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ON THE VERTEX FOLKMAN NUMBERS $F_V(\underbrace{2,\dots,2}_{R};R-1)$ AND $F_V(\underbrace{2,\dots,2}_{R};R-2)$

NEDYALKO NENOV

For a graph G the symbol $G \xrightarrow{v} (a_1,\dots,a_r)$ means that in every r-coloring of the vertices of G, for some $i \in \{1,2,\dots,r\}$ there exists a monochromatic a_i-clique of color i. The vertex Folkman numbers

$$F_v(a_1,\dots,a_r;q) = \min\{|V(G)| : G \xrightarrow{v} (a_1,\dots,a_r) \text{ and } K_q \nsubseteq G\}$$

are considered. We prove that

$$F_v(\underbrace{2,\dots,2}_{r};r-1) = r+7\,,\quad r \geq 6 \quad \text{and} \quad F_v(\underbrace{2,\dots,2}_{r};r-2) = r+9\,,\quad r \geq 8\,.$$

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1. INTRODUCTION

We consider only finite, non-oriented graphs without loops and multiple edges. We call a p-clique of the graph G a set of p vertices, each two of which are adjacent. The largest positive integer p such that the graph G contains a p-clique is denoted by $\mathrm{cl}(G)$. In this paper we shall also use the following notation:

- $V(G)$ is the vertex set of the graph G;
- $E(G)$ is the edge set of the graph G;

- $\overline{G}$ is the complement of G;
- $G[V]$, $V \subseteq V(G)$ is the subgraph of G induced by V;
- $G - V$, $V \subseteq V(G)$ is the subgraph of G induced by $V(G) \setminus V$;
- $\alpha(G)$ is the vertex independence number of G;
- $\chi(G)$ is the chromatic number of G;
- $f(G) = \chi(G) - \mathrm{cl}(G)$;
- K_n is the complete graph on n vertices;
- C_n is the simple cycle on n vertices.

Let G_1 and G_2 be two graphs without common vertices. We denote by $G_1 + G_2$ the graph G for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[x, y] : x \in V(G_1), y \in V(G_2)\}$.

The Ramsey number $R(p, q)$ is the smallest natural n such that for every n-vertex graph G either $\mathrm{cl}(G) \geq p$ or $\alpha(G) \geq q$. An exposition of the results on the Ramsey numbers is given in [25]. In Table 1.1 we list the known Ramsey numbers $R(p, 3)$ (see [25]).

p	3	4	5	6	7	8	9	10
$R(p, 3)$	6	9	14	18	23	28	36	40–43

Table 1.1: The known Ramsey numbers

Definition. *Let $a_1, \dots, a_r$ be positive integers. We say that the r-coloring*

$$V(G) = V_1 \cup \dots \cup V_r, \quad V_i \cap V_j = \emptyset, \ i \neq j$$

of the vertices of the graph G is $(a_1, \dots, a_r)$-free, if V_i does not contain an a_i-clique for each $i \in \{1, \dots, r\}$. The symbol $G \overset{v}{\rightarrow} (a_1, \dots, a_r)$ means that there is no $(a_1, \dots, a_r)$-free coloring of the vertices of G.

Let $a_1, \dots, a_r$ and q be natural numbers. Define

$$H_v(a_1, \dots, a_r; q) = \{G : G \overset{v}{\rightarrow} (a_1, \dots, a_r) \text{ and } \mathrm{cl}(G) < q\},$$
$$F_v(a_1, \dots, a_r; q) = \min\{|V(G)| : G \in H_v(a_1, \dots, a_r; q)\}.$$

The graph $G \in H_v(a_1, \dots, a_r; q)$ is said to be an extremal graph in $H_v(a_1, \dots, a_r; q)$, if $|V(G)| = F_v(a_1, \dots, a_r; q)$.

It is clear that $G \overset{v}{\rightarrow} (a_1, \dots, a_r)$ implies $\mathrm{cl}(G) \geq \max\{a_1, \dots, a_r\}$. Folkman [3] proved that there exists a graph G such that $G \overset{v}{\rightarrow} (a_1, \dots, a_r)$ and $\mathrm{cl}(G) = \max\{a_1, \dots, a_r\}$. Therefore

$$F_v(a_1, \dots, a_r; q) \text{ exists} \iff q > \max\{a_1, \dots, a_r\}. \tag{1.1}$$

The numbers $F_v(a_1, \dots, a_r; q)$ are called vertex Folkman numbers.

