-1$*, so that* C *can be obtained from an MDS-code of the same length* $2k$ *and dimension* k *by removing and adjoining appropriate words, depending explicitly on the numbers* $\mathcal{W}_C^{(d)}, \mathcal{W}_C^{(d+1)}, \ldots, \mathcal{W}_C^{(k)}$ *of the codeword of* C *of weight* $\leq k = \dim_{\mathbb{F}_q} C$.

Proof. Towards $(i) \Rightarrow (ii)$, one substitutes by $P_C(t) = (1-t)(1-qt)D_C(t) + t^g$ in (17), in order to obtain

$$(1-t)(1-qt)D_C(t) + t^g = (qt-1)(t-1)\left[D_C\left(\frac{1}{qt}\right) q^{g-1} t^{2g-2}\right] + t^g,$$

whereas (23).

Conversely, $(ii) \Rightarrow (i)$ is justified by

$$P_C(t) = (1-t)(1-qt)D_C(t) + t^g =$$

$$= (t-1)(qt-1)\left[D_C\left(\frac{1}{qt}\right) q^{g-1} t^{2g-2}\right] + t^g$$

$$= \left[\left(1 - \frac{1}{t}\right)t\right]\left[\left(1 - \frac{1}{qt}\right)qt\right]\left[D_C\left(\frac{1}{qt}\right)q^{g-1}t^{2g-2}\right] + \frac{q^g t^{2g}}{q^g t^g}$$

$$= \left[\left(1 - \frac{q}{qt}\right)\left(1 - \frac{1}{qt}\right) D_C\left(\frac{1}{qt}\right) + \frac{1}{(qt)^g}\right] q^g t^{2g} = P_C\left(\frac{1}{qt}\right) q^g t^{2g}.$$

That proves the equivalence $(i) \Leftrightarrow (ii)$.

Towards $(ii) \Leftrightarrow (iii)$, note that the functional equation of $D_C(t)$ reads as

$$\sum_{i=0}^{2g-2} c_i t^i = D_C(t) = D_C\left(\frac{1}{qt}\right) q^{g-1}t^{2g-2} = \left(\sum_{i=0}^{2g-2} \frac{c_i}{q^i t^i}\right) q^{g-1}t^{2g-2}$$
$$= \sum_{i=0}^{2g-2} c_i q^{g-1-i} t^{2g-2-i} = \sum_{j=0}^{2g-2} c_{2g-2-j} q^{-g+1+j} t^j.$$

Comparing the coefficients of the left-most and the right-most side, one expresses the formal self-duality of C by the relations

$$c_j = q^{-g+1+j} c_{2g-2-j} \quad \text{for} \quad \forall 0 \leq j \leq 2g-2.$$

Let $i := g-1-j$, in order to transform the above conditions to

$$c_{g-1+i} = q^i c_{g-1-i} \quad \text{for } \forall -g+1 \leq i \leq g-1. \tag{30}$$

For any $-g+1 \leq i \leq -1$ note that $c_{g-1+i} = q^i c_{g-1-i}$ is equivalent to $c_{g-1-i} = q^{-i} c_{g-1+i}$ and follows from (30) with $1 \leq -i \leq g-1$. In the case of $i = 0$, (30) holds trivially and (30) amounts to (24). That proves the equivalence of (ii) with (iii).

Towards $(iii) \Rightarrow (iv)$, one introduces a new variable $z := x - y$ and expresses (2) in the form

$$\mathcal{V}_C(y+z,y) := \mathcal{W}_C(y+z,y) - \mathcal{M}_{2k,k+1}(y+z,y) = (q-1)\sum_{i=0}^{2g-2} c_i \binom{2k}{d+i} y^{d+i} z^{2k-d-i}$$
$$= (q-1)\sum_{i=0}^{g-1} c_i \binom{2k}{d+i} y^{d+i} z^{2k-d-i} + (q-1)\sum_{i=g}^{2g-2} c_i \binom{2k}{d+i} y^{d+i} z^{2k-d-i}.$$

Let us change the summation index of the first sum to $0 \leq j := g-1-i \leq g-1$, put $1 \leq j := i-g+1 \leq g-1$ in the second sum and make use of $d+g = k+1$, in order to obtain

$$\mathcal{V}_C(y+z,y)$$
$$= (q-1)\sum_{j=0}^{g-1} c_{g-1-j} \binom{2k}{k-j} y^{k-j} z^{k+j} + (q-1)\sum_{j=1}^{g-1} c_{j+g-1} \binom{2k}{k+j} y^{k+j} z^{k-j}. \tag{31}$$

