Relativized Degree Spectra

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Abstract. We present a relativized version of the notion of a degree spectrum of a structure with respect to finitely many abstract structures. We study the connection to the notion of joint spectrum. We prove that some properties of the degree spectrum as a minimal pair theorem and the existence of quasi-minimal degrees are true for the relative spectrum.
Key words: enumeration degrees; forcing; degree spectra; recursive Σ_k^+ formulae.

1 Introduction

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_s)$ be a structure, where $\mathbb{N}$ is the set of all natural numbers, each R_i is a subset of $\mathbb{N}^{r_i}$ and the equality $=$ and the inequality $\neq$ are among $R_1, \ldots, R_s$.

An enumeration f of $\mathfrak{A}$ is a total mapping from $\mathbb{N}$ onto $\mathbb{N}$.

Given an enumeration f of $\mathfrak{A}$ and a subset A of $\mathbb{N}^a$ let

$$f^{-1}(A) = \{\langle x_1, \ldots, x_a\rangle \mid (f(x_1), \ldots, f(x_a)) \in A\}.$$

Denote by $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \ldots \oplus f^{-1}(R_s)$.

Given a set X of natural numbers denote by $d_{\mathrm{e}}(X)$ the enumeration degree of X and by $d_{\mathrm{T}}(X)$ the Turing degree of X.

The following notion of enumeration degree spectrum of $\mathfrak{A}$ is introduced by Soskov [10].

Definition 1. *The enumeration degree spectrum* of $\mathfrak{A}$ is the set

$$\mathrm{DS}(\mathfrak{A}) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) \mid f \text{ is an enumeration of } \mathfrak{A}\}\ .$$

Let us point out that the notion of enumeration degree spectra of a structure $\mathfrak{A}$ differs from the one usually studied in the literature [2, 6–8] where the degree spectrum of a structure $\mathfrak{A}$ is defined to be the set

$$\mathrm{DS}_{\mathrm{T}}(\mathfrak{A}) = \{d_{\mathrm{T}}(f^{-1}(\mathfrak{A})) \mid f \text{ is an injective enumeration of } \mathfrak{A}\}.$$

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We shall call $\mathrm{DS_T}(\mathfrak{A})$ the *Turing degree spectrum* of $\mathfrak{A}$. In some sense the notion of enumeration degree spectra is more general of the notion of Turing degree spectra. To see that observe that if we want to consider Turing degrees instead of enumeration degrees then it is enough to take the structure

$$\mathfrak{A}^+ = (\mathbb{N}; R_1, \dots, R_s, R_1^c, \dots, R_s^c),$$

where R_i^c is the complement of R_i. Let ι be the Roger's embedding of the Turing degrees into the enumeration degrees. Then

$$\mathrm{DS}(\mathfrak{A}^+) = \{\iota(d_\mathrm{T}(f^{-1}(\mathfrak{A}))) \mid f \text{ is an enumeration of } \mathfrak{A}\}\ .$$

Concerning the injectivity of the enumerations Soskov [10] proved that for every enumeration f of $\mathfrak{A}$ there exists a bijective enumeration g of $\mathfrak{A}$ such that $g^{-1}(\mathfrak{A}) \leq_e f^{-1}(\mathfrak{A})$. The last result shows that almost all of the known results about Turing degree spectra remain valid also for enumeration degree spectra.

Since we are going to work only with enumeration degree spectra from now on we shall call them simply degree spectra.

Soskov [10] initiated the study of the properties of the degree spectra as sets of enumeration degrees. He introduced the notion of *co-spectrum* $\mathrm{CS}(\mathfrak{A})$ of a structure $\mathfrak{A}$ as the set of all lower bounds of the elements of the degree spectra and proved several properties which show that the degree spectra behave with respect to their co-spectra very much like the cones of the enumeration degrees $\{\mathbf{x} \mid \mathbf{x} \geq \mathbf{a}\}$ behave with respect to the intervals $\{\mathbf{x} \mid \mathbf{x} \leq \mathbf{a}\}$.

Some typical properties of degree spectra and their co-spectra are the existence of minimal pair of enumeration degrees and the existence of quasi-minimal degree.

More precisely, for every degree spectrum $\mathrm{DS}(\mathfrak{A})$, there exist total enumeration degrees $\mathbf{f_0}$ and $\mathbf{f_1}$ in $\mathrm{DS}(\mathfrak{A})$ such that the set of all enumeration degrees less than or equal to both $\mathbf{f_0}$ and $\mathbf{f_1}$ is equal to $\mathrm{CS}(\mathfrak{A})$. Every such pair of degrees is called *minimal pair for* $\mathrm{DS}(\mathfrak{A})$.

For each degree spectrum $\mathrm{DS}(\mathfrak{A})$, there is an enumeration degree $\mathbf{q} \notin \mathrm{CS}(\mathfrak{A})$, called *quasi-minimal for* $\mathrm{DS}(\mathfrak{A})$ such that every total degree $\mathbf{a} \geq \mathbf{q}$ belongs to $\mathrm{DS}(\mathfrak{A})$ and every total degree $\mathbf{a} \leq \mathbf{q}$ belongs to $\mathrm{CS}(\mathfrak{A})$.

In this paper we shall relativize Soskov's approach to degree spectra by considering multi-component spectra.

The notion of relatively intrinsically Σ^0_n sets on a structure $\mathfrak{A}$, studied by Ash, Chisholm, Knight, Manasse and Slaman [3, 4], defines a kind of reducibility of a set to a structure. A set A is relatively intrinsically Σ^0_{n+1} on $\mathfrak{A}$ if for every enumeration f of $\mathfrak{A}$, $f^{-1}(A)$ is enumeration reducible to $f^{-1}(\mathfrak{A})^{(n)}$.

Soskov and Baleva [11] extended this notion in the spirit of Ash [1]. Let $B_1, \dots, B_k$ be sets of natural numbers. A set A is relatively intrinsically Σ^0_{n+1} with respect to $B_1, \dots, B_k$ if $f^{-1}(A)$ is enumeration reducible to $f^{-1}(\mathfrak{A})^{(n)}$ for every enumeration f of $\mathfrak{A}$ such that $f^{-1}(B_i)$ is enumeration reducible to $f^{-1}(\mathfrak{A})^{(i)}$ for each $i = 1, \dots, k$. In other words, in the definition above not all enumerations of $\mathfrak{A}$ are considered but only those enumerations which "assume" that each B_i is relatively Σ^0_{i+1} on $\mathfrak{A}$ for $i = 1, \dots, k$.

Consider a structure $\mathfrak{A}$ and finitely many structures $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. We will restrict the class of enumerations of $\mathfrak{A}$ to these enumerations of $\mathfrak{A}$ which "assume" that each $\mathfrak{A}_i$ is relatively Σ^0_{i+1} on $\mathfrak{A}$.

An enumeration f of $\mathfrak{A}$ is *n-acceptable* with respect to the structures $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ if $f^{-1}(\mathfrak{A}_i)$ is enumeration reducible to $f^{-1}(\mathfrak{A})^{(i)}$ for each $i \leq n$.

Definition 2. *The relative spectrum* of the structure $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is the set

$$\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \{d_\mathrm{e}(f^{-1}(\mathfrak{A})) \mid f \text{ is a } n\text{-acceptable enumeration of } \mathfrak{A}\}\ .$$

In the present paper we shall study the properties of the relative spectra. We shall show that all properties of the degree spectra obtained by Soskov [10] remain true for the relative spectra and hence these properties are not enough to specify the sets of enumeration degrees which are degree spectra. In the last section we shall compare the notion of relative spectra with another generalization of the notion of spectra — the joint spectrum of a structure $\mathfrak{A}$ with respect to the structures $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ [12–14].

2 Preliminaries

2.1 Enumeration Degrees

Intuitively a set A is enumeration reducible to a set B, denoted by $A \leq_\mathrm{e} B$, if there is an effective procedure to enumerate A given any enumeration of B. More precisely, the set A is enumeration reducible to B if there is an enumeration operator Γ_z such that $A = \Gamma_z(B)$, i.e.

$$(\forall x)(x \in A \iff (\exists v)(\langle v, x\rangle \in W_z \ \&\ D_v \subseteq B))$$

where D_v is the finite set with a canonical code v and W_z is the recursively enumerable set with index z with respect to a Gödel numbering of all r.e. sets.

The relation $\leq_\mathrm{e}$ is reflexive and transitive and induces an equivalence relation $\equiv_\mathrm{e}$ on all sets of natural numbers. The enumeration degree of the set A, denoted by $d_\mathrm{e}(A)$, is the equivalence class relatively $\equiv_\mathrm{e}$. By $\mathcal{D}_\mathrm{e}$ we denote the set of all enumeration degrees. Define $A^+ = A \oplus (\mathbb{N}\backslash A)$. A set A is *total* if $A \equiv_\mathrm{e} A^+$. An enumeration degree $\mathbf{a}$ is total if $\mathbf{a}$ contains the e-degree of a total set. Cooper [5] introduced the jump operation "$'$" for enumeration degrees.

Definition 3. Given a set A, let $K^0_A = \{\langle x, z\rangle \mid x \in \Gamma_z(A)\}$.

The *e-jump* A' of A is the set $(K^0_A)^+$.

1. $A^{(0)} = A$;
2. $A^{(n+1)} = (A^{(n)})'$.