If $a_1, \dots, a_r$ are positive integers, $r \ge 2$ and $a_i = 1$ then it is easily seen that

$$G \xrightarrow{v} (a_1, \dots, a_i, \dots, a_r) \iff G \xrightarrow{v} (a_1, \dots, a_{i-1}, a_{i+1}, a_r).$$

Thus it suffices to consider only such numbers $F_v(a_1, \dots, a_r; q)$ for which $a_i \ge 2$, $i = 1, \dots, r$. In this paper we consider the vertex Folkman numbers $F_v(2, \dots, 2; q)$. Set

$$(\underbrace{2, \dots, 2}_{r}) = (2_r) \quad \text{and} \quad F_v(\underbrace{2, \dots, 2}_{r}; q) = F_v(2_r; q).$$

By (1.1),

$$F_v(2_r; q) \text{ exists} \iff q \ge 3. \tag{1.2}$$

It is clear that

$$G \xrightarrow{v} (2_r) \iff \chi(G) \ge r + 1. \tag{1.3}$$

Since $K_{r+1} \xrightarrow{v} (2_r)$ and $K_r \overset{v}{\nrightarrow} (2_r)$, we have

$$F_v(2_r; q) = r + 1 \quad \text{if} \quad q \ge r + 2.$$

In [2] Dirac proved the following result.

Theorem 1.1. ([2]) *Let G be a graph such that $\chi(G) \ge r + 1$ and $\mathrm{cl}(G) \le r$. Then*

(a) $|V(G)| \ge r + 3$*;*

(b) *If* $|V(G)| = r + 3$, *then* $G = K_{r-3} + C_5$.

According to (1.3), Theorem 1.1 admits the following equivalent form:

Theorem 1.2. *Let $r \ge 2$ be a positive integer. Then*

(a) $F_v(2_r; r + 1) = r + 3$*;*

(b) $K_{r-2} + C_5$ *is the only extremal graph in* $H_v(2_r; r + 1)$.

In [14] Łuczak, Ruciński and Urbański defined for arbitrary positive integers $a_1, \dots, a_r$ the numbers

$$m = \sum_{i=1}^{r} (a_i - 1) + 1 \quad \text{and} \quad p = \max\{a_1, \dots, a_r\}. \tag{1.4}$$

They proved the following extension of Theorem 1.2.

Theorem 1.3. ([14]) *Let $a_1, \dots, a_r$ be positive integers and m and p be defined by (1.4). Let $m \ge p + 1$. Then*

(a) $F_v(a_1,\ldots,a_r;m) = m+p$;

(b) $K_{m-p-1} + \overline{C}_{2p+1}$ *is the only extremal graph in* $H_v(a_1,\ldots,a_r;m)$.

For another extension of Theorem 1.1 see [21].

From (1.1) it follows that the numbers $F_v(a_1,\ldots,a_r;m-1)$ exist if and only if $m \geq p+2$. The exact values of all numbers $F_v(a_1,\ldots,a_r;m-1)$ for which $p = \max\{a_1,\ldots,a_r\} \leq 4$ are known. A detailed exposition of these results was given in [13] and [23]. We do not know any exact values of $F_v(a_1,\ldots,a_r;m-1)$ in the case when $\max\{a_1,\ldots,a_r\} \geq 5$. Here we shall note only the values $F_v(a_1,\ldots,a_r;m-1)$ when $a_1 = a_2 = \cdots = a_r = 2$, i.e. of the numbers $F_v(2_r;r)$. From (1.2) these numbers exist if and only if $r \geq 3$. If $r=3$ and $r=4$ we have that

$$F_v(2_3;3) = 11; \tag{1.5}$$

$$F_v(2_4;4) = 11. \tag{1.6}$$

The inequality $F_v(2_3;3) \leq 11$ was proved in [15] and the opposite inequality $F_v(2_3;3) \geq 11$ was proved in [1]. The equality (1.6) was proved in [18] (see also [19]). If $r \geq 5$ we have the following result.