Extracting the term with $j = 0$ from the first sum, one expresses

$$\mathcal{V}_C(y+z,y) = (q-1)c_{g-1}\binom{2k}{k} y^k z^k$$
$$+\sum_{j=1}^{g-1}(q-1)\binom{2k}{k+j}\left[c_{g-1-j} y^{k-j} z^{k+j} + c_{g-1+j} y^{k+j} z^{k-j}\right] \tag{32}$$

for an arbitrary $\mathbb{F}_q$-linear code $C \subset \mathbb{F}_q^n$. If C is formally self-dual, then plugging in by (24) in (32) and making use of (26), (27), one gets

$$\mathcal{V}_C(y+z,y) = \sum_{j=0}^{g-1} c_{g-1-j} w_j(y+z,y).$$

Substituting $z := x - y$ and $\mathcal{V}_C(x,y) := \mathcal{W}_C(x,y) - \mathcal{M}_{2k,k+1}(x,y)$, one derives the equality (25) for the homogeneous weight enumerator of a formally self-dual linear code $C \subset \mathbb{F}_q^{2k}$.

In order to justify that (iv) suffices for the formal self-duality of C, we use that (25) with (26) and (27) is equivalent to

$$\begin{aligned} \mathcal{V}_C(y+z,y) = \sum_{j=1}^{g-1} c_{g-1-j}(q-1)\binom{2k}{k+j} y^{k-j} z^{k+j} \\ + c_{g-1}(q-1)\binom{2k}{k} y^k z^k + \sum_{j=1}^{g-1} c_{g-1-j}(q-1)\binom{2k}{k+j} y^{k+j} z^{k-j} \end{aligned} \tag{33}$$

Comparing the coefficients of $y^{k+j}z^{k-j}$ with $1 \leq j \leq g-1$ from (32) and (33), one concludes that

$$c_{g-1+j} = c_{g-1-j} q^j \quad \text{for} \quad \forall 1 \leq j \leq g-1.$$

These are exactly the relations (24) and imply the formal self-duality of C.

Towards $(iv) \Leftrightarrow (v)$, it suffices to put $\mathcal{E}(x,y) := \sum_{j=0}^{g-1} c_{g-1-j} w_j(x,y)$ and to derive that $\mathcal{E}(x,y) = \sum_{w=d}^{k-1} \mathcal{W}_C^{(w)} \varphi_w(x,y) + \mathcal{W}_C^{(k)}(x-y)^k y^k$. More precisely, introducing $i := g-1-j$, one expresses

$$\begin{aligned} \mathcal{E}(x,y) = \sum_{i=0}^{g-2} c_i(q-1)\binom{2k}{d+i} \left[(x-y)^{2k-d-i} y^{d+i} + q^{g-1-i}(x-y)^{d+i} y^{2k-d-i}\right] \\ + c_{g-1}(q-1)\binom{2k}{k}(x-y)^k y^k. \end{aligned}$$

Plugging in by (5) and exchanging the summation order, one gets

$$\begin{aligned} \mathcal{E}(x,y) = \sum_{w=d}^{k-1} \sum_{i=w-d}^{g-2} \binom{2k-w}{d+i-w} \mathcal{W}_C^{(w)} [(x-y)^{2k-d-i} y^{d+i} + q^{g-1-i}(x-y)^{d+i} y^{2k-d-i}] \\ + \sum_{w=d}^{k} \binom{2k-w}{k} \mathcal{W}_C^{(w)} (x-y)^k y^k. \end{aligned}$$

Introducing $s := d+i$ and extracting $\mathcal{W}_C^{(w)}$ as coefficients, one obtains

$$\mathcal{E}(x,y) = \sum_{w=d}^{k-1} \mathcal{W}_C^{(w)} \varphi_w(x,y) + \mathcal{W}_C^{(k)}(x-y)^k y^k.$$

□

Let $C \subset \mathbb{F}_q^n$ be an $\mathbb{F}_q$-linear code of genus g, whose dual $C^\perp \subset \mathbb{F}_q^n$ is of genus $g^\perp$. In [1], Dodunekov and Landgev introduce the near-MDS linear codes C as the ones with zeta polynomial $P_C(t) \in \mathbb{Q}[t]$ of degree $\deg P_C(t) := g + g^\perp = 2$. Thus, C is a near-MDS code if and only if it has constant Duursma's reduced polynomial $D_C(t) = c_0 \in \mathbb{Q}$. Kim an Hyun prove in [5]) that a near-MDS code C satisfies the Riemann Hypothesis Analogue exactly when

$$\frac{1}{(\sqrt{q}+1)^2} \leq c_0 \leq \frac{1}{(\sqrt{q}-1)^2}.$$

The next proposition characterizes the formally-self-dual codes $C \subset \mathbb{F}_q^n$ of genus 2, which satisfy the Riemann Hypothesis Analogue. By Proposition 3 (iii), C is a formally self-dual linear code of genus 2 exactly when its Duursma's reduced polynomial is

$$D_C(t) = c_0 + c_1 t + q c_0 t^2$$

for some $c_0, c_1 \in \mathbb{Q}$, $0 < c_0 < 1$.

