ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА

Том 97

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE

Tome 97

BALANCED VERTEX SETS IN GRAPHS

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Let $v_1, \dots, v_r$ be a β-sequence (Definition 1.2) in an n-vertex graph G and $v_{r+1}, \dots, v_n$ be the other vertices of G. In this paper we prove that if $v_1, \dots, v_r$ is balansed, that is

$$\frac{1}{r}(d(v_1) + \dots + d(v_r) = \frac{1}{n}(d(v_1) + \dots + d(v_n)),$$

and if the number of edges of G is big enough, then G is regular.

Keywords: saturated sequence, balanced sequence, generalized r-partite graph, generalized Turan's graph

2000 MSC: 05C35

1. INTRODUCTION

$e(G) = |E(G)|$ – the number of edges of G;
$G[M]$ – the subgraph of G, induced by M, where $M \subset V(G)$;
$\Gamma_G(M)$ –the set of all vertices of G adjacent to any vertex of M;
$d_G(v) = |\Gamma_G(v)|$ – the degree of a vertex v in G;
K_n and $\overline{K}_n$ – the complete and discrete n-vertex graphs, respectively.

Let r be an integer. A graph G is called r-partite with partition classes $V_i, i = 1, \dots, r$ if $V(G) = V_1 \cup \dots \cup V_r$, $V_i \cap V_j = \varnothing$ for $i \neq j$ and the sets V_i are independent sets in G. If every two vertices from different partition classes are adjacent, then G is called complete r-partite graph. Let G be an n-vertex r-partite graph with partition classes V_i and $p_i = |V_i|, i = 1, \dots, r$. Obviously, $d_G(v) \leq n - p_i$, for any $v \in V_i, i = 1, \dots, r$ and $d_G(v) = n - p_i$ if and only if G is a complete r-partite graph. The symbol $K(p_1, \dots, p_r)$ denotes the complete r-partite graph with partition classes $V_1, \dots, V_r$ such that $|V_i| = p_i, i = 1, \dots, r$. If $p_1, \dots, p_r$ are as equal as possible (in the sense that $|p_i - p_j| \le 1$ for all pairs $\{i, j\}$), then if $p_1 + \dots + p_r = n$, $K(p_1, \dots, p_r)$ is denoted by $T_r(n)$ and is called r-partite n-vertex Turan's graph. Clearly

$$e(K(p_1, \dots, p_r)) = \sum \{p_i p_j \,|\, 1 \le i < j \le r\}.$$

Thus, if $p_i - p_j \ge 2$, then

$$e(K(p_1 - 1, p_2 + 1, p_3, \dots, p_r)) - e(K(p_1, p_2, \dots, p_r)) = p_1 - p_2 - 1 > 0$$

This observation implies the following elementary proposition, we make shall use of later:

Lemma 1.1. *Let n and r be positive integers. Then the inequality*

$$e(K(p_1, \dots, p_r)) \le e(T_r(n))$$

holds for each r-tuple $(p_1, \dots, p_r)$ of nonnegative integers p_i such that $p_1 + \dots + p_n = n$. The equality occurs only when $K(p_1, \dots, p_r) = T_r(n)$.

Let $V_1, \dots, V_{r-1}$ be partition classes of $T_{r-1}(n), 2 \le r \le n$. Then $T_{r-1}(n)$ is r-partite graph with partition classes $V_1, \dots, V_{r-1}, \{\varnothing\}$. Since $2 \le r \le n$, $T_{r-1}(n) \ne T_r(n)$. Thus, from Lemma 1.1 it follows that

$$e(T_{r-1}(n)) < e(T_r(n)) \tag{1.1}$$

Let $V(G) = \{v_1, \dots, v_n\}$. We call the graph G regular, if

$$d_G(v_1) = d_G(v_2) = \dots = d_G(v_n).$$

A simple calculation shows that

$$e(T_r(n)) = \frac{(n^2 - \nu^2)(r-1)}{2r} + \binom{\nu}{2}, \tag{1.2}$$

where $n = kr + \nu$, $0 \le \nu \le r - 1$. $\square$

Definition 1.1 Let G be a graph and $v_1, \dots, v_r \in V(G)$ be a vertex sequence such that

$$v_i \in \Gamma_G(v_1, \dots, v_{i-1}),\ 2 \le i \le r.$$

Define $V_1 = V(G) \backslash \Gamma_G(v_1)$, $V_2 = \Gamma_G(v_1) \backslash \Gamma_G(v_2)$, $V_3 = \Gamma_G(v_1, v_2) \backslash \Gamma_G(v_3), \dots$, $V_{r-1} = \Gamma_G(v_1, \dots, v_{r-2}) \backslash \Gamma_G(v_{r-1})$, $V_r = \Gamma_G(v_1, \dots, v_{r-1})$.