It is noteworthy that the set A is Σ^B_{n+1} if $A \leq_e (B^+)^{(n)}$.

This definition can be further generalized to the notion of $A^{(\alpha)}$ for any constructive ordinal α, see [11].

Definition 4. Let $B_0, \ldots, B_n$ be arbitrary subsets of $\mathbb{N}$. Define the set $\mathcal{P}(B_0, \ldots, B_i)$ by induction on $i \leq n$ as follows:

1. $\mathcal{P}(B_0) = B_0$;
2. If $i < n$ then $\mathcal{P}(B_0, \ldots, B_{i+1}) = (\mathcal{P}(B_0, \ldots, B_i))' \oplus B_{i+1}$.

We will use the following jump inversion theorem proved by Soskov:

Theorem 5 ([9]). *Let $B_0, \ldots, B_n$ be a sequence of sets of natural numbers and $k < n$. Suppose that Q is a total set such that $\mathcal{P}(B_0, \ldots, B_n) \leq_{\mathrm{e}} Q$ and let A be a set such that $A \not\leq_{\mathrm{e}} \mathcal{P}(B_0, \ldots, B_k)$ and $A^+ \leq_{\mathrm{e}} Q$. Then there exists a total set F with the following properties:*

1. *$B_i \leq_{\mathrm{e}} F^{(i)}$ for all $i \leq n$;*
2. *$F^{(i+1)} \equiv_{\mathrm{e}} F \oplus (\mathcal{P}(B_0, \ldots, B_i))'$ for all $i \leq n$;*
3. *$A \not\leq_{\mathrm{e}} F^{(k)}$;*
4. *$F^{(n)} \equiv_{\mathrm{e}} Q$.*

Furthermore, this theorem can be modified using regular enumerations in the following way:

Theorem 6 ([14]). *For each $k \in \{0, \ldots, n-1\}$, let $\{A_r^k\}_{r \in \mathbb{N}}$ be a sequence of subsets of $\mathbb{N}$ such that for every r, $A_r^k \not\leq_{\mathrm{e}} \mathcal{P}(B_0, \ldots, B_k)$. Then there exists a total set F with the following properties:*

1. *$B_i \leq_{\mathrm{e}} F^{(i)}$ for all $i \leq n$;*
2. *$A_r^k \not\leq_{\mathrm{e}} F^{(k)}$ for all r and all $k < n$.*

Let $\mathcal{A} \subseteq \mathcal{D}_e$. Then $\mathcal{A}$ is *upwards closed* if whenever $\mathbf{b}$ is a total e-degree above an enumeration degree $\mathbf{a} \in \mathcal{A}$ then $\mathbf{b} \in \mathcal{A}$.

The *co-set of* $\mathcal{A}$ is the set $co(\mathcal{A})$ of all lower bounds of $\mathcal{A}$:

$$co(\mathcal{A}) = \{\mathbf{b} \mid \mathbf{b} \in \mathcal{D}_{\mathrm{e}} \ \& \ (\forall \mathbf{a} \in \mathcal{A})(\mathbf{b} \leq \mathbf{a})\} \ .$$

2.2 Degree Spectra and Co-Spectra

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_s)$ be a structure. Recall that the degree spectrum of $\mathfrak{A}$ is $\mathrm{DS}(\mathfrak{A}) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) \mid f \text{ is an enumeration of } \mathfrak{A}\}$.

For each natural number n, the nth jump spectrum of $\mathfrak{A}$ is the set

$$\mathrm{DS}_n(\mathfrak{A}) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})^{(n)}) \mid f \text{ is an enumeration of } \mathfrak{A}\} \ .$$

Definition 7. The co-spectrum of $\mathfrak{A}$ is the co-set of $\mathrm{DS}(\mathfrak{A})$:

$$\mathrm{CS}(\mathfrak{A}) = \{\mathbf{b} \mid (\forall \mathbf{a} \in \mathrm{DS}(\mathfrak{A}))(\mathbf{b} \leq \mathbf{a})\} \ .$$

The nth co-spectrum of $\mathfrak{A}$ is the set $\mathrm{CS}_n(\mathfrak{A}) = co(\mathrm{DS}_n(\mathfrak{A}))$.

Soskov [10] proved that every degree spectrum is an upwards closed set of enumeration degrees. Therefore the degree spectra share all properties of upwards closed sets of degrees. Here are some examples of such properties. Let $\mathcal{A}$ be an upwards closed set of degrees. Then:

1. $co(\mathcal{A}) = co(\{\mathbf{b} \in \mathcal{A} \mid \mathbf{b} \text{ is a total e-degree }\})$;
2. for each $n > 0$ and any enumeration degree $\mathbf{c} \in \mathcal{D}_e$,
   $co(\mathcal{A}) = co(\{\mathbf{b} \in \mathcal{A} \mid \mathbf{c} \leq \mathbf{b}^{(n)}\})$.

From the second property it follows that the elements of an upwards closed set $\mathcal{A}$ with arbitrary high jumps determine completely the co-set of $\mathcal{A}$.

Note that the degree spectrum $\mathrm{DS}(\mathfrak{A})$ does not necessarily contain all enumeration degrees $\mathbf{b} \geq \mathbf{a}$, for some $\mathbf{a} \in \mathrm{DS}(\mathfrak{A})$. For example, the degree spectrum of the structure $\mathfrak{A} = (\mathbb{N}; =, \neq)$ is precisely the set of all total degrees.

Further properties true of the degree spectra but not necessarily true of all upwards closed sets are:

1. the existence of a minimal pair for the degree spectrum $\mathrm{DS}(\mathfrak{A})$;
2. the existence of quasi-minimal degree for the degree spectrum $\mathrm{DS}(\mathfrak{A})$;
3. for each $n \geq 1$ and each enumeration degree $\mathbf{c} \in \mathrm{DS}_n(\mathfrak{A})$,

$$\mathrm{CS}(\mathfrak{A}) = co(\{\mathbf{b} \in \mathrm{DS}(\mathfrak{A}) \mid \mathbf{c} = \mathbf{b}^{(n)}\}) \ .$$

The third property shows that all elements of the degree spectrum $\mathrm{DS}(\mathfrak{A})$ with low jumps also determine its co-set $\mathrm{CS}(\mathfrak{A})$.

3 Relative Spectra of Structures

We shall relativize the notion of degree spectrum of $\mathfrak{A}$ by considering multi-component spectra. We start by restricting the class of enumerations of $\mathfrak{A}$, considered in the definition of relative spectra.

Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ be arbitrary abstract structures on $\mathbb{N}$.

Definition 8. Let $k \leq n$. An enumeration f of $\mathfrak{A}$ is *k-acceptable with respect to the structures* $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ if $f^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(i)}$ for each $i \leq k$.

Denote by $\mathcal{E}_k$ the class of all k-acceptable enumerations of $\mathfrak{A}$ with respect to the structures $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$.

The relative spectrum of $\mathfrak{A}$ is the set generated by all n-acceptable enumerations of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. Recall that:

Definition 9. *The relative spectrum* of the structure $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is the set

$$\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) \mid f \in \mathcal{E}_n\} \ .$$

We shall see that the relative spectrum of the structure $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is an upwards closed set of degrees and hence it has all properties of upwards closed sets.

Lemma 10. *Let f be an enumeration of $\mathfrak{A}$ and F be a total set such that $f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} F$ and $f^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} F^{(i)}$ for all $i \leq n$. Then there exists a n-acceptable enumeration g of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ such that*

1. $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F$*;*
2. for every $B \subseteq \mathbb{N}$*, it holds that* $g^{-1}(B) \leq_{\mathrm{e}} F \oplus f^{-1}(B)$ *.*

Proof. Fix two different natural numbers s and t and let x_s and x_t be such that $f(x_s) \simeq s$ and $f(x_t) \simeq t$. Define an enumeration

$$g(x) \simeq \begin{cases} f(x/2) & \text{if } x \text{ is even,} \\ s & \text{if } x = 2z+1 \text{ and } z \in F, \\ t & \text{if } x = 2z+1 \text{ and } z \notin F. \end{cases}$$

It is clear that $f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} g^{-1}(\mathfrak{A})$. Since "$=$" and "$\neq$" are among the predicates of $\mathfrak{A}$, $F \leq_e g^{-1}(\mathfrak{A})$. Hence $F \oplus f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} g^{-1}(\mathfrak{A})$.

In order to see that $g^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} F \oplus f^{-1}(\mathfrak{A})$, consider the predicate R_i of $\mathfrak{A}$. Let $x_1, \ldots, x_{r_i}$ be arbitrary natural numbers. Define the natural numbers $y_1, \ldots, y_{r_i}$ by means of the following recursive in F procedure. Consider x_j for $j \leq r_i$. If x_j is even then let $y_j = x_j/2$. If $x_j = 2z+1$ and $z \in F$ then let $y_j = x_s$. If $x_j = 2z+1$ and $z \notin F$ then let $y_j = x_t$.

Clearly $\langle x_1, \ldots, x_{r_i}\rangle \in g^{-1}(R_i) \iff \langle y_1, \ldots, y_{r_i}\rangle \in f^{-1}(R_i)$. Since the set F is total, $g^{-1}(R_i) \leq_{\mathrm{e}} F \oplus f^{-1}(\mathfrak{A})$.

It follows that $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F \oplus f^{-1}(\mathfrak{A})$. But since $f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} F$, we have that $F \oplus f^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F$ and so $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F$.