Proposition 4. *A formally self-dual linear code $C \subset \mathbb{F}_q^{2k}$ with a quadratic Duursma's reduced polynomial $D_C(t) = c_0 + c_1 t + q c_0 t^2 \in \mathbb{Q}[t]$, $0 < c_0 < 1$ satisfies the Riemann Hypothesis Analogue if and only if*

$$[(q+1)c_0 + c_1]^2 \geq 4c_0, \tag{34}$$

$$q - 4\sqrt{q} + 1 \leq \frac{c_1}{c_0} \leq q + 4\sqrt{q} + 1, \tag{35}$$

$$c_1 \leq \min\left(\frac{1}{(\sqrt{q}-1)^2} - 2\sqrt{q}c_0, \ \frac{1}{(\sqrt{q}+1)^2} + 2\sqrt{q}c_0 \right). \tag{36}$$

Proof. According to (18) from the proof of Proposition 2, the zeta polynomial

$$P_C(t) = (1-t)(1-qt)(qc_0t^2 + c_1 t + c_0) + t^2$$

satisfies the Riemann Hypothesis Analogue if and only if there exist $\varphi, \psi \in [0, 2\pi)$ with

$$P_C(t) = q^2 c_0 \left(t - \frac{e^{i\varphi}}{\sqrt{q}} \right) \left(t - \frac{e^{-i\varphi}}{\sqrt{q}} \right) \left(t - \frac{e^{i\psi}}{\sqrt{q}} \right) \left(t - \frac{e^{-i\psi}}{\sqrt{q}} \right).$$

Comparing the coefficients of t and t^2 from $P_C(t)$, one expresses this condition by the equalities

$$c_1 - (q+1)c_0 = -2\sqrt{q}c_0[\cos(\varphi) + \cos(\psi)],$$
$$1 + 2qc_0 - (q+1)c_1 = 2qc_0[1 + 2\cos(\varphi)\cos(\psi)].$$

These are equivalent to

$$\cos(\varphi) + \cos(\psi) = \frac{(q+1)c_0 - c_1}{2\sqrt{q}c_0}$$

and

$$\cos(\varphi)\cos(\psi) = \frac{1 - (q+1)c_1}{4qc_0}.$$

In other words, the quadratic equation

$$f(t) := t^2 + \frac{c_1 - (q+1)c_0}{2\sqrt{q}c_0} t + \frac{1 - (q+1)c_1}{4qc_0} \in \mathbb{Q}[t]$$

has roots $-1 \leq t_1 = \cos(\varphi) \leq t_2 = \cos(\psi) \leq 1$. This, in turn, holds exactly when the discriminant

$$D(f) = \left[\frac{c_1 - (q+1)c_0}{2\sqrt{q}c_0} \right]^2 - \frac{4[1 - (q+1)c_1]}{4qc_0} \geq 0 \tag{37}$$

is non-negative, the vertex

$$-1 \leq \frac{(q+1)c_0 - c_1}{4\sqrt{q}c_0} \leq 1 \tag{38}$$

belongs to the segment $[-1, 1]$ and the values of $f(t)$ at the ends of this segment are non-negative,

$$f(1) \geq 0, \quad f(-1) \geq 0. \tag{39}$$

The equivalence of (37) to (34) is straightforward. Since C is of minimum distance $d = k-1$ and $\mathcal{W}_C^{(k-1)} = (q-1)\binom{2k}{k-1}c_0 \in \mathbb{N}$, the constant term $c_0 > 0$ of $D_C(t)$ is a positive rational number and one can multiply (38) by $-4\sqrt{q}c_0 < 0$, add $(q+1)c_0$ to all the terms and rewrite it in the form

$$(q - 4\sqrt{q} + 1)c_0 \leq c_1 \leq (q + 4\sqrt{q} + 1)c_0.$$

Making use of $c_0 > 0$, one observes that the above inequalities are tantamount to (35). Finally,

$$4qc_0 f(1) = 4qc_0 + 2\sqrt{q}[c_1 - (q+1)c_0] + 1 - (q+1)c_1 = (-c_1 - 2\sqrt{q}c_0)(\sqrt{q}-1)^2 + 1 \geq 0$$

and

$$4qc_0 f(-1) = 4qc_0 - 2\sqrt{q}[c_1 - (q+1)c_0] + 1 - (q+1)c_1 = (2\sqrt{q}c_0 - c_1)(\sqrt{q}+1)^2 + 1 \geq 0$$

can be expressed as (36).

□

3. **Duursma's reduced polynomial of a function field.** Let $F = \mathbb{F}_q(X)$ be the function field of a curve X of genus g over $\mathbb{F}_q$ and $h_g := h(F)$ be the class number of F, i.e., the number of the linear equivalence classes of the divisors of F of degree 0. The present section introduces an additive decomposition of the Hasse-Weil polynomial $L_F(t) \in \mathbb{Z}[t]$ of F, which associates to F a sequence $\{h_i\}_{i=1}^{g-1}$ of virtual class numbers h_i of function fields of curves of genus i over $\mathbb{F}_q$.

