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Interpolation of Convex Scattered Data in $\mathbb{R}^3$ Using Edge Convex Minimum L^p-Norm Networks, $1 < p < \infty$

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Abstract. We consider the extremal problem of interpolation of scattered data in $\mathbb{R}^3$ by smooth curve networks with minimal L^p-norm of the second derivative for $1 < p < \infty$. The problem for $p = 2$ was set and solved by Nielson [7]. Andersson et al. [1] gave a new proof of Nielson's result by using a different approach. It allowed them to set and solve the constrained extremal problem of interpolation of convex scattered data in $\mathbb{R}^3$ by minimum L^2-norm networks that are convex along the edges of an associated triangulation. Partial results for the unconstrained and the constrained problems were announced without proof in [8]. The unconstrained problem for $1 < p < \infty$ was fully solved in [10]. Here we present complete characterization of the solution to the constrained problem for $1 < p < \infty$.
Keywords: Extremal scattered data interpolation, Minimum norm networks.
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INTRODUCTION

Scattered data interpolation is a fundamental problem in approximation theory and computer aided geometric design. It finds applications in a variety of fields such as automotive, aircraft and ship design, architecture, medicine, computer graphics, and more. Different methods and approaches for solving the problem were proposed and discussed, see, e. g., the surveys [4–6], and also [2, 3].

Consider the following problem: Given *scattered data* $(x_i, y_i, z_i) \in \mathbb{R}^3$, $i = 1, \ldots, n$, that is, the points $V_i = (x_i, y_i)$ are different and non-collinear, find a bivariate function $F(x, y)$ defined in a certain domain D containing the points V_i, such that F possesses continuous partial derivatives up to a given order and $F(x_i, y_i) = z_i$.

Nielson [7] proposed a three-steps method for solving the problem as follows:

Step 1. Triangulation. Construct a triangulation T of $V_i,\ i = 1, \ldots, n$. The domain D is the union of all triangles in T.

Step 2. Minimum norm network. The interpolant F and its first order partial derivatives are defined on the edges of T to satisfy an extremal property. The resulting minimum norm network is a cubic curve network, i. e. on every edge of T it is a cubic polynomial.

Step 3. Interpolation surface. The network obtained is extended to F by an appropriate *blending method*.

Andersson et al. [1] paid special attention to Step 2 of the above method, namely the construction of the minimum norm network. Using a different approach, the authors gave a new proof of Nielson's result. They constructed a system of simple linear curve networks called *basic curve networks* and then represented the second derivative of the minimum norm network as a linear combination of these basic curve networks. The new approach allows to consider and handle the case where the data are convex and we seek a convex interpolant. Andersson et al. formulate the corresponding extremal constrained interpolation problem of finding a minimum norm network that is convex along the edges of the triangulation. The extremal network is characterized as a solution of a nonlinear system of equations. The authors propose a Newton-type algorithm for solving this type of systems. The validity and convergence of the algorithm were studied further in [9].

The problem of interpolation of scattered data by minimum L^p-norms networks for $1 < p < \infty$ was considered in [8] where sufficient conditions for the solution were formulated without proof for both the unconstrained and the

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constrained problems. Recently, the unconstrained problem for $1 < p < \infty$ was fully solved in [10] where the existence and the uniqueness of the solution were proved and a complete characterization of the unique solution was presented.

In this paper we consider the constrained problem for $1 < p < \infty$. We prove the existence and the uniqueness of the solution in the case of strictly convex data and provide its complete characterization using the basic curve networks defined in [1].

PRELIMINARIES

Let $n \geq 3$ be an integer and $(x_i, y_i, z_i),\ i = 1, \dots, n$ be different points in $\mathbb{R}^3$. We call this set of points *data*. The data are *scattered* if the projections $V_i := (x_i, y_i)$ onto the plane Oxy are different and non-collinear.

Definition 1 *A collection of non-overlapping, non-degenerate triangles in Oxy is a triangulation of the points $V_i,\ i = 1, \dots, n$, if the set of the vertices of the triangles coincides with the set of the points $V_i,\ i = 1, \dots, n$.*

For a given triangulation T there is a unique continuous function $L : D \to \mathbb{R}^1$ that is linear inside each of the triangles of T and interpolates the data.

