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# ON THE VERTEX FOLKMAN NUMBERS $F_v(2, \ldots, 2; q)$*

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Abstract. For a graph $G$ the symbol $G \overset{v}{\to} (a_1, \ldots, a_r)$ means that in every $r$-coloring of the vertices of $G$ for some $i \in \{1, \ldots, r\}$ there exists a monochromatic $a_i$-clique of color $i$. The vertex Folkman numbers

$$F_v(a_1, \ldots, a_r; q) = \min\{|V(G)| : G \overset{v}{\to} (a_1, \ldots, a_r) \text{ and } K_q \nsubseteq G\}$$

are considered. In this paper we shall compute the Folkman numbers $F_v(\underbrace{2, \ldots, 2}_{r}; r - k + 1)$ when $k \leq 12$ and $r$ is sufficiently large. We prove also new bounds for some vertex and edge Folkman numbers.

**1. Introduction.** We consider only finite, non-oriented graphs without loops and multiple edges. The vertex set and the edge set of a graph $G$ will be denoted by $V(G)$ and $E(G)$, respectively. A graph $G$ is said to be an *empty graph* if $V(G) = \emptyset$. We call *a p-clique* of a graph $G$ a set of $p$ pairwise adjacent vertices.

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The largest integer $p$ such that the graph $G$ contains a $p$-clique is denoted by $\mathrm{cl}(G)$. A set of vertices of a graph is said to be *independent* if every two of them are not adjacent. We shall also use the following notations:

$\overline{G}$ is the complement of $G$;

$\alpha(G)$ is the vertex independence number of $G$, i.e., $\alpha(G) = \mathrm{cl}(\overline{G})$;

$\chi(G)$ is the chromatic number of $G$;

$f(G) = \chi(G) - \mathrm{cl}(G)$;

$K_n$ is the complete graph on $n$ vertices;

$C_n$ is the simple cycle on $n$ vertices;

$M(x, y) = \{G : |V(G)| < \chi(G) + 2f(G) - x \text{ and } f(G) \le y\}$.

The graph $G$ is a $(p, q)$*-graph* if $\mathrm{cl}(G) < p$ and $\alpha(G) < q$. The *Ramsey number* $R(p, q)$ is the smallest natural $n$ such that every graph $G$ with $|V(G)| \ge n$ is not a $(p, q)$-graph. An exposition of the results on the Ramsey numbers is given in [26]. We shall need Table 1.1 of the known Ramsey numbers $R(p, 3)$ (see [26]).

Table 1.1. Ramsey numbers $R(p, 3)$

| $p$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|
| $R(p,3)$ | 6 | 9 | 14 | 18 | 23 | 28 | 36 | 40–43 | 46–51 |

Let $G_1$ and $G_2$ be two graphs without common vertices. We denote by $G_1 + G_2$ the graph $G$ for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$ where $E' = \{[x, y], x \in V(G_1), y \in V(G_2)\}$. A graph $G$ is *separable* if $G = G_1 + G_2$, where $V(G_i) = \emptyset$, $i = 1, 2$.

**Definition 1.1.** *Let $\mathcal{M} \neq \emptyset$ be a set of graphs. We say that $G_0 \in \mathcal{M}$ is a* minimal graph *in $\mathcal{M}$ if $|V(G_0)| = \min\{|V(G)| : G \in \mathcal{M}\}$.*

**Definition 1.2.** *Let $a_1, \dots, a_r$ be positive integers. The symbol $G \xrightarrow{v} (a_1, \dots, a_r)$ means that in every $r$-coloring*

$$V(G) = V_1 \cup \cdots \cup V_r, \qquad V_i \cap V_j = \emptyset, \quad i \neq j$$

*of the vertices of $G$ for some $i \in \{1, \dots, r\}$ there exists a monochromatic $a_i$-clique $Q$ of color $i$, that is $Q \subseteq V_i$.*

*Define*

$$H_v(a_1, \dots, a_r; q) = \{G \xrightarrow{v} (a_1, \dots, a_r) \text{ and } \mathrm{cl}(G) < q\},$$
$$F_v(a_1, \dots, a_r; q) = \min\{|V(G)| : G \in H_v(a_1, \dots, a_r; q)\}.$$

It is clear that $G \overset{v}{\to} (a_1,\dots,a_r)$ implies $\mathrm{cl}(G) \ge \max\{a_1,\dots,a_r\}$. Folkman proved in [6] that there exists a graph $G$ such that $G \overset{v}{\to} (a_1,\dots,a_r)$ and $\mathrm{cl}(G) = \max\{a_1,\dots,a_r\}$. Therefore,

(1.1) $$F_v(a_1,\dots,a_r;q) \text{ exists} \iff q > \max\{a_1,\dots,a_r\}.$$

The numbers $F_v(a_1,\dots,a_r;q)$ are called *vertex Folkman numbers.* If $a_1,\dots,a_r$ are positive integers, $r \ge 2$ and $a_i = 1$ then it is easy to see that

$$G \overset{v}{\to} (a_1,\dots,a_{i-1},a_i,a_{i+1},\dots,a_r) \Rightarrow G \overset{v}{\to} (a_1,\dots,a_{i-1},a_{i+1},\dots,a_r).$$

Hence, for $a_i = 1$

$$F_v(a_1,\dots,a_r;q) = F_v(a_1,\dots,a_{i-1},a_{i+1},\dots,a_r;q).$$

Thus, it is enough to consider just such numbers $F_v(a_1,\dots,a_r;q)$ for which $a_i \ge 2$. In this paper we consider the Folkman numbers $F_v(2,\dots,2;q)$. Set

$$(\underbrace{2,\dots,2}_{r}) = (2_r) \text{ and } F_v(\underbrace{2,\dots,2}_{r};q) = F_v(2_r;q).$$

By (1.1)

(1.2) $$F_v(2_r;q) \text{ exists} \iff q \ge 3.$$

It is clear that

(1.3) $$G \overset{v}{\to} (2_r) \iff \chi(G) \ge r+1.$$

Since $K_{r+1} \overset{v}{\to} (2_r)$ and $K_r \overset{v}{\not\to} (2_r)$ we have

(1.4) $$F_v(2_r;q) = r+1 \text{ if } q \ge r+2.$$

According to (1.4) it is enough to consider just such numbers $F_v(2_r;r-k+1)$ for which $k \ge -1$. In this paper we shall prove the following results.