Lemma 3.1. *The following conditions are equivalent for a polynomial $L_g(t) \in \mathbb{Q}[t]$ of degree $\deg L_g(t) = 2g$:*

(i) $L_g(t)$ satisfies the functional equation

$$L_g(t) = L_g\left(\frac{1}{qt}\right) q^g t^{2g}$$

of the Hasse-Weil polynomial of the function field of a curve of genus g over $\mathbb{F}_q$;

(ii) $$L_{g-1}(t) := \frac{L_g(t) - L_g(1)t^g}{(1-t)(1-qt)}$$

is a polynomial with rational coefficients of degree $2g-2$, satisfying the functional equation

$$L_{g-1}(t) = L_{g-1}\left(\frac{1}{qt}\right) q^{g-1} t^{2g-2}$$

of the Hasse-Weil polynomial of the function field of a curve of genus $g-1$ over $\mathbb{F}_q$;

(iii) $$L_g(t) = \sum_{i=0}^{g} h_i t^i (1-t)^{g-i} (1-qt)^{g-i}$$

for some rational numbers $h_i \in \mathbb{Q}$.

Proof. Towards $(i) \Rightarrow (ii)$, let us note that the polynomial $M_g(t) := L_g(t) - L_g(1)t^g$ vanishes at $t = 1$, so that it is divisible by $1 - t$. Further,

$$M_g(t) = L_g(t) - L_g(1)t^g = \left[L_g\left(\frac{1}{qt}\right) - \frac{L_g(1)}{q^g t^g}\right] q^g t^{2g} = M_g\left(\frac{1}{qt}\right) q^g t^{2g}$$

satisfies the functional equation of the Hasse-Weil polynomial of the function field of a curve of genus g over $\mathbb{F}_q$. In particular, $M_g\left(\frac{1}{q}\right) = M_g(1)\frac{q^g}{q^{2g}} = 0$ and $M_g(t)$ is divisible by the linear polynomial $q\left(\frac{1}{q}-t\right)=1-qt$, which is relatively prime to $1-t$ in $\mathbb{Q}[t]$. As a result,

$$L_{g-1}(t):=\frac{M_g(t)}{(1-t)(1-qt)}\in\mathbb{Q}[t]$$

is a polynomial of degree $\deg L_{g-1}(t)=2g-2$. Straightforwardly,

$$\begin{aligned}L_{g-1}\left(\frac{1}{qt}\right)q^{g-1}t^{2g-2}&=\left[M_g\left(\frac{1}{qt}\right):\left(1-\frac{1}{qt}\right)\left(1-\frac{1}{t}\right)\right]q^{g-1}t^{2g-2}\\&=\frac{M_g(t)}{qt^2}:\frac{(qt-1)(t-1)}{qt^2}=\frac{M_g(t)}{(1-t)(1-qt)}=L_{g-1}(t)\end{aligned}$$

satisfies the functional equation of the Hasse-Weil polynomial of the function field of a curve of genus $g-1$ over $\mathbb{F}_q$.

The implication $(ii)\Rightarrow(i)$ follows from the functional equation of $L_{g-1}(t)$, applied to $L_g(t)=(1-t)(1-qt)L_{g-1}(t)+L_g(1)t^g$. Namely,

$$\begin{aligned}&L_g\left(\frac{1}{qt}\right)q^gt^{2g}\\&=\left[\left(1-\frac{1}{qt}\right)qt\right]\left[\left(1-\frac{1}{t}\right)t\right]\left[L_{g-1}\left(\frac{1}{qt}\right)q^{g-1}t^{2g-2}\right]+\frac{L_g(1)}{q^gt^g}q^gt^{2g}\\&=(qt-1)(t-1)L_{g-1}(t)+L_g(1)t^g\\&=(1-t)(1-qt)L_{g-1}(t)+L_g(1)t^g=L_g(t).\end{aligned}$$

We derive $(i)\Rightarrow(iii)$ by an induction on g, making use of (ii). More precisely, for $g=1$ one has $L_0(t):=\frac{L_1(t)-L_1(1)t}{(1-t)(1-qt)}\in\mathbb{Q}[t]$ of degree $\deg L_0(t)=0$ or $L_0\in\mathbb{Q}$. Then

$$L_1(t)=(1-t)(1-qt)L_0+L_1(1)t=\sum_{i=0}^{1}h_it^i(1-t)^{1-i}(1-qt)^{1-i}$$

with $h_0:=L_0\in\mathbb{Q}$ and $h_1:=L_1(1)\in\mathbb{Q}$. In the general case, (ii) provides a polynomial

