EFFECTIVE STRUCTURES

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ABSTRACT. The connection between some special properties of the computable functions in an abstract structure and the existence of a recursive and semi-recursive representation of the structure is studied. The considerations are based on the notion of computability on many-sorted abstract structures.

1. INTRODUCTION

In the present paper we continue the investigation of the notion of computability on many-sorted abstract structures introduced in [17]. The main features of that computability are its invariance with respect to strong homomorphisms and effectiveness, i.e. the computable functions on a structure over the natural numbers are partial recursive relative to the initial functions and predicates of the structure. We consider two kinds of sorts - "*effectively enumerable*" and "*general*" ones. During the computation of a function θ we allow a search trough the data of the effectively enumerable sorts while for the general sorts such a search is not allowed. So applied to a single-sorted structure, this computability is actually the Search computability of Moschovakis [11, 14] if the domain of the structure is assumed to be effectively enumerable and the computability by means of Recursively Enumerable Definitional Schemes of Friedman and Shepherdson [8, 13, 15], otherwise.

From the results in [17] it follows that under certain natural conditions our notion of computability is the strongest possible. Therefore, we have evidence to accept a version of the generalized Church-Turing thesis stating that every computable on a many-sorted structure function is computable in the sence of the definition from [17].

In this paper we consider the so called "*effective*" abstract structures with respect to this computability. The intuition behind this notion is the following. We can identify an abstract many-sorted structure with the initial data types of a programming

1991 *Mathematics Subject Classification.* 03D45, 03D75.

Key words and phrases. Many-sorted structure, Computability, Enumerations, Definability, Effective structure.

This work was partially supported by the Ministry of education, science and technologies, Contract I 412/94.

I thank the referee for the many errors he found and for his numerous remarks which prompted me to simplify the text in several places.

language $\mathfrak{L}$. Suppose that the control part of $\mathfrak{L}$ is rich enough so that the functions programmable in $\mathfrak{L}$ are exactly those which are computable in our sense. Here we are interested in those data types which if added as new sorts to $\mathfrak{L}$, will not increase the class of the functions of the originally given sorts of $\mathfrak{L}$, programmable in the enriched language. Let's call such data types "*effective*" with respect to $\mathfrak{L}$.

"*Effective*" structures are those which are effective with respect to all programming languages.

Following our notion of computability, we can add an abstract structure to a given many-sorted structure in two ways – as a general sort or as an effectively enumerable sort. In the first case, we shall prove that a structure is effective if and only if it admits a semi-recursive representation. In the second case, we shall show that a structure is effective if and only if it admits a semi-recursive representation with a recursively enumerable existential diagram. Several examples will be presented which show some differences between the considered notions.

Another natural problem is to characterize those effective data types for which there exists an uniform effective procedure translating each program using these new data types into an equivalent program of the original language $\mathfrak{L}$. Such data types are called *uniformly effective* with respect to $\mathfrak{L}$. Here we obtain that a data type is uniformly effective with respect to all programming languages if and only if it admits a recursive representation.

2. Preliminaries

First we shall describe the notion of computability from [17]. Let a many-sorted signature $\mathbb{L} = (\mathbb{S}, \mathbb{E}, \mathbb{F}, \mathbb{P}, \rho)$ be fixed. Here $\mathbb{S} = \{1, \dots, m\}$ is the set of sorts; $\mathbb{E} \subseteq \mathbb{S}$ is the set of the effectively enumerable sorts; $\mathbb{F} = \{f_1, \dots, f_l\}$ is the set of functional symbols; $\mathbb{P} = \{P_1, \dots, P_r\}$ is the set of predicate symbols; and ρ is a mapping which assigns to each f_i of $\mathbb{F}$ a type $\rho(f_i)$ over $\mathbb{S}$ of the form $(s_1, \dots, s_{a_i}, s)$, where $s_1, \dots, s_{a_i}$ are the sorts of the arguments, and s is the sort of the result and to each P_j of $\mathbb{P}$ a type $\rho(P_j)$ over $\mathbb{S}$ of the form $(s_1, \dots, s_{b_j})$, for some $s_1, \dots, s_{b_j}$ of $\mathbb{S}$. The only difference from the usual definition is that we include the set $\mathbb{E}$ of the effectively enumerable sorts as a part of $\mathbb{L}$.

Let $\mathfrak{A} = (A_1, A_2, \dots, A_m;\ \psi_1, \psi_2, \dots, \psi_l;\ \Delta_1, \Delta_2, \dots, \Delta_r)$ be a partial many-sorted structure of signature $\mathbb{L}$, where for all $s \in \mathbb{S}$, the basic set of sort s, A_s, is denumerable and non empty; $\psi_1, \psi_2, \dots, \psi_l$ are the initial partial functions, $\psi_i \colon A_{s_1} \times \dots \times A_{s_{a_i}} \rightharpoonup A_s$, i.e. ψ_i is a partial function on $A_{s_1} \times \dots \times A_{s_{a_i}}$ to A_s; $\Delta_1, \Delta_2, \dots, \Delta_r$ are the initial partial predicates, $\Delta_j : A_{s_1} \times \dots \times A_{s_{b_j}} \rightharpoonup \{0, 1\}$ (0 for true, 1 for false). If θ is a partial function on $A_{s_1} \times \dots \times A_{s_a}$ to A_s, for some $s_1, \dots, s_a, s \in \mathbb{S}$, then we shall say that the function θ of type $(s_1, \dots, s_a, s)$ is *correctly defined*. By $\mathfrak{F}_{\mathfrak{A}}$, we shall denote the set of all correctly defined partial functions on $\mathfrak{A}$, i. e. of a fixed type on $\mathfrak{A}$.

Denote by N the set of all natural numbers. Consider for each sort s a surjective mapping α_s from a subset of N onto A_s. Let $\mathfrak{B} = (\overline{N};\ \varphi_1, \varphi_2, \dots, \varphi_l;\ \sigma_1, \sigma_2, \dots, \sigma_r)$ be a partial many-sorted structure of signature $\mathbb{L}$ over the natural numbers, that is, each basic set of $\mathfrak{B}$ is N.

2.1. Definition. The tuple $\langle \alpha_1, \alpha_2, \dots, \alpha_m,\ \mathfrak{B}\rangle$ is called *an enumeration of* $\mathfrak{A}$ if the following conditions hold:

(i) if $x_1 \in dom(\alpha_{s_1}), \dots, x_{a_i} \in dom(\alpha_{s_{a_i}})$ and $\varphi_i(x_1, \dots, x_{a_i})$ is defined, then $\varphi_i(x_1, \dots, x_{a_i}) \in dom(\alpha_s)$;

(ii) if $x_1 \in dom(\alpha_{s_1}), \dots, x_{a_i} \in dom(\alpha_{s_{a_i}})$, then $\alpha_s(\varphi_i(x_1, \dots, x_{a_i})) \simeq \psi_i(\alpha_{s_1}(x_1), \dots, \alpha_{s_{a_i}}(x_{a_i}))$;

(iii) if $x_1 \in dom(\alpha_{s_1}), \dots, x_{b_j} \in dom(\alpha_{s_{b_j}})$, then $\sigma_j(x_1, \dots, x_{b_j}) \simeq \Delta_j(\alpha_{s_1}(x_1), \dots, \alpha_{s_{b_j}}(x_{b_j}))$;

(iv) for all effectively enumerable sorts $s \in \mathbb{E}$ it holds $dom(\alpha_s) = N$.

(Here, as usual, $\simeq$ means "is defined and equal to".)

Actually, $\langle \alpha_1, \alpha_2, \dots, \alpha_m\rangle$ is a $\mathbb{L}$ - strong homomorphism from $(dom(\alpha_1), \dots, dom(\alpha_m);\ \overline{\varphi};\ \overline{\sigma})$ onto $\mathfrak{A}$.

Let $\langle \alpha_1, \alpha_2, \dots, \alpha_m,\ \mathfrak{B}\rangle$ be an enumeration of $\mathfrak{A}$. Denote by

$$D(\mathfrak{B}) = \{\langle i, \overline{x}, y\rangle \colon \varphi_i(\overline{x}) \simeq y, 1 \le i \le l\} \cup \{\langle n+j, \overline{x}, \epsilon\rangle \colon \sigma_j(\overline{x}) \simeq \epsilon, 1 \le j \le r\},$$

the diagram of $\mathfrak{B}$. Here "$\langle , \rangle$" is a fixed one to one effective coding of the tuples of natural numbers.

A set $A \subseteq N$ is said to be *partial recursive (p. r.) in* $\mathfrak{B}$ if there exists an enumeration operator Γ such that $A = \Gamma(D(\mathfrak{B}))$ [12].

A function φ of $\mathfrak{F}_{\mathfrak{B}}$ is called *partial recursive in* $\mathfrak{B}$ if the graph of φ is partial recursive in $\mathfrak{B}$.