**Theorem 1.1.** *Let $r$ and $k$ be integers such that $-1 \le k \le 5$ and $r \ge k+2$. Then*

(a) $F_v(2_r;r-k+1) \ge r+2k+3$;

(b) $F_v(2_r;r-k+1) = r+2k+3$ *if* $k \in \{0,2,3,4,5\}$ *and* $r \ge 2k+2$ *or* $k \in \{-1,1\}$ *and* $r \ge 2k+3$.

**Theorem 1.2.** *Let $r \geq 8$ be a natural number. Then*

(a) $F_v(2_r; r-5) \geq r+14$ *and* $F_v(2_r; r-5) = r+14$ *if and only if* $r \geq 13$*;*

(b) $F_v(2_r; r-6) \geq r+16$ *if* $r \geq 9$ *and* $F_v(2_r; r-6) = r+16$ *if* $r \geq 15$*;*

(c) $F_v(2_r; r-7) \geq r+17$*,* $r \geq 10$ *and* $F_v(2_r; r-7) = r+17$ *if and only if* $r \geq 16$*;*

(d) $F_v(2_r; r-8) \geq r+18$*,* $r \geq 11$ *and* $F_v(2_r; r-8) = r+18$ *if and only if* $r \geq 17$*;*

(e) $F_v(2_r; r-9) \geq r+20$*,* $r \geq 12$ *and* $F_v(2_r; r-9) = r+20$ *if* $r \geq 19$*.*

**Theorem 1.3.** *Let $r \geq 13$ be a natural number. Then*

(a) $F_v(2_r; r-10) \geq r+21$ *and* $F_v(2_r; r-10) = r+21$ *if* $R(10,3) > 41$ *and* $r \geq 20$*;*

(b) *If* $R(10,3) \leq 41$ *we have* $F_v(2_r; r-10) \geq r+22$ *and* $F_v(2_r; r-10) = r+22$ *if* $r \geq 21$*.*

**Theorem 1.4.** *Let $r$ and $k$ be natural numbers such that $r \geq k+2$ and $k \geq 12$. Then*

(a) $F_v(2_r; r-k+1) \geq r+k+11$*;*

(b) *If* $k = 12$ *and* $r \geq 22$ *then* $F_v(2_r; r-11) = r+23$*.*

**Remark 1.1.** By (1.2) the number $F_v(2_r; r-k+1)$ exists if and only if $r \geq k+2$. Thus, the inequality $r \geq k+2$ in the statements of these Theorems is necessary.

**Remark 1.2.** The case $k = 0$ of Theorem 1.1 was proved by Dirac in [3]. It was also proved in [3] that the graph $K_{r-2} + C_5$, $r \geq 2$ is the only minimal graph in $H_v(2_r; r+1)$. The cases $k = 1$ and $k = 2$ of Theorem 1.1 were proved in [18]. It was also proved in [18] that $K_{r-5} + C_5 + C_5$, $r \geq 5$ is the only minimal graph in $H_v(2_r; r)$ (see also [23]). The case $k = 3$ was proved in [17]. We gave new proofs of the cases $k = 2$ and $k = 3$ of Theorem 1.1 in [24].

The method we use here does not allow us to compute the numbers $F_r(2_r; r-k+1)$ when $r < 2k+2$ and $1 \le k \le 5$. We know only the following numbers of this kind:

$F_v(2_3;3) = 11$, [1] and [14];

$F_v(2_4;3) = 22$, [9];

$F_v(2_r;4) = 11$, [19] (see also [20]).

We know about number $F_4(2_5;4)$ only that $12 \le F_v(2_5;4) \le 16$ (see [24]).

**Remark 1.3.** If $k \ge 2$ then there is more than one minimal graph in $H_v(2_r; r-1)$. For example, if $r \ge 8$ the graph $K_{r-8}+C_5+C_5+C_5$ is also minimal in $H_v(2_r; r-1)$ besides the minimal graph from the proof of Theorem 1.1.

**Remark 1.4.** Luczak et al. [13] proved the inequality

$$F_v(2_r; r-k+1) \le r+2k+3 \text{ if } r \ge 3k+2. \tag{1.5}$$

They also proved that (1.5) is strict when $k$ is very large (see [13]). It can be easily seen from Theorem 1.1 and Theorem 1.2 (a) that $k=6$ is the smallest value of $k$ for which the inequality (1.5) is strict.

## 2. Auxiliary results.

The following lemmas are used to prove the main results.

**Lemma 2.1.** *Let $q \ge 4$ be an integer and $G$ be a minimal graph (see Definition 1.1) in $H_v(2_r; q-1)$. Then*

$$F_v(2_r; q-1) \ge F_v(2_r;q) + \alpha(G) - 1.$$

P r o o f. Let $A \subseteq V(G)$ be an independent set of vertices of $G$ such that $|A| = \alpha(G)$. Consider the graph $G' = K_1 + (G-A)$. By (1.3), $\chi(G) \ge r+1$. Since $A$ is an independent vertex set it follows that $\chi(G-A) \ge r$ and $\chi(G') \ge r+1$. By (1.3), $G' \overset{v}{\to} (2_r)$. Since $\mathrm{cl}(G) \le q-2$ we have $\mathrm{cl}(G') \le q-1$. Hence, $G' \in H_v(2_r;q)$ and

$$F_v(2_r;q) \le |V(G')| = |V(G)| - \alpha(G) + 1.$$

Lemma 2.1 follows from this inequality because $|V(G)| = F_v(2_r; q-1)$. $\square$

**Corollary 2.1.** *Let $q$ and $r$ be integers such that $4 \le q < r+3$. Then*

(a) $F_v(2_r; q-1) \ge F_v(2_r, q) + 1$;

(b) *If* $F_v(2_r; q) + 1 \ge R(q-1, 3)$ *then the inequality* (a) *is strict.*

P r o o f. Let $G$ be a minimal graph in $H_v(2_r; q-1)$. By (1.3), $\chi(G) \ge r+1$. Since $\mathrm{cl}(G) \le q-2$ and $q < r+3$ we have

$$\mathrm{cl}(G) < r+1 \le \chi(G).$$

Thus, $\alpha(G) \ge 2$ and inequality (a) follows from Lemma 2.1.