$$L_{g-1}(t):=\frac{L_g(t)-L_g(1)t^g}{(1-t)(1-qt)},$$

subject to the functional equation

$$L_{g-1}(t)=L_{g-1}\left(\frac{1}{qt}\right)q^{g-1}t^{2g-2}$$

of the Hasse-Weil polynomial of the function field of a curve of genus $g-1$ over $\mathbb{F}_q$. By the inductional hypothesis, there exist $h'_i\in\mathbb{Q}$, $0\leq i\leq g-1$ with

$$L_{g-1}(t)=\sum_{i=0}^{g-1}h'_it^i(1-t)^{g-1-i}(1-qt)^{g-1-i}.$$

Then

$$L_g(t)=(1-t)(1-qt)L_{g-1}(t)+L_g(1)t^g=\sum_{i=0}^{g}h_it^i(1-t)^{g-i}(1-qt)^{g-i}$$

with $h_i:=h'_i\in\mathbb{Q}$ for $0\leq i\leq g-1$ and $h_g:=L_g(1)\in\mathbb{Q}$ justifies $(i)\Rightarrow(iii)$.

Towards $(iii) \Rightarrow (i)$, let us assume that $L_g(t) = \sum\limits_{i=0}^{g} h_i t^i (1-t)^{g-i}(1-qt)^{g-i}$. Then

$$L\left(\frac{1}{qt}\right) q^g t^{2g} = \left[\sum_{i=0}^{g} \frac{h_i}{q^i t^i}\left(1-\frac{1}{qt}\right)^{g-i}\left(1-\frac{1}{t}\right)^{g-i}\right] q^g t^{2g}$$

$$= \sum_{i=0}^{g}\left[\frac{h_i}{q^i t^i} q^i t^{2i}\right]\left[\left(1-\frac{1}{qt}\right) qt\right]^{g-i}\left[\left(1-\frac{1}{t}\right) t\right]^{g-i}$$

$$= \sum_{i=0}^{g} h_i t^i (qt-1)^{g-i}(t-1)^{g-i} = L_g(t)$$

satisfies the functional equation of the Hasse-Weil polynomial of the function field of a curve of genus g over $\mathbb{F}_q$.

□

Proposition 5. *Let $F = \mathbb{F}_q(X)$ be the function field of a smooth irreducible curve $X/\mathbb{F}_q \subset \mathbb{P}^N(\overline{\mathbb{F}_q})$ of genus g, defined over $\mathbb{F}_q$, with $h(F)$ linear equivalence classes of divisors of degree 0, $\mathcal{A}_i$ effective divisors of degree $i \geq 0$, Hasse-Weil polynomial $L_F(t) \in \mathbb{Q}[t]$ and Duursma's reduced polynomial $D_F(t) \in \mathbb{Q}[t]$, defined by the equality*

$$L_F(t) = (1-t)(1-qt)D_F(t) + h(F)t^g.$$

Then:

(i) $D_F(t) = \sum\limits_{i=0}^{g-2} \mathcal{A}_i (t^i + q^{g-1-i} t^{2g-2-i}) + \mathcal{A}_{g-1} t^{g-1} \in \mathbb{Z}[t]$ is a polynomial with integral coefficients, which is uniquely determined by $\mathcal{A}_0 = 1, \mathcal{A}_1, \ldots, \mathcal{A}_{g-1}$;

(ii) the equality

$$\frac{D_F(t)}{(1-t)(1-qt)} = \sum_{i=0}^{\infty} \mathcal{B}_i t^i \tag{40}$$

of formal power series of t holds for

$$\mathcal{B}_i = \sum_{j=0}^{i} \mathcal{A}_j \left(\frac{q^{i-j+1}-1}{q-1}\right) \tag{41}$$

for $0 \leq i \leq g-1$,

$$\mathcal{B}_i = \sum_{j=0}^{g-1} \mathcal{A}_j \left(\frac{q^{i-j+1}-1}{q-1}\right) + \sum_{j=g}^{i} \mathcal{A}_{2g-2-j}\left(\frac{q^{i-g+2}-q^{j-g+1}}{q-1}\right) \tag{42}$$

for $g \leq i \leq 2g-3$,

$$\mathcal{B}_i = D_F(1)\left(\frac{q^{i-g+2}-1}{q-1}\right) \tag{43}$$

for $i \geq 2g-2$;

(iii) the natural numbers $\mathcal{B}_i$, $i \geq 0$ from (ii) satisfy the relations

$$\mathcal{B}_i = q^{i-g+2}\mathcal{B}_{2g-4-i} + D_F(1)\left(\frac{q^{i-g+2}-1}{q-1}\right) \quad \text{for} \quad \forall g-1 \leq i \leq 2g-4; \tag{44}$$

$$\mathcal{B}_i = D_F(1)\left(\frac{q^{i-g+2}-1}{q-1}\right) \quad \text{for} \quad \forall i \geq 2g-3. \tag{45}$$

