The vertex Folkman numbers $F_v(a_1, ..., a_s; m-1) = m + 9$, if $\max\{a_1, ..., a_s\} = 5$

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Abstract

For a graph G the expression $G \overset{v}{\rightarrow} (a_1, ..., a_s)$ means that for any s-coloring of the vertices of G there exists $i \in \{1, ..., s\}$ such that there is a monochromatic a_i-clique of color i. The vertex Folkman numbers

$$F_v(a_1, ..., a_s; m-1) = \min\{|\operatorname{V}(G)| : G \overset{v}{\rightarrow} (a_1, ..., a_s) \text{ and } K_{m-1} \not\subseteq G\}.$$

are considered, where $m = \sum_{i=1}^{s}(a_i - 1) + 1$.

With the help of computer we show that $F_v(2, 2, 5; 6) = 16$, and then we prove

$$F_v(a_1, ..., a_s; m-1) = m + 9,$$

if $\max\{a_1, ..., a_s\} = 5$.

We also obtain the bounds

$$m + 9 \leq F_v(a_1, ..., a_s; m-1) \leq m + 10,$$

if $\max\{a_1, ..., a_s\} = 6$.

Keywords: Folkman number, Ramsey number, clique number, independence number, chromatic number

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1 Introduction

In this paper only finite, non-oriented graphs without loops and multiple edges are considered. The following notations are used:

$\mathrm{V}(G)$ - the vertex set of G;

$\mathrm{E}(G)$ - the edge set of G;

$\overline{G}$ - the complement of G;

$\omega(G)$ - the clique number of G;

$\alpha(G)$ - the independence number of G;

$\mathrm{N}(v), \mathrm{N}_G(v), v \in \mathrm{V}(G)$ - the set of all vertices of G adjacent to v;

$d(v), v \in \mathrm{V}(G)$ - the degree of the vertex v, i.e. $d(v) = |\mathrm{N}(v)|$;

$G-v, v \in \mathrm{V}(G)$ - subgraph of G obtained from G by deleting the vertex v and all edges incident to v;

$G-e, e \in \mathrm{E}(G)$ - subgraph of G obtained from G by deleting the edge e;

$G+e, e \in \mathrm{E}(\overline{G})$ - supergraph of G obtained by adding the edge e to $\mathrm{E}(G)$.

K_n - complete graph on n vertices;

C_n - simple cycle on n vertices;

Let G_1 and G_2 be two graphs without common vertices. Denote by $G_1 + G_2$ the graph G for which $\mathrm{V}(G) = \mathrm{V}(G_1) \cup \mathrm{V}(G_2)$ and $\mathrm{E}(G) = \mathrm{E}(G_1) \cup \mathrm{E}(G_2) \cup E'$, where $E' = \{[x,y] : x \in \mathrm{V}(G_1), y \in \mathrm{V}(G_2)\}$, i.e. G is obtained by making every vertex of G_1 adjacent to every vertex of G_2.

All undefined terms can be found in [29].

Let $a_1, ..., a_s$ be positive integers. The expression $G \overset{v}{\rightarrow} (a_1, ..., a_s)$ means that for any coloring of $\mathrm{V}(G)$ in s colors (s-coloring) there exists $i \in \{1, ..., s\}$ such that there is a monochromatic a_i-clique of color i. In particular, $G \overset{v}{\rightarrow} (a_1)$ means that $\omega(G) \geq a_1$. Further, for convenience instead of $G \overset{v}{\rightarrow} (\underbrace{2, ..., 2}_{r})$ we write $G \overset{v}{\rightarrow} (2_r)$ and instead of $G \overset{v}{\rightarrow} (\underbrace{2, ..., 2}_{r}, a_1, ..., a_s)$ we write $G \overset{v}{\rightarrow} (2_r, a_1, ..., a_s)$.

Define:

$\mathcal{H}(a_1, ..., a_s; q) = \left\{G : G \overset{v}{\rightarrow} (a_1, ..., a_s) \text{ and } \omega(G) < q\right\}$.

$\mathcal{H}(a_1, ..., a_s; q; n) = \{G : G \in \mathcal{H}(a_1, ..., a_s; q) \text{ and } |\mathrm{V}(G)| = n\}$.

The vertex Folkman number $F_v(a_1, ..., a_s; q)$ is defined by the equality

$F_v(a_1, ..., a_s; q) = \min\{|\mathrm{V}(G)| : G \in \mathcal{H}(a_1, ..., a_s; q)\}$.

The graph G is called an extremal graph in $\mathcal{H}(a_1, ..., a_s; q)$ if $G \in \mathcal{H}(a_1, ..., a_s; q)$ and $|\mathrm{V}(G)| = F_v(a_1, ..., a_s; q)$. The set of all extremal graphs in $\mathcal{H}(a_1, ..., a_s; q)$ is denoted by $\mathcal{H}_{extr}(a_1, ..., a_s; q)$.

The graph G is called a maximal graph in $\mathcal{H}(a_1, ..., a_s; q)$ if $G \in \mathcal{H}(a_1, ..., a_s; q)$, but $G + e \notin \mathcal{H}(a_1, ..., a_s; q), \forall e \in \mathrm{E}(\overline{G})$, i.e. $\omega(G + e) \geq$

$q, \forall e \in \mathrm{E}(\overline{G})$. Also, G is called a minimal graph in $\mathcal{H}(a_1, ..., a_s; q)$ if $G \in \mathcal{H}(a_1, ..., a_s; q)$, but $G - e \notin \mathcal{H}(a_1, ..., a_s; q), \forall e \in \mathrm{E}(G)$, i.e. $G - e \overset{v}{\nrightarrow} (a_1, ..., a_s), \forall e \in \mathrm{E}(G)$. If G is both maximal and minimal graph in $\mathcal{H}(a_1, ..., a_s; q)$, then we say that G is a bicritical graph in $\mathcal{H}(a_1, ..., a_s; q)$.

Folkman proves in [5] that

$$F_v(a_1, ..., a_s; q) \text{ exists } \Leftrightarrow q > \max\{a_1, ..., a_s\}. \tag{1.1}$$

Other proofs of (1.1) are given in [4] and [10].
Obviously $F_v(a_1, ..., a_s; q)$ is a symmetric function of $a_1, ..., a_s$ and if $a_i = 1$, then

$$F_v(a_1, ..., a_s; q) = F_v(a_1, ..., a_{i-1}, a_{i+1}, ..., a_s; q).$$

Therefore, it is enough to consider only such Folkman numbers $F_v(a_1, ..., a_s; q)$ for which

$$2 \leq a_1 \leq ... \leq a_s. \tag{1.2}$$

We call the numbers $F_v(a_1, ..., a_s; q)$ for which the inequalities (1.2) hold, canonical vertex Folkman numbers.
In [11] for arbitrary positive integers $a_1, ..., a_s$ are defined

$$m(a_1, ..., a_s) = m = \sum_{i=1}^{s}(a_i - 1) + 1 \quad \text{and} \quad p = \max\{a_1, ..., a_s\}. \tag{1.3}$$

Obviously $K_m \overset{v}{\rightarrow} (a_1, ..., a_s)$ and $K_{m-1} \overset{v}{\nrightarrow} (a_1, ..., a_s)$. Therefore,

$$F_v(a_1, ..., a_s; q) = m, \quad q \geq m + 1.$$

In accordance with (1.1),

$$F_v(a_1, ..., a_s; m) \text{ exists } \Leftrightarrow m \geq p + 1.$$

For these numbers the following theorem is true:

Theorem 1.1. *Let $a_1, ..., a_s$ be positive integers and let m and p be defined by (1.3). If $m \geq p + 1$, then:*

(a) $F_v(a_1, ..., a_s; m) = m + p$, *[11],[10].*

(b) $K_{m+p} - C_{2p+1} = K_{m-p-1} + \overline{C}_{2p+1}$

is the only extremal graph in $\mathcal{H}(a_1, ..., a_s; m)$, [10].

Other proofs of Theorem 1.1 are given in [21] and [22].

In accordance with (1.1),

$$(1.4) \qquad F_v(a_1, ..., a_s; m-1) \text{ exists } \Leftrightarrow m \geq p+2.$$

Let m and p be defined by (1.3). Then

$$(1.5) \qquad F_v(a_1, ..., a_s, m-1) = \begin{cases} m+4, & \textit{if } p=2 \textit{ and } m \geq 6, \textit{ [18]} \\ m+6, & \textit{if } p=3 \textit{ and } m \geq 6, \textit{ [23]} \\ m+7, & \textit{if } p=4 \textit{ and } m \geq 6, \textit{ [23].} \end{cases}$$

The remaining canonical numbers $F_v(a_1, ..., a_s; m-1)$, $p \leq 4$ are: $F_v(2,2,2;3) = 11$, [16] and [1], $F_v(2,2,2,2;4) = 11$, [19] (see also [20]), $F_v(2,2,3;4) = 14$, [21] and [2], $F_v(3,3;4) = 14$, [17] and [24]. From these facts it becomes clear that we know all Folkman numbers of the form $F_v(a_1, ..., a_s; m-1)$ when $\max\{a_1, ..., a_s\} \leq 4$. The only known canonical vertex Folkman number of the form $F_v(a_1, ..., a_s, m-1)$, $p \geq 5$ is $F_v(3,5;6) = 16$, [26]. Since we know all the numbers $F_v(a_1, ..., a_s; m-1)$ when $p=2$, further we shall assume that $p \geq 3$. The following bounds for these numbers are known:

$$(1.6) \qquad m+p+2 \leq F_v(a_1, ..., a_s; m-1) \leq m+3p, \quad p \geq 3.$$

The lower bound is obtained in [21] and the upper bound is obtained in [8]. It is easy to see that in the border case $m = p+2$ when $p \geq 3$ there are only two canonical numbers of the form $F_v(a_1, ..., a_s; m-1)$, namely $F_v(2,2,p;p+1)$ and $F_v(3,p;p+1)$.
With the help of the numbers $F_v(2,2,p;p+1)$, the lower bound in (1.6) can be improved.

