Interpolation of scattered data in $\mathbb{R}^3$ using minimum L_p-norm networks, $1 < p < \infty$

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Abstract

We consider the extremal problem of interpolation of scattered data in $\mathbb{R}^3$ by smooth curve networks with minimal L_p-norm of the second derivative for $1 < p < \infty$. The problem for $p = 2$ was set and solved by Nielson (1983). Andersson et al. (1995) gave a new proof of Nielson's result by using a different approach. Partial results for the problem for $1 < p < \infty$ were announced without proof in (Vlachkova, 1992). Here we present a complete characterization of the solution for $1 < p < \infty$. Numerical experiments are visualized and presented to illustrate and support our results.

Keywords: Extremal scattered data interpolation; Minimum norm network

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1. Introduction

Scattered data interpolation is a fundamental problem in approximation theory and CAGD. It finds applications in a variety of fields such as automotive, aircraft and ship design, architecture, medicine, computer graphics, and more. Recently, the problem has become particularly relevant in bioinformatics and scientific visualization. The interpolation of scattered data in $\mathbb{R}^3$ attracted a considerable amount of research. Different methods and approaches were proposed and discussed, excellent surveys are, e. g. [9, 12, 13], see also [8, 1, 3, 4, 7].

Consider the following problem: Given a set of points $(x_i, y_i, z_i) \in \mathbb{R}^3$, $i = 1, \ldots, n$, find a bivariate function $F(x, y)$ defined in a certain domain D containing points $V_i = (x_i, y_i)$, such that F possesses continuous partial derivatives up to a given order and $F(x_i, y_i) = z_i$.

Nielson [14] proposed a three steps method for solving the problem as follows:

Step 1. Triangulation. Construct a triangulation T of V_i, $i = 1, \ldots n$.

Step 2. Minimum norm network. The interpolant F and its first order partial derivatives are defined on the edges of T to satisfy an extremal property. The obtained minimum norm network is a cubic curve network, i. e. on every edge of T it is a cubic polynomial.

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Step 3. Interpolation surface. The obtained network is extended to F by an appropriate *blending method*.

Andersson et al. [2] paid special attention to Step 2 of the above method, namely the construction of the minimum norm network. Using a different approach, the authors gave a new proof of Nielson's result. They constructed a system of simple linear curve networks called *basic curve networks* and then represented the second derivative of the minimum norm network as a linear combination of these basic curve networks.

The problem of interpolation of scattered data by minimum L_p-norms networks for $1 < p < \infty$ was considered in [17] where sufficient conditions for the solution were formulated without proof. In this paper we prove the existence and the uniqueness of the solution to the problem for $1 < p < \infty$ and provide its complete characterization using the basic curve networks defined in [2].

The paper is organized as follows. In Sect. 2 we introduce notation, formulate the extremal problem for interpolation by minimum L_p-norms networks for $1 < p < \infty$, and present some related results. In Sect. 3 we prove the existence and the uniqueness of the solution to the problem for $1 < p < \infty$. In Sect. 4 we establish a full characterization of the solution. In final Sect. 4 we present the results from our experimental work. Based on numerical solving of nonlinear systems of equations we apply computer modeling and visualization tools to illustrate and support our results.

2. Preliminaries and related results

Let $n \geq 3$ be an integer and $P_i := (x_i, y_i, z_i),\ i = 1, \ldots, n$ be different points in $\mathbb{R}^3$. We call this set of points *data*. The data are *scattered*[1] if the projections $V_i := (x_i, y_i)$ onto the plane Oxy are different and non-collinear.

Definition 1. A collection of non-overlapping, non-degenerate triangles in Oxy is a *triangulation* of the points $V_i,\ i = 1, \ldots, n$, if the set of the vertices of the triangles coincides with the set of the points $V_i,\ i = 1, \ldots, n$.

Hereafter we assume that a triangulation T of the points $V_i,\ i = 1, \ldots, n$, is given and fixed. Furthermore, for the sake of simplicity, we assume that the domain D formed by the union of the triangles in T is connected. In general D is a collection of polygons with holes. The set of the edges of the triangles in T is denoted by E. If there is an edge between V_i and V_j in E, it will be referred to by e_{ij} or simply by e if no ambiguity arises.

Definition 2. A *curve network* is a collection of real-valued univariate functions $\{f_e\}_{e \in E}$ defined on the edges in E.