Let $F_v(2_r; q) + 1 \ge R(q-1, 3)$. Then we see from (a) that

$$|V(G)| = F_v(2_r; q-1) \ge R(q-1, 3).$$

Since $\mathrm{cl}(G) < q-1$, this inequality implies $\alpha(G) \ge 3$. From Lemma 2.1 we obtain

$$F_v(2_r; q-1) \ge F_v(2_r; q) + 2.$$

The Corollary 2.1 is proved. □

A graph $G$ is said to be *$k$-chromatic* if $\chi(G) = k$. A graph $G$ is defined to be *vertex-critical chromatic* if $\chi(G - v) < \chi(G)$ for all $v \in V(G)$.

**Lemma 2.2.** *Let $q \ge 3$ be an integer and let $G$ be a minimal graph in $H_v(2_r; q)$. Then*

(a) *$G$ is a vertex-critical $(r+1)$-chromatic graph;*

(b) *If $q < r+3$ then* $\mathrm{cl}(G) = q-1$.

P r o o f. *Proof of (a).* By (1.3), $\chi(G) \ge r+1$. Assume that (a) is wrong. Then there exists $v \in V(G)$ such that $\chi(G - v) \ge r+1$. Thus, according to (1.3), $G - v \in H_v(2_r; q)$. This contradicts the minimality of $G$ in $H_v(2_r; q)$.

*Proof of (b).* Assume that (b) is wrong. Then, since $\mathrm{cl}(G) \le q-1$ we have $\mathrm{cl}(G) \le q-2$. Thus, $G \in H_v(2_r; q-1)$. Hence $H_v(2_r; q-1) \ne \emptyset$ and, by (1.2), $q \ge 4$. So,

$$|V(G)| = F_v(2_r; q) \ge F_v(2_r; q-1).$$

Since $q < r+3$ this contradicts Corollary 2.1 (a). □

The following obvious equalities

$$\chi(G_1+G_2)=\chi(G_1)+\chi(G_2); \tag{2.1}$$

$$\mathrm{cl}(G_1+G_2)=\mathrm{cl}(G_1)+\mathrm{cl}(G_2) \tag{2.2}$$

are used to prove the following Lemma 2.3.

Let $f(G)=\chi(G)-\mathrm{cl}(G)$. Then it easily follows from (2.1) and (2.2) that

$$f(G_1+G_2)=f(G_1)+f(G_2). \tag{2.3}$$

**Lemma 2.3.** *Let $m$ and $k$ be positive integers such that $m\geq k+3$ and $2m-1<R(m-k,3)$. Let*

$$F_v(2_r;r-k+1)\geq r+m \text{ for any } r\geq m-1.$$

*Then*

$$F_v(2_r;r-k+1)=r+m \text{ if } r\geq m-1.$$

**Remark 2.1.** It follows from $r\geq m-1$ and $m\geq k+3$ that $r-k+1\geq 3$. Thus, by (1.2), the number $F_v(2_r;r-k+1)$ exists.

Proof. We need to prove that

$$F_v(2_r;r-k+1)\leq r+m \text{ if } r\geq m-1.$$

It follows from $0<2m-1<R(m-k,3)$ that there exists a graph $P$ such that $|V(P)|=2m-1$, $\mathrm{cl}(P)\leq m-k-1$ and $\alpha(G)<3$. Define

$$P(r)=K_{r-m+1}+P,\quad r\geq m-1.$$

Since $|V(P)|=2m-1$ and $\alpha(P)<3$ we have $\chi(P)\geq m$. From (2.1) we see that $\chi(P(r))\geq r+1$. The inequality $\mathrm{cl}(G)\leq m-k-1$ together with (2.2) implies that $\mathrm{cl}(P(r))\leq r-k$. Hence, by (1.3), $P(r)\in H_v(2_r;r-k+1)$ and

$$F_v(2_r;r-k+1)\leq |V(P(r))|=r+m \text{ if } r\geq m-1.$$

Lemma 2.3 is proved. □

**Remark 2.2.** It is clear from the proof of Lemma 2.3 that the following theorem is true:

**Theorem 2.1.** *Let $m$ and $k$ be positive integers such that*

$$2m-1 < R(m-k,3) \text{ and } m \geq k+3.$$

*Then $F_v(2_r; r-k+1) \leq r+m$ if $r \geq m-1$.*

## 3. Some properties of the minimal graphs in $M(x,y)$.

Let $x$ and $y$ be integers. Define

$$M(x,y) = \{G : |V(G)| < \chi(G) + 2f(G) - x \text{ and } f(G) \leq y\}.$$

In this section we shall prove some properties of the minimal graphs in $M(x,y)$ (see Definition 1.1). These properties will be required for the proofs of Theorem 4.1 and Theorem 4.2 in the Section 4. If $x < 0$ then the empty graph belongs to $M(x,y)$ and hence it is the only minimal graph in $M(x,y)$. That is why we shall assume $x \geq 0$.

The aim of this section is to prove the following result:

**Theorem 3.1.** *Let $M(x,y) \neq \emptyset$, $x \geq 0$ and let $G_0$ be a minimal graph in $M(x,y)$. If $G_0$ is a nonseparable graph then:*

(a) $|V(G_0)| = 4f(G_0) - 2x - 1$;

(b) $4f(G_0) - 2x - 1 < R(f(G_0) - x + 1, 3)$ *where $R(p,3)$ is the Ramsey number.*

An important result of Gallai that we shall need later is:

**Theorem 3.2** [7] (see also [8]). *Let $G$ be a vertex-critical chromatic graph and $\chi(G) \geq 2$. Then, if $|V(G)| < 2\chi(G) - 1$, the graph $G$ is separable in the sense that $G = G_1 + G_2$, where $V(G_i) \neq \emptyset$, $i = 1, 2$.*

**Remark 3.1.** In the original statement of Theorem 3.2 the graph $G$ is edge-critical (and not vertex-critical) chromatic. Since each vertex-critical chromatic graph $G$ contains an edge-critical chromatic subgraph $H$ such that $\chi(H) = \chi(G)$ and $V(H) = V(G)$ the above statement of Theorem 3.2 is equivalent to the original one.