2.2. Definition. A function $\theta \in \mathfrak{F}_{\mathfrak{A}}$ is said to be *admissible in* $\langle \alpha_1, \alpha_2, \dots, \alpha_m,\ \mathfrak{B}\rangle$ if there exists a function φ over N, partial recursive in $\mathfrak{B}$, such that if $x_1 \in dom(\alpha_{s_1})$, $\dots$, $x_a \in dom(\alpha_{s_a})$, then it holds

(i) if $\varphi(x_1, \dots, x_a)$ is defined, then $\varphi(x_1, \dots, x_a) \in dom(\alpha_s)$;

(ii) $\alpha_s(\varphi(x_1, \dots, x_a)) \simeq \theta(\alpha_{s_1}(x_1), \dots, \alpha_{s_a}(x_a))$.

2.3. Definition. θ is called *computable in* $\mathfrak{A}$ if θ is admissible in every enumeration of $\mathfrak{A}$.

We shall denote by $\mathfrak{C}(\mathfrak{A})$ the class of all computable functions in $\mathfrak{A}$.

In [17] a normal form for the computable functions in $\mathfrak{A}$ is obtained. Suppose that an infinite list of variables of sort s, for each sort s of $\mathbb{S}$, is fixed. Let for each s, P_0^s be a new unary predicate symbol (of type (s)) which is intended to represent the unary totally defined over A_s predicate $\lambda t.0$. We assume $P_0^s \in \mathbb{P}$ for each s of $\mathbb{S}$.

Terms of a given sort in $\mathbb{L}$ are defined as usual. *A termal predicate* is a finite conjunction of atomic formulae and negated atomic formulae.

2.4. Definition. Let Π be a termal predicate, τ be a term of sort s and $Y_1, \dots, Y_b, b \geq 0$, be variables of sorts in $\mathbb{E}$. Then every expression of the form $\exists Y_1 \dots \exists Y_b(\Pi \supset \tau)$ is called an *s-conditional term.*

(Note that if $\mathbb{E} = \emptyset$, then there are no existential quantifiers.)

Let $Q = \exists Y_1 \dots \exists Y_b(\Pi \supset \tau)$ be an s-conditional term with free variables among $X_1, \dots, X_a$, and $t_1, \dots, t_a \in |\mathfrak{A}|$. The value $Q_{\mathfrak{A}}(X_1/t_1, \dots, X_a/t_a)$ of Q is the set

$$\{\tau_{\mathfrak{A}}(Y_1/p_1 \dots Y_b/p_b, X_1/t_1 \dots X_a/t_a) : \Pi_{\mathfrak{A}}(Y_1/p_1 \dots Y_b/p_b, X_1/t_1 \dots X_a/t_a) \simeq 0, \text{ for some } p_1, \dots, p_b \text{ of sorts as } Y_1, \dots, Y_b\}.$$

2.5. Definition. A function θ is said to be *definable on* $\mathfrak{A}$ if for some recursively enumerable (r. e.) set $\{Q^v\}_{v \in V}$ of s-conditional terms with free variables among $Z_1, \dots, Z_c, X_1, \dots, X_a$ and for some fixed elements $q_1, \dots, q_c$ of $|\mathfrak{A}|$, the following equivalence is true

$$\theta(t_1, \dots, t_a) \simeq t \Leftrightarrow \exists v(v \in V \;\&\; t \in Q^v_{\mathfrak{A}}(Z_1/q_1, \dots, Z_c/q_c, X_1/t_1, \dots, X_a/t_a)).$$

2.6. Theorem. ***(Normal Form Theorem)*** [17] *The function θ is computable in $\mathfrak{A}$ iff θ is definable on $\mathfrak{A}$.*

Using this normal form, we obtain an explicit form of the domains of the computable in $\mathfrak{A}$ functions.

Let Π be a termal predicate and $Y_1, \dots, Y_b, b \geq 0$, variables of sorts in $\mathbb{E}$. Then the expression of the form $\exists Y_1 \dots \exists Y_b(\Pi)$ is called *a condition.*

2.7. Corollary. *A set A is a domain of a function computable in $\mathfrak{A}$ iff for some r. e. set $\{C^v\}_{v \in V}$ of conditions with free variables among $Z_1, \dots, Z_c, X_1, \dots, X_a$ and for some fixed elements $q_1, \dots, q_c$ of $|\mathfrak{A}|$, the following equivalence is true*

$$(t_1, \dots, t_a) \in A \Leftrightarrow \exists v(v \in V \;\&\; C^v_{\mathfrak{A}}(Z_1/q_1, \dots, Z_c/q_c, X_1/t_1, \dots, X_a/t_a) \simeq 0).$$

Call those sets *definable on* $\mathfrak{A}$.

3. Effective structures

Let $\mathfrak{A} = (A_1, A_2, \dots, A_m; \overline{\psi}; \overline{\Delta})$ be a partial many-sorted structure of the signature $\mathbb{L}$.

Let $\mathfrak{M} = (M; \overline{\theta}; \overline{\Sigma})$ be a partial single-sorted structure of signature $\mathbb{L}' = (\{0\}, \mathbb{E}', \mathbb{F}', \mathbb{P}', \rho')$ choosen so that the sets $\mathbb{F}$ and $\mathbb{F}'$, $\mathbb{P}$ and $\mathbb{P}'$ are disjoint. Denote by $[\mathfrak{M}, \mathfrak{A}]$ the structure $(M, A_1, \dots, A_m; \overline{\theta}, \overline{\psi}; \overline{\Sigma}, \overline{\Delta})$ of signature $\mathbb{L}' \cup \mathbb{L} = (\{0\} \cup \mathbb{S}, \mathbb{E}' \cup \mathbb{E}, \mathbb{F}' \cup \mathbb{F}, \mathbb{P}' \cup \mathbb{P}, \rho' \cup \rho)$. Thus the elements of M are of sort 0 and they are supposed to be general ones if $\mathbb{E}' = \emptyset$, and effectively enumerable if $\mathbb{E}' = \{0\}$.

Let θ be a function of type $(s_1, \dots, s_a, s)$, for $s_1, \dots, s_a, s \in \mathbb{S}$, i. e. $\theta \in \mathfrak{F}_{\mathfrak{A}}$. From the normal form of the computable in $[\mathfrak{M}, \mathfrak{A}]$ functions, we have

3.1. Proposition. *θ is computable in $[\mathfrak{M}, \mathfrak{A}]$ iff there exists an r. e. set V and recursive functions α and β such that for each $v \in V$, $C^{\alpha(v)}$ is a condition in $\mathbb{L}'$ with free variables among $Y_1, \dots, Y_c$, $Q^{\beta(v)}$ is an s-conditional term in $\mathbb{L}$ with free variables among $Z_1, \dots, Z_b, X_1, \dots, X_a$, and, for some fixed elements $q_1, \dots, q_c$ of $|\mathfrak{M}|$, $p_1, \dots, p_b$ of $|\mathfrak{A}|$, the following equivalence is true*

$$\theta(t_1, \dots, t_a) \simeq t \Leftrightarrow \exists v (v \in V \ \& \ C^{\alpha(v)}_{\mathfrak{M}}(Y_1/q_1, \dots, Y_c/q_c) \simeq 0 \ \& \\ t \in Q^{\beta(v)}_{\mathfrak{A}}(Z_1/p_1, \dots, Z_b/p_b, X_1/t_1, \dots, X_a/t_a)).$$

If the structure $\mathfrak{M}$ is finitely generated, i. e. there are finitely many elements $t_1, \dots, t_r$ of $|\mathfrak{M}|$ such that every element of $|\mathfrak{M}|$ is a value of a term on $t_1, \dots, t_r$, then the class of the functions of $\mathfrak{F}_{\mathfrak{A}}$, computable in $[\mathfrak{M}, \mathfrak{A}]$ remains the same either if $\mathfrak{M}$ is an effectively enumerable sort in $[\mathfrak{M}, \mathfrak{A}]$ or not. As a corollary of this observation and the preceding proposition, we receive the following fact.

Consider a standard structure over the natural numbers $\mathfrak{N} = (N; S, P; Z)$, where $S = \lambda x.x+1$, $P = \lambda x.x \dot{-} 1$ and Z is a predicate over N such that $Z(x) = 0$ if $x = 0$ and $Z(x) = 1$, otherwise. Let $[\mathfrak{N}, \mathfrak{A}] = (N, A_1, \dots, A_m;\ S, P, \overline{\psi};\ Z, \overline{\Delta})$.