[1] Note that this definition of scattered data slightly misuses commonly accepted meaning of the term. It allows data with some structure among points V_i. We have opted to do this in order to cover all cases where our presentation and results are valid.

With any real-valued bivariate function F defined on D we naturally associate the curve network defined as the restriction of F on the edges in E, i. e. for $e = e_{ij} \in E$,

$$\begin{aligned} f_e(t) &:= F\Big((1 - \frac{t}{\|e\|})x_i + \frac{t}{\|e\|}\,x_j,\ (1 - \frac{t}{\|e\|})y_i + \frac{t}{\|e\|}\,y_j\Big), \\ &\text{where } 0 \le t \le \|e\| \text{ and } \|e\| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}. \end{aligned} \tag{1}$$

Furthermore, according to the context F will denote either a real-valued bivariate function or a curve network defined by (1). For p, such that $1 < p < \infty$, we introduce the following class of *smooth interpolants*

$$\mathcal{F}_p := \{F(x,y)\,|\,F(x_i,y_i) = z_i,\ i = 1,\dots,n,\ \partial F/\partial x, \partial F/\partial y \in C(D),\ f'_e \in AC,\ f''_e\ \in L_p,\ e \in E\},$$

where $C(D)$ is the class of bivariate continuous functions defined in D, AC is the class of univariate absolutely continuous functions defined in $[0, \|e\|]$, and L_p is the class of univariate functions defined in $[0, \|e\|]$ whose p-th power of the absolute value is Lebesgue integrable. The restrictions on E of the functions in $\mathcal{F}_p$ form the corresponding class of so-called *smooth interpolation curve networks*

$$\mathcal{C}_p(E) := \left\{F_{|E} = \{f_e\}_{e\in E}\ |\ F(x,y) \in \mathcal{F}_p,\ e \in E\right\}. \tag{2}$$

We note that the class $\mathcal{F}_p$ is nonempty since, e. g. Clough-Tocher [5] and Powell-Sabin [15] interpolants belong to it. Hence $\mathcal{C}_p(E)$ is nonempty too. The smoothness of the interpolation curve network $F \in \mathcal{C}_p(E)$ geometrically means that at each point P_i there is a *tangent plane* to F, where a plane is *tangent* to the curve network at a point P_i if it contains the tangent vectors at P_i of the curves incident to P_i.

Inner product and L_p-norm are defined in $\mathcal{C}_p(E)$ by

$$\langle F, G\rangle = \int_E FG = \sum_{e\in E}\int_0^{\|e\|} f_e(t)g_e(t)dt,$$

$$\|F\|_p := \left(\sum_{e\in E}\int_0^{\|e\|} |f_e(t)|^p dt\right)^{1/p}, \quad 1 < p < \infty,$$

where $F \in \mathcal{C}_p(E)$ and $G := \{g_e\}_{e\in E} \in \mathcal{C}_p(E)$. We denote the networks of the second derivative of F by $F'' := \{f''_e\}_{e\in E}$ and consider the following extremal problem:

$$(\mathbf{P}_p) \qquad \textit{Find } F^* \in \mathcal{C}_p(E) \ \textit{ such that } \|F^{*\prime\prime}\|_p = \inf_{F\in\mathcal{C}_p(E)} \|F''\|_p.$$

Problem (P_p) is a generalization of the classical univariate extremal problem $(\tilde{P}_p)$ for interpolation of data in $\mathbb{R}^2$ by a univariate function with minimal L_p-norm of the second derivative. The latter was studied by Holladay [11] for $p = 2$ and by de Boor [6] in more general settings.[2] Holladay [11] proved that the natural interpolating cubic spline is the unique solution to $(\tilde{P}_2)$. Nielson's approach to construct minimum norm network can be seen as an extension of Holladay's proof [11].

[2] C. de Boor studied the more general problem of minimum L_p-norm of the k-th derivative, $k \ge 1$, $1 < p \le \infty$.