Definition 2 *Scattered data in D are convex if there exists a triangulation T of V_i such that the corresponding function L is convex. The data are strictly convex if they are convex and the gradient of L has a jump discontinuity across each edge inside D.*

Hereafter we assume that the data are convex and T is an associated triangulation of the points $V_i,\ i = 1, \dots, n$. The set of the edges of the triangles in T is denoted by E. If there is an edge between V_i and V_j in E, it will be referred to by e_{ij} or simply by e if no ambiguity arises.

Definition 3 *A curve network is a collection of real-valued univariate functions $\{f_e\}_{e\in E}$ defined on the edges in E.*

With any real-valued bivariate function F defined on D we naturally associate the curve network defined as the restriction of F on the edges in E, i. e. for $e = e_{ij} \in E$,

$$f_e(t) := F\Big(\Big(1 - \frac{t}{\|e\|}\Big)x_i + \frac{t}{\|e\|}\,x_j,\ \Big(1 - \frac{t}{\|e\|}\Big)y_i + \frac{t}{\|e\|}\,y_j\Big), \text{ where } 0 \leq t \leq \|e\| \text{ and } \|e\| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}. \quad (1)$$

Furthermore, according to the context F will denote either a real-valued bivariate function or a curve network defined by (1). For p, such that $1 < p < \infty$, we introduce the following class of *smooth interpolants*

$$\mathcal{F}_p := \Big\{F(x, y)\,|\,F(x_i, y_i) = z_i,\ i = 1, \dots, n,\ \partial F/\partial x, \partial F/\partial y \in C(D),\ f_e' \in AC_{[0,\|e\|]},\ f_e'' \in L^p_{[0,\|e\|]},\ e \in E\Big\}$$

and the corresponding class of so-called *smooth interpolation edge convex curve networks*

$$C_p(E) := \Big\{F_{|E} = \{f_e\}_{e\in E}\ |\ F(x, y) \in \mathcal{F}_p,\ f_e'' \geq 0,\ e \in E\Big\},$$

where $C(D)$ is the class of bivariate continuous functions defined in D, $AC_{[0,\|e\|]}$ is the class of univariate absolutely continuous functions defined in $[0, \|e\|]$, and $L^p_{[0,\|e\|]}$ is the corresponding Lebesgue space of univariate functions defined in $[0, \|e\|]$.

The smoothness of the interpolation curve network $F = \{f_e\}_{e\in E} \in C_p(E)$ geometrically means that if we consider the graphs of functions $\{f_e\}_{e\in E}$ as curves in $\mathbb{R}^3$ then at every point $V_i,\ i = 1, \dots, n$, they have a common tangent plane.

The L^p-norm is defined in $C_p(E)$ by

$$\|F\|_p := \left(\sum_{e\in E} \int_0^{\|e\|} |f_e(t)|^p dt\right)^{1/p}, \quad 1 < p < \infty.$$

We denote the networks of the second derivative of F by $F'' := \{f_e''\}_{e\in E}$ and consider the following extremal problem:

$$(\mathbf{P}_p) \quad \textit{Find } F^* \in C_p(E) \textit{ such that } \|F^{*\prime\prime}\| = \inf_{F\in C_p(E)} \|F''\|.$$

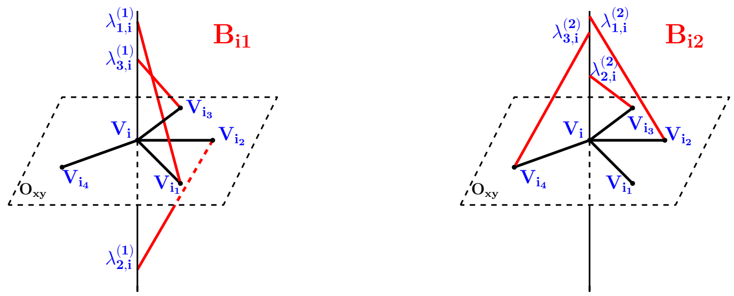

FIGURE 1. The basic curve networks for vertex V_i, $\deg(V_i) = 4$

For $i = 1, \ldots, n$, let m_i denote the degree of the vertex V_i, i. e. the number of the edges in E incident to V_i. Furthermore, let $\{e_{ii_1}, \ldots, e_{ii_{m_i}}\}$ be the edges incident to V_i listed in clockwise order around V_i. The first edge e_{ii_1} is chosen so that the coefficient $\lambda^{(s)}_{1,i}$ defined below is not zero - this is always possible. A *basic curve network* B_{is} is defined on E for any pair of indices is, such that $i = 1, \ldots, n$ and $s = 1, \ldots, m_i - 2$, as follows (see Fig. 1):

$$B_{is} := \begin{cases} \lambda^{(s)}_{r,i}\left(1 - \frac{t}{\|e_{ii_{s+r-1}}\|}\right) & \text{on } e_{ii_{s+r-1}},\ r = 1,2,3, \\ & \quad 0 \le t \le \|e_{ii_{s+r-1}}\| \\ 0 & \text{on the other edges of } E. \end{cases}$$