Definition 1.2 The sequence of vertices $v_1, \dots, v_r$ in a graph G is called β-sequence, if the following conditions are satisfied: v_1 is a vertex of maximal degree in G, and for $i \ge 2$, $v_i \in \Gamma_G(v_1, \dots, v_{i-1})$ and

$$d_G(v_i) = \max \{d_G(v) | v \in \Gamma_G(v_1, \dots, v_{i-1})\}.$$

Definition 1.3 Let G be an n-vertex graph and $v_1, \dots, v_r \in V(G)$. Then the sequence $v_1, \dots, v_r$ is called saturated, if

$$\frac{1}{r}(d_G(v_1) + \ldots + d_G(v_r)) > \frac{2e(G)}{n}.$$

This sequence is called balanced, if

$$\frac{1}{r}(d_G(v_1) + \ldots + d_G(v_r)) = \frac{2e(G)}{n}.$$

Obviously, if G is regular, then any vertex sequence in G is balanced. Let $V(G) = \{v_1, \dots, v_n\}$. Then

$$d(v) \geq \frac{2e(G)}{n} = \frac{1}{n}(d_G(v_1) + \ldots + d_G(v_n))$$

for any vertex of maximal degree in G. Thus, if $d(v) = \dfrac{2e(G)}{n}$ for some vertex of maximal degree in G, then G is regular.

Let r and n be positive integers,$2 \leq r \leq n$. Define

$$f(n,r) = \begin{cases} \dfrac{n^2(r-1)}{2r} - \dfrac{n}{2r} & \text{if } n \equiv 0 (mod\, r); \\ \dfrac{n^2(r-1)}{2r} - \dfrac{\nu n}{2r(r-1)} & \text{if } n \equiv \nu (mod\, r), 1 \leq \nu \leq r-1. \end{cases}$$

It straightforward to show that

$$f(n,r) > \frac{(r-2)n^2}{2(r-1)},\ r \geq 2$$

Since $\dfrac{(r-2)n^2}{2(r-1)} > f(n, r-1)$, we have

$$f(n, r-1) < f(n,r),\ 2 \leq r \leq n \tag{1.3}$$

Our main result is the following theorem:

Theorem 1.1 (The Main Theorem). *Let G be an n-vertex graph and r be a positive integer, $2 \leq r \leq n$, such that $e(G) > f(n,r)$. Let for some s, $1 \leq s \leq r$, there exists a balanced β-sequence $v_1, \dots, v_s \in V(G)$. Then G is regular.*

Example 1.1. Consider the graph G shown in Fig.1. The β-sequence $\{v_1, v_3\}$ is balanced, because

$$\frac{1}{2}(d_G(v_1) + d_G(v_2)) = \frac{2e(G)}{8} = \frac{5}{2}.$$

Obviously, G is not regular.

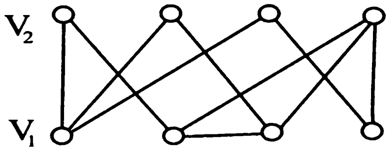

Fig. 1.

2. GENERALIZED r-PARTITE GRAPHS

Definition 2.1. ([2]) An n-vertex graph G is called generalized r-partite with partition classes V_i, $i = 1, \dots, r$, if $V(G) = V_1 \cup \dots \cup V_r$, $V_i \cap V_j = \varnothing$, $i \neq j$ and $d_G(v) \leq n - p_i$ for any $v \in V_i$, $i = 1, \dots, r$, where $p_i = |V_i|$. If $d_G(v) = n - p_i$ for any $v \in V_i$, $i = 1, \dots, r$, then G is called generalized complete r-partite graph with partition classes $V_1, \dots, V_r$. We call G generalized Turan's r-partite graph if G is a generalized complete r-partite graph with partition classes $V_1, \dots, V_r$ and $|p_i - p_j| \leq 1$ for all pairs $\{i, j\}$.

Proposition 2.1. *Let r and n be natural numbers, $1 \leq r \leq n$. Let G be an n-vertex graph, such that*

$$d(v) \leq \frac{(r-1)n}{r}, \ \forall v \in V(G).$$

Then G is generalized r-partite graph.

Proof. Let

$$V(G) = V_1 \cup \dots \cup V_r, \ V_i \cap V_j = \varnothing, \ i \neq j$$

and $\lfloor \frac{n}{2} \rfloor \leq |V_i| \leq \lceil \frac{n}{2} \rceil$, $i = 1, \dots, r$.