For $i = 1, \dots, n$ let m_i denote the degree of the vertex V_i, i. e. the number of the edges in E incident to V_i. Furthermore, let $\{e_{ii_1}, \dots, e_{ii_{m_i}}\}$ be the edges incident to V_i listed in counterclockwise order around V_i (see Fig. 1). The first edge e_{ii_1} is chosen so that the coefficient $\lambda_{1,i}^{(s)}$ defined below is not zero - this is always possible. A *basic curve network* B_{is} is defined on E for any pair of indices is, such that $i = 1, \dots, n$ and $s = 1, \dots, m_i - 2$, as follows (see Fig. 1):

$$B_{is} := \begin{cases} \lambda_{r,i}^{(s)} \left(1 - \frac{t}{\|e_{ii_{s+r-1}}\|}\right) & \text{on } e_{ii_{s+r-1}},\ r = 1, 2, 3, \\ & \quad 0 \le t \le \|e_{ii_{s+r-1}}\| \\ 0 & \text{on the other edges of } E. \end{cases} \tag{3}$$

The coefficients $\lambda_{r,i}^{(s)}$, $r = 1, 2, 3$, are uniquely determined to sum to one and to form a zero linear combination of the three unit vectors along the edges $e_{ii_{s+r-1}}$ starting at V_i.

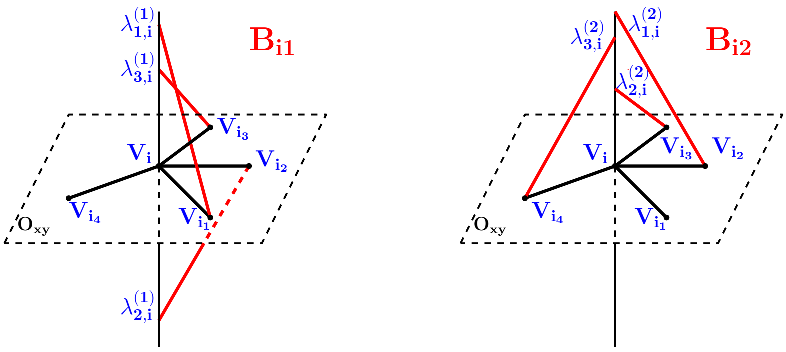

Figure 1: The basic curve networks for vertex V_i of degree $m_i = 4$

Note that basic curve networks are associated with points that have at least three edges incident to them. We denote by N_B the set of pairs of indices is for which a basic curve network is defined, i. e.,

$$N_B := \{is \mid m_i \ge 3,\ i = 1, \dots, n,\ s = 1, \dots, m_i - 2\}.$$

With each basic curve network B_{is} for $is \in N_B$ we associate a number d_{is} defined by

$$d_{is} = \frac{\lambda_{1,i}^{(s)}}{\|e_{ii_s}\|}(z_{i_s} - z_i) + \frac{\lambda_{2,i}^{(s)}}{\|e_{ii_{s+1}}\|}(z_{i_{s+1}} - z_i) + \frac{\lambda_{3,i}^{(s)}}{\|e_{ii_{s+2}}\|}(z_{i_{s+2}} - z_i),$$

which reflects the position of the data in the supporting set of B_{is}. The following two lemmas are proved in [2] for $p = 2$ but they clearly hold for any p, $1 < p < \infty$.

Lemma 1. *Functions* B_{is}, $is \in N_B$, *are linearly independent in* E.

Lemma 2. $F \in \mathcal{C}_p(E)$ *if and only if* $\langle F'', B_{is} \rangle = d_{is}$, $is \in N_B$.

3. Existence and uniqueness of the solution

In the next theorem we prove that problem (P_p) for $1 < p < \infty$ always has a unique solution which we call *optimal curve network*.

Theorem 1. *The extremal problem (P_p) for $1 < p < \infty$ always has a unique solution.*

Proof. Let $1 < p < \infty$. We recall that $\mathcal{C}_p(E) \neq \emptyset$. The set of real non-negative numbers $\{\|F''\|_p \mid F \in \mathcal{C}_p(E)\}$ is bounded from below and therefore it has a greatest lower bound $d := \inf_{F\in\mathcal{C}_p(E)} \|F''\|_p$. Let $\{F_\nu\}_{\nu=1}^{\infty}$, where $F_\nu := \{f_{e,\nu}\}_{e\in E} \in \mathcal{C}_p(E)$, be a minimizing sequence, i. e. $\lim_{\nu\to\infty} \|F''_\nu\|_p = d$. We denote