3.2. Corollary. *If $\theta \in \mathfrak{F}_{\mathfrak{A}}$, then*

$$\theta \in \mathfrak{C}[\mathfrak{N}, \mathfrak{A}]) \Longleftrightarrow \theta \in \mathfrak{C}(\mathfrak{A}).$$

We want to describe all structures which have the last property. So we come to the following definition:

3.3. Definition. The structure $\mathfrak{M}$ is called *effective with respect to* $\mathfrak{A}$ if for each function θ of $\mathfrak{F}_{\mathfrak{A}}$

$$\theta \in \mathfrak{C}\ ([\mathfrak{M}, \mathfrak{A}]) \Longleftrightarrow \theta \in \mathfrak{C}\ (\mathfrak{A}).$$

3.4. Definition. The structure $\mathfrak{M}$ is called *effective* if it is effective with respect to every $\mathfrak{A}$.

Notice that the structure $\mathfrak{N}$ defined above is an effective structure.

In order to describe the class of all effective structures we need some more definitions.

3.5. Definition. Let C be a condition in $\mathbb{L}$, and n be a natural number. The expression of the form $(C \supset n)$ is called a *conditional expression.*

3.6. Definition. Let $A \subseteq N$. The set A is called an $\mathfrak{A}$ - *definable set of natural numbers* if there exists an r. e. set $\{C^v \supset n^v\}_{v \in V}$ of conditional expressions in the signature of $\mathfrak{A}$ with free variables among $Z_1, \dots, Z_c$, and, for some fixed elements $q_1, \dots, q_c$ of $|\mathfrak{A}|$, the following equivalence is true

$$x \in A \Leftrightarrow \exists v(v \in V \ \& \ C^v_{\mathfrak{A}}(Z_1/q_1, \dots, Z_c/q_c) \simeq 0 \ \& \ x = n^v).$$

3.7. Proposition. *Let $A \subseteq N$. Then the following three properties are equivalent:*

(1) *A is a definable on $[\mathfrak{N}, \mathfrak{A}]$;*
(2) *A is an $\mathfrak{A}$ - definable set of natural numbers;*
(3) *A is a partial recursive in $\mathfrak{B}$, for every enumeration $\langle \overline{\alpha}, \mathfrak{B} \rangle$ of $\mathfrak{A}$.*

Proof. (1) $\Longrightarrow$ (2): Let A be a definable on $[\mathfrak{N}, \mathfrak{A}]$. Combining Corollary 2.7 and Proposition 3.1, we have that there is an r. e. set V and recursive functions α and β such that for each $v \in V$: $C^{\alpha(v)}$ is a condition in $\mathbb{L}'$ with free variables among $X, Y_1, \dots, Y_c$, $Q^{\beta(v)}$ is an s-conditional term in $\mathbb{L}$ with free variables among $Z_1, \dots, Z_b$, and, for some fixed elements $n_1, \dots, n_c$ of N, $p_1, \dots, p_b$ of $|\mathfrak{A}|$, it holds

$$x \in A \Leftrightarrow \exists v(v \in V \ \& \ C^{\alpha(v)}_{\mathfrak{N}}(X/x, Y_1/n_1, \dots, Y_c/n_c) \simeq 0 \ \& \\ C^{\beta(v)}_{\mathfrak{A}}(Z_1/p_1, \dots, Z_b/p_b) \simeq 0).$$

Let $U = \{\langle v, x \rangle : v \in V \ \& \ C^{\alpha(v)}_{\mathfrak{N}}(X/x, Y_1/n_1, \dots, Y_c/n_c) \simeq 0\}$.

It is clear that U is r. e. and

$$x \in A \Leftrightarrow \exists u(u \in U \ \& \ C^{\beta(L(u))}_{\mathfrak{A}}(Z_1/p_1, \dots, Z_b/p_b) \simeq 0 \ \& \ x = R(u)).$$

Here L and R are recursive functions s.t. $L(\langle v, x \rangle) = v$ and $R(\langle v, x \rangle) = x$.

Thus A is an $\mathfrak{A}$ - definable set of natural numbers.

(2) $\Longrightarrow$ (3) : Follows from the definition of an $\mathfrak{A}$ - definable set of natural numbers.

(3) $\Longrightarrow$ (1) : Suppose that the set A is p. r. in $\mathfrak{B}$ for every enumeration $\langle \overline{\alpha}, \mathfrak{B} \rangle$ of $\mathfrak{A}$. Considering the definable on $[\mathfrak{N}, \mathfrak{A}]$ sets as the domains of the definable on $[\mathfrak{N}, \mathfrak{A}]$ functions, by the normal form theorem, it is enough to prove that A is admissible in every enumeration of $[\mathfrak{N}, \mathfrak{A}]$. Let $\langle \alpha_0, \alpha_1, \dots, \alpha_m, \mathfrak{B} \rangle$, where $\mathfrak{B} = (\overline{N}; \ S_0, P_0, \varphi_1, \varphi_2, \dots, \varphi_l; \ Z_0, \sigma_1, \sigma_2, \dots, \sigma_r)$, be an enumeration of $[\mathfrak{N}, \mathfrak{A}]$. Here S_0, P_0, Z_0 are the corresponding for S, P, Z. We want to find a p. r. in $\mathfrak{B}$ set $W \subseteq N$ s.t., for each $x \in dom(\alpha_0)$, $x \in W \Leftrightarrow \alpha_0(x) \in A$. But $\langle \alpha_1, \alpha_2, \dots, \alpha_m, \mathfrak{B}_1 \rangle$, where $\mathfrak{B}_1 = (\overline{N}; \ \varphi_1, \varphi_2, \dots, \varphi_l; \ \sigma_1, \sigma_2, \dots, \sigma_r)$, is an enumeration of $\mathfrak{A}$ and $\langle \alpha_0, \mathfrak{B}_0 \rangle$, where $\mathfrak{B}_0 = (N; S_0, P_0; Z_0)$, is an enumeration of $\mathfrak{N}$. Therefore, A is p. r. in $\mathfrak{B}_1$.

Consider the predicate σ over N, defined inductively as follows:

$\sigma(0, x) \simeq 0 \Leftrightarrow Z_0(x) \simeq 0$;

$\sigma(n+1, x) \simeq 0 \Leftrightarrow Z_0(x) \simeq 1 \ \& \ Z_0(P_0(x)) \simeq 1 \ \& \dots \& \ Z_0(P^n_0(x)) \simeq 1 \ \&$ $Z_0(P^{n+1}_0(x)) \simeq 0$.

It is clear that σ is p. r. in $\mathfrak{B}_0$.

Let $x \in dom(\alpha_0)$. We shall prove that, for each $n \in N$,

$$\sigma(n, x) \simeq 0 \Leftrightarrow \alpha_0(x) \simeq n.$$

(i) Let $n = 0$. Then $\sigma(0, x) \simeq 0 \Leftrightarrow Z_0(x) \simeq 0 \Leftrightarrow Z(\alpha_0(x)) \simeq 0 \Leftrightarrow \alpha_0(x) \simeq 0$;
(ii) Let $n > 0$. Then
$\sigma(n, x) \simeq 0 \Leftrightarrow Z_0(x) \simeq 1 \ \& \ Z_0(P_0(x)) \simeq 1 \ \& \ldots \&$
$Z_0(P_0^{n-1}(x)) \simeq 1 \ \& \ Z_0(P_0^n(x)) \simeq 0 \Leftrightarrow Z(\alpha_0(x)) \simeq 1 \ \& \ Z(P(\alpha_0(x))) \simeq 1 \ \& \ldots \ \&$
$Z(P^{n-1}(\alpha_0(x))) \simeq 1 \ \& \ Z(P^n(\alpha_0(x))) \simeq 0 \ \Leftrightarrow \alpha_0(x) \simeq n$.

Let $W \subseteq N$, s. t. $x \in W \Leftrightarrow \exists n(n \in A \ \& \ \sigma(n, x) \simeq 0)$. So W is p. r. in $\mathfrak{B}$. Moreover, if $x \in dom(\alpha_0)$, then $x \in W \Leftrightarrow \alpha_0(x) \in A$, i. e. A is admissible in $[\mathfrak{N}, \mathfrak{A}]$. □

4. Generally effective structures

In this section we suppose that the structure $\mathfrak{M}$ is of signature $\mathbb{L}' = (\{0\}, \mathbb{E}', \mathbb{F}', \mathbb{P}', \rho')$, where $\mathbb{E}' = \emptyset$, i. e. the only sort of $\mathfrak{M}$ is a general sort.

If for every function θ of $\mathfrak{F}_{\mathfrak{A}}$

$$\theta \in \mathfrak{C}([\mathfrak{M}, \mathfrak{A}]) \Longleftrightarrow \theta \in \mathfrak{C}(\mathfrak{A}),$$

we shall call $\mathfrak{M}$ *generally effective with respect to* $\mathfrak{A}$. If $\mathfrak{M}$ is generally effective with respect to all $\mathfrak{A}$, then $\mathfrak{M}$ is called *generally effective.*

A corollary of the preceding proposition is the following.