Theorem 1.2. *[23] Let $a_1, ..., a_s$ be positive integers, let m and p be defined by (1.3), $p \geq 3$ and $m \geq p+2$. If $F_v(2,2,p;p+1) \geq 2p+5$, then*

$$F_v(a_1, ..., a_s; m-1) \geq m+p+3.$$

Some graphs, with which upper bounds for $F_v(3,p;p+1)$ are obtained, can be used for obtaining general upper bounds for $F_v(a_1, ..., a_s; m-1)$. For example, the graph Γ_p from [21], which witnesses the bound $F_v(3,p;p+1) \leq 4p+2, p \geq 3$, helps to obtain the upper bound in (1.6). Thus, obtaining bounds for the numbers $F_v(a_1, ..., a_s; m-1)$ and computing some of them is related with computation and obtaining bounds for the numbers $F_v(2,2,p;p+1)$ and $F_v(3,p;p+1)$. It is easy to see that $G \overset{v}{\rightarrow} (3,p) \Rightarrow G \overset{v}{\rightarrow} (2,2,p)$. Therefore, the following inequality holds:

$$(1.7) \qquad F_v(2,2,p;p+1) \leq F_v(3,p;p+1), \quad p \geq 3.$$

However, we do not know if these numbers are different. For now we do know that they are equal if $p = 3$ or $p = 4$.

Problem 1.3. *[7] Does there exist a positive integer p for which the inequality (1.7) is strict?*

In this paper we prove $F_v(2, 2, 5; 6) = 16$ and a new proof of the equality $F_v(3, 5; 6) = 16$, [26] is given. Hence for $p = 5$ there is an equality in (1.7). We find all extremal graphs in $\mathcal{H}(2, 2, 5; 6)$ and in $\mathcal{H}(3, 5; 6)$. With the help of these results, we compute the numbers $F_v(a_1, ..., a_s; m - 1) = m + 9$, when $\max\{a_1, ..., a_s\} = 5$. In the case $\max\{a_1, ..., a_s\} = 6$ we improve the bounds (1.6) by proving $m + 9 \leq F_v(a_1, ..., a_s; m - 1) \leq m + 10$. The exact formulations of the obtained results are as follows:

Theorem 1.4. $|\mathcal{H}(2, 2, 5; 6; 16)| = 147$.

In the proof of Theorem 1.4 we find all graphs in $\mathcal{H}(2, 2, 5; 6; 16)$. Some properties of these graphs are listed in Table 1. Among them there are 4 bicritical graphs, which are shown in Figure 1, and some of their properties are listed in Table 2.

Theorem 1.5. $F_v(2, 2, 5; 6) = 16$ *and the graphs from Theorem 1.4 are all the graphs in* $\mathcal{H}_{extr}(2, 2, 5; 6)$.

Corollary 1.6. *[26]* $F_v(3, 5; 6) = 16$.

Proof. From Theorem 1.5 and (1.7) we obtain $F_v(3, 5; 6) \geq 16$. Since among the graphs from Theorem 1.4 there are such, which belong to $\mathcal{H}(3, 5; 6)$ (see Figure 2), it follows that $F_v(3, 5; 6) \leq 16$. □

Theorem 1.7. $|\mathcal{H}(3, 5; 6; 16)| = 4$.

In the proof of Theorem 1.7 we find all graphs in $\mathcal{H}(3, 5; 6; 16)$. These graphs are shown in Figure 2 and some of their properties are listed in Table 3.

Theorem 1.8. *Let $a_1, ..., a_s$ be positive integers,* $m = \sum\limits_{i=1}^{s}(a_i - 1) + 1$, $\max\{a_1, ..., a_s\} = 5$ *and* $m \geq 7$. *Then*

$$F_v(a_1, ..., a_s; m - 1) = m + 9.$$

At the end of this paper as a consequence of these results and with the help of one graph (see Figure 5) from [28] we prove that

Theorem 1.9. *Let $a_1, ..., a_s$ be positive integers,* $m = \sum\limits_{i=1}^{s}(a_i - 1) + 1$, $\max\{a_1, ..., a_s\} = 6$ *and* $m \geq 8$. *Then*

$$m + 9 \leq F_v(a_1, ..., a_s; m - 1) \leq m + 10.$$

Remark. *Since* $F_v(a_1, ..., a_s; m-1)$ *exists only if* $m \geq 2 + \max\{a_1, ..., a_s\}$, *the conditions* $m \geq 7$ *in Theorem 1.8 and* $m \geq 8$ *in Theorem 1.9 are necessary.*

This paper has a previous version (arXiv:1503.08444v1). We decided to forgo publishing this first version, because we managed to improve the main result by proving the Theorem 1.8 above.

2 Proof of Theorem 1.4

We adapt Algorithm A1 from [24] to obtain all graphs in $\mathcal{H}(2, 2, 5; 6; 16)$ with the help of computer. Similar algorithms are used in [2], [30], [9] and [26]. Also, with the help of computer, results for Folkman numbers are obtained in [6], [28], [27] and [3].
The naive approach for finding all graphs in $\mathcal{H}(2, 2, 5; 6; 16)$ suggests to check all graphs of order 16 for inclusion in $\mathcal{H}(2, 2, 5; 6)$. However, this is practically impossible because the number of graphs to check is too large. The method that is described uses an algorithm for effective generation of all maximal graphs in $\mathcal{H}(2, 2, 5; 6; 16)$. The other graphs in $\mathcal{H}(2, 2, 5; 6; 16)$ are their subgraphs. The algorithm is based on the following proposition:

Proposition 2.1. *Let* G *be a maximal graph in* $\mathcal{H}(2_r, p; q; n)$ *and let* $v_1, v_2, ..., v_k$ *be independent vertices. Let* $H = G - \{v_1, v_2, ..., v_k\}$. *Then:*

(a) $H \in \mathcal{H}(2_{r-1}, p; q; n-k)$.

(b) *the addition of a new edge to* H *forms a new* $(q-1)$*-clique.*

(c) $\mathrm{N}_G(v_i)$ *is a maximal* K_{q-1}*-free subset of* $\mathrm{V}(H)$, $i = 1, ..., k$.

Proof. The proposition (a) follows from the assumption that $G \in \mathcal{H}(2_r, p; q; n)$, (b) and (c) follow from the maximality of G. □

The following algorithm, which is a modification of Algorithm A1 from [24], generates all maximal graphs in $\mathcal{H}(2_r, p; q; n)$ with independence number at least k:

Algorithm 2.2. *Generation of all maximal graphs in* $\mathcal{H}(2_r, p; q; n)$ *with independence number at least* k *by adding* k *independent vertices to the graphs from* $\mathcal{H}(2_{r-1}, p; q; n-k)$ *in which the addition of a new edge forms a new* $(q-1)$*-clique.*

1. Let $\mathcal{A} \subseteq \mathcal{H}(2_{r-1}, p; q; n-k)$ be the set of these graphs in which the addition of a new edge forms a new $(q-1)$-clique (see Proposition 2.1 (a) and (b)). The maximal graphs in $\mathcal{H}(2_r, p; q; n)$ are output in $\mathcal{B}$.

2. For each graph $H \in \mathcal{A}$:

2.1. Find the family $\mathcal{M}(H) = \{M_1, ..., M_t\}$ of all maximal K_{q-1}-free subsets of $\mathrm{V}(H)$.

2.2. Consider all the k-tuples $(M_{i_1}, M_{i_2}, ..., M_{i_k})$ of elements of $\mathcal{M}(H)$ for which $1 \leq i_1 \leq ... \leq i_k \leq t$ (in these k-tuples some subsets M_i can coincide). For every such k-tuple construct the graph $G = G(M_{i_1}, M_{i_2}, ..., M_{i_k})$ by adding to $\mathrm{V}(H)$ new independent vertices $v_1, v_2, ..., v_k$, so that $N_G(v_j) = M_{i_j}, j = 1, ..., k$ (see Proposition 2.1 (c)). If $\omega(G+e) = q, \forall e \in \mathrm{E}(\overline{G})$, then add G to $\mathcal{B}$.

3. Exclude the isomorph copies of graphs from $\mathcal{B}$.

4. Exclude from $\mathcal{B}$ all graphs which are not in $\mathcal{H}(2_r, p; q; n)$.

According to Proposition 2.1, at the end of step 4 $\mathcal{B}$ is the set of all maximal graphs in $\mathcal{H}(2_r, p; q; n)$ with independence number at least k.

Intermediate problems, that are solved, are finding all graphs in $\mathcal{H}(2, 5; 6; 13)$ and in $\mathcal{H}(5; 6; 10)$. For each of the sets $\mathcal{H}(2, 2, 5; 6; 16)$ and $\mathcal{H}(2, 5; 6; 13)$ we start by finding the maximal graphs in them. The remaining graphs are obtained by removing edges from the maximal graphs. Using Algorithm 2.2, we can obtain the maximal graphs in $\mathcal{H}(2, 2, 5; 6; 16)$ with independence number at least 3 by adding 3 independent vertices to graphs in $\mathcal{H}(2, 5; 6; 13)$. Similarly, we can obtain the maximal graphs in $\mathcal{H}(2, 5; 6; 13)$ with independence number at least 3 by adding 3 independent vertices to graphs in $\mathcal{H}(5; 6; 10)$. What remains is to find the maximal graphs in these sets with independence number 2. Let

$\mathcal{R}(p, q) = \{G : \alpha(G) < p$ and $\omega(G) < q\}$.

$\mathcal{R}(p, q; n) = \{G : G \in \mathcal{R}(p, q)$ and $|\mathrm{V}(G)| = n\}$.

The graphs $\mathcal{R}(3, 6)$ are known (see [13] and [25]). The maximal graphs in $\mathcal{H}(2, 2, 5; 6; 16)$ with independence number 2 are a subset of $\mathcal{R}(3, 6; 16)$ and the maximal graphs in $\mathcal{H}(2, 5; 6; 13)$ with independence number 2 are a subset of $\mathcal{R}(3, 6; 13)$

The *nauty* programs [12] have an important role in this work. We use them for fast generation of non-isomorphic graphs, isomorph rejection, and to determine the automorphism groups of graphs.

2.1 Finding all graphs in $\mathcal{H}(5; 6; 10)$

It is clear that $\mathcal{H}(5; 6; 10)$ is the set of 10 vertex graphs with clique number 5. The number of non-isomorphic graphs of order 10 is 12 005 168. Out of those we can easily find the graphs with clique number 5. Thus, we obtain all 1 724 440 graphs in $\mathcal{H}(5; 6; 10)$.

2.2 Finding all graphs in $\mathcal{H}(2,5;6;13)$

Algorithm 2.3. *Finding all graphs in $\mathcal{H}(2,5;6;13)$.*

1. Find all maximal graphs $G \in \mathcal{H}(2,5;6;13)$ for which $\alpha(G) \geq 3$:

1.1. Determine which of the graphs in $\mathcal{H}(5;6;10)$ have the property that the addition of a new edge forms a new 5-clique.

