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Extremal Scattered Data Interpolation in $\mathbb{R}^3$ Using Tensor Product Bézier Surfaces*

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We consider the problem of extremal scattered data interpolation in $\mathbb{R}^3$. Using our previous work on minimum L_p-norm interpolation curve networks, $1 < p \leq \infty$, we construct a bivariate interpolant F with the following properties:

(i) F is G^1-continuous;

(ii) F consists of tensor product Bézier surfaces;

(iii) Each Bézier surface satisfies the tetraharmonic equation $\Delta^4 F = 0$. Hence F minimizes the corresponding energy functional.

Keywords and Phrases: Scattered data interpolation, tensor product Bézier surfaces, tetraharmonic surfaces.

Mathematics Subject Classification 2010: 65D05, 65D07, 65D17.

1. Introduction

Scattered data interpolation is a fundamental problem in approximation theory and finds applications in various areas including geology, meteorology, cartography, medicine, computer graphics, geometric modeling etc. Different methods for solving this problem were applied and reported, excellent surveys are [5, 6, 7, 8].

The problem can be formulated as follows: Given a set of points $(x_i, y_i, z_i) \in \mathbb{R}^3$, $i = 1, \ldots, N$, find a bivariate function F possessing continuous partial derivatives up to a given order and such that $F(x_i, y_i) = z_i$, $i = 1, \ldots, N$. One of the possible approaches to solving the problem is due to Nielson [11]. The method consists of the following three steps:

Step 1: Triangulation. Construct a triangulation T of the projection points $V_i = (x_i, y_i)$, $i = 1, \ldots, N$, in the plane Oxy.

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Step 2: Minimum norm network (MNN). The interpolant F and its first order partial derivatives are defined on the edges of T so as to satisfy an extremal property. The obtained minimum norm network is a cubic curve network, i. e. on every edge of T it is a cubic polynomial.

Step 3: Interpolation surface. The obtained network is extended to F by an appropriate *blending method*.

In [1] Andersson et al. pay special attention to Step 2 of the above method – the construction of the MNN. Using a different approach, the authors give a new proof of Nielson's result. They construct a system of simple linear curve networks called *basic curve networks* and then represent the second derivative of the MNN as a linear combination of these basic curve networks. The results from [1] are extended in [14] to the class of L_p-norms for $1 < p \leq \infty$. The extremal network is characterized as a solution to a system of equations which is nonlinear except for the case $p = 2$ when it is linear. A Newton-type algorithm for solving such type of nonlinear systems has been proposed in [15] where its validity and convergence were evaluated.

In this paper we propose a solution to the scattered data interpolation problem as follows. We consider the minimal rectangular domain D with sides parallel to the axes of Oxy and define our interpolation surface on D. Instead of triangulation we use a rectangular mesh in D such that all points $V_i = (x_i, y_i)$, $i = 1, \ldots, N$, are vertices of the mesh. We define z-values for the remaining vertices of the mesh (if any) using an approach from [13] and we add the new points to our data. Our method allows to build interpolation networks that are polynomials of degree n on the edges of the mesh where $n \in \mathbb{N}$, $n \geq 3$, is chosen in advance. We obtain these networks by setting $p = \frac{n-1}{n-2}$ and then computing the MNN with respect to the L_p-norm. Hereafter we assume that n is part of our input data.

Since the MNN is a polynomial curve network it is natural to require that the interpolant F is a polynomial surface on any rectangle of D. Although the MNN is C^1-continuous at the vertices V_i, it is preferable and more appropriate to require G^1 continuity for the interpolant instead of C^1 continuity since the latter is parametrization dependent. Two surfaces with a common boundary curve are called G^1*-continuous* if they have a continuously varying tangent plane along that boundary curve.

After the MNN is computed we construct an interpolation surface $F(x, y)$ defined on D with the following properties.

(i) F consists of tensor product Bézier surfaces (patches) of degree $n \times n$. Each patch is defined on a rectangle of the mesh;

(ii) F is G^1-continuous;

(iii) F satisfies the tetraharmonic equation $\Delta^4 \mathbf{x} = 0$ a.e. for $(u, v) \in D$ where $\mathbf{x}(u, v) := \big(x(u, v), y(u, v), z(u, v)\big)$ and $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ is the Laplace operator. Hence F is a solution to the extremal problem

$$\min_{\mathbf{x}} \int_D \|\Delta^4 \mathbf{x}\|_2 dudv, \tag{1}$$

i.e. F is an extremum to the corresponding energy functional.