Theorem 1.4. ([17], see also 24]) *Let* $r \geq 5$. *Then:*

(a) $F_v(2_r;r) = r+5$;

(b) $K_{r-5} + C_5 + C_5$ *is the only extremal graph in* $H_v(2_r;r)$.

Theorem 1.4(a) was proved also in [8] and [14].

According to (1.2), the number $F_v(2_r;r-1)$ exists if and only if $r \geq 4$. In [17] we proved that

$$F_v(2_r;r-1) = r+7 \quad \text{if} \quad r \geq 8. \tag{1.7}$$

In this paper we improve (1.7) by proving the following result:

Theorem 1.5. *Let* $r \geq 4$ *be an integer. Then:*

(a) $F_v(2_r;r-1) \geq r+7$;

(b) $F_v(2_r;r-1) = r+7$, *if* $r \geq 6$;

(c) $F_v(2_5;4) \leq 16$.

In [9] Jensen and Royle showed that

$$F_v(2_4;3) = 22. \tag{1.8}$$

We see from Theorem 1.5 and (1.8) that $F_v(2_5;4)$ is the only unknown number of the kind $F(2_r;r-1)$[1].

From (1.2) it follows that the Folkman number $F(2_r;r-2)$ exists if and only if $r \geq 5$. In [16] we proved that $F_v(2_r;r-2) = r+9$ if $r \geq 11$. In this paper we improve this result as follows:

Theorem 1.6. *Let $r \geq 5$ be an integer. Then:*

(a) $F_v(2_r;r-2) \geq r+9$;

(b) $F_v(2_r;r-2) = r+9\,,\quad$ *if* $\quad r \geq 8$.

The numbers $F_v(2_r;r-2)$, $5 \leq r \leq 7$, are unknown.

2. AUXILIARY RESULTS

Let G be an arbitrary graph. Define

$$f(G) = \chi(G) - \mathrm{cl}(G).$$

Lemma 2.1. *Let G be a graph such that $f(G) \leq 2$. Then*

$$|V(G)| \geq \chi(G) + 2f(G).$$

Proof. Since $\chi(G) \geq \mathrm{cl}(G)$, we have $f(G) \geq 0$. For $f(G) = 0$ the inequality is trivial. Let $f(G) = 1$ and $\chi(G) = r+1$. Then $\mathrm{cl}(G) = r$. Note that $r \geq 2$ because of $\chi(G) \neq \mathrm{cl}(G)$. By (1.3) we have $G \in H_v(2_r;r+1)$. Thus, from Theorem 1.2(a) it follows that $|V(G)| \geq r+3 = 2f(G) + \chi(G)$. Let $f(G) = 2$ and $\chi(G) = r+1$. Then $\mathrm{cl}(G) = r-1$. Since $\chi(G) \neq \mathrm{cl}(G)$, $\mathrm{cl}(G) = r-1 \geq 2$, i.e. $r \geq 3$. From Theorem 1.4(a), (1.5) and (1.6) we obtain that $|V(G)| \geq r+5 = \chi(G) + 2f(G)$. This completes the proof of Lemma 2.1. ∎

Let $G = G_1 + G_2$. Obviously,

$$\chi(G) = \chi(G_1) + \chi(G_2); \tag{2.1}$$

$$\mathrm{cl}(G) = \mathrm{cl}(G_1) + \mathrm{cl}(G_2). \tag{2.2}$$

Hence,

$$f(G) = f(G_1) + f(G_2). \tag{2.3}$$

[1]Meanwhile, it has been proved that $F_v(2_5;4) = 16$, see J. Lathrop, S. Radziszowski, Computing the Folkman Number $F_v(2,2,2,2,2;4)$, Journal of Combinatorial Mathematics and Combinatorial Computing, 78 (2011), 213–222.

A graph G is said to be vertex-critical chromatic if $\chi(G-v) < \chi(G)$ for all $v \in V(G)$. We shall use the following result in the proof of Theorem 1.6.

Theorem 2.1. ([4], see also [5]) *Let G be a vertex-critical chromatic graph and $\chi(G) \geq 2$. If $|V(G)| < 2\chi(G) - 1$, then $G = G_1 + G_2$, where $V(G_i) \neq \emptyset$, $i = 1, 2$.*

Remark. In the original statement of Theorem 2.1 the graph G is supposed to be edge-critical chromatic (and not vertex-critical chromatic). Since each vertex-critical chromatic graph G contains an edge-critical chromatic subgraph H such that $\chi(G) = \chi(H)$ and $V(G) = V(H)$, the above statement is equivalent to the original one. It is also more convenient for the proof of Theorem 1.6.