For the proof of the second property, consider a set of natural numbers B. Let $g^{-1}(=) = \{\langle x, y\rangle : g(x) = g(y)\}$. Since $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F$ then $g^{-1}(=) \leq_{\mathrm{e}} F$. From the definition of g it follows that

$$g^{-1}(B) = \{x \mid (\exists y \in f^{-1}(B))(\langle x, 2y\rangle \in g^{-1}(=))\} \ .$$

Therefore $g^{-1}(B) \leq_e F \oplus f^{-1}(B)$.

Then for $i \in \{1, \ldots, n\}$, it holds that $g^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} F \oplus f^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} F \oplus F^{(i)} \equiv_{\mathrm{e}} F^{(i)} \equiv_{\mathrm{e}} g^{-1}(\mathfrak{A})^{(i)}$. And thus g is a n-acceptable enumeration of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$.

Proposition 11. *The relative spectrum is closed upwards, i.e. if* $\mathbf{b}$ *is a total e-degree and for some* $\mathbf{a} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$, $\mathbf{b} \geq \mathbf{a}$ *then* $\mathbf{b} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$*.*

Proof. Let $\mathbf{b}$ be a total degree, $\mathbf{b} \geq \mathbf{a}$ and $\mathbf{a} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$. Consider a total set F representing the degree $\mathbf{b}$ and let f be a n-acceptable enumeration of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ such that $f^{-1}(\mathfrak{A}) \in \mathbf{a}$. Thus $f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} F$ and $f^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(i)} \leq_{\mathrm{e}} F^{(i)}$ for all $i \leq n$. Then by the previous lemma there exists a n-acceptable enumeration g of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ such that $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F$. Hence $d_{\mathrm{e}}(F) = \mathbf{b} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

Note that if f is a k-acceptable enumeration of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ for some $k \leq n$ then $\mathcal{P}(f^{-1}(\mathfrak{A}), f^{-1}(\mathfrak{A}_1), \ldots, f^{-1}(\mathfrak{A}_k)) \equiv_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(k)}$.

Definition 12. For $k \in \{1, \ldots, n\}$, *the kth relative jump spectrum of* $\mathfrak{A}$ *with respect to* $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is the set

$$\mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \{\mathbf{a}^{(k)} \mid \mathbf{a} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)\} \ .$$

Proposition 13. *For $k \in \{1, \dots, n\}$, the kth relative jump spectrum of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ is closed upwards, i.e. if $\mathbf{b}$ is a total e-degree, $\mathbf{b} \geq \mathbf{a}^{(k)}$ and $\mathbf{a} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$ then $\mathbf{b} \in \mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.*

Proof. Let G be a total set representing the total degree $\mathbf{b}$, $\mathbf{b} \geq \mathbf{a}^{(k)}$, $\mathbf{a} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$ and let f be a n-acceptable enumeration of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ such that $f^{-1}(\mathfrak{A}) \in \mathbf{a}$. Then $f^{-1}(\mathfrak{A})^{(k)} \leq_{\mathrm{e}} G$. The enumeration f is a n-acceptable, hence $f^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(i)}$ for each $i \leq n$. Then $\mathcal{P}(f^{-1}(\mathfrak{A}), f^{-1}(\mathfrak{A}_1), \dots, f^{-1}(\mathfrak{A}_k)) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(k)} \leq_{\mathrm{e}} G$. By Theorem 5 there exists a total set F such that $f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} F$, $F^{(k)} \equiv_{\mathrm{e}} G$ and $f^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} F^{(i)}$ for $i \leq k$. By Lemma 10 there is a k-acceptable enumeration g of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_k$ so that $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F$. So, $g^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} g^{-1}(\mathfrak{A})^{(i)}$ for $i \leq k$. But again by Lemma 10 for each $j \in \{k+1, \dots, n\}$, we have $g^{-1}(\mathfrak{A}_j) \leq_{\mathrm{e}} F \oplus f^{-1}(\mathfrak{A}_j) \leq_{\mathrm{e}} F \oplus f^{-1}(\mathfrak{A})^{(j)} \leq_{\mathrm{e}} F \oplus F^{(j)} \equiv_{\mathrm{e}} F^{(j)} \equiv_{\mathrm{e}} g^{-1}(\mathfrak{A})^{(j)}$. It follows that g is a n-acceptable enumeration of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$. Thus $d_{\mathrm{e}}(g^{-1}(\mathfrak{A})) \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$ and $G \equiv_{\mathrm{e}} g^{-1}(\mathfrak{A})^{(k)}$. Then $d_{\mathrm{e}}(G) \in \mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.

4 Relative Co-spectra of Structures

Definition 14. *The relative co-spectrum of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$* is the co-set of $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$, i.e.

$$\mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \{\mathbf{b} \mid \mathbf{b} \in \mathcal{D}_{\mathrm{e}} \& (\forall \mathbf{a} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n))(\mathbf{b} \leq \mathbf{a})\} \ .$$

Define by $\mathrm{RS}_0(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$ and by $\mathrm{CRS}_0(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.

For every enumeration f of $\mathfrak{A}$ and each $k \leq n$, let

$$\mathcal{P}^f_k = \mathcal{P}(f^{-1}(\mathfrak{A}), f^{-1}(\mathfrak{A}_1), \dots, f^{-1}(\mathfrak{A}_k)) \ .$$

Definition 15. For $k \leq n$, *the kth relative co-spectrum of $\mathfrak{A}$ with respect to* $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ is the co-set of $\mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$, i.e.

$$\mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \{\mathbf{b} \mid \mathbf{b} \in \mathcal{D}_{\mathrm{e}} \& (\forall \mathbf{a} \in \mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n))(\mathbf{b} \leq \mathbf{a})\} \ .$$

We will show that the kth relative co-spectrum of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ depends actually only on the first $k+1$ structures $\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_k$.

Proposition 16. $\mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_k, \dots, \mathfrak{A}_n) = \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_k)$ *for each $k \in \{0, \dots, n\}$.*

Proof. It is clear that $\mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_k, \dots, \mathfrak{A}_n) \subseteq \mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_k)$. Thus, $\mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_k) \subseteq \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_k, \dots, \mathfrak{A}_n)$.

Fix $\mathbf{a} \in \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1 \dots \mathfrak{A}_k \dots, \mathfrak{A}_n)$ and let $A \in \mathbf{a}$. Assume that $A \not\leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(k)}$ for some k-acceptable enumeration f of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_k$. Then $A \not\leq_{\mathrm{e}} \mathcal{P}^f_k$. Hence, by Theorem 5 for $B_0 = f^{-1}(\mathfrak{A})$, $B_1 = f^{-1}(\mathfrak{A}_1)$, …, $B_n =$

$f^{-1}(\mathfrak{A}_n)$, $B_{n+1} = \mathbb{N}$, there exists a total set F such that $f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} F$, $A \not\leq_{\mathrm{e}} F^{(k)}$ and $f^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} F^{(i)}$ for each $i \leq n$. From Lemma 10 it follows that there is a k-acceptable enumeration g of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ such that $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F$. Then $A \not\leq_{\mathrm{e}} g^{-1}(\mathfrak{A})^{(k)}$ and $g^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} g^{-1}(\mathfrak{A})^{(i)}$ for $i \leq k$. But for $j \in \{k+1, \ldots, n\}$, $g^{-1}(\mathfrak{A}_j) \leq_{\mathrm{e}} F \oplus f^{-1}(\mathfrak{A}_j) \leq_{\mathrm{e}} F \oplus F^{(j)} \equiv_{\mathrm{e}} F^{(j)} \equiv_{\mathrm{e}} g^{-1}(\mathfrak{A})^{(j)}$. Thus g is a n-acceptable enumeration of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ and $A \not\leq_{\mathrm{e}} g^{-1}(\mathfrak{A})^{(k)}$ which contradicts the choice of A as $d_{\mathrm{e}}(A) \in \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_k \ldots, \mathfrak{A}_n)$.

Proposition 17. *For every $A \subseteq \mathbb{N}$ and $k \leq n$ the following are equivalent:*

1. $d_{\mathrm{e}}(A) \in \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.
2. $A \leq_{\mathrm{e}} \mathcal{P}^f_k$ *for every k-acceptable enumeration f of $\mathfrak{A}$ with respect to $\mathfrak{A}_1 \ldots \mathfrak{A}_k$.*

Proof. If f is a k-acceptable enumeration of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$ then $f^{-1}(\mathfrak{A})^{(k)} \equiv_{\mathrm{e}} \mathcal{P}^f_k$ since $f^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(i)}$ for each $i \leq k$. On the other hand by Proposition 16 $\mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_k)$. So, the equivalence of the two assertions follows from the definition of the kth relative co-spectrum of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$.

5 Forcing k-Definable Sets

In order to obtain a forcing normal form of the sets with enumeration degrees in $\mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ we shall define the notions of a forcing relation $\tau \Vdash_k F_e(x)$ and a relation $f \models_k F_e(x)$ for $k \leq n$.

Let f be an enumeration of $\mathfrak{A}$.