From $d(v) \leq \dfrac{(r-1)n}{r} = n - \dfrac{n}{r}$ it follows that $d(v) \leq n - \lceil \frac{n}{r} \rceil$, $\forall v \in V(G)$. Thus $d(v) \leq n - |V_i|$, $\forall v \in V_i$, $i = 1, \dots, r$, and G is generalized r-partite graph with partition classes $V_1, \dots, V_r$. □

Observe that, if $n \equiv 0 (mod\, r)$ and $d(v) = \dfrac{(r-1)n}{r}$, $\forall v \in V(G)$, then G is generalized r-partite Turan's graph.

We shall make use of the following result:

Theorem 2.1. ([2]) *Let G be a generalized r-partite graph with partition classes $V_1, \dots, V_r$, where $|V_i| = p_i$, $i = 1, \dots, r$. Then*

$$e(G) \leq e(K(p_1, \dots, p_r)).$$

The equality holds if and only if G is generalized complete r-partite graph with partition classes $V_1, \dots, V_r$.

Theorem 2.2. ([2]) *Let G be a generalized r-partite graph and $|V(G)| = n$. Then*

$$e(G) \leq e(T_r(n))$$

and equality occurs if and only if G is generalized r-partite Turan's graph.

Example 2.1. Consider the graph $K_3 + C_5 = K_8 - C_5$. Obviously, $e(K_3 + C_5) = 23 < e(T_4(8)) = 24$. This graph is not generalized 4-partite graph. Assume the opposite, i.e. that $K_3 + C_5$ is generalized 4-partite graph with partition classes V_1, V_2, V_3, V_4. Let $V(K_3)) = \{v_1, v_2, v_3\}$. If $v_i \in V_j$, then from $d(v_i) = 7 \leq 8 - |V_j|$ it follows that $|V_j| = 1$, i.e. $V_j = \{v_i\}$. Thus, we may assume that $V_i = \{v_i\}, i = 1, 2, 3$. Hence, $V_4 = V(C_5)$. Let $v \in V(C_5)$. Then $d(v) = 5 > 8 - |V_4| = 3$, which is a contradiction.

3. β-SEQUENCES AND GENERALIZED r-PARTITE GRAPHS

We shall use the following:

Theorem 3.1. ([2]) *Let $v_1, \ldots, v_r$ be a β-sequence in an n-vertex graph G, which is not contained in an $(r+1)$-clique. If V_i is the i-th stratum of the stratification induced by this sequence and $p_i = |V_i|$ (see Definition 1.1), then*

(a) *G is generalized r-partite graph with partition classes $V_1, \ldots, V_r$;*

(b) *$e(G) \leq e(K(p_1, \ldots, p_r))$, and the equality occurs if and only if G is a generalized complete r-partite graph with partition classes $V_1, \ldots, V_r$;*

(c) *$e(G) \leq e(T_r(n))$ and we have $e(G) = e(T_r(n))$ only when G is a generalized r-partite Turan's graph.*

The proof of the theorem 3.1, given in [2], actually establishes the following stronger statement:

Theorem 3.2. ([2]) *Let $v_1, \ldots, v_r$ be a β-sequence in an n-vertex graph G such that*

$$d_G(v_r) \leq n - |\Gamma_G(v_1, \ldots, v_{r-1})|$$

Then the statements (a), (b) *and* (c) *of the Theorem* 3.1 *hold.*

Denote by $\psi(G)$ the smallest integer r for which there exist a β-sequence $v_1, \ldots, v_r$, $r \geq 2$, in n-vertex graph G, such that

$$d_G(v_r) \leq n - |\Gamma_G(v_1, \ldots, v_{r-1})|.$$

Theorem 3.3. *Let G be an n-vertex graph and $e(G) \geq e(T_r(n))$. Then $\psi(G) \geq r$ and $\psi(G) = r$ only when G is a generalized r-partite Turan's graph.*

Proof. Let $\psi(G) = s$. By Theorem 3.2, $e(G) \leq e(T_s(n))$. Thus $e(T_r(n)) \leq e(T_s(n))$. From (1.1) it follows that $s \geq r$. If $s = r$, then $e(G) = e(T_r(n))$. According Theorem 3.2, G is a generalized r-partite Turan's graph. □

The following lemma generalizes the Proposition 2.1.

Lemma 3.1. ([3]) *Let G be a graph and $v_1, \ldots, v_r$ be a β-sequence in G such that*

$$d(v_1) + \ldots + d(v_k) \leq \frac{k(r-1)n}{r}, \text{ for some } 1 \leq k \leq r. \tag{3.1}$$

Then G is a generalized r-partite graph. If inequality (3.1) is strict, then G is not generalized r-partite Turan's graph.

Denote the smallest integer r for which there exists a a β-sequence $v_1, \ldots, v_r$ in n-vertex graph G, such that

$$d_G(v_1) + \ldots + d_G(v_r) \leq (r-1)n \tag{3.2}$$

by $\xi(G)$.