In the proof of Theorem 3.1 we shall use the following two Lemmas.

**Lemma 3.1.** *Let $M(x,y) \neq \emptyset$, $x \geq 0$ and $G_0$ be a minimal graph in $M(x,y)$. Let $A \neq \emptyset$ be an independent vertex set of $G_0$ and $G_0' = G_0 - A$. Then*

(a) $\chi(G_0') = \chi(G_0) - 1$;

(b) $\mathrm{cl}(G_0') = \mathrm{cl}(G_0)$;

(c) $|V(G_0)| = \chi(G_0) + 2f(G_0) - x - 1$.

Proof. *Proof of (a).* Since $A$ is an independent vertex set we have $\chi(G_0') = \chi(G_0) - 1$ or $\chi(G_0') = \chi(G_0)$. Assume that (a) is wrong. Then $\chi(G_0') = \chi(G_0)$. Let $\chi(G_0') = \chi(G_0) = m$ and

$$V(G_0') = V_1 \cup \dots \cup V_m, \qquad V_i \cap V_j = \emptyset, \quad i \neq j,$$

where $V_i$ are independent sets of $G_0$. Note that $\mathrm{cl}(G_0') \leq \mathrm{cl}(G) \leq m$. Thus, after adding new edges $[u,v]$, where $u$ and $v$ belong to different sets $V_i$ and $V_j$ to $E(G_0')$ we shall obtain the graph $G_0''$ such that $\mathrm{cl}(G_0'') = \mathrm{cl}(G_0)$, $\chi(G_0'') = \chi(G_0)$ and $f(G_0'') = f(G_0)$. Since $A \neq \emptyset$ we have

$$|V(G_0'')| < |V(G_0)| < \chi(G_0) + 2f(G_0) - x = \chi(G_0'') + 2f(G_0'') - x.$$

So, we obtain that $G_0'' \in M(x,y)$. This contradicts the minimality of $G_0$ in $M(x,y)$.

*Proof of (b).* It is clear that $\mathrm{cl}(G_0') = \mathrm{cl}(G)$ or $\mathrm{cl}(G_0') = \mathrm{cl}(G_0) - 1$. Assume that (b) is wrong. Then $\mathrm{cl}(G_0') = \mathrm{cl}(G_0) - 1$. By (a) we have $\chi(G_0') = \chi(G_0) - 1$. Thus, $f(G_0') = f(G_0) \leq y$. Since $|V(G_0')| < |V(G_0)|$, from the minimality of $G_0$ it follows that

$$|V(G_0')| \geq \chi(G_0') + 2f(G_0') - x = \chi(G_0) - 1 + 2f(G_0) - x.$$

From this inequality it follows that $|V(G_0)| \geq \chi(G_0) + 2f(G_0) - x$. This is a contradiction because $G_0 \in M(x,y)$.

*Proof of (c).* Assume the opposite, i.e.,

$$|V(G_0)| \leq \chi(G_0) + 2f(G_0) - x - 2. \tag{3.1}$$

Since $|V(G_0)| \geq \chi(G_0)$ and $x \geq 0$ it follows from (3.1) that $f(G_0) \neq 0$. Thus, there are two non-adjacent vertices $u, v \in V(G_0)$. Consider the subgraph $G_0' = G_0 - \{u,v\}$. By (a) and (b) we have $\chi(G_0') = \chi(G_0) - 1$ and $f(G_0') = f(G_0) - 1$. Since $|V(G_0')| = |V(G_0)| - 2$, it is easy to see from (3.1) that

$$|V(G_0')| \leq \chi(G_0) - 1 + 2f(G_0) - 2 - x - 1 < \chi(G_0') + 2f(G_0') - x.$$

This is a contradiction since $|V(G_0')| < |V(G_0)|$. □

**Lemma 3.2.** *Let $M(x,y) \neq \emptyset$, $x \geq 0$ and let $G_0$ be a minimal graph in $M(x,y)$. Then*

(a) $G_0$ *is a* $(\mathrm{cl}(G_0)+1, 3)$*-graph;*

(b) $|V(G_0)| \leq 2\chi(G_0) - 1$*;*

(c) $|V(G_0)| \geq 4f(G_0) - 2x - 1$.

Proof. *Proof of* (*a*). We need to prove that $\alpha(G_0) < 3$. Assume the opposite and let $\{u,v,w\}$ be an independent vertex set of $G_0$. Consider the subgraph $G_0' = G_0 - \{u,v,w\}$. By Lemma 3.1, we have $\chi(G_0') = \chi(G_0) - 1$ and $f(G_0') = f(G_0) - 1$. Since $f(G_0') < y$ and $|V(G_0')| < |V(G_0)|$, it follows from the minimality of $G_0$ that

$$|V(G_0')| \geq \chi(G_0') + 2f(G_0') - x.$$

As $|V(G_0)| = |V(G_0')| + 3$ it follows that $|V(G_0)| \geq \chi(G_0) + 2f(G_0) - x$. This contradicts $G_0 \in M(x,y)$.

*Proof of* (*b*). By (a), $\alpha(G_0) < 3$. Thus, we have $|V(G_0)| \leq 2\chi(G_0)$ and we need to prove that $|V(G_0)| \neq 2\chi(G_0)$. Assume the opposite, i.e., $|V(G_0)| = 2\chi(G_0)$ and let $v \in V(G_0)$. Consider the subgraph $G_0' = G_0 - v$. By Lemma 3.1 (a), $\chi(G_0') = \chi(G_0) - 1$. Since $\alpha(G_0') < 3$ it follows that $|V(G_0')| \leq 2\chi(G_0') - 2$ which is a contradiction.