Definition 18. For every $i \leq n$ and e, $x \in \mathbb{N}$, define the relations $f \models_i F_e(x)$ and $f \models_i \neg F_e(x)$ by induction on i:

$$\begin{array}{lll}
1.\ f \models_0 F_e(x) & \iff & (\exists v)(\langle v, x\rangle \in W_e \ \&\ D_v \subseteq f^{-1}(\mathfrak{A}));\\
2.\ f \models_{i+1} F_e(x) & \iff & (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(\\
 & & (u = \langle 0, e_u, x_u\rangle \ \&\ f \models_i F_{e_u}(x_u)) \vee \\
 & & (u = \langle 1, e_u, x_u\rangle \ \&\ f \models_i \neg F_{e_u}(x_u)) \vee \\
 & & (u = \langle 2, x_u\rangle \ \&\ x_u \in f^{-1}(\mathfrak{A}_{i+1}))));\\
3.\ f \models_i \neg F_e(x) & \iff & f \not\models_i F_e(x)\ .
\end{array}$$

As an immediate corollary of the definitions we receive the following:

Lemma 19. *Let $A \subseteq \mathbb{N}$ and $k \leq n$. Then*

$$A \leq_{\mathrm{e}} \mathcal{P}^f_k \iff (\exists e)(A = \{x \mid f \models_k F_e(x)\})\ .$$

The forcing conditions, called *finite parts*, are finite mappings τ of $\mathbb{N}$ to $\mathbb{N}$. We will denote the finite parts by letters δ, τ, ρ. Assume an effective coding of the finite parts. By the least finite part with a fixed property we mean the finite part with a minimal code.

Definition 20. For any $i \leq n$ and $e, x \in \mathbb{N}$ and for every finite part τ, define the forcing relations $\tau \Vdash_i F_e(x)$ and $\tau \Vdash_i \neg F_e(x)$ following the definition of relation "$\models_i$".

$$\begin{array}{lll}
1.\ \tau \Vdash_0 F_e(x) & \iff & (\exists v)(\langle v, x\rangle \in W_e \ \&\ D_v \subseteq \tau^{-1}(\mathfrak{A}));\\
2.\ \tau \Vdash_{i+1} F_e(x) & \iff & (\exists v)(\langle v, x\rangle \in W_e \ \&\ \ (\forall u \in D_v)(\\
 & & (u = \langle 0, e_u, x_u\rangle \ \&\ \tau \Vdash_i F_{e_u}(x_u)) \vee\\
 & & (u = \langle 1, e_u, x_u\rangle \ \&\ \tau \Vdash_i \neg F_{e_u}(x_u)) \vee\\
 & & (u = \langle 2, x_u\rangle \ \&\ x_u \in \tau^{-1}(\mathfrak{A}_{i+1}))));\\
3.\ \tau \Vdash_i \neg F_e(x) & \iff & (\forall \rho \supseteq \tau)(\rho \nVdash_i F_e(x)) \ .
\end{array}$$

Definition 21. Let $k \leq n+1$. An enumeration f of $\mathfrak{A}$ is *k-generic* with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ if for every $j < k$ and $e, x \in \mathbb{N}$ it holds that

$$(\forall \tau \subseteq f)(\exists \rho)(\tau \subseteq \rho \ \&\ \rho \Vdash_j F_e(x)) \Longrightarrow (\exists \tau \subseteq f)(\tau \Vdash_j F_e(x)) \ .$$

Clearly if f is a k-generic enumeration then f is a m-generic for all $m \leq k$.

The next properties follow from the definition of a k-generic enumeration:

Lemma 22. *1. Let $k \leq n$ and $e, x \in \mathbb{N}$ and let $\tau \subseteq \rho$ be finite parts. Then*

$$\tau \Vdash_k (\neg)F_e(x) \Rightarrow \rho \Vdash_k (\neg)F_e(x) \ .$$

2. If f is a k-generic enumeration of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ then

$$f \models_k F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash_k F_e(x)) \ .$$

3. If f is a $(k+1)$-generic enumeration of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ then

$$f \models_k \neg F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash_k \neg F_e(x)) \ .$$

Definition 23. Let $A \subseteq \mathbb{N}$ and $k \leq n$. The set A is *forcing k-definable* on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ if there exist a finite part δ and $e \in \mathbb{N}$ such that

$$x \in A \iff (\exists \tau \supseteq \delta)(\tau \Vdash_k F_e(x)) \ .$$

Proposition 24. *For each $k \in \{0, 1, \dots, n\}$, let $\{A^k_r\}_{r \in \mathbb{N}}$ be a sequence of subsets of $\mathbb{N}$ such that for every r, the set A^k_r be not forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$. Then there exists a $(n+1)$-generic enumeration f of $\mathfrak{A}$ such that $A^k_r \not\leq_{\mathrm{e}} \mathcal{P}^f_k$ for all r and every $k \leq n$.*

Proof. We shall construct a $(n+1)$-generic enumeration f of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ such that for each $k \leq n$ and $r \in \mathbb{N}$, $A^k_r \not\leq_{\mathrm{e}} \mathcal{P}^f_k$. We will call the last condition the omitting condition. The enumeration f will be constructed on stages. On each stage q we shall define a finite part δ_q so that $\delta_q \subseteq \delta_{q+1}$ and ultimately we will define $f = \bigcup_q \delta_q$. We shall consider three kinds of stages. On stages $q = 3r$ we shall ensure that the mapping f is total and surjective. On stages $q = 3r+1$ we shall ensure that f is $(n+1)$-generic and on stages $q = 3r+2$ we shall ensure that f satisfies the omitting condition.

Let $\delta_0 = \emptyset$. Suppose that we have already defined δ_q.

(a) Case $q = 3r$. Let x_0 be the least natural number which does not belong to $\mathrm{dom}(\delta_q)$ and let s_0 be the least natural number which does not belong to the range of δ_q. Set $\delta_{q+1}(x_0) \simeq s_0$ and $\delta_{q+1}(x) \simeq \delta_q(x)$ for $x \neq x_0$.

(b) Case $q = 3\langle e, k, x\rangle + 1$. If $k \leq n$ then consider the set $X^k_{\langle e,x\rangle} = \{\rho \mid \rho \Vdash_k F_e(x)\}$. Check whether there exists a finite part $\rho \in X^k_{\langle e,x\rangle}$ that extends δ_q. If there is then let δ_{q+1} be the least extension of δ_q that belongs to $X^k_{\langle e,x\rangle}$. Otherwise let $\delta_{q+1} = \delta_q$.

(c) Case $q = 3\langle e, k, r\rangle + 2$. If $k \leq n$ then consider the set A^k_r which is not forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. Denote by

$$C = \{x \mid (\exists \tau \supseteq \delta_q)(\tau \Vdash_k F_e(x))\} \enspace .$$

Clearly C is forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ and hence $C \neq A^k_r$.

Let x_0 be the least natural number such that

$$x_0 \in C \;\&\; x_0 \notin A^k_r \vee x_0 \notin C \;\&\; x_0 \in A^k_r \enspace .$$

(i) Suppose that $x_0 \in C$. Then there exists a finite part τ such that

$$\delta_q \subseteq \tau \;\&\; \tau \Vdash_k F_e(x_0) \enspace . \tag{1}$$

Let δ_{q+1} be the least τ satisfying (1).

(ii) If $x_0 \notin C$ then set $\delta_{q+1}(x) \simeq \delta_q(x)$. Notice that in this case we have that $\delta_{q+1} \Vdash_k \neg F_e(x_0)$.

If $k > n$ then set $\delta_{q+1} = \delta_q$.

Let $f = \bigcup_q \delta_q$. It follows from stages $3r$ that the obtained enumeration is total and surjective. Furthermore, f is $(n+1)$-generic: suppose that for every finite part $\tau \subseteq f$, there is an extension ρ of τ so that $\rho \Vdash_k F_e(x)$. Consider the stage $q = 3\langle e, k, x\rangle + 1$. Since $\delta_q \subseteq f$ then there is a finite part $\rho \supseteq \delta_q$ such that $\rho \Vdash_k F_e(x)$. From the construction we have that $\delta_{q+1} \Vdash_k F_e(x)$ and $\delta_{q+1} \subseteq f$.

To prove that the enumeration f satisfies the omitting condition, let the set A^k_r be one of the given sets, not forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$, and suppose for a contradiction that $A^k_r \leq_e \mathcal{P}^f_k$. Then $A^k_r = \{x \mid f \models_k F_e(x)\}$ for some e. Since the enumeration f is $(n+1)$-generic and hence $(k+1)$-generic by Lemma 22 we have

$$f \models_k (\neg)F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash_k (\neg)F_e(x)) \tag{2}$$

for each number x.

Consider the stage $q = 3\langle e, k, r\rangle + 2$. From the construction there is a x_0 such that one of the following two cases holds:

(i) $x_0 \notin A^k_r$ & $\delta_{q+1} \Vdash_k F_e(x_0)$. By (2) $f \models_k F_e(x_0)$ and hence $x_0 \in A^k_r$. A contradiction.

(ii) $x_0 \in A^k_r$ and $(\forall \rho \supseteq \delta_q)(\rho \nVdash_k F_e(x_0))$. Then $\delta_q \Vdash_k \neg F_e(x_0)$. So by (2), $f \not\models_k F_e(x_0)$ and hence $x_0 \notin A^k_r$. A contradiction.

Thus we have obtained the desired enumeration of $\mathfrak{A}$.