The harmonic and biharmonic Bézier surfaces were studied by Monterde and Ugail [9]. Their method was extended to general 4th-order PDE Bézier surfaces in [10]. Here we use a result by Centella et al. [3] to generate tetraharmonic tensor product Bézier surfaces from given boundary curves and tangent conditions along them.

The paper is organised as follows. In Section 2 we introduce the notation, present some related results from [1, 14], and propose our Algorithm 1 for solving the scattered data interpolation problem. In Section 3 we investigate the G^1 continuity conditions for adjacent Bézier patches and prove that they correctly apply to our problem. The construction of surface F is considered in Section 4. In Section 5 we present our concluding remarks.

2. Preliminaries and Description of the Algorithm

Let $N \geq 3$ be an integer and $P_i := (x_i, y_i, z_i)$, $i = 1, \ldots, N$, be different points in $\mathbb{R}^3$. We call this set of points *data*. The data are *scattered* if the projections $V_i := (x_i, y_i)$ onto the plane Oxy are different and non-collinear.

Definition 1. A collection of non-overlapping, non-degenerate quadrangles in $\mathbb{R}^2$ is a *quadrangulation* of the points V_i, $i = 1, \ldots, N$, if the set of the vertices of the quadrangles coincides with the set of the points V_i, $i = 1, \ldots, N$.

We construct rectangular quadrangulation Q of the points V_i, $i = 1, \ldots, N$, using lines parallel to the axes Ox and Oy, as shown in Figure 1.

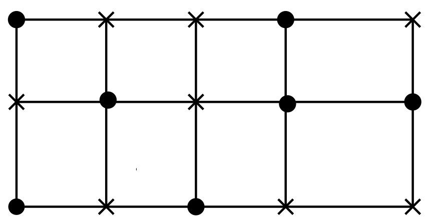

Figure 1. Rectangular quadrangulation of the projection points V_i, $i = 1, \ldots, N$, where $\bullet$ denotes old (given) points, and $\times$ denotes new (added) points.

Obviously Q may introduce new vertices. To sample z_i-values of the corresponding new points P_i we use Renka's algorithm 790 [13] for constructing a smooth bivariate function that interpolates the data. This method achieves cubic precision, C^2 continuity, and is one of the most accurate available. The software can be found in the TOMS subdirectory of the NETLIB web cite [16]. Furthermore, we suppose that V_i, $i = 1, \dots, N$, are all vertices of Q.

Let D be the rectangular domain that is the union of all rectangles in Q. The set of the edges of the rectangles in Q is denoted by E. If there is an edge in E joining V_i and V_j, it will be referred to by e_{ij} or simply by e if no ambiguity arises.

Definition 2. A *curve network* is a collection of real-valued univariate functions $\{f_e\}_{e\in E}$ defined on the edges in E.

With any real-valued bivariate function F defined on D we naturally associate the curve network defined as the restriction of F on the edges in E, i.e. for $e = e_{ij} \in E$,

$$\begin{gathered} f_e(t) := F\Big(\big(1 - \frac{t}{\|e\|}\big)x_i + \frac{t}{\|e\|}\, x_j,\ \big(1 - \frac{t}{\|e\|}\big)y_i + \frac{t}{\|e\|}\, y_j\Big), \\ \text{where } 0 \le t \le \|e\| \text{ and } \|e\| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}. \end{gathered} \tag{2}$$

In our presentation, according to the context, F will denote either a real-valued bivariate function or a curve network defined by (2). Let $1 < p < \infty$. We introduce the following class of functions defined on D

$$\begin{aligned} \mathcal{F}_p := \big\{F(x,y) \in C^1(D) \ :\ & F(x_i, y_i) = z_i,\ i = 1, \dots, N, \\ & f_e' \in AC[0, \|e\|],\ f_e'' \in L^p[0, \|e\|],\ e \in E\big\}, \end{aligned}$$

and the corresponding class of the so-called *smooth interpolation curve networks*

$$\mathcal{C}_p(E) := \big\{F|_E = \{f_e\}_{e\in E} \ :\ F(x,y) \in \mathcal{F}_p\big\}.$$