*Proof of* (*c*). From (b) and Lemma 3.1 (c) we obtain

$$\chi(G_0) \geq 2f(G_0) - x.$$

By this inequality and Lemma 3.1 (c) we see that

$$|V(G_0)| \geq 4f(G_0) - 2x - 1.$$

□

Proof of Theorem 3.1. *Proof of* (*a*). According to Lemma 3.1 (a) $G_0$ is a vertex-critical chromatic graph. Since $G_0$ is nonseparable, it follows from Lemma 3.2 (b) and Theorem 3.2 that

$$|V(G_0)| = 2\chi(G_0) - 1. \tag{3.2}$$

By (3.2) and Lemma 3.1 (c) we obtain

(3.3) $\chi(G_0) = 2f(G_0) - x, \quad \mathrm{cl}(G_0) = f(G_0) - x$ and
$$|V(G_0)| = 4f(G_0) - 2x - 1.$$

*Proof of* (*b*). According to Lemma 3.2 (a) we have

$$|V(G_0)| < R(\mathrm{cl}(G_0) + 1, 3).$$

From this inequality and (3.3) it follows (b).

Theorem 3.1 is proved. □

**4. A lower bound for $|V(G)|$ when $f(G) \le 13$.** In this section our goal is to prove the following two theorems.

**Theorem 4.1.** *Let $G$ be a graph such that $f(G) \le 11$. Then*

(a) $|V(G)| \ge \chi(G) + 2f(G)$ *if* $f(G) \le 6$;

(b) $|V(G)| \ge \chi(G) + 2f(G) - 1$ *if* $f(G) = 7$ *or* $f(G) = 8$;

(c) $|V(G)| \ge \chi(G) + 16$ *if* $f(G) = 9$;

(d) $|V(G)| \ge \chi(G) + 2f(G) - 3$ *if* $f(G) = 10$ *or* $f(G) = 11$.

**Theorem 4.2.** *Let $G$ be a graph such that $f(G) \le 13$. Then*

(a) $|V(G)| \ge \chi(G) + 2f(G) - 4$;

(b) *If* $f(G) = 12$ *and* $R(10,3) \le 41$ *then the inequality* (a) *is strict.*

**Remark 4.1.** If $f(G) \ge 7$ then the inequality (a) of Theorem 4.1 is not true. For example if $G$ is a minimal graph in $H_v(2_r; r-5)$ we have from Lemma 2.2 that $\chi(G) = r+1$, $\mathrm{cl}(G) = r-6$ and $f(G) = 7$. By Theorem 1.2 we see that
$$|V(G)| = r + 14 < \chi(G) + 2f(G) \text{ if } r \ge 13.$$

In the same way we also see that the conditions for $f(G)$ in the statements (b), (c) and (d) of Theorem 4.1 are necessary.

**Remark 4.2.** If $f(G) \le 6$ the inequality (a) of Theorem 4.1 is exact. Indeed, if $G$ is a minimal graph in $H_v(2_r; r-k+1)$ where $-1 \le k \le 5$, by

Lemma 2.2 we have $\chi(G) = r + 1$, $\mathrm{cl}(G) = r - k$ and $f(G) = k + 1 \leq 6$. When $r$ is large enough we have according to Theorem 1.1

$$|V(G)| = r + 2k + 3 = \chi(G) + 2f(G).$$

In the same way (using Theorem 1.2) we see that the inequalities (b), (c) and (d) are exact.

**Remark 4.3.** If $f(G) = 13$ the inequality (a) of Theorem 4.2 is exact by Theorem 1.4 (b). If $f(G) = 12$ and $R(10, 3) \geq 42$ this inequality is exact according to Theorem 1.3 (a).

We shall use the following two lemmas in the proof of Theorem 4.1 and Theorem 4.2.

**Lemma 4.1.** *Let $M(0, y) \neq \emptyset$. Then every minimal graph in $M(0, y)$ is nonseparable.*

Proof. Assume the opposite and let $G_0$ be a minimal graph in $M(0, y)$ such that $G_0 = G_1 + G_2$, where $V(G_i) \neq \emptyset$, $i = 1, 2$. Since $|V(G_i)| < |V(G_0)|$ we have $G_i \notin M(0, y)$. Since $f(G_i) \leq f(G) \leq y$ it follows that

$$|V(G_i)| \geq \chi(G_i) + 2f(G_i), \quad i = 1, 2.$$

Summing these two inequalities we obtain, by (2.1) and (2.3), that

$$|V(G_0)| \geq \chi(G_0) + 2f(G_0)$$

a contradiction.

**Corollary 4.1.** $M(0, y) = \emptyset$ *if* $y \leq 6$.

Proof. Assume the opposite, i.e., $M(0, y) \neq \emptyset$ for some $y \leq 6$. Let $G_0$ be minimal in $M(0, y)$. Then $f(G_0) \leq 6$. According to Lemma 4.1 $G_0$ is nonseparable. Thus, by Theorem 3.1 (b) ($x = 0$) we have

$$4f(G_0) - 1 < R(f(G_0) + 1, 3)$$

for $f(G_0) \leq 6$ which is a contradiction (see Table 1.1). □

**Corollary 4.2.** *Let $G$ be a graph such that*

$$|V(G)| < \chi(G) + 2f(G).$$

*Then* $|V(G)| \geq 27$.

Proof. Since $G \in M(0, f(G))$ we have $M(0, f(G)) \neq \emptyset$. Let $G_0$ be a minimal graph in $M(0, f(G))$. By Corollary 4.1, $f(G_0) \geq 7$. Thus, it follows from Lemma 3.2 (c) that $|V(G)| \geq |V(G_0)| \geq 27$. □

**Lemma 4.2.** *Let $M(x, y) \neq \emptyset$ where $x \geq 0$ and $y \leq 13$. Then every minimal graph in $M(x, y)$ is nonseparable.*

Proof. Assume the opposite and let $G_0$ be a minimal graph in $M(x, y)$ such that $G_0 = G_1 + G_2$, $V(G_i) \neq \emptyset$, $i = 1, 2$. Let $f(G_1) \leq f(G_2)$. Then $f(G_1) \leq 6$ because $f(G_1) + f(G_2) = f(G_0) \leq 13$. By Corollary 4.1 we obtain that

$$|V(G_1)| \geq \chi(G_1) + 2f(G_1). \tag{4.1}$$

Since $G_2 \notin M(x, y)$ and $f(G_2) \leq y$ we have that

$$|V(G_2)| \geq \chi(G_2) + 2f(G_2) - x. \tag{4.2}$$

Summing the inequalities (4.1) and (4.2) we obtain by (2.1) and (2.3) that

$$|V(G_0)| \geq \chi(G_0) + 2f(G_0) - x,$$

which is a contradiction. □

Proof of Theorem 4.1. Statement (a) follows immediately from Corollary 4.1.