Corollary 25. *For each $k \in \{0, 1, \ldots, n\}$, let $\{A_r^k\}_{r\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{N}$ such that for all r, the set A_r^k is not forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. Then there exists a n-acceptable enumeration g of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ such that the enumeration degree of $g^{-1}(\mathfrak{A})$ is total and $A_r^k \not\leq_{\mathrm{e}} g^{-1}(\mathfrak{A})^{(k)}$ for all $k \leq n$ and $r \in \mathbb{N}$.*

Proof. We know from the previous proposition that there is a $(n+1)$-generic enumeration f of $\mathfrak{A}$ such that $A_r^k \not\leq_{\mathrm{e}} \mathcal{P}_k^f$ for all $k \leq n$ and $r \in \mathbb{N}$.

Let $B_0 = f^{-1}(\mathfrak{A})$, $B_1 = f^{-1}(\mathfrak{A}_1)$, …, $B_n = f^{-1}(\mathfrak{A}_n)$, $B_{n+1} = \mathbb{N}$. For each $k \leq n$ and all r, the set $A_r^k \not\leq_{\mathrm{e}} \mathcal{P}_k^f = \mathcal{P}(B_0, \ldots, B_k)$. By Theorem 6 there exists a total set F such that $f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} F$ and for each $i \leq n$, $f^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} F^{(i)}$ and moreover $A_r^k \not\leq_{\mathrm{e}} F^{(k)}$ for all $k \leq n$ and $r \in \mathbb{N}$. From Lemma 10 it follows that there is a n-acceptable enumeration g of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ such that $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F$. Then the enumeration degree of $g^{-1}(\mathfrak{A})$ is total and $A_r^k \not\leq_{\mathrm{e}} g^{-1}(\mathfrak{A})^{(k)}$ for all $k \leq n$ and $r \in \mathbb{N}$.

Theorem 26. *For every $A \subseteq \mathbb{N}$ and $k \leq n$ if $d_{\mathrm{e}}(A) \in \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ then A is forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$.*

Proof. If a set A is not forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ then there exists a n-acceptable enumeration g of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ such that $A \not\leq_{\mathrm{e}} g^{-1}(\mathfrak{A})^{(k)}$. Hence $A \not\leq_{\mathrm{e}} \mathcal{P}_k^g$ and according to Proposition 17, $d_{\mathrm{e}}(A) \notin \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

The proof of the existence of such a n-acceptable enumeration g is similar to the one in Corollary 25. First we construct a $(n+1)$-generic enumeration f of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ which omits the set A, i.e. $A \not\leq_{\mathrm{e}} \mathcal{P}_k^f$. Then we apply Theorem 5 and Lemma 10 in order to find a n-acceptable enumeration g of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ such that $A \not\leq_{\mathrm{e}} g^{-1}(\mathfrak{A})^{(k)}$.

We will see in the next section that the opposite is also true.

Next we will give an abstract version of Theorem 5. We shall examine the construction from Proposition 24 in order to get the complexity of the constructed enumeration. By $\mathcal{D}(\mathfrak{A})$ we denote the diagram of the structure $\mathfrak{A}$, i.e. $\mathcal{D}(\mathfrak{A}) = f^{-1}(\mathfrak{A})$ for $f = \lambda x.x$ and by $\mathcal{D}(\mathfrak{A}_i)$ — the diagram of the structure $\mathfrak{A}_i$ for $i = 1, \ldots, n$. Let $\mathcal{P}_n = \mathcal{P}(\mathcal{D}(\mathfrak{A}), \mathcal{D}(\mathfrak{A}_1), \ldots, \mathcal{D}(\mathfrak{A}_n))$.

Theorem 27. *Let $k < n$ and let $A \subseteq \mathbb{N}$ be not forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. Suppose that Q is a total set such that $A^+ \oplus \mathcal{P}_n \leq_{\mathrm{e}} Q$. Then there exists an enumeration g of $\mathfrak{A}$ satisfying the following conditions:*

1. *The enumeration degree of $g^{-1}(\mathfrak{A})$ is total;*
2. *g is a n-acceptable enumeration of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$;*
3. *$A \not\leq_{\mathrm{e}} g^{-1}(\mathfrak{A})^{(k)}$;*
4. *$g^{-1}(\mathfrak{A})^{(n)} \equiv_e Q$.*

Proof. First we define a n-generic enumeration f of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ which omits the set A, i.e. $A \not\leq_{\mathrm{e}} \mathcal{P}_k^f$. Note that since $k < n$, n-genericity suffices. On stages $q = 3\langle e, k, x\rangle + 1$ for $k < n$ we ensure that f is n-generic. On stages $q = 3r+2$ we ensure that f satisfies the only omitting condition $A \not\leq_{\mathrm{e}} \mathcal{P}^f_k$. From the construction and from the definition of the forcing relation it follows that the enumeration f will be enumeration reducible to $A^+ \oplus \mathcal{P}_n$.

Let $i \leq n$. Then there exists an e such that $\mathcal{P}^f_i = \{x : f \models_i F_e(x)\}$. Since f is n-generic, we can rewrite this as $\mathcal{P}^f_i = \{x : (\exists \tau \subseteq f)(\tau \Vdash_i F_e(x))\}$. Then $\mathcal{P}^f_i \leq_e f \oplus \mathcal{P}_n$.

Let $B_0 = f^{-1}(\mathfrak{A})$, $B_1 = f^{-1}(\mathfrak{A}_1)$, …, $B_n = f^{-1}(\mathfrak{A}_n)$. Then $\mathcal{P}(B_0, B_1, \ldots, B_n) \leq_{\mathrm{e}} f \oplus \mathcal{P}_n \leq_{\mathrm{e}} A^+ \oplus \mathcal{P}_n \leq_{\mathrm{e}} Q$. We also have that $A \not\leq_{\mathrm{e}} \mathcal{P}(B_0, B_1, \ldots, B_k) = \mathcal{P}^f_k$. By Theorem 5, there exists a total set F such that $f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} F$, $f^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} F^{(i)}$ for each $i \leq n$, $A \not\leq_{\mathrm{e}} F^{(k)}$ and $F^{(n)} \equiv_{\mathrm{e}} Q$. By Lemma 10 it follows that there is a n-acceptable enumeration g of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ such that $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F$. Then the enumeration degree of $g^{-1}(\mathfrak{A})$ is total, $A \not\leq_{\mathrm{e}} g^{-1}(\mathfrak{A})^{(k)}$ and $g^{-1}(\mathfrak{A})^{(n)} \equiv_{\mathrm{e}} Q$.

Proposition 28. *Let $k < n$ and $A \subseteq \mathbb{N}$ be not forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. Suppose that f is a n-acceptable enumeration of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ and Q is a total set such that $f^{-1}(\mathfrak{A})^{(n)} \oplus A^+ \leq_{\mathrm{e}} Q$. Then there exists a n-acceptable enumeration g of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ such that $g^{-1}(\mathfrak{A})^{(n)} \equiv_{\mathrm{e}} Q$ and $A \not\leq_{\mathrm{e}} g^{-1}(\mathfrak{A})^{(k)}$.*

Proof. We first construct a bijective enumeration h of $\mathfrak{A}$ such that $h^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})$. Let $f^{-1}(=) = \{\langle x, y\rangle : f(x) = f(y)\}$. Since $\neq$ is also among the predicates of $\mathfrak{A}$, $f^{-1}(=)^+ \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})$. Define the function m using primitive recursion relative $f^{-1}(\mathfrak{A})$ as follows:

$$\begin{aligned} m(0) &\simeq 0; \\ m(i+1) &\simeq \mu z[(\forall k \leq i)(\langle m(k), z\rangle \notin f^{-1}(=))] \ . \end{aligned}$$

Set $h = \lambda x. f(m(x))$. The enumeration h is bijective and $h^{-1}(\mathfrak{A}) \oplus f^{-1}(=)^+ \equiv_{\mathrm{e}} f^{-1}(\mathfrak{A})$. Moreover if B is an arbitrary set of natural numbers then $h^{-1}(B) \oplus f^{-1}(=)^+ \equiv_{\mathrm{e}} f^{-1}(B)$. Hence $h^{-1}(\mathfrak{A}_i) \oplus f^{-1}(=)^+ \equiv_{\mathrm{e}} f^{-1}(\mathfrak{A}_i)$ for each $i \leq n$. Since f is a n-acceptable enumeration of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$, for each $i \leq n$, $h^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(i)}$. Then $\mathcal{P}^h_n \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(n)} \leq_{\mathrm{e}} Q$.