For $F \in \mathcal{C}_p(E)$ we denote the curve network of second derivatives of F by $F'' := \{f_e''\}_{e\in E}$. The L_p-norm of F'' is defined by

$$\|F''\|_p := \left(\sum_{e\in E} \int_0^{\|e\|} |f_e''(t)|^p dt\right)^{1/p}.$$

We consider the following extremal problem.

$$(\mathbf{P}_p) \qquad \textit{Find } F^* \in \mathcal{C}_p(E) \ \textit{ such that } \ \|F^{*\prime\prime}\|_p = \inf_{F\in\mathcal{C}_p(E)} \|F''\|_p.$$

The degree of all vertices in Q, i.e. the number of the edges in E incident to each vertex, is four. Let $\{e_{ii_1}, \dots, e_{ii_4}\}$ be the edges incident to V_i listed in clockwise order around V_i. A *basic curve network* B_{is} is defined on E for $i = 1, \dots, N$, and $s = 1, 2$. The support of the basic curve network B_{is} consists of the two collinear edges e_{ii_s} and $e_{ii_{s+2}}$ where B_{is} is linear. More precisely, B_{is} is defined by

$$B_{is} := \begin{cases} 1 - \frac{t}{\|e_{ii_{s+r}}\|} & \text{on } e_{ii_{s+r}},\ 0 \le t \le \|e_{ii_{s+r}}\|,\ r = 0, 2, \\ 0 & \text{on the other edges of } E. \end{cases}$$

Note that basic curve networks are associated with points that have at least two collinear edges incident to them. Thus, no basic curve network is associated with the four corner points on the boundary of Q. We denote by N_B the set of pairs of indices is for which a basic curve network is defined. With each basic curve network B_{is} for $is \in N_B$ we associate a number d_{is} defined by $d_{is} = (z_{i_s} - z_i)/\|e_{ii_s}\| + (z_{i_{s+2}} - z_i)/\|e_{ii_{s+2}}\|$.

The next theorem characterizes the solution to problem $(\mathbf{P}_p)$.

Theorem 1 ([1, 14]). *Problem $(\mathbf{P}_p)$, $1 < p < \infty$, has a unique solution F^*. The second derivative of the solution $F^{*\prime\prime}$ has the form*

$$F^{*\prime\prime} = \left(\sum_{is \in N_B} \alpha_{is} B_{is} \right)_{\pm}^{q-1},$$

where $(x)_{\pm}^{r} := |x|^r \operatorname{sign}(x)$, x, $r \in \mathbb{R}$, $1/p + 1/q = 1$, and the coefficients α_{is} satisfy the following system of equations

$$\int_E \left(\sum_{is \in N_B} \alpha_{is} B_{is} \right)_{\pm}^{q-1} B_{kl}\, dt = d_{kl}, \qquad kl \in N_B. \tag{3}$$

The basic curve networks B_{is} are the univariate B-splines defined along every line and every row of the quadrangulation Q and the numbers d_{is} are the univariate second-order divided differences. The solution to $(\mathbf{P}_p)$ decomposes to $n_1 + n_2$ solutions to the problem in the univariate case along every row and every column of Q where n_1, n_2 are the numbers of the rows and columns of Q, respectively and $n_1 n_2 = N$. The problem in the univariate case is solved by de Boor [2] for $1 < p < \infty$. For $p = 2$ the solution is the natural interpolating cubic spline. Hence for problem $(\mathbf{P}_p)$ the corresponding MNN decomposes to $n_1 + n_2$ solutions to the univariate problem along every row and every column of Q. For $p = 2$ we obtain $n_1 + n_2$ natural interpolating cubic splines.

To find the solution to $(\mathbf{P}_p)$ we can solve either the nonlinear system (3) using Newton's method [14] (for $p = 2$ the system is linear) or the corresponding $n_1 + n_2$ problems in the univariate case. In the case where $q \in \mathbb{N}$, $q > 1$, then F^* is a C^1-continuous polynomial curve network and the degree of the polynomials is $q + 1$. Note that $n = q + 1$. Further on, we consider the polynomials in its Bézier form. To obtain a polynomial curve network of degree n we can proceed in one of the following ways.

1. Solve problem $(\mathbf{P}_p)$ for $p = \frac{q}{q-1} = \frac{n-1}{n-2}$.

2. Solve problem $(\mathbf{P}_2)$. Then F^* is a cubic network. We reach the required degree n after performing degree elevation $n-3$ times. The advantage of this method is that the system we have to solve to obtain F^* is linear.