(iv) the number $h(F)$ of the linear equivalence classes of the divisors of F of degree 0 satisfies the inequilities

$$(\sqrt{q}-1)^{2g} \leq h(F) \leq (\sqrt{q}+1)^{2g}$$

Proof. (i) By Theorem 4.1.6. (ii) and Theorem 4.1.11 from [6], the Hasse-Weil zeta function of F is the generating function

$$Z_F(t) = \frac{L_F(t)}{(1-t)(1-qt)} = \sum_{j=0}^{\infty} \mathcal{A}_j t^j$$

of the sequence $\{\mathcal{A}_i\}_{i=0}^{\infty}$. According to Lemma 3.1 and $L_F(1) = h(F)$,

$$D_F(t) := \frac{L_F(t) - h(F)t^g}{(1-t)(1-qt)}$$

is a polynomial of $\deg D_F(t) = 2g-2$, subject to the functional equation of the Hasse-Weil polynomial of the function field of a curve of genus $g-1$ over $\mathbb{F}_q$. Thus,

$$Z_F(t) = D_F(t) + \frac{h(F)t^g}{(1-t)(1-qt)} = \sum_{j=0}^{\infty} \mathcal{A}_j t^j. \tag{46}$$

Let $l(G)$ is the dimension of the space $H^0(X, \mathcal{O}_X(G))$ of the global holomorphic sections of the line bundle $\mathcal{O}_X(G) \to X$, associated with a divisor $G \in \mathrm{Div}(F)$. Riemann-Roch Theorem asserts that

$$l(G) = l(K_X - G) + \deg(G) - g + 1$$

for a canonical divisor K_X of X. For any $j \geq g-1$, suppose that $G_1, \ldots, G_{h(F)} \in \mathrm{Div}(F)$ is a complete set of representatives of the linear equivalence classes of the divisors of F of degree j. Then

$$\mathcal{A}_j = \sum_{\nu=1}^{h(F)} \frac{q^{l(G_\nu)}-1}{q-1} = q^{j-g+1} \sum_{\nu=1}^{h(F)} \left(\frac{q^{l(K_Y - G_\nu)}-1}{q-1} \right) + h(F) \left(\frac{q^{j-g+1}-1}{q-1} \right) \tag{47}$$

for $g \leq j \leq 2g-2$ and

$$\mathcal{A}_j = h(F) \left(\frac{q^{j-g+1}-1}{q-1} \right) \quad \text{for} \quad \forall j \geq 2g-1. \tag{48}$$

Note that $K_Y - G_1, \ldots, K_Y - G_{h(F)}$ is a complete set of representatives of the linear equivalence classes of the divisors of F of degree $2g-2-j$, so that

$$\mathcal{A}_{2g-2-j} = \sum_{\nu=1}^{h(F)} \frac{q^{l(K_Y - G_\nu)}-1}{q-1}. \tag{49}$$

Plugging in by (49) in (47), one obtains

$$\mathcal{A}_j = q^{j-g+1} \mathcal{A}_{2g-2-j} + h(F) \left(\frac{q^{j-g+1}-1}{q-1} \right) \quad \text{for} \quad g \leq j \leq 2g-2, \tag{50}$$

whereas

$$Z_F(t) = \sum_{j=0}^{g-1} \mathcal{A}_j t^j + \sum_{j=g}^{2g-2} q^{j-g+1} \mathcal{A}_{2g-2-j} t^j + h(F) \sum_{j=g}^{\infty} \left(\frac{q^{j-g+1}-1}{q-1} \right) t^j,$$

Putting $i := 2g-2-j$ in the second sum and $i := j-g$ in the third sum, one expresses

$$Z_F(t) = \sum_{i=0}^{g-2} \mathcal{A}_i (t^i + q^{g-1-i} t^{2g-2-i}) + \mathcal{A}_{g-1} t^{g-1}$$
$$+h(F) \left[\frac{qt^g}{q-1} \left(\sum_{i=0}^{\infty} q^i t^i \right) - \frac{t^g}{q-1} \left(\sum_{i=0}^{\infty} t^i \right) \right],$$

Summing up the geometric progressions

$$\sum_{i=0}^{\infty} q^i t^i = \frac{1}{1-qt}, \quad \sum_{i=0}^{\infty} t^i = \frac{1}{1-t},$$

one derives

$$Z_F(t) = \sum_{i=0}^{g-2} \mathcal{A}_i (t^i + q^{g-1-i} t^{2g-2-i}) + \mathcal{A}_{g-1} t^{g-1} + h(F) \frac{t^g}{(1-t)(1-qt)},$$

whereas

$$D_F(t) = \sum_{i=0}^{g-2} \mathcal{A}_i (t^i + q^{g-1-i} t^{2g-2-i}) + \mathcal{A}_{g-1} t^{g-1}.$$

In particular, $D_F(t) \in \mathbb{Z}[t]$ has integral coefficients.