4.1. Proposition. *The following properties are equivalent*

(i) $\mathfrak{M}$ *is generally effective;*
(ii) $\mathfrak{M}$ *is generally effective with respect to* $\mathfrak{N}$*;*
(iii) *Everey* $\mathfrak{M}$ *- definable set of natural numbers is r. e.*

Proof. Let $\mathfrak{M}$ be generally effective with respect to $\mathfrak{N}$ and A be an $\mathfrak{M}$ - definable set of natural numbers. By proposition 3.7, A is definable on $[\mathfrak{M}, \mathfrak{N}]$. But $A \subseteq N$. Since $\mathfrak{M}$ is generally effective with respect to $\mathfrak{N}$, then A is definable on $\mathfrak{N}$ and thus A is r. e..

Let all $\mathfrak{M}$ - definable sets of natural numbers be r. e.. Consider an arbitrary many-sorted structure $\mathfrak{A}$. By the normal form of the computable functions in $[\mathfrak{M}, \mathfrak{A}]$, which are correctly defined on $\mathfrak{A}$, and by Proposition 3.1, it follows that all those functions are computable in $\mathfrak{A}$, too. □

Now, we are ready to give a characterization of the generally effective structures.

4.2. Definition. An enumeration $\langle \alpha, \ \mathfrak{B} \rangle$ of $\mathfrak{M}$ is said to be *effective* if all initial functions and predicates of $\mathfrak{B}$ are partial recursive.

4.3. Theorem. *A structure* $\mathfrak{M}$ *is generally effective iff there is an effective enumeration of* $\mathfrak{M}$*.*

Proof. If $\mathfrak{M}$ admits an effective enumeration, it is clear that each $\mathfrak{M}$ - definable set of natural numbers is r. e. and by the preceding proposition, $\mathfrak{M}$ is generally effective.

To prove the other direction, suppose that $\mathfrak{M}$ is generally effective. Then the $\mathfrak{M}$ - definable sets of natural numbers are r. e.. Using this fact, we shall construct an enumeration $\langle \alpha, \ \mathfrak{B} \rangle$ of $\mathfrak{M}$, in which all initial functions and predicates are p. r.. Note that since the only sort of $\mathfrak{M}$ is supposed to be a general one, then the only obligation that we have (for $dom(\alpha)$) is that the domain of α should be closed under the functions in $\mathfrak{B}$.

Let $\mathfrak{M} = (M; \ \theta_1, \theta_2, \dots, \theta_n; \ \Sigma_0, \Sigma_1, \dots, \Sigma_k)$ and $\mathbb{L}' = (\{0\}, \emptyset, \{g_1, \dots, g_n\}, \{T_0, T_1, \dots, T_k\}, \rho')$, where $\Sigma_0 = \lambda t.0$ represents T_0 on $\mathfrak{M}$.

For each $i, 1 \leq i \leq n$, consider the function $\varphi_i^* = \lambda x_1, \dots, x_{a_i}.\langle i, x_1, \dots, x_{a_i} \rangle$ in N. Let $N^0 = N \setminus (\text{range}(\varphi_1^*) \cup \dots \cup \text{range}(\varphi_n^*))$.

Let α_0 be a partial surjective mapping from N^0 onto M and α be defined as follows:

(1) if $x \in N^0$, then $\alpha(x) \simeq \alpha_0(x)$;

(2) if $x = \langle i, x_1, \dots, x_{a_i} \rangle$, then $\alpha(x) \simeq \theta_i(\alpha(x_1), \dots \alpha(x_{a_i}))$.

Let $\varphi_1, \varphi_2, \dots, \varphi_n$ be partial functions such that, if $x_1, \dots, x_{a_i} \in dom(\alpha)$, then

$$\varphi_i(x_1, \dots, x_{a_i}) \simeq \begin{cases} \langle i, x_1, \dots, x_{a_i} \rangle & \text{if } (\alpha(x_1), \dots \alpha(x_{a_i})) \in dom(\theta_i), \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Let $\sigma_1, \sigma_2, \dots, \sigma_k$ be partial predicates on N s.t., if $x_1, \dots, x_{b_j} \in dom(\alpha)$, then

$$\sigma_j(x_1, \dots, x_{b_j}) \simeq \Sigma_j(\alpha(x_1), \dots \alpha(x_{b_j})).$$

Each $\langle \alpha, \ \mathfrak{B} \rangle$, $\mathfrak{B} = (N; \ \varphi_1, \varphi_2, \dots, \varphi_n; \ \sigma_1, \sigma_2, \dots, \sigma_k)$ with the above properties, we shall call *a standard enumeration* [18] . In fact, it is an enumeration of $\mathfrak{M}$. By the above definitions in order to construct a standard enumeration $\langle \alpha, \ \mathfrak{B} \rangle$, it is sufficient to define the partial mapping α_0, the domains of $\varphi_1, \varphi_2, \dots, \varphi_n$ and $\sigma_1, \sigma_2, \dots, \sigma_k$. Moreover, we want the initial functions and predicates of $\mathfrak{B}$ to be partial recursive.

Consider first some properties of the standard enumerations. Let $\mathfrak{B}^* = (N; \varphi_1^*, \dots, \varphi_n^*)$.

4.4. Lemma. *If* $\tau(Y_1, \dots, Y_b)$ *is a term and* $y_1, \dots, y_b \in dom(\alpha)$, *then*

$$\alpha(\tau_{\mathfrak{B}^*}(Y_1/y_1, \dots, Y_b/y_b)) \simeq \tau_{\mathfrak{M}}(Y_1/\alpha(y_1), \dots, Y_b/\alpha(y_b)).$$

Suppose that an effective coding of the finite sets of natural numbers is fixed. By E_v we shall denote the finite set with code v. Consider a recursive function g with values - the codes of the finite sets s.t., if $x \in N^0$, then $E_{g(x)} = \{x\}$; and if $x = \langle i, x_1, \dots, x_{a_i} \rangle$, then $E_{g(x)} = E_{g(x_1)} \cup \dots \cup E_{g(x_{a_i})}$.

It is clear that if $x \in dom(\alpha)$, then $E_{g(x)} \subseteq dom(\alpha_0)$.

Let *var* be a mapping from N^0 onto all variables. The mapping *term* is defined inductively: if $x \in N^0$, then $term(x) = var(x)$, and if $x = \langle i, x_1, \dots, x_{a_i} \rangle$, then

$term(x) = g_i(term(x_1), \dots, term(x_{a_i}))$. Note that any number x we consider as a code of a term over some elements of N^0.

4.5. Lemma. *If $x_1, \dots, x_a$ are natural numbers and $term(x_1) = \tau^1, \dots, term(x_a) = \tau^a$ and $y_1, \dots, y_b$ are the all elements of $E_{g(x_1)} \cup \dots \cup E_{g(x_a)}$, $var(y_1) = Y_1, \dots, var(y_b) = Y_b$, then*

(1) *$\tau^1, \dots, \tau^a$ are terms with variables among $Y_1, \dots, Y_b$;*
(2) *$x_i = \tau^i_{\mathfrak{B}^*}(Y_1/y_1, \dots, Y_b/y_b)$, $1 \le i \le a$;*
(3) *if $y_1, \dots, y_b \in dom(\alpha)$, then $\alpha(x_i) \simeq \tau^i_{\mathfrak{M}}(Y_1/\alpha(y_1), \dots Y_b/\alpha(y_b))$.*

The construction. Let $M = \{t_0, t_1, \dots, t_m, \dots\}$ be an arbitrary enumeration of the elements of M.

Let $N^0_0 = \{\langle 0, m, c\rangle : m, c \in N\}$. Then $N^0_0 \subseteq N^0$. Let st be a recursive function s.t. $st(\langle 0, m, c\rangle) = m$ and $st(n) = 0$ for $n \notin N^0_0$. For each m, we shall choose only one $x_m \in N^0_0$, with $st(x_m) = m$, and we shall put $\alpha_0(x_m) = t_m$.

Let $X_0, \dots, X_m, \dots$ be a sequence of all variables in the signature. Define $var(y) = X_{st(y)}$, for each $y \in N^0$. Suppose that π is an effective coding of all formulae of the signature and a natural number m is fixed. Denote by P_m the set of all codes of termal predicates with variables among $X_0, \dots, X_m$. Let ξ_m be a partial function s.t.:

If $v \notin P_m$, then $\xi_m(v)$ is undefined;

If $v \in P_m$ and $v = \pi(\Pi)$, then $\xi_m(v) \simeq \Pi_{\mathfrak{M}}(X_0/t_0, \dots, X_m/t_m)$.