1.2. Using Algorithm 2.2, add three independent vertices to the graphs from step 1.1. to obtain the graphs wanted in step 1.

2. Find all maximal graphs $G \in \mathcal{H}(2,5;6;13)$ for which $\alpha(G) = 2$:

2.1. In order to do so, check which of the graphs in $\mathcal{R}(3,6;13)$ are maximal graphs in $\mathcal{H}(2,5;6;13)$.

3. The union of the graphs from steps 1. and 2. gives all maximal graphs in $\mathcal{H}(2,5;6;13)$. By removing edges from them, the remaining graphs in $\mathcal{H}(2,5;6;13)$ are obtained.

Results of computations:

Step 1: Among all the graphs in $\mathcal{H}(5;6;10)$ exactly 3633 have the property that the addition of a new edge forms a new 5-clique. By adding three independent vertices to them, we obtain 326 maximal graphs in $\mathcal{H}(2,5;6;13)$.

Step 2: The number of graphs in $\mathcal{R}(3,6;13)$ is 275 086 [14]. Among them 61 are maximal graphs in $\mathcal{H}(2,5;6;13)$.

Step 3: The union of the graphs from steps 1. and 2. gives all 387 maximal graphs in $\mathcal{H}(2,5;6;13)$. By removing edges from them, we obtain all 20 013 726 graphs in $\mathcal{H}(2,5;6;13)$.

2.3 Finding all graphs in $\mathcal{H}(2,2,5;6;16)$

Algorithm 2.4. *Finding all graphs in $\mathcal{H}(2,2,5;6;16)$.*

1. Find all maximal graphs $G \in \mathcal{H}(2,2,5;6;16)$ for which $\alpha(G) \geq 3$:

1.1. Determine which of the graphs in $\mathcal{H}(2,5;6;13)$ have the property that the addition of a new edge forms a new 5-clique.

1.2. Using Algorithm 2.2, add three independent vertices to the graphs from step 1.1. to obtain the graphs wanted in step 1.

2. Find all maximal graphs $G \in \mathcal{H}(2,2,5;6;16)$ for which $\alpha(G) = 2$:

2.1. In order to do so, check which of the graphs in $\mathcal{R}(3,6;16)$ are maximal graphs in $\mathcal{H}(2,2,5;6;16)$.

3. The union of the graphs from steps 1. and 2. gives all maximal graphs in $\mathcal{H}(2,2,5;6;16)$. By removing edges from them, the remaining graphs in $\mathcal{H}(2,2,5;6;16)$ are obtained.

Results of computations:

Step 1: Among all the graphs in $\mathcal{H}(2,5;6;13)$ exactly 2 265 005 have the property that the addition of a new edge forms a new 5-clique. By adding three independent vertices to them, we obtain 32 maximal graphs $\mathcal{H}(2,2,5;6;16)$.

Step 2: The number of graphs in $\mathcal{R}(3,6;16)$ is 2576 [15]. Among them 5 are maximal graphs in $\mathcal{H}(2,2,5;6;16)$.

Step 3: The union of the graphs from steps 1. and 2. gives all 37 maximal graphs in $\mathcal{H}(2,2,5;6;16)$. By removing edges from them, we obtain all 147 graphs in $\mathcal{H}(2,2,5;6;16)$.

Thus, we finished the proof of Theorem 1.4.

We denote by $G_1, ..., G_{147}$ the graphs in $\mathcal{H}(2,2,5;6;16)$. The indexes correspond to the defined order in the *nauty* programs. In Table 1 are listed some properties of the graphs in $\mathcal{H}(2,2,5;6;16)$. Among them there are 37 maximal, 41 minimal and 4 bicritical graphs (see Figure 1). The properties of the bicritical graphs are listed in Table 2.

All computations were done on a personal computer. The slowest part was step 1.2 of Algorithm 2.4, which took several days to complete.

Note that to find all graphs in $\mathcal{H}(2,2,5;6;16)$ it is enough to find only these graphs from the sets $\mathcal{H}(2,5;6;13)$ and $\mathcal{H}(5;6;10)$ for which the addition of a new edge forms a new 5-clique. In this case that does not save us much of the time needed for computer work, but later, in the proof of Theorem 6.1, we use that possibility.

$\lvert E(G)\rvert$	#	$\delta(G)$	#	$\Delta(G)$	#	$\alpha(G)$	#	$\chi(G)$	#	$\lvert Aut(G)\rvert$	#
83	7	7	2	11	24	2	21	7	65	1	84
84	25	8	36	12	123	3	126	8	82	2	44
85	42	9	61							3	1
86	37	10	47							4	8
87	29	11	1							6	8
88	6									8	1
89	1									96	1

Table 1: Properties of the graphs in $\mathcal{H}(2,2,5;6;16)$

Graph	$\lvert E(G)\rvert$	$\delta(G)$	$\Delta(G)$	$\alpha(G)$	$\chi(G)$	$\lvert Aut(G)\rvert$
G_{74}	86	9	12	3	7	1
G_{78}	87	10	12	3	7	2
G_{134}	85	9	12	3	7	2
G_{135}	85	9	12	3	7	1

Table 2: Properties of the bicritical graphs in $\mathcal{H}(2,2,5;6;16)$

3 Proof of Theorem 1.5 and Theorem 1.7

Proof of Theorem 1.5

Since $\mathcal{H}(2,2,5;6;16) \neq \emptyset$, it follows that $F_v(2,2,5;6) \leq 16$. With a simple algorithm, which removes a vertex from each graph in $\mathcal{H}(2,2,5;6;16)$ and checks for inclusion in $\mathcal{H}(2,2,5;6)$, we obtain $\mathcal{H}(2,2,5;6;15) = \emptyset$, which proves $F_v(2,2,5;6) \geq 16$. Thus, the theorem is proved.

Remark. *The lower bound $F_v(2,2,5;6) \geq 16$ can be proved simpler in terms of time needed for computer work. The result $\mathcal{H}(2,2,5;6;15) = \emptyset$ can be obtained with a method similar to the one used to find all graphs in $\mathcal{H}(2,2,5;6;16)$, but in the slowest step we add 3 vertices to appropriately chosen 12-vertex graphs instead of 13-vertex graphs. A similar approach is used in the proof of the bound $F_v(3,5;6) \geq 16$ in [26].*

Proof of Theorem 1.7

Using that $\mathcal{H}(3,5;6;16) \subseteq \mathcal{H}(2,2,5;6;16)$, by checking the graphs from Theorem 1.4 with computer we obtain $|\mathcal{H}(3,5;6;16)| = 4$.

Thus, the theorem is proved.

The graphs from $\mathcal{H}(3,5;6;16)$ are shown in Figure 2. Some properties of these graphs are listed in Table 3.

It is interesting to note that for all these graphs the inequality (4.4) is strict. The graphs G_{50} and G_{146} are maximal and the other two graphs G_{51} and G_{81} are their subgraphs and are obtained by removing one edge. In [28] the inequality $F_v(3,5;6) \leq 16$ is proved with the help of the graph G_{146}. We shall note that $|Aut(G_{146})| = 96$ and among all graphs in $\mathcal{H}(2,2,5;6;16)$ it has the most automorphisms.

Graph	$\lvert E(G)\rvert$	$\delta(G)$	$\Delta(G)$	$\alpha(G)$	$\chi(G)$	$\lvert Aut(G)\rvert$
G_{50}	87	10	12	3	8	6
G_{51}	86	9	11	3	8	6
G_{81}	87	10	11	2	8	6
G_{146}	88	11	11	2	8	96

Table 3: Properties of the graphs in $\mathcal{H}(3,5;6;16)$

4 Bounds for the numbers $F_v(a_1, ..., a_s; q)$

First, we define a modification of the vertex Folkman numbers $F_v(a_1, ..., a_s; q)$ with the help of which we obtain upper bound for these numbers.

Definition 4.1. *Let G be a graph and let m and p be positive integers. The expression*

$$G \overset{v}{\rightarrow} m\big|_p$$

means that for every choice of positive integers $a_1, ..., a_s$ (s is not fixed), such that $m = \sum\limits_{i=1}^{s}(a_i - 1) + 1$ and $\max\{a_1, ..., a_s\} \leq p$, we have

$$G \overset{v}{\rightarrow} (a_1, ..., a_s).$$

Example 4.2. $K_m \overset{v}{\rightarrow} m\big|_p, \quad \forall p$ *(obviously).*

Example 4.3. *[10] Let us notice that $\overline{C}_{2p+1} \overset{v}{\rightarrow} (p+1)\big|_p$. Indeed, let $b_1, ..., b_s$ be positive integers, such that $\sum\limits_{i=1}^{s}(b_i - 1) + 1 = p + 1$ and $\max\{b_1, ..., b_s\} \leq p$. Assume that there exists s-coloring $\mathrm{V}(G) = V_1 \cup ... \cup V_s$, such that V_i does not contain a b_i-clique. Then $|V_i| \leq 2b_i - 2$ and $|\mathrm{V}(G)| = \sum\limits_{i=1}^{s}|V_i| \leq 2\sum\limits_{i=1}^{s}(b_i - 1) = 2p$, which is a contradiction.*

Define:
$\widetilde{\mathcal{H}}(m\big|_p; q) = \left\{G : G \overset{v}{\rightarrow} m\big|_p \text{ and } \omega(G) < q\right\}$.
$\widetilde{F}_v(m\big|_p; q) = \min\left\{|\mathrm{V}(G)| : G \in \widetilde{\mathcal{H}}(m\big|_p; q)\right\}$.

Proposition 4.4. $\widetilde{\mathcal{H}}(m\big|_p; q) \neq \emptyset$, *i.e.* $\widetilde{F}_v(m\big|_p; q)$ *exists* $\Leftrightarrow q > \min\{m, p\}$.

Proof. Let $\widetilde{\mathcal{H}}(m\big|_p; q) \neq \emptyset$ and $G \in \widetilde{\mathcal{H}}(m\big|_p; q)$. If $m \leq p$, then $G \overset{v}{\rightarrow} (m)$, and it follows $\omega(G) \geq m$. Since $\omega(G) < q$, we obtain $q > m$. Let $m > p$. Then there exist positive integers $a_1, ..., a_s$, such that $m = \sum_{i=1}^{s}(a_i - 1) + 1$ and $p = \max\{a_1, ..., a_s\}$, for example $a_1 = ... = a_{m-p} = 2$ and $a_{m-p+1} = p$. Since $G \overset{v}{\rightarrow} (a_1, ..., a_s)$, it follows that $\omega(G) \geq p$ and $q > p$. Therefore, if $\mathcal{H}(m; p; q) \neq \emptyset$, then $q > \min\{m, p\}$.