$$L_p(E) := \{G = \{g_e\}_{e\in E} \ : \ g_e \in L_p, \ e \in E\}. \tag{4}$$

Next we prove that $\{F''_\nu\}_{\nu=1}^{\infty}$ is a fundamental sequence in $L_p(E)$, i. e. $\lim_{\nu,\ \mu\to\infty} \|F''_\nu - F''_\mu\|_p = 0$. For this purpose we use the following Clarkson's inequalities (see [10], pp. 225, 227) which hold for $f, g \in L_p, \ 1 < p < \infty, \ 1/p + 1/q = 1$,

$$\begin{aligned} \|\frac{f+g}{2}\|_p^p + \|\frac{f-g}{2}\|_p^p &\leq \frac{1}{2}\left(\|f\|_p^p + \|g\|_p^p\right), \text{ if } p \geq 2, \\ \|\frac{f+g}{2}\|_p^q + \|\frac{f-g}{2}\|_p^q &\leq (\frac{1}{2}(\|f\|_p^p + \|g\|_p^p))^{1/(p-1)}, \text{ if } 1 < p < 2. \end{aligned} \tag{5}$$

Since $F_\nu, F_\mu \in \mathcal{C}_p(E)$ then $(F_\nu + F_\mu)/2 \in \mathcal{C}_p(E)$. Hence $\|(F''_\nu + F''_\mu)/2\|_p \geq d$. Let $p \geq 2$. From the first inequality in (5) we obtain

$$\begin{aligned} \|\frac{F''_\nu - F''_\mu}{2}\|_p^p &\leq \frac{1}{2}(\|F''_\nu\|_p^p + \|F''_\mu\|_p^p) - \|\frac{F''_\nu + F''_\mu}{2}\|_p^p \\ &\leq \frac{1}{2}(\|F''_\nu\|_p^p + \|F''_\mu\|_p^p) - d^p \to \frac{1}{2}(2d^p) - d^p = 0 \text{ when } \nu, \mu \to \infty. \end{aligned}$$

For $1 < p < 2$ from the second inequality in (5) we obtain

$$\begin{aligned} \|\frac{F''_\nu - F''_\mu}{2}\|_p^q &\leq (\frac{1}{2}(\|F''_\nu\|_p^p + \|F''_\mu\|_p^p))^{1/(p-1)} - \|\frac{F''_\nu + F''_\mu}{2}\|_p^q \\ &\leq (\frac{1}{2}(\|F''_\nu\|_p^p + \|F''_\mu\|_p^p))^{1/(p-1)} - d^q \\ &\quad \to (\frac{1}{2}(2d^p))^{1/(p-1)} - d^q = 0 \text{ when } \nu, \mu \to \infty. \end{aligned}$$

Therefore $\lim_{\nu,\ \mu\to\infty} \|F''_\nu - F''_\mu\|_p = 0$ for every $p, \ 1 < p < \infty$, i. e. $\{F''_\nu\}_{\nu=1}^{\infty}$ is a fundamental sequence. Since L_p is a complete space then there exists curve network $G = \{g_e\}_{e\in E} \in L_p(E)$ such that $g_e \in L_p$ for every $e \in E$ and

$$\lim_{\nu\to\infty} \|F''_\nu - G\|_p = 0. \tag{6}$$

From (6) it follows that there exists a subsequence of $\{F''_\nu\}_{\nu=1}^{\infty}$ that converges pointwise almost everywhere (a.e.) to G. For simplicity we assume that $\{F''_\nu\}_{\nu=1}^{\infty}$ is that subsequence, i. e. $\lim_{\nu\to\infty} f''_{e,\nu}(t) =$

$g_e(t)$ a.e. in $[0, \|e\|]$. Moreover, from the continuity of the norm we have $\lim_{\nu\to\infty} \|F''_\nu\|_p = \|G\|_p$, and hence $\|G\|_p = d$.

Let $F = \{f_e\}_{e\in E}$ be the unique curve network that satisfies the interpolation conditions $F(V_i) = z_i,\ i = 1, \dots, n$ and its second derivative F'' coincides a.e. with G. To prove the existence of the solution to the problem, next we show that $F = \{f_e\}_{e\in E}$ belongs to $\mathcal{C}_p(E)$. We have to show that for every vertex V_i there exists a tangent plane to the curve network F. First, we prove that

$$\lim_{\nu\to\infty} f'_{e,\nu}(t) = f'_e(t) \text{ for every } \ t \in [0, \|e\|]. \tag{7}$$