Theorem 3.4. *Let G be an n-vertex graph and $e(G) \geq e(T_r(n))$. Then $\xi(G) \geq r$ and $\xi(G) = r$ only when G is generalized r-partite Turan's graph.*

Proof. Let $\xi(G) = s$ and let $v_1, \ldots, v_s$ be a β-sequence in G, such that

$$d_G(v_1) + \ldots + d_G(v_s) \leq (s-1)n$$

By Lemma 3.1 ($r = k = s$), the graph G is generalized r-partite. According to Theorem 2.2 $e(G) \leq e(T_s(n))$. Thus, the inequality $e(G) \geq e(T_r(n))$ implies $e(T_s(n)) \geq e(T_r(n))$. By (1.1) we have $s \geq r$.

Let $s = r$. Then $e(G) = e(T_r(n))$ and from the Theorem 2.2 it follows that G is a generalized r-partite Turan's graph. □

4. SATURATED AND BALANCED β-SEQUENCES

The following results were proved by us:

Theorem 4.1. ([3]) *Let G be an n-vertex graph and $v_1, \ldots, v_r$ be a β-sequence in G, which is not balanced and not saturated. Then G is generalized r-partite graph, which is not a generalized r-partite Turan's graph. Thus $e(G) < e(T_r(n))$.*

Theorem 4.2. ([3]) *Let G be an n-vertex graph and let $v_1, \ldots, v_r$ be a β-sequence in G, $r \geq 2$, which is not balanced and not saturated. Then*

$$d(v_1) + \ldots + d(v_{r-1}) < \frac{(r-1)^2}{r} n.$$

In this section we improve Theorem 4.2.

Theorem 4.3. *Let G be an n-vertex graph and $v_1, \ldots, v_r$ $r \geq 2$ be a β-sequence in G, which is not saturated but $v_1, \ldots, v_{r-1}$ is saturated. Then*

$$d(v_1) + \ldots + d(v_{r-1}) \leq \frac{(r-1)^2}{r} n. \tag{4.1}$$

If there is equality in (4.1), *then:*
(a) $v_1, \ldots, v_r$ *is balanced;*
(b) $n \equiv 0 (mod\, r)$ *and* G *is a generalized (noncomplete)* r*-partite graph with partition classes* $V_1', \ldots, V_r'$, *such that* $|V_i'| = \frac{n}{r}$, $i = 1, \ldots, r$ *and*

$$d(v) = \frac{r-1}{r} n, \ \forall v \in \bigcup_{i=1}^{r-1} V_i'$$

$$d(v) = \frac{2e(G)r}{n} - \frac{(r-1)^2 n}{r}, \ \forall v \in V_r';$$

(c) $\dfrac{(r-1)^2 n^2}{r^2} + \dfrac{r-1}{2r} n \le e(G) \le \dfrac{(r-1)n^2}{2r} - \dfrac{n}{2r}$.

Proof. Since $(r-2)n < \frac{(r-1)^2 n}{r}$, in case $d(v_1) + \ldots + d(v_{r-1}) \le (r-2)n$ the inequality (4.1) holds. Therefore, we shall assume that

$$d(v_1) + \ldots + d(v_{r-1}) > (r-2)n. \tag{4.2}$$

Let V_i be the i-stratum of the stratification, induced by sequence $v_1, \ldots, v_r$. Obviously, $v_i \in V_i$, $i = 1, \ldots, r$ and

$$V(G) = V_1 \cup \ldots \cup V_r, \ V_i \cap V_j = \varnothing, \ i \neq j. \tag{4.3}$$

Since $V_i \subset V(G) \backslash \Gamma(v_i)$, $i = 1, \ldots, r-1$, we have

$$|V_i| \le n - d(v_i), \ i = 1, \ldots, r-1. \tag{4.4}$$

It follows from (4.3), (4.4) and (4.2) that

$$|V_r| = n - \sum_{i=1}^{r-1} |V_i| \ge \sum_{i=1}^{r-1} d(v_i) - (r-2)n > 0.$$

Thus $V_r \neq \varnothing$. Let V_r' be a subset of V_r such that

$$|V_r'| = \sum_{i=1}^{r-1} d(v_i) - (r-2)n. \tag{4.5}$$

Define $W = V(G) \backslash V_r'$. By (4.5) we have

$$|W| = \sum_{i=1}^{r-1} (n - d(v_i)). \tag{4.6}$$

Since $V_i \subset W$, $i = 1, \ldots, r-1$, from (4.3), (4.4) and (4.6) it follows that there exist disjoint sets V_i', $i = 1, \ldots, r-1$, such that $V_i \subseteq V_i' \subset W$ and $|V_i'| = n - d(v_i)$.