Let G be a graph and $A \subseteq V(G)$ be an independent set of vertices of the graph G. It is easy to see that

$$G \overset{v}{\rightarrow} (2_r),\ r \geq 2 \ \Rightarrow\ G - A \overset{v}{\rightarrow} (2_{r-1}). \tag{2.4}$$

Lemma 2.2. *Let $G \in H_v(2_r; q)$, $q \geq 3$ and $|V(G)| = F_v(2_r; q)$. Then*

(a) *G is a vertex-critical $(r+1)$-chromatic graph;*

(b) *If $q < r + 3$, then $\mathrm{cl}(G) = q - 1$.*

Proof. By (1.3), $\chi(G) \geq r + 1$. Assume that (a) is false. Then there would exist $v \in V(G)$ such that $\chi(G - v) \geq r + 1$. According to (1.3), $G - v \in H_v(2_r; q)$. This contradicts the equality $|V(G)| = F_v(2_r; q)$.

Assume that (b) is false, i.e. $\mathrm{cl}(G) \leq q - 2$. Then from $q < r + 3$ it follows that $\mathrm{cl}(G) < r + 1$. Since $\chi(G) \geq r + 1$ there are $a, b \in V(G)$ such that $[a, b] \notin E(G)$. Consider the subgraph $G_1 = G - \{a, b\}$. We have $r \geq 2$, because $\chi(G) \neq \mathrm{cl}(G)$. Thus, from (2.4) and $\mathrm{cl}(G) \leq q - 2$ it follows that $G_1 \in H_v(2_{r-1}; q - 1)$. Obviously, $G_1 \in H_v(2_{r-1}; q - 1)$ leads to $K_1 + G_1 \in H_v(2_r; q)$. This contradicts the equality $|V(G)| = F_v(2_r; q)$, because $|V(K_1 + G_1)| = |V(G)| - 1$. Lemma 2.2 is proved. □

Lemma 2.3. *Let $G \in H_v(2_r; q)$, $r \geq 2$. Then*

$$|V(G)| \geq F_v(2_{r-1}; q) + \alpha(G).$$

Proof. Let $A \subseteq V(G)$ be an independent set such that $|A| = \alpha(G)$. Consider the subgraph $G_1 = G - A$. According to (2.4), $G_1 \in H_v(2_{r-1}; q)$. Hence $|V(G_1)| \geq F_v(2_{r-1}; q)$. Since $|V(G)| = |V(G_1)| + \alpha(G)$, Lemma 2.3 is proved. □

We shall use also the following three results:

$$F_v(2, 2, p; p + 1) \geq 2p + 4, \qquad \text{see [20] ;} \tag{2.5}$$

$$F_v(2, 2, 4; 5) = 13, \qquad \text{see [22].} \tag{2.6}$$

Theorem 2.2. ([12]) *Let G be a graph,* $\mathrm{cl}(G) \le p$ *and* $|V(G)| \ge p+2$*,* $p \ge 2$*. Let G also possess the following two properties:*

(i) $G \overset{v}{\nrightarrow} (2,2,p)$*;*

(ii) *If* $V(G) = V_1 \cup V_2 \cup V_3$ *is a* $(2,2,p)$*-free 3-coloring, then* $|V_1| + |V_2| \le 3$*.*

Then $G = K_1 + G_1$*.*

3. AN UPPER BOUND FOR THE NUMBERS $F_v(2_r; q)$

Consider the graph P whose complementary graph $\overline{P}$ is depicted in Figure 1. This graph is a well-known construction of Greenwood and Gleason [6], which