*Proof of* (b). Assume the opposite. Then $M(1, 8) \neq \emptyset$. Let $G_0$ be a minimal graph in $M(1, 8)$. It is easy to see that

$$G_0 \in M(1, 8) \Rightarrow G_0 \in M(0, 8).$$

Thus, by Corollary 4.1, we have $f(G_0) \geq 7$, i.e., $f(G_0) = 7$ or $f(G_0) = 8$. According to Lemma 4.2 $G_0$ is nonseparable. Thus, from Theorem 3.1 ($x = 1$), it follows that

$$4f(G_0) - 3 < R(f(G_0), 3),$$

where $f(G_0) = 7$ or $f(G_0) = 8$, which is a contradiction.

The proofs of statements (c) and (d) are completely similar to that of statement (b).

Theorem 4.1 is proved. □

Proof of Theorem 4.2. *Proof of* (a). Assume the opposite. Then $M(4, 13) \neq \emptyset$. Let $G_0$ be a minimal graph in $M(4, 13)$. It is clear that

$$G_0 \in M(4, 13) \Rightarrow G_0 \in M(3, 13).$$

Thus, it follows from Theorem 4.1 that $f(G_0) \geq 12$. Hence $f(G_0) = 12$ or $f(G_0) = 13$. By Lemma 4.2, $G_0$ is nonseparable. Thus, Theorem 3.1 (b) ($x = 4$) implies

$$4f(G_0) - 9 < R(f(G_0) - 3, 3),$$

where $f(G_0) = 12$ or $f(G_0) = 13$ which is a contradiction.

*Proof of* (b). Assume the opposite. Then $M(3, 12) \neq \emptyset$. Let $G_0$ be a minimal graph in $M(3, 12)$. From Theorem 4.1 it follows that $f(G_0) = 12$. Since $G_0$, by Lemma 4.2, is nonseparable it follows from Theorem 3.1 (b) that

$$4f(G_0) - 7 < R(f(G_0) - 2, 3),$$

where $f(G_0) = 12$ which is a contradiction, by our assumption $R(10, 3) \leq 41$. □

## 5. Proof of Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. *Proof of* (a). Let $G$ be a minimal graph in $H_v(2_r; r - k + 1)$. By Lemma 2.2 $\chi(G) = r + 1$, $\mathrm{cl}(G) = r - k$ and $f(G) = k + 1$. Since $k \leq 5$ we have $f(G) \leq 6$. Thus, from Theorem 4.1 (a) it follows that

$$F_v(2_r; r - k + 1) = |V(G)| \geq r + 2k + 3.$$

*Proof of* (b). We shall consider the following three cases.

Case 1. $k = -1$. In this case (b) follows from (1.4).

Case 2. $k \in \{0, 2, 3, 4, 5\}$. By Table 1.1 in this case the following inequality

$$2(2k + 3) - 1 < R(k + 3, 3).$$

holds. Thus, by Lemma 2.3 we obtain $F_v(2_r; r - k + 1) = r + 2k + 3$ if $r \geq 2k + 2$.

Case 3. $k = 1$. We need to prove that $F_v(2_r; r) \leq r + 5$ if $r \geq 5$. Define

$$P(r) = K_{r-5} + C_5 + C_5, \quad r \geq 5.$$

By (2.1) and (2.2) we have $\chi(P(r)) = r + 1$ and $\mathrm{cl}(P(r)) = r - 1$. Thus, from (1.3) it follows that $P(r) \in H_v(2_r; r)$. Hence

$$F_v(2_r; r) \leq |V(P(r))| = r + 5, \quad r \geq 5$$

and Theorem 1.1 is proved. □

Proof of Theorem 1.2. *Proof of* (a). Let $G$ be a minimal graph in $H_v(2_r; r - 5)$. Then, by Lemma 2.2, $\chi(G) = r + 1$, $\mathrm{cl}(G) = r - 6$ and $f(G) = 7$. Thus, from Theorem 4.1 (b) it follows

$$F_v(2_r; r - 5) = |V(G)| \geq r + 14.$$

Applying Lemma 2.3 ($k = 6$, $m = 14$) we obtain

$$F_v(2_r; r-5) = r + 14 \text{ if } r \geq 13.$$

Let $8 \leq r \leq 12$. From Table 1.1 we see that $R(r-5,3) \leq r+14$. By Theorem 1.1 ($k = 5$) we have $F_v(2_r; r-4) \geq r+13$ and thus $F_v(2_r; r-4)+1 \geq R(r-5,3)$. According to Corollary 2.1 (b) ($q = r-4$), $F_v(2_r; r-5) \geq r+15$.

*Proof of* (b). Let $G$ be a minimal graph in $H_v(2_r; r-6)$. By Lemma 2.2, $\chi(G) = r+1$ and $f(G) = 8$. From Theorem 4.1 (b) it follows that

$$F_v(2_r; r-6) = |V(G)| \geq r + 16.$$

Thus, Lemma 2.3 ($k = 7$, $m = 16$) implies $F_v(2_r; r-6) = r+16$ if $r \geq 15$.

*Proof of* (c). Let $G$ be a minimal graph in $H_v(2_r; r-7)$. By Lemma 2.2, $\chi(G) = r+1$ and $f(G) = 9$. Thus, from Theorem 4.1 (c) it follows that $F_v(2_r; r-7) \geq r+17$, $r \geq 10$. From this inequality and Lemma 2.3 ($k = 8$, $m = 17$) we see that $F_v(2_r; r-7) = r+17$ if $r \geq 16$.

Let $10 \leq r \leq 15$. By Table 1.1 we have that $R(r-7,3) < r+17$. Since, by (b), $F_v(2_r; r-6)+1 \geq r+17$ we have $F_v(2_r; r-6)+1 > R(r-7,3)$. From Corollary 2.1 (b), the inequality $F_v(2_r; r-7) \geq r+18$ holds.