Let $q > \min\{m, p\}$. If $m \geq p$, then $q > p$. According to (1.1), for every choice of positive integers $a_1, ..., a_s$, such that $m = \sum_{i=1}^{s}(a_i - 1) + 1$ and $\max\{a_1, ..., a_s\} \leq p$ there exists a graph $G(a_1, ..., a_s) \in \mathcal{H}(a_1, ..., a_s; q)$. Let G be the union of all graphs $G(a_1, ..., a_s)$. It is clear that $G \in \widetilde{\mathcal{H}}(m\big|_p; q)$. If $m \leq p$, then $m < q$, and therefore $K_m \in \widetilde{\mathcal{H}}(m\big|_p; q)$. $\square$

The following theorem gives bounds for the numbers $F_v(a_1, ..., a_s; q)$:

Theorem 4.5. *Let $a_1, ..., a_s$ be positive integers and let m and p be defined by (1.3), $q > p$. Then*

$$F_v(2_{m-p}, p; q) \leq F_v(a_1, ..., a_s; q) \leq \widetilde{F}_v(m\big|_p; q).$$

Proof. The right inequality follows from the inclusion

$$\widetilde{\mathcal{H}}(m\big|_p; q) \subseteq \mathcal{H}(a_1, ..., a_s; q).$$

In order to prove the left inequality, let us notice that if $a_i \geq 3$, then

$$G \overset{v}{\rightarrow} (a_1, ..., a_s) \Rightarrow G \overset{v}{\rightarrow} (a_1, ..., a_{i-1}, 2, a_i - 1, ..., a_s). \tag{4.1}$$

Since $m(a_1, ..., a_s) = m(a_1, ..., a_{i-1}, 2, a_i - 1, ..., a_s)$, by successively applying (4.1) we obtain

$$G \overset{v}{\rightarrow} (a_1, ..., a_s) \Rightarrow G \overset{v}{\rightarrow} (2_{m-p}, p), \tag{4.2}$$

$$G \overset{v}{\rightarrow} (a_1, ..., a_s) \Rightarrow G \overset{v}{\rightarrow} (2_{m-1}). \tag{4.3}$$

From (4.2) it follows

$$F_v(a_1, ..., a_s; q) \geq F_v(2_{m-p}, p; q). \qquad \square$$

Since $G \overset{v}{\rightarrow} (2_{m-1}) \Leftrightarrow \chi(G) \geq m$, from (4.3) it becomes clear that

$$G \overset{v}{\rightarrow} (a_1, ..., a_s) \Rightarrow \chi(G) \geq m, \quad [22]. \tag{4.4}$$

This fact is used later in the proof of Theorem 5.2.

The bounds from Theorem 4.5 are useful because in general they are easier to estimate and compute than the numbers $F_v(a_1, ..., a_s)$ themselves. Later, we compute the exact value of the numbers $F_v(2_{m-5}, 5; m-1)$ (see Corollary 6.3) and the numbers $\widetilde{F}_v(m\big|_5; m-1)$ (see Theorem 7.4). This way, with the help of Theorem 4.5, Theorem 1.8 is proved. Similarly, we obtain the bounds of Theorem 1.9

Remark. *It is easy to see that if $q > m$, then $F_v(a_1, ..., a_s; q) = \widetilde{F}_v(m\big|_p; q)$ $= m$. From Theorem 1.1 it follows $F_v(a_1, ..., a_s; m) = \widetilde{F}_v(m\big|_p; m) = m + p$. If $q = m - 1$ and $p \leq 4$, according to (1.5), we also have $F_v(a_1, ..., a_s; q) = \widetilde{F}_v(m\big|_p; q)$. The first case in which the upper bound in Theorem 4.5 is not reached is $m = 7, p = 5, q = 6$, since $\widetilde{F}_v(7\big|_5; 6) = 17$ (see Theorem 7.4) and the corresponding numbers $F_v(a_1, ..., a_s; q) \leq 16$.*

5 Some necessary results for the numbers $F_v(2_r, p, r+p-1),\ p \geq 2$

In this section we prove that the computation of the lower bound in Theorem 4.5 in the case $q = m-1$, i.e. computation of the numbers $F_v(2_r, p, r+p-1)$ where p is fixed, is reduced to the computation of a finite number of these numbers (Theorem 5.2).
It is easy to prove that

$$G \overset{v}{\rightarrow} (a_1, ..., a_s) \Rightarrow K_1 + G \overset{v}{\rightarrow} (2, a_1, ..., a_s).$$

Therefore, it is true that

$$G \overset{v}{\rightarrow} (a_1, ..., a_s) \Rightarrow K_t + G \overset{v}{\rightarrow} (2_t, a_1, ..., a_s). \tag{5.1}$$

Lemma 5.1. *Let $2 \leq s \leq r$. Then*

$$F_v(2_r, p; r+p-1) \leq F_v(2_s, p; s+p-1) + r - s.$$

Proof. Let G be an extremal graph in $\mathcal{H}(2_s, p; s+p-1)$. Consider $G_1 = K_{r-s} + G$. According to (5.1), $G_1 \overset{v}{\rightarrow} (2_r, p)$. Since $\omega(G_1) = r - s + \omega(G) < r + p - 1$, it follows that $G_1 \in \mathcal{H}(2_r, p; r+p-1)$. Therefore,
$F_v(2_r, p; r+p-1) \leq |\mathrm{V}(G_1)| = F_v(2_s, p; s+p-1) + r - s.$ □

Theorem 5.2. *Let $r_0(p) = r_0$ be the smallest positive integer for which*

$$\min_{r \geq 2} \{F_v(2_r, p; r+p-1) - r\} = F_v(2_{r_0}, p; r_0 + p - 1) - r_0.$$

Then:

(a) $F_v(2_r, p; r+p-1) = F(2_{r_0}, p; r_0 + p - 1) + r - r_0, \quad r \geq r_0.$

(b) *if $r_0 = 2$, then*

$F_v(2_r, p; r+p-1) = F_v(2, 2, p; p+1) + r - 2, \quad r \geq 2.$

(c) *if $r_0 > 2$ and G is an extremal graph in $\mathcal{H}(2_{r_0}, p; r_0 + p - 1)$, then*

$G \overset{v}{\rightarrow} (2, r_0 + p - 2).$

(d) $r_0 < F_v(2, 2, p; p+1) - 2p.$

In particular, for $p = 5$ we have $r_0(5) \leq 5$.

Proof. (a) According to the definition of $r_0 = r_0(p)$, if $r \geq 2$, then

$F_v(2_r, p; r+p-1) - r \geq F_v(2_{r_0}, p; r_0+p-1) - r_0,$

i.e.

$F_v(2_r, p; r+p-1) \geq F_v(2_{r_0}, p; r_0+p-1) + r - r_0.$

If $r \geq r_0$, according to Lemma 5.1, the opposite inequality is also true.

(b) This equality is the special case $r_0 = 2$ of the equality (a).

(c) Suppose the opposite is true and let G be an extremal graph in $\mathcal{H}(2_{r_0}, p; r_0+p-1)$ and $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, where V_1 is an independent set and V_2 does not contain an (r_0+p-2)-clique. We can suppose that $V_1 \neq \emptyset$. Let $G_1 = \mathrm{G}[V_2]$. Then $\omega(G_1) < r+p-2$, and from $G \overset{v}{\rightarrow} (2_{r_0}, p)$ it follows $G_1 \overset{v}{\rightarrow} (2_{r_0-1}, p)$. Therefore, $G_1 \in \mathcal{H}(2_{r_0-1}, p; r_0+p-2)$ and

$|\mathrm{V}(G_1)| \geq F_v(2_{r_0-1}, p; r_0+p-2).$

Since $|\mathrm{V}(G)| = F_v(2_{r_0}, p; r_0+p-1)$ and $|\mathrm{V}(G_1)| \leq |\mathrm{V}(G)| - 1$, we obtain

$F_v(2_{r_0-1}, p; r_0+p-2) - (r_0-1) \leq F_v(2_{r_0}, p; r_0+p-1) - r_0,$

which contradicts the definition of $r_0 = r_0(p)$.

(d) According to (1.6), $F_v(2,2,p;p+1) \geq 2p+4$. Therefore, if $r_0 = 2$, the inequality (d) is obvious.

Let $r_0 \geq 3$ and G be an extremal graph in $\mathcal{H}(2_{r_0}, p; r_0+p-1)$. According to (c) and Theorem 1.1, $|\mathrm{V}(G)| \geq 2r_0+2p-3$. Let us notice that $\chi(\overline{C}_{2r_0+2p-3}) = r_0+p-1$ and $\chi(G) \geq r_0+p = m$ by (4.4). Therefore, $G \neq \overline{C}_{2r_0+2p-3}$ and from Theorem 1.1 we obtain

$|\mathrm{V}(G)| = F_v(2_{r_0}, p; r_0+p-1) \geq 2r_0+2p-2.$

Since $r_0 \geq 3$, from the definition of r_0 we have

$F_v(2_{r_0}, p; r_0+p-1) < F_v(2,2,p;p+1) + r_0 - 2.$

Thus, we proved that

$2r_0+2p-2 < F_v(2,2,p;p+1) + r_0 - 2$, i.e.

$r_0 < F_v(2,2,p;p+1) - 2p.$

□

Remark. *Since we suppose that $r \geq 2$, according to (1.1) all Folkman numbers in the proof of Theorem 5.2 exist.*

Example 5.3. *From (1.5) and $F_v(2,2,2;3) = F_v(2,2,2,2;4) = 11$ it follows $r_0(2) = 4$, and from (1.5) and $F_v(2,2,3;4) = 14$ it follows $r_0(3) = 3$. Also, from (1.5) we see that $r_0(4) = 2$.*

We suppose that the following is true:

Conjecture 5.4. *If $p \geq 4$, then*

$$\min_{r \geq 2} \{F_v(2_r, p; r+p-1) - r\} = F_v(2,2,p;p+1) - 2,$$

and therefore

$$F_v(2_r, p; r+p-1) = F_v(2,2,p;p+1) + r - 2, \quad r \geq 2.$$

In this paper we prove this conjecture in the case $p = 5$ (see Theorem 6.1 and Corollary 6.3).