Since the definable on $\mathfrak{M}$ sets of natural numbers are r. e., then the function ξ_m is p. r., as we can see from the following equivalence:

$$\xi_m(n) \simeq 0 \Leftrightarrow \exists v(v \in P_m \ \&\ \Pi^v_{\mathfrak{M}}(X_0/t_0, \dots, X_m/t_m) \simeq 0 \ \&\ n = v).$$

Let c_m be a fixed index of ξ_m. Define $x_m = \langle 0, m, c_m\rangle$ and $\alpha_0(x_m) = t_m$. Thus the definition of α is completed.

Let E be a finite subset of N. E is called *correct* if $E \subseteq N^0_0$ and, for each $x, y \in E$, $st(x) = st(y)$ implies $x = y$.

The function φ_i is defined as follows. Let $x_1, \dots, x_{a_i} \in N$.

If $E_{g(x_1)} \cup \dots \cup E_{g(x_{a_i})}$ is not correct, then $\varphi_i(x_1, \dots, x_{a_i})$ is undefined.

Otherwise, let $y_1, \dots, y_b$ be all different elements of $E_{g(x_1)} \cup \dots \cup E_{g(x_{a_i})}$, $st(y_1) = m_1, \dots, st(y_b) = m_b$ so that $m_1 < \dots < m_b$ and $y_b = \langle 0, m_b, c_b\rangle$.
Let $term(x_1) = \tau^1, \dots, term(x_{a_i}) = \tau^{a_i}$ and $u = \pi(T_0(g_i(\tau^1, \dots, \tau^{a_i})))$. Let χ be the p. r. function with index c_b. Define

$$\varphi_i(x_1, \dots, x_{a_i}) \simeq \begin{cases} \langle i, x_1, \dots, x_{a_i}\rangle & \text{if } \chi(u) \simeq 0, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

It is clear that φ_i is a p. r. function. From the definition of α, it remains to prove that if $x_1, \dots, x_{a_i} \in dom(\alpha)$, then

$$\varphi_i(x_1, \dots, x_{a_i}) \text{ is defined } \iff \theta_i(\alpha(x_1), \dots, \alpha(x_{a_i})) \text{ is defined.}$$

Let $x_1, \ldots, x_{a_i} \in dom(\alpha)$. Then $E_{g(x_1)} \cup \cdots \cup E_{g(x_{a_i})} \subseteq dom(\alpha_0)$ and it is correct. Moreover, by Lemma 4.5, $\tau^1, \ldots, \tau^{a_i}$ are terms with variables among $X_{m_1}, \ldots X_{m_b}$, $x_i = \tau^i_{\mathfrak{B}^*}(X_{m_1}/y_1, \ldots, X_{m_b}/y_b)$, for $1 \leq i \leq a_i$, and $\alpha(x_i) \simeq \tau^i_{\mathfrak{M}}(X_{m_1}/\alpha(y_1), \ldots X_{m_b}/\alpha(y_b))$.

Using Lemma 4.4, we have:

$\theta_i(\alpha(x_1), \ldots, \alpha(x_{a_i}))$ is defined $\Leftrightarrow$

$\theta_i(\alpha(\tau^1_{\mathfrak{B}^*}(X_{m_1}/y_1, \ldots, X_{m_b}/y_b)), \ldots, \alpha(\tau^{a_i}_{\mathfrak{B}^*}(X_{m_1}/y_1, \ldots, X_{m_b}/y_b)))$ is defined $\Leftrightarrow$

$\theta_i(\tau^1_{\mathfrak{M}}(X_{m_1}/t_{m_1}, \ldots, X_{m_b}/t_{m_b}), \ldots, \tau^{a_i}_{\mathfrak{M}}(X_{m_1}/t_{m_1}, \ldots, X_{m_b}/t_{m_b}))$ is defined $\Leftrightarrow$

$\Sigma_0(\theta_i(\tau^1_{\mathfrak{M}}(X_{m_1}/t_{m_1}, \ldots, X_{m_b}/t_{m_b}), \ldots, \tau^{a_i}_{\mathfrak{M}}(X_{m_1}/t_{m_1}, \ldots, X_{m_b}/t_{m_b}))) \simeq 0 \Leftrightarrow$

$\chi(u) \simeq 0 \Leftrightarrow \varphi_i(x_1, \ldots, x_{a_i})$ is defined .

The definition of $\sigma_1, \sigma_2, \ldots, \sigma_k$ is similar. Let $1 \leq j \leq k$ and $x_1, \ldots, x_{b_j} \in N$.

If $E_{g(x_1)} \cup \cdots \cup E_{g(x_{b_j})}$ is not correct, then $\sigma_j(x_1, \ldots, x_{b_j})$ is undefined.

Otherwise, let $y_1, \ldots, y_b$ be all different elements of $E_{g(x_1)} \cup \cdots \cup E_{g(x_{b_j})}$, $st(y_1) = m_1, \ldots, st(y_b) = m_b$, so that $m_1 < \cdots < m_b$, and $y_b = \langle 0, m_b, c_b \rangle$. Let $term(x_1) = \tau^1, \ldots, term(x_{b_j}) = \tau^{b_j}$ and $u_0 = \pi(T_j(\tau^1, \ldots, \tau^{b_j}))$, $u_1 = \pi(\neg T_j(\tau^1, \ldots, \tau^{b_j}))$. Let χ be the p. r. function with index c_b. Define:

$$\sigma_j(x_1, \ldots, x_{b_j}) \simeq \begin{cases} 0 & \text{if } \chi(u_0) \simeq 0, \\ 1 & \text{if } \chi(u_1) \simeq 0, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

It is clear that σ_j is p. r. but it is not always single-valued. If $x_1, \ldots, x_{b_j} \in dom(\alpha)$, then it is. Thus we can use the single-valuedness theorem [12] to make σ_j single-valued. In the same way, we can prove that if $x_1, \ldots, x_{b_j} \in dom(\alpha)$, then

$$\sigma_j(x_1, \ldots, x_{b_j}) \simeq \Sigma_j(\alpha(x_1), \ldots, \alpha(x_{b_j})).$$

□

5. Strongly effective structures.

In this section we suppose that the structure $\mathfrak{M}$ is of signature $\mathbb{L}' = (\{0\}, \mathbb{E}', \mathbb{F}', \mathbb{P}', \rho')$, where $\mathbb{E}' = \{0\}$, i. e. the only sort of $\mathfrak{M}$ is an effectively enumerable sort.

If for each function θ of $\mathcal{F}_{\mathfrak{A}}$

$$\theta \in \mathcal{C}([\mathfrak{M}, \mathfrak{A}]) \Longleftrightarrow \theta \in \mathcal{C}(\mathfrak{A})$$

we shall call $\mathfrak{M}$ *strongly effective with respect to* $\mathfrak{A}$. If $\mathfrak{M}$ is strongly effective with respect to all $\mathfrak{A}$, then $\mathfrak{M}$ is called *strongly effective.*

We have the similar characterization as in Proposition 4.1.

5.1. Proposition. *The following properties are equivalent*

(i) $\mathfrak{M}$ *is strongly effective;*

(ii) $\mathfrak{M}$ *is strongly effective with respect to* $\mathfrak{N}$*;*
(iii) *Every* $\mathfrak{M}$ *- definable sets of natural numbers is r. e.*

5.2. Definition. An effective enumeration $\langle \alpha,\ \mathfrak{B}\rangle$, $\mathfrak{B} = (N;\ \overline{\varphi};\overline{\sigma})$, is called *strongly effective* if the existential diagram of the structure $(dom(\alpha);\overline{\varphi};\overline{\sigma})$ is r. e..

We want to point out that by a strongly effective enumeration of $\mathfrak{M}$ we mean an effective enumeration $\langle \alpha,\ \mathfrak{B}\rangle$, where the domain of α is not necessarily N. Moreover, in the next section it will be given an example where $\mathfrak{M}$ is strongly effective but there is no total effective enumeration of $\mathfrak{M}$.

5.3. Theorem. *A structure* $\mathfrak{M}$ *is strongly effective iff there is a strongly effective enumeration of* $\mathfrak{M}$*.*

Proof. We shall prove the following:

$\mathfrak{M}$ is strongly effective iff there is an effective enumeration $\langle \alpha,\ \mathfrak{B}\rangle$ of $\mathfrak{M}$ and an r. e. set $W \subseteq N$ such that, for any termal predicate $\Pi(X_1,\dots,X_a,Y_1,\dots,Y_b)$ with code v, if $x_1,\dots,x_a \in dom(\alpha)$, then

(i) if $y_1,\dots,y_b \in dom(\alpha)$, then

$$\Pi_{\mathfrak{B}}(X_1/x_1,\dots,X_a/x_a,Y_1/y_1,\dots,Y_b/y_b) \simeq 0$$
$$\Leftrightarrow \langle v,x_1,\dots,x_a,y_1,\dots,y_b\rangle \in W;$$

(ii) if for some $y_1,\dots,y_b$, $\langle v,x_1,\dots,x_a,y_1,\dots,y_b\rangle \in W$, then there exist $z_1,\dots,z_b \in dom(\alpha)$ s.t.

$$\Pi_{\mathfrak{B}}(X_1/x_1,\dots,X_a/x_a,Y_1/z_1,\dots,Y_b/z_b) \simeq 0.$$

If $\langle \alpha,\ \mathfrak{B}\rangle$ is an effective enumeration of $\mathfrak{M}$ and W is r. e. with the above properties, it follows from the Proposition 5.1 that $\mathfrak{M}$ is strongly effective.