Since $V_i \subseteq V_i'$, we have $v_i \in V_i'$, $i = 1, \ldots, r-1$. From (4.6) it follows that $W = \bigcup_{i=1}^{r-1} V_i'$. Hence,

$$V(G) = V_1' \cup \ldots \cup V_r', \ V_i' \cap V_j' = \varnothing, \ i \neq j. \tag{4.7}$$

Observe that

$$V_i' \backslash V_i \subset V_r = \Gamma(v_1, \ldots, v_{r-1}) \subset \Gamma(v_1, \ldots, v_{i-1})$$

and $V_i \subset \Gamma(v_1, \ldots, v_{i-1})$. Thus $V_i' \subset \Gamma(v_1, \ldots, v_{i-1})$, $i = 1, \ldots, r-1$ and $d(v) \leq d(v_i)$, $\forall v \in V_i'$, $i = 1, \ldots, r-1$. From the inclusion $V_r' \subset V_r$ it follows that $d(v) \leq d(v_r)$, $\forall v \in V_r'$. So, we have

$$d(v) \leq d(v_i), \ \forall v \in V_i', \ i = 1, \ldots, r. \tag{4.8}$$

By (4.7), we have

$$2e(G) = \sum_{v \in V(G)} d(v) = \sum_{v \in V_1'} d(v) + \ldots + \sum_{v \in V_r'} d(v).$$

Let $d(v_i) = d_i$, $i = 1, \ldots, r$. From $|V_i'| = n - d_i$, $i = 1, \ldots, r-1$, (4.8) and (4.5) it follows that

$$2e(G) \leq \sum_{i=1}^{r-1} d_i(n - d_i) + \Big(\sum_{i=1}^{r-1} d_i - (r-2)n \Big) d_r. \tag{4.9}$$

The equality in (4.9) occurs if and only if

$$d(v) = d_i, \ \forall v \in V_i', \ i = 1, \ldots, r.$$

Let $\sigma = d_1 + \ldots + d_{r-1}$. We have $\dfrac{\sigma + d_r}{r} \leq \dfrac{2e(G)}{n}$ because the sequence $v_1, \ldots, v_r$ is not saturated. Thus,

$$d_r \leq \frac{2re(G)}{n} - \sigma. \tag{4.10}$$

By the Caushy-Schwarz inequality $\left(\sum x_i y_i\right)^2 \leq \sum x_i^2 \sum y_i^2$, applied to $x_i = d_i$, $y_i = 1$, we have

$$\sum_{i=1}^{r-1} d_i^2 \geq \frac{\sigma^2}{r-1}. \tag{4.11}$$

and the equality holds if and only if $d_1 = \ldots = d_{r-1}$. We obtain by (4.10) and (4.11)

$$2e(G) \leq n\sigma - \frac{\sigma^2}{r-1} + (\sigma - (r-2)n)\Big(\frac{2re(G)}{n} - \sigma \Big).$$

This inequality is equivalent to

$$\frac{2e(G)}{n}\left((r-1)^2n - r\sigma\right) \le \frac{\sigma}{r-1}\left((r-1)^2n - r\sigma\right). \tag{4.12}$$

The equality in (4.12) occurs simultaneously with the equalities in (4.9), (4.10) and (4.11), i.e. when

$$d(v) = d_i = d_1,\ \forall v \in V_i',\ i = 1,\dots,r-1 \text{ and} \tag{4.13}$$

$$d(v) = d_r = \frac{2re(G)}{n} - \sigma,\ \forall v \in V_r'.$$

Since $v_1,\dots,v_{r-1}$ is saturated, we have

$$\frac{\sigma}{r-1} > \frac{2e(G)}{n}.$$

Thus, (4.12) is equivalent to the inequality $\sigma \le \dfrac{(r-1)^2n}{r}$. The inequality (4.1) is proved.

It remains to examine the case of the equality in (4.1). Assume, that

$$\sigma = \frac{(r-1)^2n}{r}. \tag{4.14}$$

Then $n \equiv 0(mod\ r)$ and the equality holds in (4.12), i.e. (4.13) is realized. From (4.14) and (4.13) it follows that

$$d(v) = d_1 = \ldots = d_{r-1} = \frac{(r-1)n}{r},\ \forall v \in V_i',\ i = 1,\dots,r-1 \tag{4.15}$$

and

$$d(v) = d_r = \frac{2re(G)}{n} - \frac{(r-1)^2}{r}n,\ \forall v \in V_r'. \tag{4.16}$$

By (4.15) and (4.16) it follows that

$$\frac{d_1 + \ldots + d_r}{r} = \frac{2e(G)}{n},$$

i.e. $v_1,\dots,v_r$ is balansed. Since $v_1,\dots,v_{r-1}$ is saturated, we have

$$\frac{d_1 + \ldots + d_{r-1}}{r-1} > \frac{2e(G)}{n} = \frac{d_1 + \ldots + d_r}{r},$$

Hence $d_r < d_1 = \dfrac{r-1}{r}n$.Thus

$$d(v) = d_r < \frac{r-1}{r}n,\ v \in V_r'. \tag{4.17}$$

Since $|V_i'| = n - d_i$, $i = 1, \dots, r-1$ and $|V_r'| = \sum_{i=1}^{r-1} d_i - (r-2)n$, we obtain by (4.15)

$$|V_i'| = \frac{n}{r},\ i = 1, \dots, r$$

Thus, from (4.15) and (4.17) it follows that G generalized (noncomplete) r-partite graph with equal partite classes $V_1', \dots, V_r'$.