Corollary 5.5. *Let $a_1, ..., a_s$ be positive integers, let m and p be defined by (1.3), $m \geq p+2$ and $r = m - p \geq r_0(p)$. Then*

$$F_v(a_1, ...a_s; m-1) \geq F_v(2_{r_0}, p; r_0 + p - 1) + r - r_0.$$

In particular, if $r_0 = 2$, then

$$F_v(a_1, ..., a_s; m-1) \geq F_v(2,2,p;p+1) + r - 2.$$

Proof. According to Theorem 4.5,
$F_v(a_1, ..., a_s; m-1) \geq F_v(2_r, p; r+p-1)$.
From this inequality and Theorem 5.2(a) we obtain the desired inequality. □

6 Computation of $r_0(5)$

In this section we prove the following

Theorem 6.1. $r_0(5) = 2$

Proof. From Theorem 5.2(d) we have $r_0(5) \leq 5$. Therefore, we have to prove that $r_0(5) \neq 3$, $r_0(5) \neq 4$ and $r_0(5) \neq 5$, i.e. we have to prove the inequalities $F_v(2,2,2,5;7) > 16$, $F_v(2,2,2,2,5;8) > 17$, $F_v(2,2,2,2,2,5;9) > 18$.

The proof of each of these three inequalities consists of several steps, similarly to the proof of Theorem 1.4. Since not all graphs in $\mathcal{R}(3,7)$ are known to us, this time in the process of extending graphs to maximal ones we are adding two independent vertices instead of three.

Algorithm 6.2. *Finding all maximal graphs in $\mathcal{H}(2_r, 5; q; n)$ starting from all maximal graphs in $\mathcal{H}(2_{r-1}, 5; q; n-2)$.*

1. By removing edges from the maximal graphs in $\mathcal{H}(2_{r-1}, 5; q; n-2)$, find all graphs in this set which have the property that the addition of a new edge forms a new $(q-1)$-clique.

2. Using Algorithm 2.2, add two independent vertices to the graphs from step 1. to obtain all maximal graphs in $\mathcal{H}(2_r, 5; q; n)$.

6.1 Proof of $F_v(2,2,2,5;7) > 16$

By checking all 10-vertex graphs, we find the maximal graphs in $\mathcal{H}(5;7;10)$. Starting from them, by successively applying Algorithm 6.2($n = 12, 14, 16$; $q = 7; r = 1, 2, 3$) we obtain the maximal graphs in the sets $\mathcal{H}(2,5;7;12)$, $\mathcal{H}(2,2,5;7;14)$ and $\mathcal{H}(2,2,2,5;7;16)$. The results are described in Table 4. There we can see that $\mathcal{H}(2,2,2,5;7;16) = \emptyset$ and therefore $F_v(2,2,2,5;7) > 16$.

6.2 Proof of $F_v(2,2,2,2,5;8) > 17$

By checking all 9-vertex graphs, we find the maximal graphs in $\mathcal{H}(5;8;9)$. Starting from them, by successively applying Algorithm 6.2($n = 11, 13, 15, 17; q = 8; r = 1, 2, 3, 4$) we obtain the maximal graphs in the sets $\mathcal{H}(2,5;8;11)$, $\mathcal{H}(2,2,5;8;13)$, $\mathcal{H}(2,2,2,5;8;15)$ and $\mathcal{H}(2,2,2,2,5;8;17)$. The results are described in Table 5. There we can see that $\mathcal{H}(2,2,2,2,5;8;17) = \emptyset$ and therefore $F_v(2,2,2,2,5;8) > 17$.

6.3 Proof of $F_v(2,2,2,2,2,5;9) > 18$

By checking all 10-vertex graphs, we find the maximal graphs in $\mathcal{H}(2,5;9;10)$. Starting from them, by successively applying Algorithm 6.2($n = 12, 14, 16, 18; q = 9; r = 2, 3, 4, 5$) we obtain the maximal graphs in the sets $\mathcal{H}(2,2,5;9;12)$, $\mathcal{H}(2,2,2,5;9;14)$, $\mathcal{H}(2,2,2,2,5;9;16)$ and $\mathcal{H}(2,2,2,2,2,5;9;18)$. The results are described in Table 6. There we can see that $\mathcal{H}(2,2,2,2,2,5;9;18) = \emptyset$ and therefore $F_v(2,2,2,2,2,5;9) > 18$.

Thus, the proof of Theorem 6.1 is finished. $\square$

All computations were done on a personal computer. The slowest part was the proof of $F_v(2,2,2,2,2,5;9) > 18$, which took several days to complete.

From Theorem 6.1 and Theorem 5.2(b) we obtain

Corollary 6.3. $F_v(2_r, 5; r+4) = r + 14, \quad r \geq 2.$

set	maximal graphs	graphs in which the addition of an edge forms a new 6-clique
$\mathcal{H}(5;7;10)$	8	324
$\mathcal{H}(2,5;7;12)$	56	104 283
$\mathcal{H}(2,2,5;7;14)$	420	2 614 547
$\mathcal{H}(2,2,2,5;7;16)$	0	0

Table 4: Steps in the proof of $F_v(2,2,2,5;7) > 16$

set	maximal graphs	graphs in which the addition of an edge forms a new 7-clique
$\mathcal{H}(5;8;9)$	2	13
$\mathcal{H}(2,5;8;11)$	8	326
$\mathcal{H}(2,2,5;8;13)$	56	105 138
$\mathcal{H}(2,2,2,5;8;15)$	423	2 616 723
$\mathcal{H}(2,2,2,2,5;8;17)$	0	0

Table 5: Steps in the proof of $F_v(2,2,2,2,5;8) > 17$

set	maximal graphs	graphs in which the addition of an edge forms a new 8-clique
$\mathcal{H}(2,5;9;10)$	2	13
$\mathcal{H}(2,2,5;9;12)$	8	327
$\mathcal{H}(2,2,2,5;9;14)$	56	105 294
$\mathcal{H}(2,2,2,2,5;9;16)$	423	2 616 741
$\mathcal{H}(2,2,2,2,2,5;9;18)$	0	0

Table 6: Steps in the proof of $F_v(2,2,2,2,2,5;9) > 18$

7 Computation of the numbers $\widetilde{F}_v(m\big|_5; m-1)$ and proof of Theorem 1.8

Let us remind that $\widetilde{\mathcal{H}}(m\big|_p; q)$ and $\widetilde{F}_v(m\big|_p; q)$ are defined in Section 4.

We need the following

Lemma 7.1. *[23] Let m_0 and p be positive integers and $G \overset{v}{\to} m_0\big|_p$. Then for every positive integer $m \geq m_0$ it is true that $K_{m-m_0} + G \overset{v}{\to} m\big|_p$.*

This lemma is formulated in an obviously equivalent way and is proved by induction with respect to $m \geq m_0$ in [23] as Lemma 3.

Theorem 7.2. *Let m, m_0, p and q be positive integers, $m \geq m_0$ and $q > \min\{m_0, p\}$. Then*

$$\widetilde{F}_v(m\big|_p; m - m_0 + q) \leq \widetilde{F}_v(m_0\big|_p; q) + m - m_0.$$

Proof. Let $G_0 \in \widetilde{\mathcal{H}}(m_0\big|_p; q)$, $|V(G_0)| = \widetilde{F}_v(m_0\big|_p; q)$ and $G = K_{m-m_0} + G_0$. According to Lemma 7.1, $G \xrightarrow{v} m\big|_p$. Since $\omega(G) = m - m_0 + \omega(G_0) < m - m_0 + q$, it follows that $G \in \widetilde{\mathcal{H}}(m\big|_p; m - m_0 + q)$. Therefore, $\widetilde{F}_v(m\big|_p; m - m_0 + q) \leq |\mathrm{V}(G)| = \widetilde{F}_v(m_0\big|_p; q) + m - m_0$. □

The following obvious proposition will be used in the proof of Theorem 7.4:

Proposition 7.3. *Let $a_1, ..., a_s$ be positive integers, $a_i \geq k$ and $G \xrightarrow{v} (a_1, ..., a_s)$. Then*

$$G \xrightarrow{v} (a_1, ..., a_{i-1}, k, a_i - k + 1, a_{i+1}, ..., a_s).$$

According to proposition 4.4, we have

$$\widetilde{F}_v(m\big|_5; m - 1) \text{ exists } \Leftrightarrow m \geq 7. \tag{7.1}$$

We prove the following

Theorem 7.4. *The following equalities are true:*

$$\widetilde{F}_v(m\big|_5; m-1) = \begin{cases} 17, & \text{if } m = 7 \\ m + 9, & \text{if } m \geq 8. \end{cases}$$

Proof. Case 1. $m = 7$. According to Theorem 4.5 and Theorem 1.5, $\widetilde{F}_v(7\big|_5; 6) \geq F_v(2, 2, 5; 6) = 16$. With the help of the computer we check that none of the 4 graphs in $\mathcal{H}(3, 5; 6; 16)$ (see Figure 2) belongs to $\mathcal{H}(4, 4; 6; 16)$. Therefore, $\widetilde{F}_v(7\big|_5; 6) \geq 17$.