We propose the following Algorithm 1 for solving the scattered data interpolation problem.

Algorithm 1 Extremal Scattered Data Interpolation

Input:	Scattered data $P_i = (x_i, y_i, z_i) \in \mathbb{R}^3$, $i = 1, \ldots, N$; $n \in \mathbb{N}$, $n \geq 3$.
Output:	Interpolation surface F with certain extremal property
Step 1.	Construct rectangular quadrangulation Q of the projection points $V_i = (x_i, y_i)$, $i = 1, \ldots, N$, using lines parallel to the axes of Oxy.
Step 2.	Add new input points to the data if necessary.
Step 3.	Solve $(\mathbf{P}_p)$ for $p = \frac{n-1}{n-2}$.
Step 4.	For each rectangle in Q find nearest to the boundary control points that satisfy G^1 continuity conditions.
Step 5.	Find the remaining inner control points so that the tensor product Bézier surface for each rectangle satisfies the tehraharmonic equation $\Delta^4 F = 0$.

Step 4 and Step 5 of Algorithm 1 are discussed in detail in Section 3 and Section 4, respectively.

3. The G^1 Continuity Conditions

3.1. Control Points That Are Nearest to a Boundary Curve

Let B_1 and B_2 be tensor product Bézier patches whose common boundary is the polynomial $q(t)$ of degree n, $n \in \mathbb{N}$. Let

$$q(t) = \sum_{i=0}^{n} \mathbf{q}_i B_i^n(t),$$

where $\mathbf{q}_i$, $i = 0, \ldots, n$, are the control points of $q(t)$, and $B_i^n(t)$ are the Bernstein polynomials defined for $0 \leq t \leq 1$ as follows:

$$B_i^n(t) := \binom{n}{i} t^i (1-t)^{n-i}, \qquad \binom{n}{i} = \begin{cases} \frac{n!}{i!(n-i)!}, & \text{for } i = 0, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathbf{p}_i$ and $\mathbf{r}_i$, $i = 0, \ldots, n$, be nearest to the boundary control points of B_1 and B_2, respectively. Let us degree elevate $q(t)$ to a polynomial of degree $n+1$

and denote the degree elevated control points by $\hat{\mathbf{q}}_i$, $i = 0, \ldots, n+1$. Then

$$q(t) = \sum_{i=0}^{n+1} \hat{\mathbf{q}}_i B_i^{n+1}(t),$$

where

$$\hat{\mathbf{q}}_i = \frac{i}{n+1}\,\mathbf{q}_{i-1} + \Big(1 - \frac{i}{n+1}\Big)\mathbf{q}_i, \qquad i = 0, \ldots, n+1. \tag{4}$$

Farin [4] proposed the following sufficient conditions for G^1 continuity between B_1 and B_2:

$$\frac{i}{n+1} d_{i,n+1} + \Big(1 - \frac{i}{n+1}\Big) d_{i,0} = 0, \qquad i = 0, \ldots, n+1, \tag{5}$$

where

$$\begin{aligned} d_{i,0} &= \alpha_0 \mathbf{p}_i + (1-\alpha_0)\mathbf{r}_i - \big(\beta_0 \hat{\mathbf{q}}_i + (1-\beta_0)\hat{\mathbf{q}}_{i+1}\big), \\ d_{i,n+1} &= \alpha_1 \mathbf{p}_{i-1} + (1-\alpha_1)\mathbf{r}_{i-1} - \big(\beta_1 \hat{\mathbf{q}}_{i-1} + (1-\beta_1)\hat{\mathbf{q}}_i\big), \end{aligned}$$

and $0 < \alpha_0, \alpha_1 < 1$. Next we shall prove that system (5) always has a solution. From (5) for $i = 0$ and $i = n+1$ we obtain

$$d_{0,0} = 0 \quad \Rightarrow \quad \alpha_0 \mathbf{p}_0 + (1-\alpha_0)\mathbf{r}_0 = \beta_0 \hat{\mathbf{q}}_0 + (1-\beta_0)\hat{\mathbf{q}}_1, \tag{6}$$

$$d_{n+1,n+1} = 0 \quad \Rightarrow \quad \alpha_1 \mathbf{p}_n + (1-\alpha_1)\mathbf{r}_n = \beta_1 \hat{\mathbf{q}}_n + (1-\beta_1)\hat{\mathbf{q}}_{n+1}. \tag{7}$$