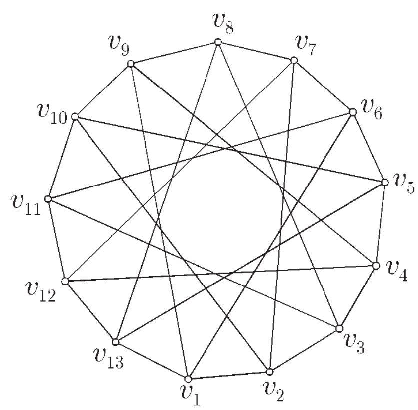

Figure 1: Graph $\overline{P}$

shows that $R(5,3) \ge 14$, since $|V(P)| = 13$ and

$$\alpha(P) = 2; \tag{3.1}$$

$$\mathrm{cl}(P) = 4 \qquad \text{(see [6])}. \tag{3.2}$$

From $|V(P)| = 13$ and (3.1) it follows that $\chi(P) \ge 7$. Since $\{v_1\} \cup \{v_2, v_3\} \cup \cdots \cup \{v_{12}, v_{13}\}$ is a 7-chromatic partition of $V(P)$, we have

$$\chi(P) = 7. \tag{3.3}$$

Let r and s be non-negative integers and $r \ge 3s + 6$. Define

$$\tilde{P} = K_{r-3s-6} + P + \underbrace{C_5 + \cdots + C_5}_{s}.$$

From (2.1), (2.2), (3.2) and (3.3) we obtain that $\chi(\tilde{P}) = r+1$ and $\mathrm{cl}(\tilde{P}) = r-s-2$. By (1.3), it follows that $\tilde{P} \in H_v(2_r; r-s-1)$ and thus

$$F_v(2_r; r-s-1) \leq |V(\tilde{P})|.$$

Since $|V(\tilde{P})| = r + 2s + 7$, we proved the following

Theorem 3.1. *Let r and s be non-negative integers and $r \geq 3s+6$. Then*

$$F_v(2_r; r-s-1) \leq r + 2s + 7.$$

Remark. Since $r \geq 3s+6$ we have $r-s-1 > 2$. Thus, according to (1.2), the numbers $F_v(2_r; r-s-1)$ exist.

4. PROOF OF THEOREM 1.5

Proof of Theorem 1.5(a) Let $G \in H_v(2_r; r-1)$. We need to show that $|V(G)| \geq r+7$. From Lemma 2.3 we have

$$|V(G)| \geq F_v(2_{r-1}; r-1) + \alpha(G).$$

By (1.5), (1.6) and Theorem 1.4(a) we deduce $F_v(2_{r-1}; r-1) \geq r+4$. Hence

$$|V(G)| \geq r + 4 + \alpha(G). \tag{4.1}$$

We prove the inequality $|V(G)| \geq r+7$ by induction with respect to r. From Table 1.1 we see that

$$R(r-1,3) < r+6 \quad \text{if } r=4 \quad \text{or} \quad r=5. \tag{4.2}$$

Obviously, from $G \in H_v(2_r; r-1)$ it follows that $\chi(G) \neq \mathrm{cl}(G)$. Thus, $\alpha(G) \geq 2$. From (4.1) we obtain $|V(G)| \geq r+6$. From this inequality and (4.2) we see that $|V(G)| > R(r-1,3)$ if $r=4$ or $r=5$. Since $\mathrm{cl}(G) < r-1$, it follows that $\alpha(G) \geq 3$. Now from (4.1) we obtain that $|V(G)| \geq r+7$ if $r=4$ or $r=5$.

Let $r \geq 6$. We shall consider separately two cases:

Case 1. $G \overset{v}{\nrightarrow} (2,2,r-2)$. From Theorem 2.2 we see that only following two subcases are possible:

Subcase 1a. $G = K_1 + G_1$. From $G \in H_v(2_r, r-1)$ it follows that $G_1 \in H_v(2_{r-1}; r-2)$. By the induction hypothesis, $|V(G_1)| \geq r+6$. Therefore, $|V(G)| \geq r+7$.

Subcase 1b. There is a $(2,2,r-2)$-free 3-coloring $V(G) = V_1 \cup V_2 \cup V_3$ such that $|V_1| + |V_2| \geq 4$. Let us consider the subgraph $\tilde{G} = G[V_3]$. By assumption $\tilde{G}$ does not contain an $(r-2)$-clique, i.e. $\mathrm{cl}(\tilde{G}) < r-2$. Since V_1 and V_2 are independent sets and $G \overset{v}{\to} (2_r)$, it follows from (2.4) that $\tilde{G} \overset{v}{\to} (2_{r-2})$. Thus, $\tilde{G} \in H_v(2_{r-2}; r-2)$. By (1.6) and Theorem 1.4(a), $|V(\tilde{G})| \geq r+3$. As $|V_1|+|V_2| \geq 4$, we have $|V(G)| \geq r+7$.