By adding one vertex to the graphs from $\mathcal{H}(2, 2, 5; 6; 16)$, and then removing edges from the obtained 17 vertex graphs, we find 353 graphs which belong to both $\mathcal{H}(3, 5; 6; 17)$ and $\mathcal{H}(4, 4; 6; 17)$. The graph Γ_1, given on Figure 3, is one of these graphs (it is the only one with independence number 4). We will prove that $\Gamma_1 \in \widetilde{\mathcal{H}}(7\big|_5; 6)$. Since $\omega(\Gamma_1) = 5$, it remains to be proved that if $2 \leq b_1 \leq ... \leq b_s \leq 5$ are positive integers, such that $\sum_{i=1}^{s}(b_1 - 1) + 1 = 7$, then $\Gamma_1 \xrightarrow{v} (b_1, ..., b_s)$. The following cases are possible:

$s = 2, b_1 = 3, b_2 = 5.$
$s = 2, b_1 = b_2 = 4.$
$s = 3, b_1 = b_2 = 2, b_3 = 5.$
$s = 3, b_1 = 2, b_2 = 3, b_3 = 4.$
$s = 3, b_1 = b_2 = b_3 = 3.$
$s = 4, b_1 = b_2 = b_3 = 2, b_4 = 4.$
$s = 4, b_1 = b_2 = 2, b_3 = b_4 = 3.$
$s = 5, b_1 = b_2 = b_3 = b_4 = 2, b_5 = 3.$
$s = 6, b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = 2.$

By construction, $\Gamma_1 \overset{v}{\to} (3,5)$ and $\Gamma_1 \overset{v}{\to} (4,4)$. From Proposition 7.3 and $\Gamma_1 \overset{v}{\to} (3,5)$ it follows $\Gamma_1 \overset{v}{\to} (2,2,5)$, $\Gamma_1 \overset{v}{\to} (2,3,4)$ and $\Gamma_1 \overset{v}{\to} (3,3,3)$. Consequently, we have

$\Gamma_1 \overset{v}{\to} (3,3,3) \Rightarrow \Gamma_1 \overset{v}{\to} (2,2,3,3),$
$\Gamma_1 \overset{v}{\to} (2,2,5) \Rightarrow \Gamma_1 \overset{v}{\to} (2,2,2,4),$
$\Gamma_1 \overset{v}{\to} (2,2,2,4) \Rightarrow \Gamma_1 \overset{v}{\to} (2,2,2,2,3),$
$\Gamma_1 \overset{v}{\to} (2,2,2,2,3) \Rightarrow \Gamma_1 \overset{v}{\to} (2,2,2,2,2,2).$

We proved that $\Gamma_1 \in \widetilde{\mathcal{H}}(7\big|_5; 6)$. Therefore, $\widetilde{F}_v(7\big|_5; 6) \leq |\mathrm{V}(\Gamma_1)| = 17$.

Case 2. $m = 8$. According to Theorem 4.5 and Corollary 6.3, $\widetilde{F}_v(8\big|_5; 7) \geq F_v(2,2,2,5;7) = 17$. To prove the upper bound, consider the 17-vertex graph $\Gamma_2 \in \mathcal{H}(4,5;7;17)$, which is shown on Figure 4. Appendix A describes the method to obtain this graph. By construction, $\omega(\Gamma_2) = 6$ and $\Gamma_2 \overset{v}{\to} (4,5)$. As in Case 1., we prove that from $\Gamma_2 \overset{v}{\to} (4,5)$ it follows $\Gamma_2 \overset{v}{\to} 8\big|_5$. Therefore, $\Gamma_2 \in \widetilde{\mathcal{H}}(8\big|_5; 7)$ and $\widetilde{F}_v(8\big|_5; 7) \leq |\mathrm{V}(\Gamma_2)| = 17$.

Case 3. $m > 8$. From Theorem 4.5 and Corollary 6.3 it follows $\widetilde{F}_v(m\big|_5; m-1) \geq m + 9$. From Theorem 7.2($m_0 = 8, p = 5, q = 7$) and $\widetilde{F}_v(8\big|_5; 7) = 17$ it follows $\widetilde{F}_v(m\big|_5; m-1) \leq m + 9$. □

Proof of Theorem 1.8

Since $m \geq 7$, only the following two cases are possible:

Case 1. $m = 7$. In this case $F_v(2,2,5;6)$ and $F_v(3,5;6)$ are the only canonical vertex Folkman numbers of the form $F_v(a_1, ..., a_s; m-1)$. The equality $F_v(2,2,5;6) = 16$ is proved in this work as Theorem 1.5, and the equality $F_v(3,5;6) = 16$ is proved in [26] (see also Corollary 1.6).

Case 2. $m \geq 8$. In this case Theorem 1.8 follows easily from Theorem 7.4, Theorem 4.5($q = m - 1$) and Corollary 6.3.

8 Proof of Theorem 1.9

According to Corollary 6.3, $F_v(2,2,2,5;7) = 17$. From Proposition 7.3 it follows $F_v(2,2,6;7) \geq F_v(2,2,2,5;7)$. Therefore, $F_v(2,2,6;7) \geq 17$. Now, from Theorem 1.2 (p = 6) we obtain the lower bound

$F_v(a_1,...,a_s;m-1) \geq m+9, \quad m \geq 8.$

To prove the upper bound consider the 18 vertex graph Γ_3 (Figure 5) with the help of which in [28] they prove the inequality $F_v(3,6;7) \leq 18$. In addition to the property $\Gamma_3 \overset{v}{\rightarrow} (3,6)$, the graph Γ_3 also has the property $\Gamma_3 \overset{v}{\rightarrow} (4,5)$. By repeating the arguments in the proof of Theorem 7.4 Case 1, we see that from $\Gamma_3 \overset{v}{\rightarrow} (3,6)$ and $\Gamma_3 \overset{v}{\rightarrow} (4,5)$ it follows $\Gamma_3 \overset{v}{\rightarrow} 8\big|_6$. Since $\omega(\Gamma_3) = 6$, we obtain $\Gamma_3 \in \widetilde{\mathcal{H}}(8\big|_6;7)$ and $\widetilde{F}_v(8\big|_6;7) \leq |\mathrm{V}(\Gamma_3)| = 18$. From this inequality and Theorem 7.2 ($m_0 = 8, p = 6; q = 7$) it follows $\widetilde{F}_v(m\big|_6;m-1) \leq m+10,\ m \geq 8$. At last, according to Theorem 4.5

$F_v(a_1,...,a_s;m-1) \leq \widetilde{F}_v(m\big|_6;m-1) \leq m+10, \quad m \geq 8,$

which finishes the proof of Theorem 1.9

9 Concluding remarks

In this paper we presented a method to compute and bound the Folkman numbers of the form $F_v(a_1,...,a_s;m-1)$ with the help of the border numbers $F_v(2,2,p;p+1)$ and $F_v(3,p;p+1)$. With this method new results for other Folkman numbers can be obtained. For example, we will show how, in some special situations, one can strengthen the inequality:

(9.1) $$F_v(p,p;p+1) \geq 4p-1, \text{ [31]}.$$

Let $G \in \mathcal{H}(2_r,p;p+1)$ and $A \subseteq \mathrm{V}(G)$ be an independent set. Then, obviously, $G - A \in \mathcal{H}(2_{r-1},p;p+1)$, and therefore

(9.2) $$F_v(2_r,p;p+1) \geq F_v(2_{r-1},p;p+1) + \alpha(r,p),\ r \geq 2,$$

where $\alpha(r,p) = \max\{\alpha(G) : G \in \mathcal{H}_{extr}(2_r,p;p+1)\}$.
From (9.2) it follows easily

(9.3) $$F_v(2_r,p;p+1) \geq F_v(2,2,p;p+1) + \sum_{i=3}^{r} \alpha(i,p),\ r \geq 3.$$

Since $\alpha(i,p) \geq 2$, from (9.3) we obtain

(9.4) $$F_v(2_r,p;p+1) \geq F_v(2,2,p;p+1) + 2(r-2),\ r \geq 3.$$

From (9.4) and Theorem 4.5 we see that

(9.5) $F_v(p,p;p+1) \geq F_v(2_{p-1},p;p+1) \geq F_v(2,2,p;p+1)+2p-6,\ p \geq 3.$

According to (1.6), $F_v(2,2,p;p+1) \geq 2p+4$. If $F_v(2,2,p;p+1) = 2p+4$, then the inequality (9.1) gives a better bound for $F_v(p,p;p+1)$ than the inequality (9.5). It is interesting to note that it is not known whether the equality $F_v(2,2,p;p+1) = 2p+4$ holds for any p. If $F_v(2,2,p;p+1) = 2p+5$, then the bounds for $F_v(p,p;p+1)$ from (9.1) and (9.5) coincide, and if $F_v(2,2,p;p+1) > 2p+5$, then the inequality (9.5) gives a better bound for $F_v(p,p;p+1)$.

In the case $p = 5$, by using the graphs from Theorem 1.4, one can obtain an even better bound for $F_v(5,5;6)$. From $F_v(2,2,5;6) = 16$ and (9.4) it follows that $F_v(2_r,5;6) \geq 18,\ r \geq 3$. Since the Ramsey number $R(3,6) = 18$, we have $\alpha(r,5) \geq 3,\ r \geq 3$. Now, from (9.3) we obtain

$$F_v(2_r,5;6) \geq F_v(2,2,5;6) + 3(r-2) = 10 + 3r.$$

From this inequality we see that $F_v(2,2,2,5;6) \geq 19$. By adding 3 independent vertices to the graphs from $\mathcal{H}(2,2,5;6;16)$ with the help of Algorithm 2.2, we find all maximal graphs in $\mathcal{H}(2,2,2,5;6;19)$ with independence number at least 3, and since $\alpha(3,5) \geq 3$, that would be all maximal graphs in $\mathcal{H}(2,2,2,5;6;19)$. We obtain $\mathcal{H}(2,2,2,5;6;19) = \emptyset$, and therefore $F_v(2,2,2,5;6) \geq 20$. Since $\alpha(4,5) \geq 3$, from (9.2) ($r = 4, p = 5$) and Theorem 4.5 it follows that

$$F_v(5,5;6) \geq F_v(2,2,2,2,5;6) \geq 23.$$

Obviously, $F_v(2,2;3) = 5$. As mentioned in the introduction of this paper, $F_v(3,3;4) = 14$, and in [30] it is proved that $17 \leq F_v(4,4;5) \leq 23$. For now, there is no good upper bound for $F_v(5,5;6)$. We suppose that by successively extending graphs from $\mathcal{H}(2,2,5;16)$, one can obtain a good bound for $F_v(5,5;6)$.

Appendix A Obtaining the graph $\Gamma_2 \in \mathcal{H}(4, 5; 7; 17)$

Consider the 18-vertex graph Γ_3 (Figure 5). As mentioned, this is the graph with the help of which in [28] they prove the inequality $F_v(3, 6; 7) \leq 18$. With the help of the computer we check that Γ_3 is maximal in $\mathcal{H}(4, 5; 7; 18)$. We use the following procedure to obtain other maximal graphs in $\mathcal{H}(4, 5; 7; 18)$:

Procedure A.1. *Extending a set of maximal graphs in $\mathcal{H}(a_1, ..., a_s; q; n)$.*

1. Let $\mathcal{A}$ be a set of maximal graphs in $\mathcal{H}(a_1, ..., a_s; q; n)$.

2. By removing edges from the graphs in $\mathcal{A}$, find all their subgraphs which are in $\mathcal{H}(a_1, ..., a_s; q; n)$. This way a set of non-maximal graphs in $\mathcal{H}(a_1, ..., a_s; q; n)$ is obtained.