Since $f''_e \in L_p$ then $f'_e \in AC$ and since $\lim_{\nu\to\infty} f''_{e,\nu}(t) = f''_e(t)$ a.e. in $[0, \|e\|]$, we have

$$\lim_{\nu\to\infty} \int_0^t f''_{e,\nu}(u)du = \int_0^t f''_e(u)du \text{ for every } t \in [0, \|e\|],$$

hence

$$\lim_{\nu\to\infty} \left(f'_{e,\nu}(t) - f'_{e,\nu}(0)\right) = f'_e(t) - f'_e(0). \tag{8}$$

From (8) after integration we obtain

$$\lim_{\nu\to\infty} \left(f_{e,\nu}(t) - f_{e,\nu}(0) - tf'_{e,\nu}(0)\right) = f_e(t) - f_e(0) - tf'_e(0),$$

hence from the interpolation conditions we have

$$\lim_{\nu\to\infty} \left(f_{e,\nu}(t) - tf'_{e,\nu}(0)\right) = f_e(t) - tf'_e(0) \text{ for every } t \in [0, \|e\|].$$

In particular, for $t = \|e\|$ we obtain $\lim_{\nu\to\infty} f'_{e,\nu}(0) = f'_e(0)$ and (7) follows from (8).

Further on, if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are vectors in $\mathbb{R}^3$, we denote by $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ their scalar triple product.

Functions $f_e,\ e \in E$, defined by (1) are parametric curves in $\mathbb{R}^3$ represented by $(x = x_e(t),\ y = y_e(t),\ z = z_e(t))$. Then vector $\mathbf{t_e} \in \mathbb{R}^3$ defined by $\mathbf{t_e} := (\frac{x_j - x_i}{\|e_{ij}\|}, \frac{y_j - y_i}{\|e_{ij}\|}, f'_{ij}(0))$ is a tangent vector to curve f_e at point P_i. Let V_i be a vertex in T of degree $m_i \geq 3$ (if $m_i = 2$ then a tangent plane at P_i always exists). Let e_1, e_2, and e_3 be three arbitrary edges incident to V_i. Since $F_\nu \in \mathcal{C}_p(E)$ then F_ν has a tangent plane at P_i. A necessary and sufficient condition for the existence of such a plane is

$$(\mathbf{t_{e_1}}_{,\nu}, \mathbf{t_{e_2}}_{,\nu}, \mathbf{t_{e_3}}_{,\nu}) = 0, \tag{9}$$

where $\mathbf{t_{e_1}}_{,\nu}$, $\mathbf{t_{e_2}}_{,\nu}$, and $\mathbf{t_{e_3}}_{,\nu}$ are the three tangent vectors. We take the limit in (9) for $\nu \to \infty$, use (7), and obtain that the scalar triple product of the limit vectors $\mathbf{t_{e_1}}, \mathbf{t_{e_2}}, \mathbf{t_{e_3}}$ is zero too. The three edges e_1, e_2, and e_3 are arbitrarily chosen, hence the curve network F has a tangent plane at point P_i, which has been arbitrarily chosen too. Therefore $F = \{f_e\}_{e\in E}$ belongs to $\mathcal{C}_p(E)$ and solves problem (P_p).

It remains to prove uniqueness of the solution. Let F_0 and F_1 be two solutions of (P_p). Then $\frac{1}{2}(F_0 + F_1) \in \mathcal{C}_p(E)$. From Minkowski's inequality it follows

$$\|\frac{1}{2}(F''_0 + F''_1)\|_p \leq \frac{1}{2}\|F''_0\|_p + \frac{1}{2}\|F''_1\|_p = \|F''_0\|_p = \|F''_1\|_p. \tag{10}$$

Hence, in (10) we have equality which holds if and only if $aF_0'' = bF_1''$ a.e., where a and b are non-negative real numbers such that $a^2 + b^2 > 0$. From the equality of the norms it follows $a = b = 1$, i. e. $F_0'' = F_1''$ a.e. which means $f_{0,e}''(t) = f_{1,e}''(t)$ a.e. in $[0, \|e\|]$ for every $e \in E$. Since $f_{0,e}(t)$ and $f_{1,e}(t)$ coincide at the endpoints of the edge e then $f_{0,e}(t) = f_{1,e}(t)$ for every $t \in [0, \|e\|]$. Therefore $F_0 \equiv F_1$. □

4. Characterization of the solution

In this section we provide a full characterization of the solution F^* to the extremal problem (P_p) for $1 < p < \infty$. Its existence and uniqueness have been already established in Theorem 1. Further, for simplicity we use the notation $(x)_\pm^r := |x|^r \mathrm{sign}(x),\ r \in \mathbb{R},\ x \in \mathbb{R}$. Next we prove that $(F^{*\prime\prime})_\pm^{p-1}$ can be represented as a linear combination of the basis curve networks defined by (3). Finding of F^* reduces to the unique solution of a system of equations. The following theorem holds.