Case 2. $G \overset{v}{\to} (2, 2, r-2)$. Since $\mathrm{cl}(G) < r-1$, $G \in H_v(2, 2, r-2; r-1)$. From (2.5) it follows that $|V(G)| \geq 2(r-2)+4 = 2r$. Hence, if $2r \geq r+7$, i.e. $r \geq 7$, then $|V(G)| \geq r+7$. Let $r = 6$. Then $G \in H_v(2, 2, 4; 5)$. By (2.6) we conclude that $|V(G)| \geq 13$.

Proof of Theorem 1.5(b) Let $r \geq 6$. According to Theorem 1.5(a) we have $F_v(2_r; r-1) \geq r+7$. From Theorem 3.1 ($s = 0$) we obtain the opposite inequality $F_v(2_r; r-1) \leq r+7$.

Proof of Theorem 1.5(c) There is a 16-vertex graph G such that $\alpha(G) = 3$ and $\mathrm{cl}(G) = 3$, because $R(4,4) = 18$ (see [6]). From $|V(G)| = 16$ and $\alpha(G) = 3$ obviously it follows that $\chi(G) \geq 6$. By (1.3), $G \overset{v}{\to} (2_5)$. So, $G \in H_v(2_5; 4)$. Hence $F_v(2_5; 4) \leq |V(G)| = 16$.

Theorem 1.5 is proved. □

Corollary 4.1 *Let G be a graph such that $f(G) \leq 3$. Then*

$$|V(G)| \geq \chi(G) + 2f(G).$$

Proof. If $f(G) \leq 2$, then Corollary 4.1 follows from Lemma 2.1. Let $f(G) = 3$ and $\chi(G) = r+1$, then $\mathrm{cl}(G) = r-2$. Since $\chi(G) \neq \mathrm{cl}(G)$, it follows that $\mathrm{cl}(G) \geq 2$. Thus, $r \geq 4$. By (1.3) we get $G \in H_v(2_r; r-1)$. From Theorem 1.5(a) we obtain $|V(G)| \geq r+7 = \chi(G) + 2f(G)$. □

Remark. In $H_v(2_r; r-1)$, $r \geq 8$, there are more than one extremal graph. For instance, in $H_v(2_8; 7)$ besides $K_2 + P$ (see Theorem 3.1), the graph $C_5 + C_5 + C_5$ is extremal, too.

5. PROOF OF THEOREM 1.6

Proof of Theorem 1.6(a) Let $G \in H_v(2_r; r-2)$. We need to show that $|V(G)| \geq r+9$. From Lemma 2.3 we have

$$|V(G)| \geq F_v(2_{r-1}; r-2) + \alpha(G).$$

By Theorem 1.5(a), $F_v(2_{r-1}; r-2) \geq r+6$. Thus,

$$|V(G)| \geq r + 6 + \alpha(G). \tag{5.1}$$

We prove the inequality $|V(G)| \geq r+9$ by induction with respect to r. From Table 1.1 we see that

$$R(r-2,3) < r+8, \quad 5 \leq r \leq 7. \tag{5.2}$$

Obviously, from $G \in H_v(2_r; r-2)$ it follows that $\chi(G) \neq \mathrm{cl}(G)$. Thus, $\alpha(G) \geq 2$. From (5.1) we obtain $|V(G)| \geq r+8$. This, together with (5.2), implies $|V(G)| > R(r-2,3)$ if $5 \leq r \leq 7$. Since $\mathrm{cl}(G) < r-2$, $\alpha(G) \geq 3$. By the inequality (5.1), $|V(G)| \geq r+9$, $5 \leq r \leq 7$.

Let $r \geq 8$. Obviously, it suffices to consider only the situation when

$$|V(G)| = F_v(2_r; r-2). \tag{5.3}$$

By (5.3) and Lemma 2.2 we have that

$$G \textit{ is a vertex-critical } (r+1)\textit{-chromatic graph;} \tag{5.4}$$

and

$$\mathrm{cl}(G) = r-3. \tag{5.5}$$

From (5.4) and (5.5) it follows that

$$f(G) = 4. \tag{5.6}$$

We shall consider separately two cases.