So, (a) and (b) are proved. It remains to prove (c). The number $\frac{(r-1)n}{r}$ is integer, because $n \equiv 0 (mod\, r)$ and consequently from (4.17) it follows that

$$d_r \le \frac{(r-1)n}{r} - 1.$$

Since $v_1, \dots, v_r$ is balanced, by this inequality and (4.15) we have

$$\frac{2e(G)}{n} = \frac{d_1 + \dots + d_r}{r} \le \frac{\frac{(r-1)^2 n}{r} + \frac{(r-1)n}{r} - 1}{r} = \frac{(r-1)n - 1}{r}.$$

Thus, $e(G) \le \frac{(r-1)}{2r} n^2 - \frac{n}{2r}$.

Since $v_r \in \Gamma_G(v_1, \dots, v_{r-1})$, $d(v_r) \ge r - 1$. From this inequality and (4.16) we conclude that

$$e(G) \ge \frac{(r-1)^2}{2r^2} n^2 + \frac{r-1}{2r} n.$$

The proof of (c) is over and Theorem 4.3 is proved. □

Corollary 4.1. *Let G be an n-vertex graph and r be integer, $1 \le r \le n$. Let $e(G) \ge e(T_r(n))$ and for some s, $1 \le s \le r$ there exists a balanced β-sequence $v_1, \dots, v_s \in V(G)$. Then G is regular.*

Proof. We prove this corollary by induction on s. The base $s = 1$ is clear, since $d(v_1) = \frac{2e(G)}{n}$ implies that G is regular.

Let $s \ge 2$. Since $\frac{d(v_1) + \dots + d(v_s)}{s} = \frac{2e(G)}{n}$, from $d(v_1) \ge d(v_2) \ge \dots \ge d(v_s)$ it follows that

$$\frac{d(v_1) + \dots + d(v_{s-1})}{s-1} \ge \frac{2e(G)}{n},$$

i.e. $v_1, \dots, v_{s-1}$ is balanced or saturated. We prove that $v_1, \dots, v_{s-1}$ is balanced. Assume the opposite.

Since $v_1, \dots, v_s$ is not saturated, by Theorem 4.3

$$d(v_1) + \dots + d(v_{s-1}) \le \frac{(s-1)^2 n}{s}. \tag{4.18}$$

By Lemma 3.1, G is a generalized s-partite graph. From Theorem 2.2 it follows $e(G) \le e(T_s(n))$.

Thus, we have $e(T_r(n)) \le e(G) \le e(T_s(n))$. Since $s \le r$, (1.1) implies that $s = r$ and $e(G) = e(T_s(n))$. According to Lemma 3.1, there is equality in (4.18). Thus, Theorem 4.3 implies that $n \equiv 0(mod\ s)$ and $e(G) \le \frac{(s-1)n^2}{2s} - \frac{n}{2s}$. This contradicts the equality $e(G) = e(T_s(n)) = \frac{(s-1)n^2}{2s}$.

So, $v_1, \dots, v_{s-1}$ is balanced. By inductive hypothesis, G is regular and the proof of Corollary 4.1 is over. □

5. PROOF OF THE MAIN THEOREM

We prove that G is regular by induction on s. The base $s = 1$ is clear, since $d(v_1) = \frac{2e(G)}{n}$ implies that G is regular.

Let $s \ge 2$. From $d(v_1) \ge \dots \ge d(v_s)$ it follows that

$$\frac{d(v_1) + \dots + d(v_{s-1})}{s-1} \ge \frac{2e(G)}{n}.$$

Hence, $v_1, \dots, v_{s-1}$ is balanced or saturated. We prove that $v_1, \dots, v_{s-1}$ is balanced. Assume the opposite. Then

$$\frac{d(v_1) + \dots + d(v_{s-1})}{s-1} > \frac{2e(G)}{n}. \tag{5.1}$$

By Theorem 4.3, the inequality (4.18) holds. If there is equality in (4.18), then according to Theorem 4.3, $n \equiv 0(mod\ s)$ and $e(G) \le \frac{(s-1)n^2}{2s} - \frac{n}{2s} = f(n, s)$. But $f(n, s) \le f(n, r)$, because $s \le r$ (see (1.3)). Therefore, $e(G) \le f(n, r)$ which is a contradiction. Assume that (4.18) is strict.