In the other direction, using the fact that the $\mathfrak{M}$ - definable sets of natural numbers are r. e., we shall construct a standard enumeration of $\mathfrak{M}$, with p. r. initial functions and predicates and an r. e. set W with the above properties. We have to define first α_0.

The construction. Let $M = \{t_0,t_1,\dots,t_m,\dots\}$ be an arbitrary enumeration of the elements of M.

For each m we shall choose one $x_m = \langle 0,m,c_0,\dots,c_m\rangle \in N_0$ and put $\alpha_0(x_m) = t_m$.

Consider a recursive function η with the following property: if v is a code of the condition $\exists Y_1\cdots Y_b(\Pi)$, then $\eta(v) = \pi(\Pi)$. Denote by $\mathfrak{C}$ the set of the codes of all conditions without free variables, and by $\mathfrak{C}_m$ the set of the codes of all conditions with free variables among $X_0,\dots,X_m$.

Let ξ be defined so that

$$\xi(n) \simeq 0 \Leftrightarrow \exists v(v \in \mathfrak{C}\ \&\ C^v_{\mathfrak{M}} \simeq 0\ \&\ n = \eta(v)).$$

Let ξ_m be defined so that

$$\xi_m(n) \simeq 0 \Leftrightarrow \exists v(v \in \mathfrak{C} \ \& \ C^v_{\mathfrak{M}}(X_0/t_0, \dots, X_m/t_m) \simeq 0 \ \& \ n = \eta(v)).$$

It is clear that the so defined functions are p. r. Let c_i be a fixed index for ξ_i, $0 \leq i \leq m$. Define $x_m = \langle 0, m, c_0, \dots c_m \rangle$. Thus the definition of α is completed.

Consider some properties of the functions $\xi, \xi_1, \dots, \xi_m, \dots$.

5.4. Lemma. *If u is the code of the termal predicate Π with variables among $X_0, \dots, X_m$, then it holds*

(1) $\xi_m(u) \simeq 0 \Leftrightarrow \Pi_{\mathfrak{M}}(X_0/t_0, \dots, X_m/t_m) \simeq 0$;
(2) *if* $n < m, k = m - n$, *then*
$\xi_n(u) \simeq 0 \Leftrightarrow \exists p_1 \cdots \exists p_k(\Pi_{\mathfrak{M}}(X_0/t_0, \dots, X_n/t_n, X_{n+1}/p_1, \dots, X_m/p_k) \simeq 0)$;
(3) *if* $n \geq m$, *then* $\xi_n(u) \simeq \xi_m(u)$;
(4) $\xi(u) \simeq 0 \Leftrightarrow \exists p_0 \cdots \exists p_m(\Pi_{\mathfrak{M}}(X_0/p_0, \dots, X_m/p_m) \simeq 0)$.

Let $x, y \in N^0$ and $x = \langle 0, n, a_0, \dots, a_n \rangle, y = \langle 0, m, b_0, \dots, b_m \rangle$. Denote by $x \subset y$ the fact that $n \leq m \ \& \ a_0 = b_0 \cdots a_n = b_n$. A finite set of natural numbers E is called *correct* if for all $x \in E$, x is of the form $x = \langle 0, m, c_0, \dots, c_m \rangle$, and for any $x, y \in E$, we have $x \subset y$ or $y \subset x$.

In order to define the function φ_i, we shall use some notations of the previous theorem 4.3. Let $x_1, \dots, x_{a_i} \in N$.

If $E_{g(x_1)} \cup \dots \cup E_{g(x_{a_i})}$ is not correct, then $\varphi_i(x_1, \dots, x_{a_i})$ is undefined.

Otherwise, let $y_1, \dots, y_b$ be all different elements of $E_{g(x_1)} \cup \dots \cup E_{g(x_{a_i})}$, so that $y_1 \subset y_2 \subset \dots \subset y_b$, where $y_b = \langle 0, m_b, c_0, \dots, c_{m_b} \rangle$. Denote by $\chi_0, \dots, \chi_{m_b}$ the p. r. functions with indices $c_0, \dots, c_{m_b}$. Let $term(x_1) = \tau^1, \dots, term(x_{a_i}) = \tau^{a_i}$ and $u = \pi(T_0(g_i(\tau^1, \dots, \tau^{a_i})))$. Define

$$\varphi_i(x_1, \dots, x_{a_i}) \simeq \begin{cases} \langle i, x_1, \dots, x_{a_i} \rangle & \text{if } \xi(u) \simeq 0 \ \& \ \chi_0(u) \simeq 0 \ \& \cdots \& \ \chi_{m_b}(u) \simeq 0, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Using Lemma 4.5 and Lemma 5.4, we can easily check that φ_i has the desired properties.

Let $x_1, \dots, x_{b_j} \in N$.

If $E_{g(x_1)} \cup \dots \cup E_{g(x_{b_j})}$ is not correct, then $\sigma_j(x_1, \dots, x_{b_j})$ is undefined.

Otherwise, let $y_1, \dots, y_b$ be all different elements of $E_{g(x_1)} \cup \dots \cup E_{g(x_{b_j})}$, so that $y_1 \subset \dots \subset y_b$, and $y_b = \langle 0, m_b, c_0, \dots, c_{m_b} \rangle$ and $term(x_1) = \tau^1, \dots, term(x_{b_j}) = \tau^{b_j}$, $u_0 = \pi(T_j(\tau^1, \dots, \tau^{b_j}))$, $u_1 = \pi(\neg T_j(\tau^1, \dots, \tau^{b_j}))$. The only difference from the previous theorem in defining σ_j is that we use all p. r. functions $\chi_0, \dots \chi_{m_b}$ with indices $c_0, \dots c_{m_b}$. Thus

$$\sigma_j(x_1, \dots, x_{b_j}) \simeq \begin{cases} 0 & \text{if } \xi(u_0) \simeq 0 \ \& \ \chi_0(u_0) \simeq 0 \ \& \cdots \& \ \chi_{m_b}(u_0) \simeq 0, \\ 1 & \text{if } \xi(u_1) \simeq 0 \ \& \ \chi_0(u_1) \simeq 0 \ \& \cdots \& \ \chi_{m_b}(u_1) \simeq 0, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

$\langle\alpha,\ \mathfrak{B}\rangle$ is an effective enumeration of $\mathfrak{M}$.

We are going to define an r. e. W with the properties (i) and (ii).

Let lh be a recursive function s.t. if $v = \langle\pi(\Pi), \pi(X_1, \ldots, X_a)\rangle$ is the code of the termal predicate Π, with a fixed sequence of variables $X_1, \ldots, X_a$, then $lh(v) = a$. Denote by V the set of the codes of all termal predicates with a fixed sequence of variables. Define $U \subseteq N$ by

$$u \in U \Leftrightarrow \exists v \exists x_1 \cdots \exists x_{lh(v)} (v \in V \ \&\ u = \langle v, x_1, \ldots, x_{lh(v)}\rangle \ \&\ E_{g(x_1)} \cup \cdots \cup E_{g(x_{lh(v)})} \text{ is correct }).$$

The set U is recursive.

Consider a recursive function ν s.t. if $u \in U, u = \langle v, x_1, \ldots, x_a\rangle, lh(v) = a$, $v = \langle\pi(\Pi), \pi(X_1, \ldots, X_a)\rangle$ and $term(x_1) = \tau^1, \ldots, term(x_a) = \tau^a$, then $\nu(u) = \pi(\Pi(X_1/\tau^1, \ldots, X_a/\tau^a))$. Then W has the following definition

$W \subseteq U$;

If $u = \langle v, x_1, \ldots, x_a\rangle \in U$, and $y_1, \ldots, y_b$ are all different elements of $E_{g(x_1)} \cup \cdots \cup E_{g(x_a)}$, $y_1 \subset y_2 \subset \cdots \subset y_b$, $y_{m_b} = \langle 0, m_b, c_0, \ldots c_{m_b}\rangle$, then if $\chi_0, \ldots \chi_{m_b}$ are p. r. functions with indices $c_0, \ldots c_{m_b}$, then

$$u \in W \Leftrightarrow \xi(\nu(u)) \simeq 0 \ \&\ \chi_0(\nu(u)) \simeq 0 \ \&\cdots\&\ \chi_{m_b}(\nu(u)) \simeq 0.$$

It is clear that W is r. e.. The proof that the conditions (i) and (ii) are satisfied by W is technical. It uses Lemma 5.4 and the fact that $\langle\alpha,\ \mathfrak{B}\rangle$ is a standard enumeration of $\mathfrak{M}$. □

6. Some examples.

In this section we shall point out that the obtained results concerning the existence of a semi-recursive representation of the structure $\mathfrak{M}$ are in some sense the best possible. In [18, 6, 16, 7], several examples are considered which represent the difference between certain notions of effective enumeration. The usual notion of effective enumerations in computable algebras [9, 4, 10, 3, 19] and in recursive model theory [2, 1, 5] is recursive representation, i. e., an effective enumeration $\langle\alpha,\ \mathfrak{B}\rangle$ with $dom(\alpha) = N$. Call such an enumeration *total effective*. A characterization of the class of structures which admits total effective enumerations will be presented in the next Section.