Case 1. $|V(G)| < 2r+1$. By (5.4) and Theorem 2.1 we obtain that

$$G = G_1 + G_2. \tag{5.7}$$

From (5.7), (2.1) and (5.4) obviously it follows that

$$G_i,\ i=1,2 \quad \textit{is a vertex-critical chromatic graph.} \tag{5.8}$$

Let $f(G_1) = 0$. Then, according to (5.8) G_1 is a complete graph. Thus, it follows from (5.7) that $G = K_1 + G'$. It is clear that

$$G \in H_v(2_r; r-2) \Rightarrow G' \in H_v(2_{r-1}; r-3).$$

By the induction hypothesis, $|V(G')| \geq r+8$. Hence, $|V(G)| \geq r+9$. Let $f(G_i) \neq 0$, $i=1,2$. We see from (5.7), (2.3) and (5.6) that $f(G_i) \leq 3$, $i=1,2$. By Corollary 4.1 we conclude that

$$|V(G_i)| \geq \chi(G_i) + 2f(G_i), \quad i=1,2.$$

Summing these inequalities and using (2.1) and (2.3) we obtain

$$|V(G)| \geq \chi(G) + 2f(G). \tag{5.9}$$

According to (5.4), $\chi(G) = r+1$. Finally, from (5.9) and (5.6) it follows that $|V(G)| \geq r+9$.

Case 2. $|V(G)| \geq 2r+1$. Since $r \geq 8$, then $2r+1 \geq r+9$. Hence $|V(G)| \geq r+9$.

Proof of Theorem 1.6(b) By Theorem 1.6(a), $F_v(2_r; r-2) \geq r+9$. Therefore, we need to prove the opposite inequality $F_v(2_r; r-2) \leq r+9$ if $r \geq 8$. If $r \geq 9$, this inequality follows from Theorem 3.1 ($s = 1$). Let $r = 8$. By $R(6,3) = 18$ [11] (see also [7]), there is a graph Q such that $|V(Q)| = 17$, $\alpha(Q) = 2$ and $\mathrm{cl}(Q) = 5$. From $|V(Q)| = 17$ and $\alpha(Q) = 2$ obviously it follows that $\chi(Q) \geq 9$. Thus, by (1.3), $Q \overset{v}{\rightarrow} (2_8)$. Hence $Q \in H_v(2_8; 6)$ and $F_v(2_8; 6) \leq |V(Q)| = 17$. Theorem 1.6 is proved. □

Corollary 5.1. *Let G be a graph such that $f(G) \leq 4$. Then*

$$|V(G)| \geq \chi(G) + 2f(G).$$

Proof. If $f(G) \leq 3$, then Corollary 5.1 follows from Corollary 4.1. Let $f(G) = 4$ and $\chi(G) = r+1$, then $\mathrm{cl}(G) = r-3$. Since $\chi(G) \neq \mathrm{cl}(G)$, we have $\mathrm{cl}(G) \geq 2$, and consequently, $r \geq 5$. By (1.3), $G \in H_v(2_r; r-2)$. Using Theorem 1.6(a), we get $|V(G)| \geq r+9 = \chi(G) + 2f(G)$. □

Let $r \geq 3s+8$. Define

$$\tilde{Q} = K_{r-3s-8} + Q + \underbrace{C_5 + \cdots + C_5}_{s},$$

where graph Q is given in the proof of Theorem 1.6(b). Since $\mathrm{cl}(Q) = 5$ and $\chi(Q) \geq 9$, we have by (2.1) and (2.2) that $\mathrm{cl}(\tilde{Q}) = r-s-3$ and $\chi(\tilde{Q}) \geq r+1$. According to (1.3), $\tilde{Q} \in H_v(2_r; r-s-2)$. Thus, $F_v(2_r; r-s-2) \leq |V(\tilde{Q})|$. Since $|V(\tilde{Q})| = r+2s+9$, we obtain the following

Theorem 5.1. *Let r and s be non-negative integers and $r \geq 3s+8$. Then*

$$F_v(2_r; r-s-2) \leq r+2s+9.$$

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Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"
5, J. Bourchier Blvd., BG-1164 Sofia
BULGARIA
e-mail: `nenov@fmi.uni-sofia.bg`