Denote by $\mathfrak{B}$ the structure $(\mathbb{N}; h^{-1}(R_1), \ldots, h^{-1}(R_s))$ and if $\mathfrak{A}_i = (\mathbb{N}; R^i_1, \ldots, R^i_{s_i})$ then $\mathfrak{B}_i = (\mathbb{N}; h^{-1}(R^i_1), \ldots, h^{-1}(R^i_{s_i}))$. Thus $\mathcal{D}(\mathfrak{B}) \equiv_{\mathrm{e}} h^{-1}(\mathfrak{A})$ and $\mathcal{D}(\mathfrak{B}_i) \equiv_{\mathrm{e}} h^{-1}(\mathfrak{A}_i)$ for $i \leq n$. Hence $\mathcal{P}(\mathcal{D}(\mathfrak{B}), \mathcal{D}(\mathfrak{B}_1), \ldots, \mathcal{D}(\mathfrak{B}_n)) \leq_{\mathrm{e}} Q$. Since A is not forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ then from the definition the forcing k-definable sets it follows that A is not forcing k-definable on $\mathfrak{B}$ with respect to $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$. By Theorem 27 there exists a n-acceptable enumeration r of $\mathfrak{B}$ with respect to $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$ such that the enumeration degree of $r^{-1}(\mathfrak{B})$ is total, $r^{-1}(\mathfrak{B})^{(n)} \equiv_{\mathrm{e}} Q$ and $A \not\leq_{\mathrm{e}} r^{-1}(\mathfrak{B})^{(k)}$. Finally we can define $g = \lambda x. h(r(x))$. Then $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} r^{-1}(\mathfrak{B})$ and $g^{-1}(\mathfrak{A}_i) \equiv_{\mathrm{e}} r^{-1}(\mathfrak{B}_i)$ for $i \leq n$. Thus g is a n-acceptable enumeration on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$, $g^{-1}(\mathfrak{A})^{(n)} \equiv_{\mathrm{e}} Q$ and $A \not\leq_{\mathrm{e}} g^{-1}(\mathfrak{A})^{(k)}$.

6 Normal Form Theorem

In this section we shall give an explicit form of the sets which are forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ by means of recursive Σ_k^+ formulae. These formulae can be considered as a modification of Ash's formulae [1] appropriate for their use on abstract structures and they are first used by Soskov and Baleva [11].

6.1 Recursive Σ_k^+ Formulae

Let $\mathcal{L} = \{T_1, \ldots, T_s\}$ be the first order language of the structure $\mathfrak{A}$. For each $i \leq n$, let $\mathcal{L}_i = \{T_1^i, \ldots, T_{s_i}^i\}$ be the language of $\mathfrak{A}_i$ where every T_j^i is a r_j^i-ary predicate symbol. Without loss of generality we may assume that the languages are disjoined. Consider a fixed sequence $\{\mathbb{X}_i\}_{i \in \mathbb{N}}$ of variables.

Definition 29. (1) An elementary Σ_0^+ formula with free variables among W_1, …, W_r is an existential formula of the form

$$\exists Y_1 \ldots \exists Y_m \Phi(W_1, \ldots, W_r, Y_1, \ldots, Y_m),$$

where Φ is a finite conjunction of atomic formulae in $\mathcal{L}$;

(2) A Σ_i^+ formula with free variables among W_1, …, W_r is a recursively enumerable disjunction of elementary Σ_i^+ formulae with free variables among $W_1, \ldots, W_r$;

(3) An elementary Σ_{i+1}^+ formula with free variables among $W_1, \ldots, W_r$ is a formula of the form

$$\exists Y_1 \ldots \exists Y_m \Phi(W_1, \ldots, W_r, Y_1, \ldots, Y_m),$$

where Φ is a finite conjunction of atoms in $\mathcal{L}_{i+1}$, Σ_i^+ formulae or negations of Σ_i^+ formulae.

Let Φ be a Σ_i^+ formula with free variables among $W_1, \ldots, W_r$ and let $t_1, \ldots, t_r$ be elements of $\mathbb{N}$. Then by $(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) \models \Phi(W_1/t_1, \ldots, W_r/t_r)$ we denote that Φ is true in the structure, obtained from $\mathfrak{A}$ by adding the predicates from $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$, under the variable assignment v such that $v(W_1) = t_1, \ldots, v(W_n) = t_n$. More precisely we have the following definition:

Definition 30. (1) If Φ is an elementary Σ_0^+ formula then $(\mathfrak{A}) \models \Phi(W_1/t_1, \ldots, W_r/t_r)$ if Φ is true under the variable assignment v such that $v(W_1) = t_1, \ldots, v(W_n) = t_n$.

(2) If $\Phi = \exists Y_1 \ldots \exists Y_m \Psi(W_1, \ldots, W_r, Y_1, \ldots, Y_m)$ is an elementary Σ_{i+1}^+ formula, $\Psi = (\varphi \mathbin{\&} \alpha)$ where φ is a conjunction of Σ_i^+ formulae or negations of Σ_i^+ formulae and α is a conjunction of atoms in $\mathcal{L}_{i+1}$ then

$$\begin{aligned}
&(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_{i+1}) \models \Phi(W_1/t_1, \ldots, W_r/t_r) \iff \\
&\exists s_1 \ldots \exists s_m (\, (\mathfrak{A}, \ldots, \mathfrak{A}_i) \models \varphi(W_1/t_1, \ldots, W_r/t_r, Y_1/s_1, \ldots, Y_m/s_m) \mathbin{\&} \\
&\qquad (\mathfrak{A}_{i+1}) \models \alpha(W_1/t_1, \ldots, W_r/t_r, Y_1/s_1, \ldots, Y_m/s_m)) \ .
\end{aligned}$$

6.2 Formally k-Definable Sets

Definition 31. Let $A \subseteq \mathbb{N}$ and let $k \leq n$. The set A is *formally k-definable* on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ if there exists a recursive sequence $\{\Phi^{\gamma(x)}\}$ of Σ_k^+ formulae with free variables among $W_1, \ldots, W_r$ and elements $t_1, \ldots, t_r$ of $\mathbb{N}$ such that for every $x \in \mathbb{N}$, the following equivalence holds:

$$x \in A \iff (\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) \models \Phi^{\gamma(x)}(W_1/t_1, \ldots, W_r/t_r) \ .$$

We shall show that the forcing k-definable sets coincide with the formally k-definable sets.

Let var be an effective bijection between the natural numbers and the variables. Given a natural number x, by X we shall denote the variable $var(x)$. Let Q be one of the quantifiers $\exists$ or $\forall$, $E = \{y_1, y_2, \ldots, y_m\}$, where $y_1 < y_2 < \ldots < y_m$ and let Φ be an arbitrary formula. Then by $Q(y : y \in E)\Phi$ we shall denote the formula $QY_1 \ldots QY_m\Phi$.

Lemma 32. *Let $E = \{w_1, \ldots, w_r\} \subseteq \mathbb{N}$ and let $k, x, e \in \mathbb{N}$. There exists a uniform effective way to construct a Σ_k^+ formula $\Phi^k_{E,e,x}$ with free variables among $W_1, \ldots, W_r$ such that for every finite part δ with $\operatorname{dom}(\delta) = E$, the following equivalence is true:*

$$(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_k) \models \Phi^k_{E,e,x}(W_1/\delta(w_1), \ldots, W_r/\delta(w_r)) \iff \delta \Vdash_k F_e(x) \ .$$

Proof. We shall construct the formula $\Phi^k_{E,e,x}$ by induction on k following the definition of the forcing relation $\Vdash_k$.

(1) Let $k = 0$. Consider an element v of the set $V = \{v : \langle v, x\rangle \in W_e\}$. For every $u \in D_v$, define the atom Π_u as follows:

(a) If $u = \langle j, x_1, \ldots, x_{r_j}\rangle$, $j \leq s$ and all $x_1, \ldots, x_{r_j}$ are elements of E then let $\Pi_u = T_j(X_1, \ldots, X_{r_j})$.
(b) Let Π_u be $X_0 \neq X_0$ in the other cases.

Set $\Pi_v = \bigwedge_{u \in D_v} \Pi_u$ and $\Phi^0_{E,e,x} = \bigvee_{v \in V} \Pi_v$.

(2) Let $k = i + 1$. Let $V = \{v : \langle v, x\rangle \in W_e\}$ and $v \in V$.

For every $u \in D_v$, define the formula Π_u as follows:

(a) If $u = \langle 0, e_u, x_u\rangle$ then let $\Pi_u = \Phi^i_{E,e_u,x_u}$.
(b) If $u = \langle 1, e_u, x_u\rangle$ then let $\Pi_u = \neg(\bigvee_{E^* \supseteq E}(\exists y \in E^* \setminus E)\Phi^i_{E^*,e_u,x_u})$.
(c) If $u = \langle 2, x_u\rangle$, $x_u = \langle j, x_1, \ldots, x_{r_j^{i+1}}\rangle$, $j \leq s_{i+1}$ and $x_1, \ldots, x_{r_j^{i+1}} \in E$ then let $\Pi_u = T_j^{i+1}(X_1, \ldots, X_{r_j^{i+1}})$.
(d) Let $\Pi_u = \Phi^i_{\{0\},0,0} \wedge \neg\Phi^i_{\{0\},0,0}$ in any other case.

Now let $\Pi_v = \bigwedge_{u \in D_v} \Pi_u$ and set $\Phi^{i+1}_{E,e,x} = \bigvee_{v \in V} \Pi_v$.

An induction on k shows that for every k the Σ_k^+ formula $\Phi^k_{E,e,x}$ satisfies the requirements of the Lemma.