*Proof of* (d). If $G$ be a minimal graph in $H_v(2_r; r-8)$ then, by Lemma 2.2, $\chi(G) = r+1$ and $f(G) = 10$. From Theorem 4.1 (d) it follows that

$$|V(G)| = F_v(2_r; r-8) \geq r + 18, \quad r \geq 11.$$

Applying Lemma 2.3 ($k = 9$, $m = 18$) we obtain $F_v(2_r; r-8) = r+18$ if $r \geq 17$.

Let $11 \leq r \leq 16$. In this case we have $R(r-8,3) \leq r+18$. By (c), $F_v(2_r; r-7) \geq r+17$. Thus, $F_v(2_r; r-7)+1 \geq R(r-8,3)$ and, by Corollary 2.1 (b), $F_v(2_r; r-8) \geq r+19$.

*Proof of* (e). Let $G$ be a minimal graph in $H_v(2_r; r-9)$. According to Lemma 2.2 we have $\chi(G) = r+1$ and $f(G) = 11$. By Theorem 4.1 (d) we obtain

$$|V(G)| = F_v(2_r; r-9) \geq r + 20.$$

This inequality and Lemma 2.3 ($k = 10$, $m = 20$) imply that $F_v(2_r; r-9) = r+20$ if $r \geq 19$.

Theorem 1.2 is proved. □

## 6. Proof of Theorem 1.3 and Theorem 1.4.

Proof of Theorem 1.3. Let $G$ be a minimal graph in $H_v(2_r; r-10)$. According to Lemma 2.2 we have $\chi(G) = r+1$ and $f(G) = 12$. Thus, by Theorem 4.2 (a) it follows that

$$|V(G)| = F_v(2_r; r-10) \geq r+21, \quad r \geq 13.$$

Let $R(10,3) > 41$. Then, by Lemma 2.3 ($k = 11$, $m = 21$) it follows that

$$F_v(2_r; r-10) = r+21 \text{ if } r \geq 20.$$

Let $R(10,3) \leq 41$. From Theorem 4.2 (b) we obtain $|V(G)| = F_v(2_r; r-10) \geq r+22$. Applying Lemma 2.3 ($k = 11$, $m = 22$) we deduce that $F_v(2_r; r-10) = r+22$ if $r \geq 21$ because $43 < R(11,3)$ (see [26]). □

Proof of Theorem 1.4. *Proof of* (a). The proof is by induction on $k$ with induction base $k = 12$. Let $G$ be a minimal graph in $H_v(2_r; r-11)$. Then, by Theorem 4.2 (a) we obtain

$$|V(G)| = F_v(2_r; r-11) \geq r+23. \tag{6.1}$$

We are done with the base $k = 12$. Let $k \geq 13$ and

$$F_v(2_r; r-k+2) \geq r+k+10.$$

Then, by Corollary 2.1 (a) it follows that

$$F_v(2_r; r-k+1) \geq r+k+11.$$

*Proof of* (b). From (6.1) and Lemma 2.3 ($k = 12$, $m = 23$) we deduce that $F_v(2_r; r-11) = r+23$ if $r \geq 22$ because $R(11,3) > 45$ (see [26]).

Theorem 1.4 is proved. □

## 7. Lower bounds for arbitrary vertex Folkman numbers.

Let $a_1, \ldots, a_r$ be positive integers. Define

$$m(a_1, \ldots, a_r) = m = \sum_{i=1}^{r}(a_i - 1) + 1. \tag{7.1}$$

It is easy to see that $K_m \overset{v}{\rightarrow} (a_1, \ldots, a_r)$ and $K_{m-1} \overset{v}{\nrightarrow} (a_1, \ldots, a_r)$. Therefore

$$F_v(a_1, \ldots, a_r; q) = m \text{ if } q > m.$$

By (1.1). the Folkman number $F_v(a_1,\dots,a_r;m)$ exists only when $m \geq \max\{a_1,\dots,a_r\}+1$. It was proved in [13] that

$$F_v(a_1,\dots,a_r;m) = m + \max\{a_1,\dots,a_r\}.$$

The exact values of all numbers $F_v(a_1,\dots,a_r;m-1)$ for which $\max\{a_1,\dots,a_r\} \leq 4$ are known. A detailed exposition of these results was given in [22]. We must add the equality $F_v(2,2,3;4) = 14$ obtained in [2]. We do not know any exact values of $F_v(a_1,\dots,a_r;m-1)$ in the case when $\max\{a_1,\dots,a_r\} \geq 5$.

In this section we shall use the following result [21]

$$G \xrightarrow{v} (a_1,\dots,a_r) \Rightarrow \chi(G) \geq m. \tag{7.2}$$

Let $G$ be a minimal graph in $H_v(a_1,\dots,a_r;q)$. Then, by (7.2) and (1.3) it follows that $G \in H_v(2_{m-1};q)$. Thus we have $|V(G)| \geq F_v(2_{m-1};q)$. So, we obtain

$$F_v(a_1,\dots,a_r;q) \geq F_v(2_{m-1};q), \tag{7.3}$$

where $m$ is defined by the equality (7.1). From (7.3), Theorem 1.1, Theorem 1.2, Theorem 1.3 and Theorem 1.4 we easily get the following theorem:

**Theorem 7.1.** *Let $a_1,\dots,a_r$ be integers, $a_i \geq 2$, $i=1,\dots,r$ and $m = \sum_{i=1}^{r}(a_i-1)+1$. Let $k$ be an integer such that*

$$m - k > \max\{a_1,\dots,a_r\}. \tag{7.4}$$

*Then the following inequalities hold:*

$$F_v(a_1,\dots,a_r;m-k) \geq m+2k+2 \text{ if } -1 \leq k \leq 5;$$

$$F_v(a_1,\dots,a_r;m-6) \geq m+13;$$

$$F_v(a_1,\dots,a_r;m-7) \geq m+15;$$

$$F_v(a_1,\dots,a_r;m-8) \geq m+16;$$

$$F_v(a_1,\dots,a_r;m-9) \geq m+17;$$

$$F_v(a_1,\dots,a_r;m-10) \geq m+19;$$

$$F_v(a_1,\dots,a_r;m-11) \geq m+20;$$

$$F_v(a_1,\dots,a_r;m-11) \geq m+21 \text{ if } R(10,3) \leq 41;$$

$$F_v(a_1,\dots,a_r;m-k) \geq m+k+10 \text{ if } k \geq 12.$$

**Remark 7.1.** According to (1.1) the inequality (7.4) in the statement of Theorem 7.1 is necessary.

Proof. Since all inequalities are proved in the same way, we shall prove the last one only. By Theorem 1.4 we have

$$F_v(2_r; r-k+1) \geq r+k+11, \quad r \geq k+2. \tag{7.5}$$

As $\max\{a_1, \dots, a_r\} \geq 2$, it follows from (7.4) that $m-1 \geq k+2$. Thus, the inequality (7.5) is true for $r = m-1$, i.e.,

$$F_v(2_{m-1}; m-k) \geq m+k+10. \tag{7.6}$$

We obtain from (7.6) and (7.3) that

$$F_v(a_1, \dots, a_r; m-k) \geq m+k+10. \qquad \square$$

**Remark 7.2.** Dudek and Rödl [4] proved that

$$F_v(a_1, \dots, a_r; q) \leq cp^3 \log^3 p,$$

where $p = \max\{a_1, \dots, a_r\}$ and $c$ is a constant depending only on $r$.

**8. Lower bounds for edge Folkman numbers.** Let $a_1 \dots, a_r$ be integers, $a_i \geq 2$. The symbol $G \overset{e}{\to} (a_1, \dots, a_r)$ denotes that in every $r$-coloring of the edge set $E(G)$ there exists a monochromatic $a_i$-clique of color $i$ for some $i \in \{1, \dots, r\}$. Define

$$H_e(a_1, \dots, a_r; q) = \{G : G \overset{e}{\to} (a_1, \dots, a_r) \text{ and } \mathrm{cl}(G) < q\},$$
$$F_e(a_1, \dots, a_r; q) = \min\{|V(G)| : G \in H_e(a_1, \dots, a_r; q)\}.$$

It is clear that from $G \overset{e}{\to} (a_1, \dots, a_r)$ it follows $\mathrm{cl}(G) \geq \max\{a_1, \dots, a_r\}$. There exists a graph $G \overset{e}{\to} (a_1, \dots, a_r)$ and $\mathrm{cl}(G) = \max\{a_1, \dots, a_r\}$. In the case $r = 2$ this was proved in [6] and the general case in [25]. Thus, we have

$$F_e(a_1, \dots, a_r; q) \text{ exists } \iff q > \max\{a_1, \dots, a_r\}. \tag{8.1}$$

The numbers $F_e(a_1, \dots, a_r; q)$ are called *edge Folkman numbers.*

From definition of Ramsey number $R(a_1,\dots,a_r)$ it follows that

$$F_e(a_1,\dots,a_r;q) = R(a_1,\dots,a_r) \text{ if } q > R(a_1,\dots,a_r).$$

Thus, we consider only numbers $F_e(a_1,\dots,a_r;R(a_1,\dots,a_r)-k)$, where $k \geq -1$. An exposition of the known edge Folkman numbers is given in [10]. We must add the new upper bounds for the number $F_e(3,3;4)$ obtained in [5] and [12].

In this section we shall use the following result obtained by S. Lin [11]

$$G \xrightarrow{e} (a_1,\dots,a_r) \Rightarrow \chi(G) \geq R(a_1,\dots,a_r). \tag{8.2}$$

From (8.2) and (1.3) we see that

$$G \in H_e(a_1,\dots,a_r;q) \Rightarrow G \in H_v(2_{R-1};q),$$

where $R = R(a_1,\dots,a_r)$. Thus, we have

$$F_e(a_1,\dots,a_r;q) \geq F_v(2_{R-1};q). \tag{8.3}$$

From (8.3), Theorem 1.1, Theorem 1.2, Theorem 1.3 and Theorem 1.4 it easily follows the following statement.

**Theorem 8.1.** *Let $a_1,\dots,a_r$ be integers, $a_i \geq 2$, $i=1,\dots,r$. Let*

$$R-k > \max\{a_1,\dots,a_r\},$$

*where $k \geq -1$ is integer and $R = R(a_1,\dots,a_r)$. Then*

$$F_e(a_1,\dots,a_r;R-k) \geq R+2k+2 \text{ if } -1 \leq k \leq 5;$$

$$F_e(a_1,\dots,a_r;R-6) \geq R+13;$$

$$F_e(a_1,\dots,a_r;R-7) \geq R+15;$$

$$F_e(a_1,\dots,a_r;R-8) \geq R+16;$$

$$F_e(a_1,\dots,a_r;R-9) \geq R+17;$$

$$F_e(a_1,\dots,a_r;R-10) \geq R+19;$$

$$F_e(a_1,\dots,a_r;R-11) \geq R+20;$$

$$F_e(a_1,\dots,a_r;R-11) \geq R+21 \text{ if } R(10,3) \leq 41;$$

$$F_e(a_1,\dots,a_r;R-k) \geq R+k+10 \text{ if } k \geq 12.$$

**Remark 8.1.** According to (8.1) the inequality

$$R - k > \max\{a_1, \dots, a_r\}$$

in the statement of Theorem 8.1 is necessary.

**Remark 8.2.** In the particular cases $k = 0$ and $k = 1$ Theorem 8.1 was proved by S. Lin [11]. Lin [11] also proved that when $k = 0$ the respective inequality in Theorem 8.1 is exact and the conjecture was raised that if $k = 1$ the first inequality in Theorem 8.1 is strict. This Lin's hypothesis was disproved in [15], where the equality $F_e(3, 3, 3; 16) = 21$ was established. The particular cases $k = 2$ and $k = 3$ of Theorem 8.1 were proved in [16] and [17], respectively. In [16] and [17] it was also proved that if $k = 2$ and $k = 3$ then respective inequalities of Theorem 8.1 are exact. The other inequalities are new. We do not know whether these inequalities are exact.

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