Theorem 2. *Smooth interpolation curve network $F^* = \{f_e^*\}_{e\in E} \in \mathcal{C}_p(E)$ is a solution to problem (P_p), $1 < p < \infty$, if and only if*

$$F^{*\prime\prime} = \left(\sum_{is\in N_B} \alpha_{is} B_{is} \right)_\pm^{q-1},$$

where α_{is} are real numbers and $1/p + 1/q = 1$.

Proof. Let us consider the set of interpolation curve networks

$$\Gamma(E) := \{F = \{f_e\}_{e\in E} : F(V_i) = z_i,\ i = 1, \ldots, n,\ f_e' \in AC, f_e'' \in L_p,\ e \in E\}$$

and the mapping

$$\Gamma(E) \ni F = \{f_e\}_{e\in E} \ \mapsto\ \{g_e\}_{e\in E} = G \in L_p(E), \tag{11}$$

where $G = \{g_e\}_{e\in E}$ is such that $g_e = f_e''$, and the class $L_p(E)$ is defined by (4). If $F \in \Gamma(E)$ then obviously $F'' = \{f_e''\}_{e\in E} \in L_p(E)$. Now let $G = \{g_e\}_{e\in E}$ belong to $L_p(E)$. We integrate twice the function $g_e,\ e = e_{ij} \in E$, use the two interpolation conditions $F(V_i) = z_i$ and $F(V_j) = z_j$ at the end of the interval $[0, \|e\|]$, and obtain curve network $F = \{f_e\}_{e\in E}$ such that $F'' = G$ and $F \in \Gamma(E)$. Therefore the mapping (11) is a bijection. According to Lemma 2, (11) maps the set $\mathcal{C}_p(E) \subset \Gamma(E)$ defined by (2) onto the following subset of $L_p(E)$:

$$\{G\ :\ G \in L_p(E),\ \int_E G B_{is} dt = d_{is},\ is \in N_B\}.$$

Thus, problem (P_p), $1 < p < \infty$ is equivalent to the following problem

$$(\mathbf{P}_p) \qquad \begin{array}{c} \textit{Find } \tilde{G} \in \mathcal{C}_p(E) \ \ \textit{such that } \ \|\tilde{G}\|_p = \inf\limits_{G\in L_p(E)} \|G\|_p, \\ \textit{under the conditions } \displaystyle\int_E G(t) B_{is}(t) dt = d_{is},\ is \in N_B. \end{array} \tag{12}$$

Using the Lagrange multipliers (see, e.g, [16], pp. 113) we obtain that $\tilde{G}$ ($F^{*\prime\prime}$, respectively) is a solution to problem (P_p) for $1 < p < \infty$ if and only if there exist real numbers $\lambda_{is},\ is \in N_B$ such that $\tilde{G}$ is a solution to the problem

$$\inf_{G \in L_p(E)} \left(\int_E (|G(t)|^p - \sum_{is \in N_B} \lambda_{is} G(t) B_{is}(t)) dt + \sum_{is \in N_B} \lambda_{is} d_{is} \right). \tag{13}$$

Moreover, the partial derivative w.r.t. G of the expression in the integral in (13) is zero for the extremal function $\tilde{G}$ (it follows from the Euler equation, see [16], pp. 94). Hence,

$$p|\tilde{G}|^{p-1}\mathrm{sign}(\tilde{G}) - \sum_{is \in N_B} \lambda_{is} B_{is} = 0.$$

From the last inequality it follows that the solution $\tilde{G}$ ($F^{*\prime\prime}$, respectively) has the form

$$F^{*\prime\prime} = \tilde{G} = \left(\sum_{is \in N_B} \alpha_{is} B_{is} \right)_{\pm}^{q-1}, \tag{14}$$

where $\alpha_{is} = \lambda_{is}/p,\ is \in N_B$, are real numbers and $1/p + 1/q = 1$. Moreover, the representation (14) is unique. □

As a consequence, we can formulate the following theorem.