(ii) Let us expand

$$\frac{1}{1-t} = \sum_{i=0}^{\infty} t^i, \quad \frac{1}{1-qt} = \sum_{i=0}^{\infty} q^i t^i$$

as sums of geometric progressions and note that

$$\frac{1}{(1-t)(1-qt)} = \sum_{i=0}^{\infty} (1+q+\ldots+q^i) t^i = \sum_{i=0}^{\infty} \left(\frac{q^{i+1}-1}{q-1} \right) t^i.$$

Then represent Duursma's reduced polynomial in the form

$$D_F(t) = \sum_{j=0}^{g-1} \mathcal{A}_j t^j + \sum_{j=g}^{2g-2} \mathcal{A}_{2g-2-j} q^{j-g+1} t^j. \tag{51}$$

Now, the comparison of the coefficients of t^i, $i \geq 0$ from the left hand side and the right hand side of (40) provides (41), (42) and

$$\mathcal{B}_i = \sum_{j=0}^{g-1} \mathcal{A}_j \left(\frac{q^{i-j+1}-1}{q-1} \right) + \sum_{j=g}^{2g-2} \mathcal{A}_{2g-2-j} q^{j-g+1} \left(\frac{q^{i-j+1}-1}{q-1} \right) \quad \text{for } i \geq 2g-2.$$

The last formula can be expressed in the form

$$\mathcal{B}_i = \frac{q^{i+1}}{q-1} \left(\sum_{j=0}^{g-1} \mathcal{A}_j q^{-j} + \sum_{j=g}^{2g-2} \mathcal{A}_{2g-2-j} q^{j-g+1} q^{-j} \right) - \frac{1}{q-1} \left(\sum_{j=0}^{g-1} \mathcal{A}_j + \sum_{j=g}^{2g-2} \mathcal{A}_{2g-2} q^{j-g+1} \right)$$
$$= \frac{q^{i+1}}{q-1} D_F \left(\frac{1}{q} \right) - \frac{1}{q-1} D_F(1).$$

According to Lemma 3.1 $(i) \Rightarrow (ii)$, Duursma's reduced polynomial of F satisfies the functional equation $D_F(t) = D_F\left(\frac{1}{qt}\right) q^{g-1} t^{2g-2}$. In particular, $D_F(1) = D_F\left(\frac{1}{q}\right) q^{g-1}$ and there follows (43).

(iii) Due to $\mathcal{A}_i \geq 0$ for $\forall i \geq 0$, $\mathcal{B}_i$ are sums of non-negative integers. Moreover, $\mathcal{B}_i \geq \mathcal{A}_i \left(\frac{q^{i+1}}{q-1}\right) \geq \mathcal{A}_0 = 1 > 0$ for $\forall i \geq 0$ reveals that all $\mathcal{B}_i$ are natural numbers. Towards (44), let us introduce the polynomial $\psi(t) := \sum\limits_{j=0}^{g-2} \mathcal{A}_j t^j \in \mathbb{Z}[t]$ and express

$$D_F(t) = \sum_{j=0}^{g-2} \mathcal{A}_j t^j + q^{g-1} t^{2g-2} \left[\sum_{j=0}^{g-2} \mathcal{A}_j (qt)^{-j} \right] + \mathcal{A}_{g-1} t^{g-1}$$
$$= \psi(t) + \psi\left(\frac{1}{qt}\right) q^{g-1} t^{2g-2} + \mathcal{A}_{g-1} t^{g-1}.$$

In particular,

$$D_F(1) = \psi(1) + \psi\left(\frac{1}{q}\right) q^{g-1} + \mathcal{A}_{g-1}. \tag{52}$$

Straightforwardly,

$$\mathcal{B}_{g-1} - q\mathcal{B}_{g-3}$$
$$= \frac{q^g}{q-1}\left(\sum_{j=0}^{g-2} \mathcal{A}_j q^{-j}\right) - \frac{1}{q-1}\left(\sum_{j=0}^{g-2} \mathcal{A}_j\right) + \mathcal{A}_{g-1} - \frac{q^{g-1}}{q-1}\left(\sum_{j=0}^{g-2} \mathcal{A}_j q^{-j}\right) + \frac{q}{q-1}\left(\sum_{j=0}^{g-2} \mathcal{A}_j\right)$$
$$= \psi\left(\frac{1}{q}\right) q^{g-1} + \psi(1) + \mathcal{A}_{g-1} = D_F(1).$$

That proves (44) for $i = g-1$. In the case of $g \leq i \leq 2g-4$ note that $0 \leq 2g-4-i \leq g-4$ and

$$(q-1)(\mathcal{B}_i - q^{i-g+2}\mathcal{B}_{2g-4-i})$$
$$= \sum_{j=0}^{g-1} \mathcal{A}_j (q^{i-j+1} - 1) + \sum_{j=g}^{i} \mathcal{A}_{2g-2-j} (q^{i-g+2} - q^{j-g+1}) - \sum_{j=0}^{2g-4-i} \mathcal{A}_j (q^{g-1-j} - q^{i-g+2}).$$