The coefficients $\lambda^{(s)}_{r,i}$, $r = 1, 2, 3$, are uniquely determined to sum to one and to form a zero linear combination of the three unit vectors along the edges $e_{ii_{s+r-1}}$ starting at V_i. Note that basic curve networks are associated with points that have at least three edges incident to them. We denote by N_B the set of pairs of indices is for which a basic curve network is defined, i. e., $N_B := \{is \mid m_i \ge 3,\ i = 1, \ldots, n,\ s = 1, \ldots, m_i - 2\}$.

With each basic curve network B_{is} for $is \in N_B$ we associate a number d_{is} defined by

$$d_{is} = \frac{\lambda^{(s)}_{1,i}}{\|e_{ii_s}\|}(z_{i_s} - z_i) + \frac{\lambda^{(s)}_{2,i}}{\|e_{ii_{s+1}}\|}(z_{i_{s+1}} - z_i) + \frac{\lambda^{(s)}_{3,i}}{\|e_{ii_{s+2}}\|}(z_{i_{s+2}} - z_i),$$

which reflects the position of the data in the supporting set of B_{is}.

MAIN RESULT

Andersson et al. [1] proved a theorem that characterizes the solution to $(\mathbf{P}_p)$ for $p = 2$ by using results from the theory of convex functionals in Hilbert spaces. This approach can not be generalized directly for arbitrary p, $1 < p < \infty$, due to the fact that L^p are not Hilbert spaces for $p \ne 2$. We apply a different approach and obtain the following theorem which provides a full characterization of the solution to $(\mathbf{P}_p)$.

Theorem 1 *In the case of strictly convex data problem* $(\mathbf{P}_p)$, $1 < p < \infty$, *has a unique solution* F^*. *The second derivative of the solution* $F^{*\prime\prime}$ *has the form*

$$F^{*\prime\prime} = \left(\sum_{is \in N_B} \alpha_{is} B_{is}\right)_+^{q-1}$$

where $1/p + 1/q = 1$, $(x)_+ := \max(x, 0)$ *and the coefficients* α_{is} *satisfy the following nonlinear system of equations*

$$\int_E \left(\sum_{is \in N_B} \alpha_{is} B_{is}\right)_+^{q-1} B_{kl}\,dt = d_{kl},\ \textit{for } kl \in N_B.$$

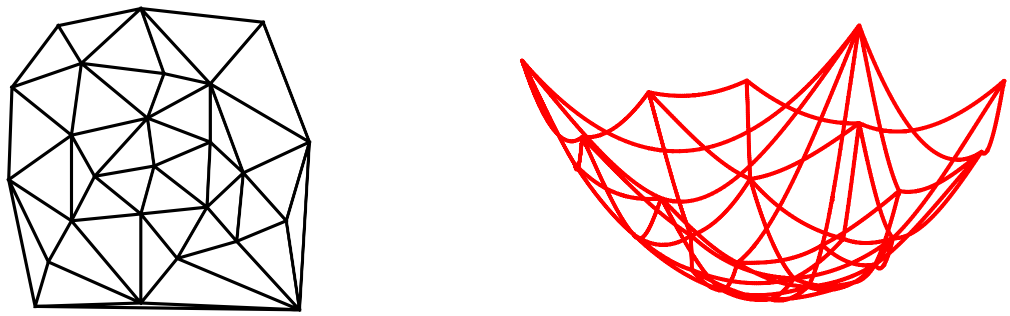

FIGURE 2. Triangulation T and the corresponding solution to $(\mathbf{P}_2)$ for $n = 25$ and data sampled from $F(x, y) = 5 * \exp((x-0.5)^2 + (y-0.5)^2)$. The computation and visualization are obtained using *Mathematica* package.

NUMERICAL EXPERIMENTS

Numerical experiments are presented and visualized to illustrate and support our results. The computation and visualization are obtained using Mathematica package. An example for the case where $p = 2$ and $n = 25$ is shown in Fig. 2. The corresponding data are sampled from the function $F(x, y) = 5 * \exp((x-0.5)^2 + (y-0.5)^2)$.

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