Points $\hat{\mathbf{q}}_0$, $\hat{\mathbf{q}}_1$, $\mathbf{p}_0$, and $\mathbf{r}_0$ are coplanar since they lie on the tangent plane at $\hat{\mathbf{q}}_0$. Hence α_0 and β_0 are uniquely determined from (6) by the intersection point $\hat{\mathbf{q}}_0$ of segments $\mathbf{p}_0\mathbf{r}_0$ and $\hat{\mathbf{q}}_0\hat{\mathbf{q}}_1$. We have

$$\hat{\mathbf{q}}_0 \equiv \mathbf{q}_0 = \alpha_0 \mathbf{p}_0 + (1-\alpha_0)\mathbf{r}_0 = \beta_0 \hat{\mathbf{q}}_0 + (1-\beta_0)\hat{\mathbf{q}}_1 \;\Rightarrow\; \beta_0 = 1.$$

Analogously, α_1 and β_1 are uniquely determined by (7) and we have

$$\hat{\mathbf{q}}_{n+1} \equiv \mathbf{q}_n = \alpha_1 \mathbf{p}_n + (1-\alpha_1)\mathbf{r}_n = \beta_1 \hat{\mathbf{q}}_n + (1-\beta_1)\hat{\mathbf{q}}_{n+1} \;\Rightarrow\; \beta_1 = 0.$$

Therefore system (5) has n equations for $i = 1, \ldots, n$ and $2(n-1)$ unknowns $\mathbf{p}_1, \mathbf{r}_1, \ldots, \mathbf{p}_{n-1}, \mathbf{r}_{n-1}$. The augmented matrix M of (5) becomes

$$\left(\begin{array}{ccccccccc|c} \alpha_0 & 1-\alpha_0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & s_1 \\ \frac{2}{n-1}\alpha_1 & \frac{2}{n-1}(1-\alpha_1) & \alpha_0 & 1-\alpha_0 & 0 & 0 & \cdots & 0 & 0 & s_2 \\ 0 & 0 & \frac{3}{n-2}\alpha_1 & \frac{3}{n-2}(1-\alpha_1) & \alpha_0 & 1-\alpha_0 & \cdots & 0 & 0 & s_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & n\alpha_1 & n(1-\alpha_1) & s_n \end{array}\right)$$

where

$$\begin{aligned} s_1 &= \frac{n+1}{n}\,\hat{\mathbf{q}}_1 - \frac{1}{n}\,(\alpha_1 - \alpha_0)(\mathbf{p}_0 - \mathbf{r}_0) - \frac{1}{n}\,\hat{\mathbf{q}}_0, \\ s_i &= \frac{n+1}{n-i+1}\,\hat{\mathbf{q}}_i, \qquad i = 2, \ldots, n-1, \\ s_n &= (n+1)\hat{\mathbf{q}}_n - \alpha_0 \mathbf{p}_n - (1-\alpha_0)\mathbf{r}_n. \end{aligned} \tag{8}$$

Next we prove the following

Lemma 1. *System* (5) *with augmented matrix* M *always has a solution.*

Proof. Since F^* is a polynomial function on every $e \in E$ then the abscissae of points $\hat{\mathbf{q}}_i$, $i = 1, \dots, n$, are uniformly distributed. Moreover, B_1 and B_2 are rectangles and therefore $\alpha_0 = \alpha_1$. Then we have $s_1 = \frac{n+1}{n}\hat{\mathbf{q}}_1 - \frac{1}{n}\hat{\mathbf{q}}_0$ and $s_n = (n+1)\hat{\mathbf{q}}_n - \hat{\mathbf{q}}_{n+1}$. By Gauss elimination the elements of the first row of M become zeros and the right hand side becomes

$$\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}\hat{\mathbf{q}}_i.$$

Using (4) we obtain consecutively

$$\begin{aligned}
\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}\hat{\mathbf{q}}_i &= \sum_{i=1}^{n+1}(-1)^i\binom{n+1}{i}\frac{i}{n+1}\,\mathbf{q}_{i-1} \\
&\quad + \sum_{i=0}^{n}(-1)^i\binom{n+1}{i}\frac{n-i+1}{n+1}\,\mathbf{q}_i \\
&= \sum_{i=1}^{n+1}(-1)^i\binom{n}{i-1}\mathbf{q}_{i-1} + \sum_{i=0}^{n}(-1)^i\binom{n}{i}\mathbf{q}_i \\
&= \sum_{i=0}^{n}(-1)^{i+1}\binom{n}{i}\mathbf{q}_i + \sum_{i=0}^{n}(-1)^i\binom{n}{i}\mathbf{q}_i \\
&= \mathbf{0}.
\end{aligned}$$