It is easy to find a structure which does not admit an effective enumeration. For example consider a structure over N, with a not p. r. function ψ among the initial functions, and rich enough to represent the graph of ψ as a definable set of natural numbers [18].

If the structure $\mathfrak{M}$ is finitely generated and generally effective, then it admits a recursive representation. In [18, 6], an example of a structure is presented that admits an effective enumeration, but there is no total effective one for it.

The following example shows that there is a structure which admits an effective enumeration but does not admit a strongly effective one.

Example 1. Let $\mathfrak{M} = (N; \theta; \Sigma)$ be a structure, where θ is a unary total function, Σ is a unary partial predicate. Denote by $N^n = \{t \ : \ t \in N \ \& \ (t)_0 = n\}$ and let $\{t_0^n, \dots, t_k^n, \ \dots\}$ be an effective sequence of the elements of N^n. Then $\theta(t_i^n) = t_{i+1}^n$.

Let M be a not r. e. subset of N and $M = \{m_0, \dots, m_n, \dots\}$. For each n, $\Sigma(t_0^n) = 0$, $\Sigma(t_k^n) = 1 \Leftrightarrow 1 \leq k \leq m_n + 1$, $\Sigma(t_{m_n+2}^n) = 0$, and undefined for all other elements of N^n.

If the only sort of $\mathfrak{M}$ is considered as general one, then the $\mathfrak{M}$ - definable sets of natural numbers are r. e. Hence $\mathfrak{M}$ admits an effective enumeration by Theorem 4.3 and Proposition 4.1. But, if the sort of $\mathfrak{M}$ is effectively enumerable, then the set: $\{\exists t(\Sigma(t) \ \& \neg\Sigma(\theta(t)) \cdots \& \neg\Sigma(\theta^{k+1}(t)) \& \Sigma(\theta^{k+2}(t))) \simeq 0 \supset k\}_{k \in N}$ is M, so it is not r. e.. And by Theorem 5.3 and Proposition 5.1, there is no strongly effective enumeration of $\mathfrak{M}$.

The next example shows that there is a structure with a strongly effective enumeration which does not admit a total effective one.

Example 2. Consider the structure $\mathfrak{M} = (N; \theta; \Sigma)$, where θ is the same as in Example 1, but Σ is total unary predicate. Suppose that an effective coding of all finite non empty sequences of 0 and 1 is fixed, so that for each code b of such a sequence, we can effectively find the length of b. Let $b_0 < b_1 < \cdots < b_n \cdots$ be all such codes.

Consider a subset M of N which is in Π_2^0 but not in Σ_2^0 in the arithmetical hierarchy and let $M = \{m_0, \dots, m_n, \dots\}$.

Suppose that Σ is defined as follows:

For each $n \in N$, if $b_n = \beta_0 \cdots \beta_l$ for $\beta_0 \cdots \beta_l \in \{0, 1\}$, then

(i) $\Sigma(t_0^n) = \beta_0, \Sigma(t_1^n) = \beta_1, \dots, \Sigma(t_l^n) = \beta_l$;

(ii) if $n \in M$, then $\Sigma(t_{l+1}^n) = 0, \Sigma(t_{l+2}^n) = 1$ and for all $m > l + 2$, $\Sigma(t_m^n) = 0$;

(iii) if $n \notin M$, then $\Sigma(t_{l+1}^n) = 1, \Sigma(t_{l+2}^n) = 0$ and for all $m > l + 2$, $\Sigma(t_m^n) = 1$.

Notice that

$$n \in M \Leftrightarrow \exists l \forall m (m > l \Rightarrow \Sigma(\theta^m(t_0^n)) = 0).$$

Suppose that $\langle \alpha, \ \mathfrak{B} \rangle$, $\mathfrak{B} = (N; \ \varphi; \ \sigma)$ is a total effective enumeration of $\mathfrak{M}$. Because of $dom(\alpha) = N$, φ is recursive. Then it holds

$$\begin{aligned} n \in M \Leftrightarrow \exists x (x \notin range(\varphi) \ \& \ \text{ if } b_n = \beta_0 \cdots \beta_{l_n} \text{ then } \forall i (0 \leq i \leq l_n \Rightarrow \sigma(\varphi^i(x)) = \beta_i) \\ \& \ \sigma(\varphi^{l_n+1}(x)) = 0 \ \& \ \sigma(\varphi^{l_n+2}(x)) = 1 \ \& \ \forall m (m > l_n + 2 \Rightarrow \sigma(\varphi^m(x)) = 0)). \end{aligned}$$

This is in contradiction with the choice of M to be not in Σ_2^0.

On the other hand, it is easy to check that all $\mathfrak{M}$ - definable sets of natural numbers are r. e. if the only sort is effectively enumerable. Then by Theorem 5.3 and Proposition 5.1, there is a strongly effective enumeration of $\mathfrak{M}$.

7. Uniformly effective stuctures.

The above formulated notions of effectiveness of a single-sorted structure $\mathfrak{M}$ seems to be not quite satisfactory from a more practical point of view. It may happen that $\mathfrak{M}$ is effective with respect to $\mathfrak{A}$ but there exists no effective way to transform each program on $[\mathfrak{M}, \mathfrak{A}]$ into an equivalent program on $\mathfrak{A}$.

To formulate this precisely, we have to define the notion of a program on $\mathfrak{A}$. The only problem is that the description of each definable on $\mathfrak{A}$ function depends of some constants. So, we have to add constants to our signature.

For any sort s, let $\mathbb{C}_s$ be a countable set of new constants, $\mathbb{C} = \bigcup\limits_s \mathbb{C}_s$ and $\bar{\mathbb{L}} = \mathbb{L} \cup \mathbb{C}$. Suppose that each constant of sort s is a term of sort s in $\bar{\mathbb{L}}$. Thus the notion of an s-conditional term is extended to the signature $\bar{\mathbb{L}}$ in a natural way.

7.1. Definition. A *program* is an r. e. set $\{Q^v\}_{v \in V}$ of s-conditional terms in $\bar{\mathbb{L}}$ with finitely many free variables and constants.

Let I be a mapping of $\mathbb{C}$ onto $|\mathfrak{A}|$, and $\mathfrak{A}_I$ be the enrichment of $\mathfrak{A}$ to $\bar{\mathbb{L}}$, where each $c \in \mathbb{C}$ is interpreted as $I(c)$.

Let $\mathcal{P} = \{Q^v(X_1, \dots, X_a)\}_{v \in V}$ be a program. Then

$$\mathcal{P}_{\mathfrak{A}_I}(X_1/t_1, \dots X_a/t_a) \simeq t \Leftrightarrow \exists v (v \in V \ \& \ t \in Q^v_{\mathfrak{A}_I}(X_1/t_1, \dots X_a/t_a)).$$

It is clear that a function θ is computable in $\mathfrak{A}$ iff there exists a program $\mathcal{P}$ such that for all $t_1, \dots, t_a \in |\mathfrak{A}|$,

$$\theta(t_1, \dots, t_a) \simeq \mathcal{P}_{\mathfrak{A}_I}(X_1/t_1, \dots X_a/t_a).$$

The type of the program $\mathcal{P}$ we identify with the type of the function computable by it.

Let a single-sorted structure $\mathfrak{M}$ of signature $\mathbb{L}'$ be given, $\mathbb{C}'$ be a new countable set of constants for the elements of $|\mathfrak{M}|$, and $\bar{\mathbb{L}}' = \mathbb{L}' \cup \mathbb{C}'$.

7.2. Definition. $\mathfrak{M}$ is called *uniformly effective with respect to* $\mathfrak{A}$ if for every interpretation I of the constants in $\mathbb{C}$, there exist a mapping I' of $\mathbb{C}'$ onto $|\mathfrak{M}|$, and an effective operator γ, such that for each program $\mathcal{P}$ in $\bar{\mathbb{L}}' \cup \bar{\mathbb{L}}$ with type in $\mathfrak{A}$, $\gamma(\mathcal{P})$ is a program in $\bar{\mathbb{L}}$ of the same type s.t.

$$\mathcal{P}_{[\mathfrak{M}_{I'}, \mathfrak{A}_I]}(X_1/t_1, \dots X_a/t_a) \simeq \gamma(\mathcal{P})_{\mathfrak{A}_I}(X_1/t_1, \dots X_a/t_a).$$

Here we have supposed that the programs in $\bar{\mathbb{L}}' \cup \bar{\mathbb{L}}$ are coded and γ is a recursive function which transforms their codes.