3. Add edges to the non-maximal graphs to find all their supergraphs which are maximal in $\mathcal{H}(a_1, ..., a_s; q; n)$. Extend the set $\mathcal{A}$ by adding the new maximal graphs.

Starting from a set containing a single element the graph Γ_3 and executing Procedure A.1, we find 12 new maximal graphs $\mathcal{H}(4, 5; 7; 18)$. Again, we execute Procedure A.1 on the new set to find 110 more maximal graphs in $\mathcal{H}(4, 5; 7; 18)$. By removing one vertex from these graphs, we obtain 17-vertex graphs, one of which is $\Gamma_2 \in \mathcal{H}(4, 5; 7; 17)$ shown on Figure 4.

Appendix B Graphs

```
0 0 1 0 0 0 1 0 1 0 1 1 1 1 1 1
0 0 0 1 0 1 0 1 0 1 0 1 1 1 1 1
1 0 0 0 1 0 1 1 1 1 1 0 0 1 1 1
0 1 0 0 1 1 1 1 1 1 1 0 1 0 0 1
0 0 1 1 0 1 1 1 1 1 1 1 0 1 1 0
0 1 0 1 1 0 0 1 1 1 1 1 1 1 1 0
1 0 1 1 1 0 0 1 1 0 1 1 1 1 0 1
0 1 1 1 1 1 1 0 0 1 1 1 0 1 0 1
1 0 1 1 1 1 1 0 0 1 1 0 1 0 1 1
0 1 1 1 1 1 0 1 1 0 0 0 1 1 1 1
1 0 1 1 1 1 1 1 1 0 0 1 1 0 1 0
1 1 0 0 1 1 1 1 0 0 1 0 1 1 1 1
1 1 0 1 0 1 1 0 1 1 1 1 0 0 1 1
1 1 1 0 1 1 1 1 0 1 0 1 0 0 1 1
1 1 1 0 1 1 0 0 1 1 1 1 1 1 0 1
1 1 1 1 0 0 1 1 1 1 0 1 1 1 1 0
```

G_{74}

```
0 0 1 0 0 0 1 1 0 1 1 1 1 1 1 1
0 0 0 1 0 1 0 0 1 1 1 1 1 1 1 1
1 0 0 1 1 1 0 1 1 1 0 1 0 1 1 0
0 1 1 0 1 0 1 1 1 0 1 0 1 1 0 1
0 0 1 1 0 1 1 1 1 1 1 0 0 1 1 1
0 1 1 0 1 0 1 1 1 1 0 1 1 0 1 1
1 0 0 1 1 1 0 1 1 0 1 1 1 0 1 1
1 0 1 1 1 1 1 0 0 1 0 1 1 1 0 1
0 1 1 1 1 1 1 0 0 0 1 1 1 1 1 0
1 1 1 0 1 1 0 1 0 0 0 1 1 1 1 1
1 1 0 1 1 0 1 0 1 0 0 1 1 1 1 1
1 1 1 0 0 1 1 1 1 1 1 0 1 0 1 0
1 1 0 1 0 1 1 1 1 1 1 1 0 0 0 1
1 1 1 1 1 0 0 1 1 1 1 0 0 0 1 1
1 1 1 0 1 1 1 0 1 1 1 1 0 1 0 1
1 1 0 1 1 1 1 1 0 1 1 0 1 1 1 0
```

G_{78}

```
0 1 1 1 1 0 0 0 1 1 0 0 0 1 1 1
1 0 0 1 0 0 1 0 1 1 1 1 1 0 1 1
1 0 0 0 1 1 0 0 1 1 1 1 1 1 0 1
1 1 0 0 1 1 0 1 1 0 1 0 0 1 1 1
1 0 1 1 0 0 1 1 0 1 0 1 0 1 1 1
0 0 1 1 0 0 0 1 1 0 1 1 1 1 1 1
0 1 0 0 1 0 0 1 0 1 1 1 1 1 1 1
0 0 0 1 1 1 1 0 0 0 1 1 1 1 1 1
1 1 1 1 0 1 0 0 0 1 1 0 1 1 1 1
1 1 1 0 1 0 1 0 1 0 0 1 1 1 1 1
0 1 1 1 0 1 1 1 1 0 0 1 1 0 1 1
0 1 1 0 1 1 1 1 0 1 1 0 1 1 0 1
0 1 1 0 0 1 1 1 1 1 1 1 0 1 1 0
1 0 1 1 1 1 1 1 1 1 0 1 1 0 1 0
1 1 0 1 1 1 1 1 1 1 1 0 1 1 0 0
1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0
```

G_{134}

```
0 1 1 1 0 0 0 0 1 1 1 0 0 1 1 1
1 0 0 1 0 1 1 0 1 1 1 0 1 0 1 1
1 0 0 1 1 0 1 1 0 1 0 1 1 1 0 1
1 1 1 0 0 0 0 1 0 0 1 1 1 1 1 1
0 0 1 0 0 1 1 1 1 1 1 1 0 1 0 1
0 1 0 0 1 0 1 1 1 0 1 1 1 1 1 0
0 1 1 0 1 1 0 0 1 1 0 1 1 0 1 1
0 0 1 1 1 1 0 0 0 1 1 1 1 1 1 0
1 1 0 0 1 1 1 0 0 1 1 1 0 1 1 1
1 1 1 0 1 0 1 1 1 0 1 0 1 1 0 1
1 1 0 1 1 1 0 1 1 1 0 0 1 1 1 0
0 0 1 1 1 1 1 1 1 0 0 0 1 1 1 1
0 1 1 1 0 1 1 1 0 1 1 1 0 0 1 1
1 0 1 1 1 1 0 1 1 1 1 1 0 0 1 1
1 1 0 1 0 1 1 1 1 0 1 1 1 1 0 1
1 1 1 1 1 0 1 0 1 1 0 1 1 1 1 0
```

G_{135}

Figure 1: All 4 bicritical graphs in $\mathcal{H}(2, 2, 5; 6; 16)$

```
0 0 0 1 1 0 1 1 1 0 1 0 1 1 1 1
0 0 0 1 0 1 1 1 0 1 0 1 1 1 1 1
0 0 0 0 1 1 0 0 1 1 1 1 1 1 1 1
1 1 0 0 1 1 1 1 1 1 0 0 1 1 0 1
1 0 1 1 0 0 0 1 1 1 1 1 1 0 1 1
0 1 1 1 0 0 1 0 1 1 1 1 0 1 1 1
1 1 0 1 0 1 0 1 1 0 1 1 0 1 1 1
1 1 0 1 1 0 1 0 0 1 1 1 1 0 1 1
1 0 1 1 1 1 1 0 0 1 1 0 1 1 0 1
0 1 1 1 1 1 0 1 1 0 0 1 1 1 0 1
1 0 1 0 1 1 1 1 1 0 0 1 0 1 1 1
0 1 1 0 1 1 1 1 0 1 1 0 1 0 1 1
1 1 1 1 1 0 0 1 1 1 0 1 0 1 1 0
1 1 1 1 0 1 1 0 1 1 1 0 1 0 1 0
1 1 1 0 1 1 1 1 0 0 1 1 1 1 0 0
1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0
```

G_{50}

```
0 0 0 1 1 1 1 1 1 1 1 1 0 0 0 0
0 0 0 1 1 0 1 0 0 1 1 1 1 1 1 1
0 0 0 0 0 1 0 1 1 1 1 1 1 1 1 1
1 1 0 0 1 1 1 1 0 1 1 0 1 1 1 0
1 1 0 1 0 0 1 1 1 0 1 1 1 0 1 1
1 0 1 1 0 0 1 1 1 1 1 0 1 1 0 1
1 1 0 1 1 1 0 0 1 1 0 1 1 1 0 1
1 0 1 1 1 1 0 0 1 0 1 1 1 1 1 0
1 0 1 0 1 1 1 1 0 1 0 1 1 0 1 1
1 1 1 1 0 1 1 0 1 0 1 1 0 1 0 1
1 1 1 1 1 1 0 1 0 1 0 1 0 1 1 0
1 1 1 0 1 0 1 1 1 1 1 0 0 0 1 1
0 1 1 1 1 1 1 1 1 0 0 0 0 1 1 1
0 1 1 1 0 1 1 1 0 1 1 0 1 0 1 1
0 1 1 1 1 0 0 1 1 0 1 1 1 1 0 1
0 1 1 0 1 1 1 0 1 1 0 1 1 1 1 0
```

G_{51}

```
0 0 1 0 1 1 1 0 0 0 1 1 1 1 1 1
0 0 0 1 0 0 0 1 1 1 1 1 1 1 1 1
1 0 0 1 1 1 1 1 1 1 0 0 0 1 1 1
0 1 1 0 1 1 1 1 1 1 1 1 1 0 0 0
1 0 1 1 0 1 1 1 1 0 1 1 0 1 1 0
1 0 1 1 1 0 1 0 1 1 0 1 1 0 1 1
1 0 1 1 1 1 0 1 0 1 1 0 1 1 0 1
0 1 1 1 1 0 1 0 1 1 1 0 1 1 0 1
0 1 1 1 1 1 0 1 0 1 1 1 0 1 1 0
0 1 1 1 0 1 1 1 1 0 0 1 1 0 1 1
1 1 0 1 1 0 1 1 1 0 0 1 1 1 0 1
1 1 0 1 1 1 0 0 1 1 1 0 1 1 1 0
1 1 0 1 0 1 1 1 0 1 1 1 0 0 1 1
1 1 1 0 1 0 1 1 1 0 1 1 0 0 1 1
1 1 1 0 1 1 0 0 1 1 0 1 1 1 0 1
1 1 1 0 0 1 1 1 0 1 1 0 1 1 1 0
```

G_{81}

```
0 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0
1 0 1 1 1 0 1 1 0 1 1 1 0 1 0 1
1 1 0 1 1 1 1 0 1 0 1 1 0 1 1 0
1 1 1 0 0 1 0 1 1 1 1 1 0 0 1 1
1 1 1 0 0 1 1 1 1 0 1 0 1 1 1 0
1 0 1 1 1 0 0 1 1 1 1 0 1 0 1 1
1 1 1 0 1 0 0 1 1 1 0 1 1 1 0 1
1 1 0 1 1 1 1 0 0 1 1 0 1 1 0 1
1 0 1 1 1 1 1 0 0 1 0 1 1 1 1 0
1 1 0 1 0 1 1 1 1 0 0 1 1 0 1 1
1 1 1 1 1 1 0 1 0 0 0 0 1 1 1 1
1 1 1 1 0 0 1 0 1 1 0 0 1 1 1 1
0 0 0 0 1 1 1 1 1 1 1 1 0 1 1 1
0 1 1 0 1 0 1 1 1 0 1 1 1 0 1 1
0 0 1 1 1 1 0 0 1 1 1 1 1 1 0 1
0 1 0 1 0 1 1 1 0 1 1 1 1 1 1 0
```