Hence system (5) has $n-1$ linearly independent rows and $2(n-1)$ columns. Moreover it is compatible and therefore has a solution. □

3.2. The Vertex Enclosure Problem

Let $\mathbf{q}_0 = V_i$ for some i where V_i is an inner vertex of Q. Let B_k, $k = 1, \dots, 4$, be the four Bézier patches incident to $\mathbf{q}_0$ and $q^k(t) = \sum_{i=0}^{n+1}\hat{\mathbf{q}}_i^k B_i^{n+1}(t)$, $k = 1, \dots, 4$, be the four curves emanating from $\mathbf{q}_0$ with corresponding α_i^k, β_i^k, $i = 0, 1$. Let us denote $\lambda := \alpha_0^1 = 1 - \alpha_0^3$ and $\mu := \alpha_0^2 = 1 - \alpha_0^4$. We also have $\alpha_0^k = \alpha_1^k$, $\beta_0^k = 1$, $\beta_1^k = 0$, $k = 1, \dots, 4$. Let $\mathbf{t}_k$ be nearest to $\mathbf{q}_0$ inner control point of B_k, $k = 1, \dots, 4$, see Figure 2.

We apply (5) for $i = 1$ to B_k, $k = 1, \dots, 4$, and obtain the following linear system for $\mathbf{t}_k$, $k = 1, \dots, 4$,

$$\left|\begin{array}{lllll}
(1-\mu)\mathbf{t}_1 & +\mu\mathbf{t}_2 & & & = \mathbf{q}_1^2 \\
 & \lambda\mathbf{t}_2 & +(1-\lambda)\mathbf{t}_3 & & = \mathbf{q}_1^3 \\
 & & \mu\mathbf{t}_3 & +(1-\mu)\mathbf{t}_4 & = \mathbf{q}_1^4 \\
\lambda\mathbf{t}_1 & & & +(1-\lambda)\mathbf{t}_4 & = \mathbf{q}_1^1.
\end{array}\right. \tag{9}$$

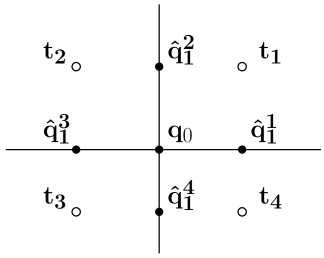

Figure 2. The vertex enclosure problem: points $\mathbf{t}_i$, $i = 1, \dots, 4$, must satisfy a linear system of equations.

The existence of a solution to system (9) is known as the *vertex enclosure problem*, see [12]. Next, we show that system (9) always has a solution.

By Gauss elimination the augmented matrix of (9) becomes

$$\left(\begin{array}{cccc|c} 1-\mu & \mu & 0 & 0 & \mathbf{q}_1^2 \\ 0 & \lambda & 1-\lambda & 0 & \mathbf{q}_1^3 \\ 0 & 0 & \mu & 1-\mu & \mathbf{q}_1^4 \\ 0 & 0 & 0 & 0 & (1-\mu)\mathbf{q}_1^1 - \lambda\mathbf{q}_1^2 + \mu\mathbf{q}_1^3 - (1-\lambda)\mathbf{q}_1^4 \end{array}\right).$$

System (9) has a solution if and only if $(1-\mu)\mathbf{q}_1^1 - \lambda\mathbf{q}_1^2 + \mu\mathbf{q}_1^3 - (1-\lambda)\mathbf{q}_1^4 = \mathbf{0}$ which is equivalent to $(1-\mu)\mathbf{q}_1^1 + \mu\mathbf{q}_1^3 - (\lambda\mathbf{q}_1^2 + (1-\lambda)\mathbf{q}_1^4) = \mathbf{q}_0 - \mathbf{q}_0 = \mathbf{0}$. Hence (9) always has a solution.