It is easy to see that if the structure $\mathfrak{M}$ has a total effective enumeration, then $\mathfrak{M}$ is uniformly effective with respect to each many-sorted structure $\mathfrak{A}$. This follows directly from the normal form of the computable functions in $[\mathfrak{M}, \mathfrak{A}]$.

To prove the opposite, we need some additional conditions. Considering the structure $\mathfrak{A}$ as a representation of a given programming language, it is natural to suppose that $\mathfrak{A}$ consists the natural numbers ($\mathfrak{N}$) as a given sort, and that $\mathfrak{A}$ has an effective representation, i. e. $\mathfrak{A}$ admits a total effective enumeration. Those structures are called *standard.*

7.3. Theorem. *$\mathfrak{M}$ is uniformly effective with respect to the standard many-sorted structure $\mathfrak{A}$ iff $\mathfrak{M}$ admits a total effective enumeration.*

Proof. Suppose that $\mathfrak{A}$ is standard and $\mathfrak{M}$ is uniformly effective with respect to $\mathfrak{A}$. Then the computable functions in $[\mathfrak{M},\mathfrak{N}]$, elements of $\mathcal{F}_{\mathfrak{N}}$, are partial recursive. Therefore, the $\mathfrak{M}$ - definable sets of natural numbers are r. e.. Moreover, there is a uniform effective way for their computation. Using these facts and the technique of the standard enumerations, we shall construct a total effective enumeration of $\mathfrak{M}$.

The enumeration $\langle \alpha,\ \mathfrak{B}\rangle$ will be standard enumeration with $dom(\alpha_0) = N^0$. Let $x_0, x_1, \dots, x_m, \dots$ be an effective sequence of the elements of N^0. Define $\alpha_0(x_m) = I'(c'_m)$. So, the definition of α is completed. In fact, if $\overline{\varphi}$ are p. r., then $dom(\alpha)$ will be r. e.. In addition, if we construct $\overline{\sigma}$ p. r., then it is clear that $\mathfrak{M}$ will admit and a total effective enumeration.

In order to define the domains of $\overline{\varphi}$ and the predicates $\overline{\sigma}$, we shall use the fact that $\mathfrak{A}$ is standard and there is a total effective enumeration on $\mathfrak{A}$. We shall identify $\mathfrak{A}$ with this enumeration. We may suppose that the first sort of $\mathfrak{A}$ is $\mathfrak{N} = (N;\ S, P; Z)$. Let p be the functional symbol which represents the function P in $\mathbb{L}$ and T be the predicate symbol for Z.

Define $\rho^0(X) = T(X)$, $\rho^n(X) = \neg T(X) \ \& \cdots \& \ \neg T(p^{n-1}(X)) \ \& \ T(p^n(X))$, for $n > 0$. As in Proposition 3.7, we have $\rho^n_{\mathfrak{A}}(X/t) \simeq 0 \Leftrightarrow t = n$.

Consider the constants $c'_0, \dots, c'_m$. Let V^i_m be the recursive set of all codes of termal predicates in $\overline{\Sigma'}$ of the form $T_0(g_i(\tau^1, \dots, \tau^{a_i}))$, where $\tau^1, \dots, \tau^{a_i}$ are closed terms in $\overline{\mathbb{L}'}$, over the constants $c'_0, \dots, c'_m$. Denote by $\mathcal{P}^i_m = \{(\Pi^u \ \& \ \rho^u(X) \supset X) : u \in V^i_m\}$, where by Π^u we denote the termal predicate with code u. So, $\mathcal{P}^i_m$ is a program of the type $(1,1)$. Moreover, if $v \in N$, then

$$\mathcal{P}^i_{m[\mathfrak{M}_{I'},\mathfrak{A}_I]}(X/v) \simeq t \Leftrightarrow \exists u(u \in V^i_m \ \& \ \Pi^u_{\mathfrak{M}_{I'}} \simeq 0 \ \& \rho^u_{\mathfrak{A}_I}(X/v) \simeq 0 \ \& \ t = v).$$

From the choice of ρ^u and the fact that $\mathfrak{M}$ is uniformly effective with respect to $\mathfrak{A}$, we have

$$v \in V^i_m \ \& \ \Pi^v_{\mathfrak{M}_{I'}} \simeq 0 \Leftrightarrow \mathcal{P}^i_{m[\mathfrak{M}_{I'},\mathfrak{A}_I]}(X/v) \simeq v \Leftrightarrow \gamma(\mathcal{P}^i_m)_{\mathfrak{A}_I}(X/v) \simeq v.$$

Then the set $D^i = \bigcup_{m\in N}\{v : v \in V^i_m \ \& \ \Pi^v_{\mathfrak{M}_{I'}} \simeq 0\}$ is r. e., since γ is recursive and $\mathfrak{A}$ is effective. Notice that this set has the following property if $v = \pi(T_0(g_i(\tau^1, \dots \tau^{a_i})))$, then

$$v \in D^i \Leftrightarrow \exists m(\gamma(\mathcal{P}^i_m)_{\mathfrak{A}_I}(X/v) \simeq v) \Leftrightarrow \exists m(\mathcal{P}^i_{m[\mathfrak{M}_{I'},\mathfrak{A}_I]}(X/v) \simeq v) \Leftrightarrow$$
$$\Sigma_0(\theta_i(\tau^1_{\mathfrak{M}_{I'}}, \dots \tau^{a_i}{}_{\mathfrak{M}_{I'}})) \simeq 0 \Leftrightarrow \theta_i(\tau^1_{\mathfrak{M}_{I'}}, \dots \tau^{a_i}{}_{\mathfrak{M}_{I'}}) \text{ is defined.}$$

We shall define φ_i by D^i. Consider a recursive function st such that $st(x_m) = m$ and a mapping $const$, defined by $const(x_m) = I'(c'_m)$. Then $term$ is a mapping defined as follows if $x \in N^0$, then $term(x) = const(x)$, and if $x = \langle i, x_1, \dots , x_{a_i}\rangle$, then $term(x) = g_i(term(x_1), \dots , term(x_{a_i}))$.

Notice that for each natural number x, if $term(x) = \tau$, then $\alpha(x) \simeq \tau_{\mathfrak{M}_{I'}}$.

Let $x_1, \dots , x_{a_i}$ be natural numbers and $v = \pi(T_0(g_i(term(x_1), \dots , term(x_{a_i}))))$. Define

$$\varphi_i(x_1, \dots , x_{a_i}) \simeq \begin{cases} \langle i, x_1, \dots , x_{a_i}\rangle & \text{if } v \in D^i, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

It is clear that φ_i is a p. r. function and if $x_1, \dots , x_{a_i} \in dom(\alpha)$, then

$$\varphi_i(x_1, \dots , x_{a_i}) \text{ is defined } \iff \theta_i(\alpha(x_1), \dots , \alpha(x_{a_i})) \text{ is defined.}$$

In the same way, we shall define the predicate σ_j. Denote by W^j the set of all codes of termal predicates of the form $T_j(\tau_1, \dots , \tau_{b_j})$ or $\neg T_j(\tau_1, \dots , \tau_{b_j})$. Let $G_j = \{v : v \in W_j \ \& \ \Pi^v_{\mathfrak{M}_{I'}} \simeq 0\}$. G_i is r. e., since $\mathfrak{M}$ is uniformly effective with respect to $\mathfrak{A}$.

Let $x_1, \dots , x_{b_j} \in N$ and $v = \pi(T_j(term(x_1), \dots , term(x_{b_j})))$, $w = \pi(\neg T_j(term(x_1), \dots , term(x_{b_j})))$. Define

$$\sigma_j(x_1, \dots , x_{b_j}) \simeq \begin{cases} 0 & \text{if } v \in G^j, \\ 1 & \text{if } w \in G^j, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

It is clear that σ_j is p. r., and if $x_1, \dots , x_{b_j} \in dom(\alpha)$, then

$$\sigma_j(x_1, \dots , x_{b_j}) \simeq \Sigma_j(\alpha(x_1), \dots , \alpha(x_{b_j})).$$

□

7.4. Definition. The structure $\mathfrak{M}$ is called *uniformly effective* if it is uniformly effective with respect to each many-sorted structure.

7.5. Corollary. *A structure $\mathfrak{M}$ is uniformly effective iff $\mathfrak{M}$ admits a total effective enumeration.*

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