G_{146}

Figure 2: All 4 graphs in $\mathcal{H}(3,5;6;16)$

```
0 0 0 0 1 0 0 0 1 1 1 1 1 1 1 0 1
0 0 0 0 1 1 1 1 1 1 1 0 1 0 0 1 1
0 0 0 0 0 1 1 1 1 1 0 1 0 1 1 1 1
0 0 0 0 1 1 1 1 0 0 1 1 1 1 1 1 0
1 1 0 1 0 1 0 1 0 0 1 1 1 1 1 0 1
0 1 1 1 1 0 1 1 0 0 1 1 0 1 0 1 1
0 1 1 1 0 1 0 0 1 0 1 1 0 1 1 1 1
0 1 1 1 1 1 0 0 0 1 0 1 1 1 0 1 1
1 1 1 0 0 0 1 0 0 1 1 0 1 1 1 1 1
1 1 1 0 0 0 0 1 1 0 0 1 1 1 1 1 1
1 1 0 1 1 1 1 0 1 0 0 0 1 1 1 1 1
1 0 1 1 1 1 1 1 0 1 0 0 1 0 1 1 1
1 1 0 1 1 0 0 1 1 1 1 1 0 0 1 1 1
1 0 1 1 1 1 1 1 1 1 1 0 0 0 1 0 1
1 0 1 1 1 0 1 0 1 1 1 1 1 1 0 1 0
0 1 1 1 0 1 1 1 1 1 1 1 1 0 1 0 0
1 1 1 0 1 1 1 1 1 1 1 1 1 1 0 0 0
```

Γ_1

Figure 3:
$\Gamma_1 \in \mathcal{H}(3,5;6;17) \cap \mathcal{H}(4,4;6;17)$

```
0 1 1 1 1 1 1 1 1 0 0 1 1 0 0 1 1
1 0 1 1 1 1 0 0 0 1 1 0 1 1 1 1 1
1 1 0 1 1 0 1 0 0 1 1 1 0 1 1 1 1
1 1 1 0 0 1 1 1 0 1 0 1 0 1 1 1 1
1 1 1 0 0 1 1 0 1 0 1 0 1 1 1 1 1
1 1 0 1 1 0 0 1 1 1 0 1 1 0 1 1 1
1 0 1 1 1 0 0 1 1 0 1 1 1 1 0 1 1
1 0 0 1 0 1 1 0 1 1 0 1 1 1 1 1 1
1 0 0 0 1 1 1 1 0 0 1 1 1 1 1 1 1
0 1 1 1 0 1 0 1 0 0 1 1 1 1 1 1 1
0 1 1 0 1 0 1 0 1 1 0 1 1 1 1 1 1
1 0 1 1 0 1 1 1 1 1 1 0 1 1 1 0 1
1 1 0 0 1 1 1 1 1 1 1 1 0 1 1 1 0
0 1 1 1 1 0 1 1 1 1 1 1 1 0 1 0 1
0 1 1 1 1 1 0 1 1 1 1 1 1 1 0 1 0
1 1 1 1 1 1 1 1 1 1 1 0 1 0 1 0 1
1 1 1 1 1 1 1 1 1 1 1 1 0 1 0 1 0
```

Γ_2

Figure 4:
$\Gamma_2 \in \mathcal{H}(4,5;7;17)$

```
0 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0
1 0 0 1 1 0 1 1 1 1 1 1 1 1 1 0 0 1
1 0 0 1 1 1 0 1 1 1 1 1 1 1 0 1 1 0
1 1 1 0 1 1 1 1 1 1 0 1 0 1 1 0 1 0
1 1 1 1 0 1 1 1 1 0 1 0 1 1 0 1 0 1
1 0 1 1 1 0 1 0 1 1 1 0 1 1 0 1 1 1
1 1 0 1 1 1 0 1 0 1 1 1 0 1 1 0 1 1
1 1 1 1 1 0 1 0 1 0 1 1 1 0 1 1 0 1
1 1 1 1 1 1 0 1 0 1 0 1 1 0 1 1 1 0
1 1 1 1 0 1 1 0 1 0 0 1 0 1 1 1 1 1
1 1 1 0 1 1 1 1 0 0 0 0 1 1 1 1 1 1
1 1 1 1 0 0 1 1 1 1 0 0 1 0 1 1 1 1
1 1 1 0 1 1 0 1 1 0 1 1 0 0 1 1 1 1
1 1 1 1 1 1 1 0 0 1 1 0 0 0 1 1 1 1
0 1 0 1 0 0 1 1 1 1 1 1 1 1 0 1 1 1
0 0 1 0 1 1 0 1 1 1 1 1 1 1 1 0 1 1
0 0 1 1 0 1 1 0 1 1 1 1 1 1 1 1 0 1
0 1 0 0 1 1 1 1 0 1 1 1 1 1 1 1 1 0
```

Γ_3

Figure 5:
$\Gamma_3 \in \mathcal{H}(3,6;7;18) \cap \mathcal{H}(4,5;7;18)$

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References

[1] V. Chvátal. The minimality of the mycielski graph. *Lecture Notes in Mathematics*, 406:243–246, 1979.

[2] J. Coles and S. Radziszowski. Computing the Folkman number $F_v(2,2,3;4)$. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 58:13–22, 2006.

[3] F. Deng, M. Liang, Z. Shao, and X. Xu. Upper bounds for the vertex Folkman number $F_v(3,3,3;4)$ and $F_v(3,3,3;5)$. *ARS Combinatoria*, 112:249–256, 2013.

[4] A. Dudek and V. Rödl. New upper bound on vertex Folkman numbers. *Lecture Notes in Computer Science*, 4557:473–478, 2008.

[5] J. Folkman. Graphs with monochromatic complete subgraphs in every edge coloring. *SIAM Journal on Applied Mathematics*, 18:19–24, 1970.

[6] T. Jensen and G. Royle. Small graphs with chromatic number 5: a computer research. *Journal of Graph Theory*, 19:107–116, 1995.

[7] N. Kolev and N. Nenov. New recurrent inequality on a class of vertex Folkman numbers. In *Proceedings of the 35th Spring Conference of the Union of Bulgarian Mathematicians*, pages 164–168, April 2006.

[8] N. Kolev and N. Nenov. New upper bound for a class of vertex Folkman numbers. *The Electronic Journal of Combinatorics*, 13, 2006.

[9] J. Lathrop and S. Radziszowski. Computing the Folkman number $F_v(2,2,2,2,2;4)$. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 78:213–222, 2011.

[10] T. Luczak, A. Ruciński, and S. Urbański. On minimal vertex Folkman graphs. *Discrete Mathematics*, 236:245–262, 2001.

[11] T. Luczak and S. Urbański. A note on restricted vertex Ramsey numbers. *Periodica Mathematica Hungarica*, 33:101–103, 1996.

[12] B. McKay. nauty user's guide (version 2.4). Technical report, Department of Computer Science, Australian National University, 1990. The latest version of the software is available at `http://cs.anu.edu.au/~bdm/nauty/`.

[13] B.D. McKay. `http://cs.anu.edu.au/~bdm/data/{R}amsey.html`.

[14] B.D. McKay. `http://cs.anu.edu.au/~bdm/data/r36_13.g6.gz`.

[15] B.D. McKay. `http://cs.anu.edu.au/~bdm/data/r36_16.g6.gz`.

[16] J. Mycielski. Sur le coloriage des graphes. *Colloquium Mathematicum*, 3:161–162, 1955.

[17] N. Nenov. An example of a 15-vertex (3, 3)-Ramsey graph with clique number 4. *Comptes rendus de l'Academie bulgare des Sciences*, 34(11):1487–1489, 1981. (in Russian).

[18] N. Nenov. On the Zykov numbers and some its applications to Ramsey theory. *Serdica Bulgariacae Mathematicae*, 9:161–167, 1983. (in Russian).

[19] N. Nenov. The chromatic number of any 10-vertex graph without 4-cliques is at most 4. *Comptes rendus de l'Academie bulgare des Sciences*, 37:301–304, 1984. (in Russian).

[20] N. Nenov. On the small graphs with chromatic number 5 without 4-cliques. *Discrete Mathematics*, 188:297–298, 1998.

[21] N. Nenov. On a class of vertex Folkman graphs. *Ann. Univ. Sofia Fac. Math. Inform.*, 94:15–25, 2000.

[22] N. Nenov. A generalization of a result of Dirac. *Ann. Univ. Sofia Fac. Math. Inform.*, 95:59–69, 2001.

[23] N. Nenov. On a class of vertex Folkman numbers. *Serdica Mathematical Journal*, 28:219–232, 2002.

[24] K. Piwakowski, S. Radziszowski, and S. Urbanski. Computation of the Folkman number $F_e(3, 3; 5)$. *Journal of Graph Theory*, 32:41–49, 1999.

[25] S. Radziszowski. Small Ramsey numbers. *The Electronic Journal of Combinatorics*, Dynamic Survey revision 14, January 12 2014.

[26] Z. Shao, M. Liang, L. Pan, and X. Xu. Computation of the Folkman number $F_v(3, 5; 6)$. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 81:11–17, 2012.

[27] Z. Shao, X. Xu, and H. Luo. Bounds for two multicolor vertex Folkman numbers. *Application Research of Computers*, 3:834–835, 2009. (in Chinese).

[28] Z. Shao, X. Xu, and L. Pan. New upper bounds for vertex Folkman numbers $F_v(3, k; k+1)$. *Utilitas Mathematica*, 80:91–96, 2009.

[29] D. West. *Introduction to Graph Theory.* Prentice Hall, Inc., Upper Saddle River, 2 edition, 2001.

[30] X. Xu, H. Luo, and Z. Shao. Upper and lower bounds for $F_v(4, 4; 5)$. *Electronic Journal of Combinatorics*, 17, 2010.

[31] X. Xu and Z. Shao. On the lower bound for $F_v(k, k; k+1)$ and $F_e(3, 4; 5)$. *Utilitas Mathematica*, 81:187–192, 2010.