4. Construction of the Bézier Patches

4.1. Choosing Nearest to the Boundary Control Points

First, we shall prove the following

Lemma 2. *Points* $\mathbf{p_i}$, $\mathbf{r}_i$, $i = 1, \dots, n-1$, *are a solution to system* (5) *if and only if point* $\mathbf{q_i}$ *divides segment* $[\mathbf{p_i}, \mathbf{r_i}]$ *in ratio* $1-\alpha : \alpha$ *for* $i = 1, \dots, n-1$, *see Figure 3.*

Proof. Let $\mathbf{p_i}$, $\mathbf{r}_i$, $i = 1, \dots, n-1$, be a solution to system (5). From (4) we have for $i = 0, \dots, n+1$,

$$\left(\hat{\mathbf{q}}_i - \frac{i}{n+1}\mathbf{q}_{i-1}\right)\frac{n+1}{n-i+1} = \mathbf{q}_i \;\Rightarrow\; \hat{\mathbf{q}}_i - \frac{i}{n-i+1}(\mathbf{q}_{i-1} - \hat{\mathbf{q}}_i) = \mathbf{q}_i. \quad (10)$$

Since $d_{i,0} = \alpha\mathbf{p}_i + (1-\alpha)\mathbf{r}_i - \hat{\mathbf{q}}_i$ and $d_{i,n+1} = \alpha\mathbf{p}_{i-1} + (1-\alpha)\mathbf{r}_{i-1} - \hat{\mathbf{q}}_i$ then it follows from (5) that

$$\frac{i}{n+1}\big(\alpha\mathbf{p}_{i-1} + (1-\alpha)\mathbf{r}_{i-1} - \hat{\mathbf{q}}_i\big) + \frac{n-i+1}{n+1}\big(\alpha\mathbf{p}_i + (1-\alpha)\mathbf{r}_i - \hat{\mathbf{q}}_i\big) = 0.$$

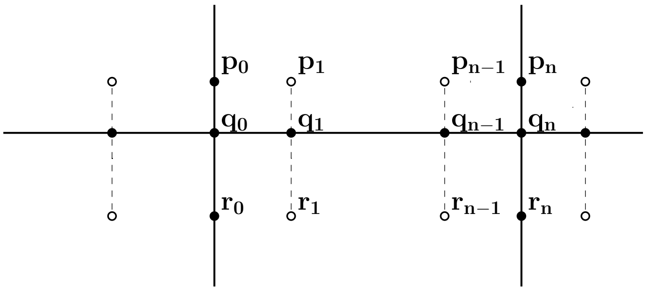

Figure 3. Point $\mathbf{q}_i$ divides segment $[\mathbf{p}_i, \mathbf{r}_i]$ in ratio $1-\alpha : \alpha$ for $i = 0, \ldots, n$.

Therefore

$$\alpha \mathbf{p}_i + (1-\alpha)\mathbf{r}_i = \hat{\mathbf{q}}_i - \frac{i}{n-i+1}\big(\alpha \mathbf{p}_{i-1} + (1-\alpha)\mathbf{r}_{i-1} - \hat{\mathbf{q}}_i\big). \tag{11}$$

We have $\hat{\mathbf{q}}_0 = \mathbf{q}_0 = \alpha \mathbf{p}_0 + (1-\alpha)\mathbf{r}_0$. Suppose that $\alpha \mathbf{p}_{i-1} + (1-\alpha)\mathbf{r}_{i-1} = \mathbf{q}_{i-1}$. Then from (11) and (10) we obtain $\alpha \mathbf{p}_i + (1-\alpha)\mathbf{r}_i = \hat{\mathbf{q}}_i - \frac{i}{n-i+1}(\mathbf{q}_{i-1} - \hat{\mathbf{q}}_i) = \mathbf{q}_i$. It follows consecutively that $\alpha \mathbf{p}_i + (1-\alpha)\mathbf{r}_i = \mathbf{q}_i$ for $i = 0, \ldots, n+1$. Therefore point $\mathbf{q}_i$ divides segment $[\mathbf{p}_i, \mathbf{r}_i]$ in ratio $1-\alpha : \alpha$ for $i = 0, \ldots, n+1$.