Theorem 3. *Curve network $F \in \mathcal{C}_p(E)$ solves problem (P_p) for $1 < p < \infty$ if and only if $F'' = \left(\sum_{is \in N_B} \alpha_{is} B_{is}\right)_{\pm}^{q-1}$. The coefficients α_{is} are the unique solution to the following system of equations*

$$\int_E \left(\sum_{is \in N_B} \alpha_{is} B_{is} \right)_{\pm}^{q-1} B_{kl} dt = d_{kl},\ kl \in N_B. \tag{15}$$

Proof. From Theorem 2 it follows that there exist real numbers $\alpha_{is},\ is \in N_B$ such that the second derivative of the unique solution F to problem (P_p) for $1 < p < \infty$ is $F'' = (\sum_{is \in N_B} \alpha_{is} B_{is})_{\pm}^{q-1}$. On the other hand, according to Lemma 2, F is a smooth interpolation curve network if and only if $\langle F'', B_{kl} \rangle = d_{kl}$ for every $kl \in N_B$. Therefore numbers α_{is} are a solution to system (15).

The uniqueness of the solution to system (15) follows from the uniqueness of the optimal curve network and from the linear independence of the basic curve networks $B_{is},\ is \in N_B$, provided by Lemma 1. □

5. Examples and results

To find the minimum L_p-norm networks for $1 < p < \infty$ we have to solve system (15) which is nonlinear except in the case where $p = 2$ when it is linear. We have adopted the Newton-type algorithm from [18] to solve this type of systems. Its validity and convergence have been studied in detail there. It was shown that under certain conditions the method achieves superlinear (quadratic for $1 < p \leq 2$) convergence.

We use Mathematica package to visualize the extremal curve networks. Below we present results of our experiments where solutions of (P_p) were computed and visualized for different p on two small data sets.

Example 1. We consider data obtained from a regular triangular pyramid. We have $n = 4$, $V_1 = (-1/2, -\sqrt{3}/6)$, $V_2 = (1/2, -\sqrt{3}/6)$, $V_3 = (0, \sqrt{3}/3)$, $V_4 = (0, 0)$, and $z_i = 0$, $i = 1, 2, 3$, $z_4 = -1/2$. The set of indices defining the edges of the corresponding triangulation is $N_B = \{12, 23, 31, 41, 42, 43\}$. We have $m_i = 3$ for $i = 1, \ldots, 4$ and four basic curve networks B_{is}, $i = 1, \ldots, 4$, $s = 1$, are defined. The triangulation, the minimum L_p-norms network F_p, and the corresponding L_p-norms of the second derivatives $\|F_p''\|_p$ for $p = 2$, 3, and 6 are shown in Fig. 2 (left).

Example 2. We have $n = 7$ and the data are $P_1 = (-2, 0, 0)$, $P_2 = (-1.6, 0, -2)$, $P_3 = (0, 0, -3)$, $P_4 = (1.6, 0, -2.5)$, $P_5 = (2, 0, 0)$, $P_6 = (-0.5, 2.3, -1.7)$, and $P_7 = (0.5, -2, -1.9)$. The set of indices defining the edges of the corresponding triangulation T_2 is $N_B = \{17, 12, 16, 27, 23, 26, 37, 34, 36, 45, 46, 47, 56, 57\}$. We have $m_1 = m_5 = 3$, $m_2 = m_3 = m_4 = 4$, $m_6 = m_7 = 5$, and hence, the number of the basic curve networks B_{is} is fourteen. The triangulation, the minimum L_p-norms network F_p, and the corresponding L_p-norms of the second derivatives $\|F_p''\|_p$ for $p = 2$, 3, and 6 are shown in Fig. 2 (right).

6. Conclusions and future work

In this paper we considered the extremal problem of interpolation of scattered data in $\mathbb{R}^3$ by smooth minimum L_p-norm networks for $1 < p < \infty$. We proved the existence and the uniqueness of the solution for $1 < p < \infty$ and provided its complete characterization. We presented numerical experiments and gave examples to visualize and support the obtained results.

The case $p = \infty$ is not completely understood and needs to be studied further. First of all it is known that the solution in this case is not unique. Second, the approach based on Lagrange multipliers can not be applied directly to the case $p = \infty$. Another interesting question that arises is whether the sequence of solutions for $1 < p < \infty$ converges as $p \to \infty$, and if yes, what is the limit?