Theorem 33. *If a set $A \subseteq \mathbb{N}$ is forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ then A is formally k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$.*

Proof. Let A be forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1 \dots, \mathfrak{A}_n$. Then there exist a finite part δ and $e \in \mathbb{N}$ such that

$$x \in A \iff (\exists \tau \supseteq \delta)(\tau \Vdash_k F_e(x)) \ .$$

Let $E = \mathrm{dom}(\delta) = \{w_1, \dots, w_r\}$ and let $\delta(w_j) = t_j$, $j = 1, \dots, r$. For each $\tau \supseteq \delta$, denote by $E^\tau = \mathrm{dom}(\tau)$. Then from the previous lemma we know that:

$$(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_k) \models \bigvee\nolimits_{\tau \supseteq \delta} \exists (y \in E^\tau \setminus E) \Phi^k_{E^\tau, e, x}(W_1/t_1, \dots, W_r/t_r) \iff$$
$$(\exists \tau \supseteq \delta)(\tau \Vdash_k F_e(x)) \ .$$

Then for all $x \in \mathbb{N}$ the following equivalence is true:

$$x \in A \iff (\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_k) \models \bigvee_{E^* \supseteq E} \exists (y \in E^* \setminus E) \Phi^k_{E^*, e, x}(W_1/t_1, \dots, W_r/t_r)$$

where E^* denotes any finite extension of E.

It is clear that A is formally k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$.

Corollary 34. *Let $A \subseteq \mathbb{N}$ and let $k \leq n$. Then the following are equivalent:*

(1) $d_\mathrm{e}(A) \in \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.
(2) *A is forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$.*
(3) *A is formally k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$.*

Proof. The implication (1) $\Rightarrow$ (2) follows from the Theorem 26.

The implication (2) $\Rightarrow$ (3) follows from the previous theorem.

The implication (3) $\Rightarrow$ (1) could be proved easily by induction on k using Proposition 17.

7 Properties of the Relative Spectra

By Proposition 11 the relative spectra are upwards closed sets of degrees. So they possess all properties of upwards closed sets of degrees proved by Soskov [10]:

1. The total e-degrees in the relative spectrum determine completely its co-set, i.e.

$\mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = co(\{\mathbf{b} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) \mid \mathbf{b} \text{ is a total e-degree }\})$.

2. The members of the relative spectrum with high jumps also determine its co-set, i.e. for $p \geq 1$ and $\mathbf{c} \in \mathcal{D}_e$,

$\mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = co(\{\mathbf{b} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) \mid \mathbf{c} \leq \mathbf{b}^{(p)}\})$.

We shall show in this section that the relative spectra have all other properties of the degree spectra proved by Soskov [10]: the minimal pair theorem, the existence of quasi-minimal degree and a third property which shows that the members of the relative spectrum with low jumps also determine its co-set.

We start with an analogue of the third property.

Proposition 35. *Let $k < n$ and let $\mathbf{c} \in \mathrm{RS}_n(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$. Then*

$$\mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = co(\{\mathbf{b}^{(k)} \mid \mathbf{b} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) \ \&\ \mathbf{b}^{(n)} = \mathbf{c}\}) \ .$$

Proof. Denote by $\mathcal{A}_k = \{\mathbf{b}^{(k)} \mid \mathbf{b} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) \ \&\ \mathbf{b}^{(n)} = \mathbf{c}\}$. It is clear that $\mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) \subseteq co(\mathcal{A}_k)$.

Since $\mathbf{c} \in \mathrm{RS}_n(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ there is a n-acceptable enumeration f of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ such that $f^{-1}(\mathfrak{A})^{(n)} \in \mathbf{c}$. Denote by $Q = f^{-1}(\mathfrak{A})^{(n)}$. Q is a total set as $n > 0$.

Let $A \in \mathbf{a}$ and $\mathbf{a} \in co(\mathcal{A}_k)$. Then for every n-acceptable enumeration h of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ such that $h^{-1}(\mathfrak{A})^{(n)} \equiv_{\mathrm{e}} Q$, the set $A \leq_{\mathrm{e}} h^{-1}(\mathfrak{A})^{(k)}$. Since f is such an enumeration $A \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(k)}$. Then by the monotonicity of the enumeration jump $A^+ \leq_{\mathrm{e}} A' \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(k+1)} \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(n)}$. So $A^+ \oplus f^{-1}(\mathfrak{A})^{(n)} \leq_{\mathrm{e}} Q$.

According to Corollary 34 it is enough to show that the set A is forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. Assume for a contradiction that A is not forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. By Proposition 28 there exists a n-acceptable enumeration g of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ such that $g^{-1}(\mathfrak{A})^{(n)} \equiv_{\mathrm{e}} Q$ and $A \not\leq_{\mathrm{e}} g^{-1}(\mathfrak{A})^{(k)}$. This contradicts the choice of A.

Corollary 36. *Let $n > 0$ and $\mathbf{c} \in \mathrm{RS}_n(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$. Then*

$$\mathrm{CRS}(\mathfrak{A}) = \mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = co(\{\mathbf{b} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) \mid \mathbf{b}^{(n)} = \mathbf{c}\}) \ .$$

7.1 Minimal Pair Theorem

Soskov [10] proved a minimal pair theorem for the degree spectrum of a structure $\mathfrak{A}$. For each constructive ordinal α, there exist elements $\mathbf{f}$ and $\mathbf{g}$ of $\mathrm{DS}(\mathfrak{A})$ such that for any enumeration degree $\mathbf{a}$ and any $\beta + 1 < \alpha$,

$$\mathbf{a} \leq \mathbf{f}^{(\beta)} \ \&\ \mathbf{a} \leq \mathbf{g}^{(\beta)} \Rightarrow \mathbf{a} \in \mathrm{CS}_\beta(\mathfrak{A}) \ .$$

We shall prove an analogue of this minimal pair theorem for the relative spectrum.

Theorem 37. *For any structures $\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$, there exist total enumeration degrees $\mathbf{f}$ and $\mathbf{g}$ in $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ such that for any enumeration degree $\mathbf{a}$ and $k \leq n$:*

$$\mathbf{a} \leq \mathbf{f}^{(k)} \ \&\ \mathbf{a} \leq \mathbf{g}^{(k)} \Rightarrow \mathbf{a} \in \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) \ .$$

Proof. Let h be an arbitrary enumeration of $\mathfrak{A}$. Consider a total set Q such that $\mathcal{P}^h_n \leq_{\mathrm{e}} Q$. By Theorem 5 there exists a total set F with the following properties: $h^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} F$, $h^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} F^{(i)}$ for all $i \leq n$ and $F^{(n)} \equiv_{\mathrm{e}} Q$. By Lemma 10 there exists a n-acceptable enumeration f of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ such that $F \equiv_{\mathrm{e}} f^{-1}(\mathfrak{A})$ and so $h^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})$. Then the enumeration degree of $f^{-1}(\mathfrak{A})$ is total and $d_{\mathrm{e}}(F) \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$. Let $k \leq n$. Since f is a n-acceptable enumeration of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$, we have that $F^{(k)} \equiv_e \mathcal{P}^f_k$. Denote by $\{X^k_r\}_{r\in\mathbb{N}}$ the sequence of all sets enumeration reducible to $\mathcal{P}^f_k$.

For each $k \leq n$, consider the sequence $\{A^k_r\}_{r\in\mathbb{N}}$ of these sets among $\{X^k_r\}_{r\in\mathbb{N}}$ which are not forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. By Corollary 25 there is a n-acceptable enumeration g such that for all r and all $k = 0, \ldots, n$, $A^k_r \not\leq_e g^{-1}(\mathfrak{A})^{(k)}$ and the enumeration degree of $g^{-1}(\mathfrak{A})$ is total. Let $G = g^{-1}(\mathfrak{A})$. It is clear that $d_e(G) \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

Suppose now that $k \leq n$ and X is a set such that $X \leq_e F^{(k)}$ and $X \leq_e G^{(k)}$. From $X \leq_e F^{(k)}$ and $F^{(k)} \equiv_e \mathcal{P}^f_k$, it follows that $X = X^k_r$ for some r. If we assume that X is not forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ then $X = A^k_l$ for some l and $X \not\leq_e G^{(k)}$.

Hence X is forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. By Corollary 34, $d_e(X) \in \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$. Then by setting $\mathbf{f} = d_e(F)$ and $\mathbf{g} = d_e(G)$ we obtain the desired minimal pair.

7.2 Quasi-Minimal Degree

Let $\mathcal{A}$ be a set of enumeration degrees and $co(\mathcal{A})$ be the co-set of $\mathcal{A}$. The degree $\mathbf{q}$ is *quasi-minimal with respect to* $\mathcal{A}$ if the following conditions hold ([10]):

1. $\mathbf{q} \notin co(\mathcal{A})$.
2. If $\mathbf{a}$ is a total degree and $\mathbf{a} \geq \mathbf{q}$ then $\mathbf{a} \in \mathcal{A}$.
3. If $\mathbf{a}$ is a total degree and $\mathbf{a} \leq \mathbf{q}$ then $\mathbf{a} \in co(\mathcal{A})$.

Soskov [10] showed that for any structure $\mathfrak{A}$, there is a quasi-minimal degree $\mathbf{q}$ with respect to $\mathrm{DS}(\mathfrak{A})$, i.e. $\mathbf{q} \notin \mathrm{CS}(\mathfrak{A})$ and for every total degree $\mathbf{a}$: if $\mathbf{a} \geq \mathbf{q}$ then $\mathbf{a} \in \mathrm{DS}(\mathfrak{A})$ and if $\mathbf{a} \leq \mathbf{q}$ then $\mathbf{a} \in \mathrm{CS}(\mathfrak{A})$. It is clear that the quasi-minimal degree $\mathbf{q}$ with respect to $\mathrm{DS}(\mathfrak{A})$ is not a total enumeration degree. Soskov constructed it as $d_e(f^{-1}(\mathfrak{A}))$ for some partial generic enumeration of $\mathfrak{A}$. For example, consider the structure $\mathfrak{A} = (\mathbb{N}; =, \neq)$. Then $\mathrm{DS}(\mathfrak{A})$ consists of all total degrees, $\mathrm{CS}(\mathfrak{A}) = \{\mathbf{0}\}$ and quasi-minimal degree with respect to $\mathrm{DS}(\mathfrak{A})$ is each quasi-minimal enumeration degree, i.e. $\mathbf{q} > 0$ and for each total $\mathbf{a} \leq \mathbf{q}$ it holds that $\mathbf{a} = \mathbf{0}$. In this case the quasi-minimal degrees are outside $\mathrm{DS}(\mathfrak{A})$.