Case 1. $n \equiv 0(mod\ s)$. Since (4.18) is strict, it follows that

$$d(v_1) + \dots + d(v_{s-1}) \le \frac{(s-1)^2 n}{s} - 1. \tag{5.2}$$

From (5.1) and (5.2) it follows that

$$e(G) < \frac{(s-1)n^2}{2s} - \frac{n}{2(s-1)} < f(n, s).$$

By $s \le r$ and (1.3), $f(n, s) \le f(n, r)$. Hence $e(G) < f(n, r)$, which is a contradiction.

Case 2. $n \equiv \nu(mod\ s)$, $1 \le \nu \le s - 1$. Since (4.18) is strict, we have

$$d(v_1) + \dots + d(v_{s-1}) \le \lfloor \frac{(s-1)^2 n}{s} \rfloor = \frac{(n-\nu)(s-1)^2}{s} + \nu(s-2). \tag{5.3}$$

From (5.1) and (5.3) it follows

$$e(G) \le f(n,s) \le f(n,r),$$

which is a contradiction.

The Main Theorem is proved.

Remark. If $n \equiv 0(mod\ r)$, then $f(n,r) < e(T_r(n)) = \dfrac{n^2(r-1)}{2r}$. Therefore, in this case the Corollary 4.1 follows from Main Theorem. Let $n \equiv \nu(mod\ r)$, $1 \le \nu \le r-1$. From (1.2) it follows that

$$e(T_r(n)) = \frac{n^2(r-1)}{2r} - \frac{\nu(r-\nu)}{2r}. \tag{5.4}$$

The equality (5.4) implies, that if

$$\frac{\nu(r-\nu)}{2r} < \frac{\nu n}{2r(r-1)},$$

i.e. $n > (n-\nu)(r-1)$, then $f(n,r) < e(T_r(n))$. Hence, if $n > (r-\nu)(r-1)$, Corollary 4.1 follows from the Main Theorem.

6. α-SEQUENCES IN GRAPHS

Let G be a graph and $v_1, \dots, v_r \in V(G)$. Define $\Gamma_0 = V(G)$ and $\Gamma_i = \Gamma_G(v_1, \dots, v_i)$, $i = 1, \dots, r-1$. In our articles [4] and [5] we introduced the following concept:

Definition 6.1. The sequence $v_1, \dots, v_r \in V(G)$ is called α-sequences if $v_i \in \Gamma_{i-1}$ and v_i has maximal degree in the graph $G[\Gamma_{i-1}]$, $i = 1, \dots, r$.

α-sequences appears later in [7-10] under the name "degree-greedy algorithm" and in [11] under the name "s-stable algorithm".

The following result was proved by us:

Theorem 6.1. ([2]) *Let $v_1, \dots, v_r$ be a α-sequence in an n-vertex graph G, which is not contained in an $(r+1)$-qlique. If V_i is the i-th stratum of the stratification induced by this sequence and $p_i = |V_i|$, $i = 1, \dots, r$ (see Definition 1.1), then*

(a) *G is generalized r-partite graph with partition classes $V_1, \dots, V_r$ and*

$$e(G) \le e(K(p_1, \dots, p_r)); \tag{6.1}$$

(b) *There is equality in* (6.1) *only when $G = K(p_1, \dots, p_r)$.*

The proof of Theorem 6.1, given in [2], actually establishes the following statement:

Theorem 6.2. *Let $v_1, \ldots, v_r$ be an α-sequence in an n-vertex graph G such that*

$$d(v) \leq n - |\Gamma_{r-1}|, \ \forall v \in \Gamma_{r-1}. \tag{6.2}$$

If V_i is the i-th stratum of the stratification induced by this sequence and $p_i = |V_i|$, $i = 1, \ldots, r$, then

(a) *G is generalized r-partite graph with partition classes $V_1, \ldots, V_r$ and inequality* (6.1) *holds;*

(b) *There is equality in* (6.1) *only when G is generalized complete r-partite graph with partition classes $V_1, \ldots, V_r$.*

Denote by $\varphi(G)$ the smallest integer r for which there exists an α-sequence $v_1, \ldots, v_r \in V(G)$, such that (6.2) holds.