The proof of the other part of the lemma is straightforward. □

Obviously there are many possibilities to choose nearest to the boundary control points. We choose $\mathbf{p}_i$ and $\mathbf{r}_i$ for $i = 1, \ldots, n-1$ in the following way. We translate segments $[\mathbf{p}_0, \mathbf{r}_0]$ and $[\mathbf{p}_n, \mathbf{r}_n]$ to segments $[\mathbf{p}'_i, \mathbf{r}'_i]$ and $[\mathbf{p}''_i, \mathbf{r}''_i]$ respectively which pass through $\mathbf{q}_i$ in such a way that $\mathbf{q}_0 \to \mathbf{q}_i$ and $\mathbf{q}_n \to \mathbf{q}_i$. Then we choose $\mathbf{p}_i = (1 - \frac{i}{n})\mathbf{p}'_i + \frac{i}{n}\mathbf{p}''_i$ and $\mathbf{r}_i = (1 - \frac{i}{n})\mathbf{r}'_i + \frac{i}{n}\mathbf{r}''_i$.

4.2. Computing the Remaining Control Points

It remains to compute the rest of the control points for each tensor product Bézier patch B_i so that B_i is a solution to the tetraharmonic equation. We use a result by Centella et al. [3] who proved the following theorem.

Theorem 2 ([3]). *Given the boundary control points and those adjacent to them of an $(n+1) \times (n+1)$ net there exists a unique tetraharmonic Bézier surface whose control net has those points as boundary control points and those adjacent to them.*

To prove Theorem 2 Centella et al. [3] used the standard polynomial power basis instead of Bernstein basis. They proved that a polynomial surface

$$p(u,v) = \sum_{i,j=0}^{n} \frac{\mathbf{a}_{ij}}{i!j!} u^i v^j, \qquad \mathbf{a}_{ij} \in \mathbb{R}^3,$$

satisfies the tetraharmonic equation $\Delta^4 p(u,v) = 0$ if and only if

$$\mathbf{a}_{i+8,j} + 4\mathbf{a}_{i+6,j+2} + 6\mathbf{a}_{i+4,j+4} + 4\mathbf{a}_{i+2,j+6} + \mathbf{a}_{i,j+8} = 0, \qquad i,j \in \mathbb{N}, \tag{12}$$

where $\mathbf{a}_{ij} = \mathbf{0}$ for $i > n$ or $j > n$.

We proceed as follows. First, we find the unique solution to (12). Then we convert the polynomial basis to Bernstein basis and compute the remaining control points of the Bézier patch. Using Algorithm 1 we construct a G^1-continuous surface $F(u,v)$ defined on D which consists of tensor product Bézier patches of degree $n \times n$ and interpolates F^*. The next theorem states the extremal properties of F.

Theorem 3. *The interpolation G^1-continuous surface $F(u,v)$ consists of tensor product Bézier patches of degree $n \times n$ and satisfies the tetraharmonic equation $\Delta^4\mathbf{x} = 0$ for $(u,v) \in D \setminus E$. Consequently F is a solution to the extremal problem* (1) *and hence F is an extremum to the energy functional $\Phi(\mathbf{x}) = \frac{1}{2}\int_D \|\Delta^4\mathbf{x}\|^2 dudv$.*

Remark 1. For $n = 3$ we obtain all control points from the G^1 continuity conditions. The corresponding bicubic Bézier patches satisfy the tetraharmonic equation. For $n = 4$ we have to find exactly one inner control point from (12) for each biquartic Bézier patch.

Remark 2. To obtain G^1 continuity we do not need the adjacent control points to the boundary of Q. Nevertheless we need them to solve system (12), i.e. to find the solution to the tetraharmonic equation.

5. Conclusions and Future Work

We have presented an algorithm for interpolating scattered data in $\mathbb{R}^3$ based on MNN and G^1-continuous tensor product Bézier patches of degree $n \times n$ where n can be chosen in advance. The patches satisfy the tetraharmonic equation and consequently the obtained interpolation surface minimizes the related energy functional. Our method extends to the case where the Bézier patches satisfy the nonhomogeneous biharmonic equation $\Delta^2 F = w$ where w is a biharmonic load, i.e. $\Delta^2 w = 0$, see [3]. It would be instructive to compare the two types of interpolation surfaces.

It is an open question how to compute G^1-continuous tetraharmonic triangular Bézier patches in the case where the underlying mesh is a triangulation.

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