Changing the summation index of the second sum to $s := 2g-2-j$, one obtains

$$(q-1)(\mathcal{B}_i - q^{i-g+2}\mathcal{B}_{2g-4-i})$$
$$= q^{i+1}\left(\sum_{j=0}^{g-1} \mathcal{A}_j q^{-j}\right) - \left(\sum_{j=0}^{g-1} \mathcal{A}_j\right) + q^{i-g+2}\left(\sum_{s=2g-2-i}^{g-2} \mathcal{A}_s\right)$$
$$-q^{g-1}\left(\sum_{s=2g-2-i}^{g-2} \mathcal{A}_s q^{-s}\right) - q^{g-1}\left(\sum_{j=0}^{2g-4-i} \mathcal{A}_j q^{-j}\right) + q^{i-g+2}\left(\sum_{j=0}^{2g-4-i} \mathcal{A}_j\right).$$

An appropriate grouping of the sums yields

$$(q-1)(\mathcal{B}_i - q^{i-g+2}\mathcal{B}_{2g-4-i})$$
$$= \psi\left(\frac{1}{q}\right)q^{i+1} + \mathcal{A}_{g-1}q^{i-g+2} - \psi(1) - \mathcal{A}_{g-1} + \psi(1)q^{i-g+2} - \psi\left(\frac{1}{q}\right)q^{g-1}$$
$$= (q^{i-g+2}-1)\left[\psi(1) + \psi\left(\frac{1}{q}\right)q^{g-1} + \mathcal{A}_{g-1}\right] = D_F(1)(q^{i-g+2}-1).$$

That justifies (44).

Note that (45) with $i \geq 2g-2$ coincides with (43). In the case of $i = 2g-3$,

$$(q-1)\mathcal{B}_{2g-3} = \sum_{j=0}^{g-1}\mathcal{A}_j(q^{2g-2-j}-1) + \sum_{s=1}^{g-2}\mathcal{A}_s(q^{g-1}-q^{g-1-s}),$$

after changing the summation index of the second sum to $s := 2g-2-j$. Then

$$(q-1)\mathcal{B}_{2g-3}$$
$$= q^{2g-2}\left(\sum_{j=0}^{g-2}\mathcal{A}_j q^{-j}\right) - \left(\sum_{j=0}^{g-2}\mathcal{A}_j\right) + \mathcal{A}_{g-1}(q^{g-1}-1) + q^{g-1}\left(\sum_{j=0}^{g-2}\mathcal{A}_j\right) - q^{g-1}\left(\sum_{j=0}^{g-2}\mathcal{A}_j q^{-j}\right)$$
$$= (q^{g-1}-1)\left[\psi(1) + \psi\left(\frac{1}{q}\right)q^{g-1} + \mathcal{A}_{g-1}\right] = D_F(1)(q^{g-1}-1),$$

which is tantamount to (45) with $i = 2g-3$.

(iv) By the Hasse-Weil Theorem, all the roots of $L_F(t)$ belong to the circle $S\left(\frac{1}{\sqrt{q}}\right) = \left\{z \in \mathbb{C} \mid |z| = \frac{1}{\sqrt{q}}\right\}$. The proof of Proposition 2 specifies that

$$L_F(t) = a_{2g}\prod_{j=1}^{g}\left(t - \frac{e^{i\varphi_j}}{\sqrt{q}}\right)\left(t - \frac{e^{-i\varphi_j}}{\sqrt{q}}\right)$$

for some $\varphi_j \in [0, 2\pi)$. The functional equation $L_F(t) = L_F\left(\frac{1}{qt}\right)q^g t^{2g}$ implies that $a_{2g} = q^g a_0$. Combining with $a_0 = L_F(0) = 1$, one gets

$$L_F(t) = \prod_{j=1}^{g}(\sqrt{q}t - e^{i\varphi_j})(\sqrt{q}t - e^{-i\varphi_j}) = \prod_{j=1}^{g}(qt^2 - 2\sqrt{q}\cos\varphi_j t + 1).$$

The substitution $t = 1$ provides

$$h(F) = L_F(1) = \prod_{j=1}^{g}(q - 2\sqrt{q}\cos\varphi_j + 1).$$

However, $\cos\varphi_j \in [-1, 1]$ requires

$$(\sqrt{q}-1)^2 \leq q - 2\sqrt{q}\cos\varphi_j + 1 \leq (\sqrt{q}+1)^2,$$

whereas

$$(\sqrt{q}-1)^{2g} \leq h(F) = L_F(1) = \prod_{j=1}^{g}(q - 2\sqrt{q}\cos\varphi_j + 1) \leq (\sqrt{q}+1)^{2g}.$$

□

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