Theorem 6.3. *Let G be an n-vertex graph, such that $e(G) \geq e(T_r(n))$, $1 \leq r \leq n$. Then $\varphi(G) \geq r$ and $\varphi(G) = r$ only when G is generalized r-partite Turan's graph.*

Proof. Let $\varphi(G) = s$ and $v_1, \ldots, v_s$ be α-sequence in G, such that $d(v) \leq n - |\Gamma_{s-1}|$, $\forall v \in \Gamma_{s-1}$. By Theorem 6.2 and Theorem 2.2, we have $e(T_r(n)) \leq e(T_s(n))$. From (1.1) it follows $s \geq r$. If $s = r$, then $e(G) = e(T_r(n))$. According to Theorem 2.2(c), G is generalized r-partite Turan's graph. This completes the proof of Theorem 6.3. □

Let $v_1, \ldots, v_r$ be α-sequence in graph G, and $G_{i-1} = G[\Gamma_{i-1}]$, $i = 1, \ldots, r$, where Γ_i, $i = 1, \ldots, r-1$ are defined above. Define

$$d_1' = d_G(v_1),\ d_2' = d_{G_1}(v_2), \ldots, d_r' = d_{G_{r-1}}(v_r).$$

Theorem 6.4. *Let G be an n-vertex graph and $v_1, \ldots, v_r$ be α-sequence in G, such that for some s, $1 \leq s \leq r$,*

$$d_1' + \ldots + d_s' \leq \frac{n}{r}\left(\binom{r}{2} - \binom{r-s}{2}\right). \tag{6.3}$$

Then G is generalized r-partite graph.

Proof. We prove Theorem 6.4 by induction on s. The induction base is $s = 1$. From (6.3) it follows that $d_1' \leq \dfrac{(r-1)n}{r}$. Since $d_1 = d_G(v_1)$ and v_1 has maximal degree in G, we have $d(v) \leq \dfrac{(r-1)n}{r}$, $\forall v \in V(G)$. By Proposition 1.1, G is generalized r-partite graph.

Let $s \geq 2$ and suppose, that assertion is true for $s-1$.

Case 1. $d_2' + \ldots + d_s' \leq \dfrac{d_1'}{r-1}\left(\binom{r-1}{2} - \binom{r-s}{2}\right)$.

Obviously $v_2, \ldots, v_r$ be α-sequence in $G_1 = G[\Gamma_G(v_1)]$. By inductive hypothesis, we may assume that G_1 is generalized $(r-1)$-partite graph with partition classes $W_2, \ldots, W_r$. Thus, G is generalized r-partite graph with partition classes $W_1 = V(G) \backslash \Gamma_G(v_1), W_2, \ldots, W_r$.

Case 2. $d'_2 + \ldots + d'_s > \frac{d'_1}{r-1}\left(\binom{r-1}{2} - \binom{r-s}{2}\right)$.

From (6.3) it follows that

$$d'_1 + \frac{d'_1}{r-1}\left(\binom{r-1}{2} - \binom{r-s}{2}\right) < \frac{n}{r}\left(\binom{r}{2} - \binom{r-s}{2}\right).$$

Hence

$$d'_1 \leq \frac{n}{r}A, \text{ where } A = \frac{\binom{r}{2} - \binom{r-s}{2}}{1 + \frac{1}{r-1}\left(\binom{r-1}{2} - \binom{r-s}{2}\right)}. \tag{6.4}$$

Note that $A = r - 1$. Thus, by (6.4), we have $d'_1 \leq \frac{n}{r}(r-1)$. Hence $d(v) \leq \frac{n(r-1)}{r}$, $\forall v \in V(G)$. By Proposition 2.1, G is generalized r-partite graph. □

Theorem 6.5. *Let G be an n-vertex graph and $v_1, \ldots, v_k$ be α-sequence in G, such that*

$$d'_1 + \ldots + d'_k \leq \frac{ke(G)}{n}.$$

Then G is generalized k-partite graph.

Proof. If $k = 1$, then $d'_1 \leq \frac{e(G)}{n}$. Since $e(G) \leq \frac{d'_1 n}{2}$, it follows that $d'_1 = 0$. Thus, $E(G) = \varnothing$ and G is 1-partite graph.

Let $k \geq 2$. Then

$$d'_2 + \ldots + d'_k \leq \frac{ke(G)}{n} - d'_1.$$

From this inequality and $e(G) \leq \frac{nd'_1}{2}$, it follows that

$$d'_2 + \ldots + d'_k \leq \frac{(k-2)d'_1}{2} = \frac{d'_1}{k-1}\binom{k-1}{2}.$$

Since $v_2, \ldots, v_k$ is an α-sequence in $G_1 = G[\Gamma_G(v_1)]$, by this inequality and Theorem 6.4 (with $r = s = k-1$), it follows that the graph G_1 is generalized $(k-1)$-partite graph. Let $W_2, \ldots, W_k$ be partition classes of G_1. Then G is generalized r-partite graph with partition classes $W_1 = V(G) \backslash \Gamma_G(v_1), W_2, \ldots, W_k$.

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Received September 15, 2004

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