As it was pointed out in the introduction minimum norm interpolation networks combined with an appropriate blending method can be used to construct approximations to an unknown sampled surface (function). It would be useful to study, analyze and evaluate the quality of such approximations under certain assumptions on the sampled function.

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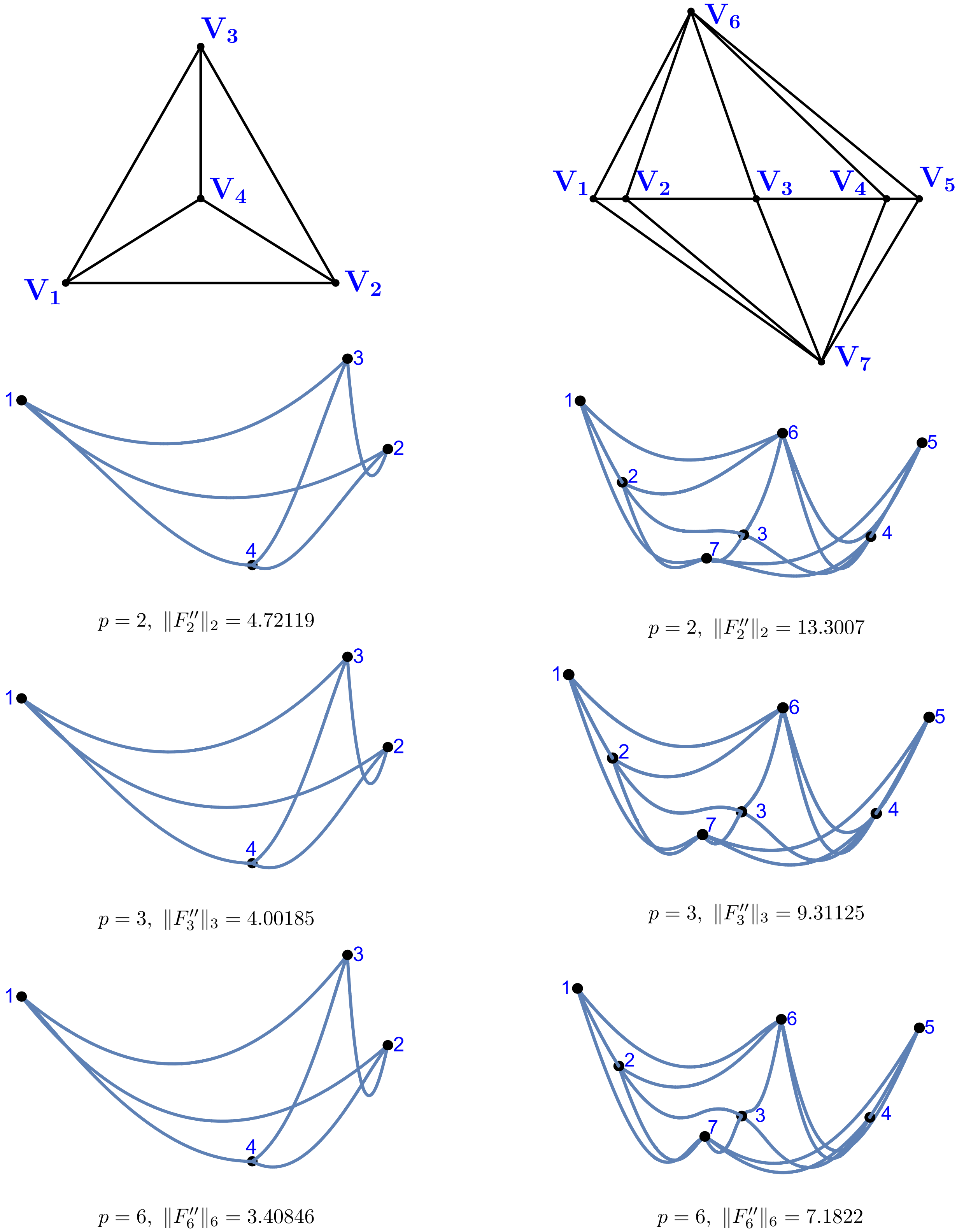

Figure 2: The triangulation, the minimum L_p-norm networks F_p, and the corresponding L_p-norms $\|F_p''\|_p$ for $p = 2$, 3, and 6 for Example 1 (left), and Example 2 (right).