Theorem 38. *For any structures $\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$, there exists an enumeration degree $\mathbf{q}$ such that:*

1. $\mathbf{q} \notin \mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$;
2. *If $\mathbf{a}$ is a total degree and $\mathbf{a} \geq \mathbf{q}$ then $\mathbf{a} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$;*
3. *If $\mathbf{a}$ is a total degree and $\mathbf{a} \leq \mathbf{q}$ then $\mathbf{a} \in \mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.*

Proof. Let f be a partial generic enumeration of $\mathfrak{A}$ [10]. Soskov proved that $d_e(f^{-1}(\mathfrak{A}))$ is quasi-minimal with respect to $\mathrm{DS}(\mathfrak{A})$. By Theorem 4 from [13] there is a set F such that $f^{-1}(\mathfrak{A}) <_e F$, $f^{-1}(\mathfrak{A}_i) \leq_e F^{(i)}$ for $i \leq n$ and for any total set A, if $A \leq_e F$ then $A \leq_e f^{-1}(\mathfrak{A})$. We call the set F quasi-minimal over $f^{-1}(\mathfrak{A})$ with respect to $f^{-1}(\mathfrak{A}_1)$,..., $f^{-1}(\mathfrak{A}_n)$. The set F is constructed as a partial regular enumeration. Set $\mathbf{q} = d_e(F)$. We will prove that $\mathbf{q}$ has the desired properties.

Since $d_e(f^{-1}(\mathfrak{A})) \notin \mathrm{CS}(\mathfrak{A})$ and $d_e(f^{-1}(\mathfrak{A})) < \mathbf{q}$ then $\mathbf{q} \notin \mathrm{CS}(\mathfrak{A})$. But $\mathrm{CS}(\mathfrak{A}) = \mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ and hence $\mathbf{q} \notin \mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

Let X be a total set.

If $X \leq_e F$ then $X \leq_e f^{-1}(\mathfrak{A})$ as F is quasi-minimal over $f^{-1}(\mathfrak{A})$. Thus $d_e(X) \in \mathrm{CS}(\mathfrak{A}) = \mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ by the choice of $f^{-1}(\mathfrak{A})$.

If $X \geq_e F$ then $X \geq_e f^{-1}(\mathfrak{A})$. Since "=" is among the predicates of $\mathfrak{A}$, $\mathrm{dom}(f) \leq_e X$ and since X is a total set, $\mathrm{dom}(f)$ is r.e. in X. Let ρ be a recursive in X enumeration of $\mathrm{dom}(f)$. Set $h = \lambda n.f(\rho(n))$. Thus $h^{-1}(\mathfrak{A}) \leq_e X$ and $h^{-1}(\mathfrak{A}_i) \leq_e X^{(i)}$ for $i \leq n$. By Lemma 10 there is a n-acceptable enumeration g of $\mathfrak{A}$ such that $g^{-1}(\mathfrak{A}) \equiv_e X$. And then $d_e(X) \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

8 The Connection with the Joint Spectra

In this section we will consider the connection of the relative spectra with the joint spectra [12].

Definition 39. *The joint spectrum of* $\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is the set

$$\mathrm{DS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \{\mathbf{a} \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A}), \mathbf{a}' \in \mathrm{DS}(\mathfrak{A}_1), \ldots, \mathbf{a}^{(n)} \in \mathrm{DS}(\mathfrak{A}_n)\} \ .$$

The co-set of $\mathrm{DS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ is denoted by $\mathrm{CS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$. *The kth jump spectrum of* $\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is the set $\mathrm{DS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ of all kth jumps of the members of the joint spectrum $\mathrm{DS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$. The co-set of $\mathrm{DS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ is denoted by $\mathrm{CS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

The properties of both notions of spectra are very similar [13, 14], for example the joint spectra are closed upwards, the kth co-spectrum depends only on the first k structures. By Proposition 16, $\mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \mathrm{CSR}(\mathfrak{A}) = \mathrm{CS}(\mathfrak{A})$. It is known [13, 14] that $\mathrm{CS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \mathrm{CS}(\mathfrak{A})$. Therefore we have the following proposition:

Proposition 40. $\mathrm{CS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

The difference is in the kth co-spectrum for $k \geq 1$. Firstly, we know by Proposition 17 that for any set $A \subseteq \mathbb{N}$,

$$d_e(A) \in \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) \iff A \leq_e \mathcal{P}(f^{-1}(\mathfrak{A}), f^{-1}(\mathfrak{A}_1), \ldots, f^{-1}(\mathfrak{A}_k))$$

for every k-acceptable enumeration f of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$. Whereas for the kth joint co-spectra we have

$$d_e(A) \in \mathrm{CS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) \iff A \leq_e \mathcal{P}(f^{-1}(\mathfrak{A}), f_1^{-1}(\mathfrak{A}_1), \ldots, f_k^{-1}(\mathfrak{A}_k))$$

for every enumerations f of $\mathfrak{A}$, f_1 of $\mathfrak{A}_1$, ..., f_k of $\mathfrak{A}_k$.

Secondly, the normal form of forcing k-definable sets for the joint spectra uses a different definition of Σ_k^+ formulae. Namely, in the induction step (3) the existential quantifier over the atomic predicates must be independent from the rest. More precisely, the third clause in the definition of Σ_k^+ formulae is:

(3) *An elementary* Σ^+_{i+1} *formula* with free variables among $\bar{W}^0, \ldots, \bar{W}^{i+1}$ is a formula of the form

$$\exists \bar{Y}^0 \ldots \exists \bar{Y}^{i+1} \Phi(\bar{W}^0, \ldots, \bar{W}^{i+1}, \bar{Y}^0, \ldots, \bar{Y}^{i+1})$$

where Φ is a finite conjunction of Σ^+_i formulae and negations of Σ^+_i formulae with free variables among $\bar{Y}^0, \ldots, \bar{Y}^i, \bar{W}^0, \ldots, \bar{W}^i$ and atoms in $\mathcal{L}_{i+1}$ with variables among $\bar{W}^{i+1}, \bar{Y}^{i+1}$;

Notice that the variables for each structure are different. Moreover, when we define the value of a Σ^+_n formula in $(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ under an assignment then we treat the structure $(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ as a many-sorted structure with disjoint sorts.

These differences are essential and will enable us to give an example of structures $\mathfrak{A}$ and $\mathfrak{A}_1$ for which $\mathrm{CS}_1(\mathfrak{A}, \mathfrak{A}_1) \neq \mathrm{CRS}_1(\mathfrak{A}, \mathfrak{A}_1)$.

Example 41. Fix an effective bijective coding of the pairs of natural numbers. Denote by $\langle i, j\rangle$ the code of the ordered pair (i, j). Let R and S be binary predicates defined as follows: for every $i, j \in \mathbb{N}$, $R(\langle i, j\rangle, \langle i+1, j\rangle)$, i.e. R is the graph of the successor function for the first coordinate. For every $i, j \in \mathbb{N}$, $S(\langle i, j\rangle, \langle i, j+1\rangle)$, i.e. S is the graph of the successor function for the second coordinate. Let $\mathfrak{A} = (\mathbb{N}, R, S, =, \neq)$.

Consider a set M which is Σ^0_3, but not Σ^0_2 in the arithmetical hierarchy. Fix an enumeration of the elements of M, $M = \{j_0, \ldots, j_i, \ldots\}$.

Define $\mathfrak{A}_1 = (\mathbb{N}, P, =, \neq)$, where $P(\langle i, j_i\rangle) \iff j_i \in M$.

Claim: $d_{\mathrm{e}}(M) \in \mathrm{CRS}_1(\mathfrak{A}, \mathfrak{A}_1)$ and $d_{\mathrm{e}}(M) \notin \mathrm{CS}_1(\mathfrak{A}, \mathfrak{A}_1)$.

Let $t_0 = \langle 0, 0\rangle$. Then $d_{\mathrm{e}}(M) \in \mathrm{CRS}_1(\mathfrak{A}, \mathfrak{A}_1)$, since

$$\begin{aligned} j \in M \iff & \exists Y_0 \ldots \exists Y_i \exists Z_0 \ldots \exists Z_j (Y_0 = t_0 \ \&\ R(Y_0, Y_1) \ \&\ldots\ \&\ R(Y_{i-1}, Y_i) \\ & \&\ Y_i = Z_0 \ \&\ S(Z_0, Z_1) \ \&\ldots\ \&\ S(Z_{j-1}, Z_j) \ \&\ P(Z_j))\ . \end{aligned}$$

On the other hand if $A \subseteq \mathbb{N}$ and $d_{\mathrm{e}}(A) \in \mathrm{CS}_1(\mathfrak{A}, \mathfrak{A}_1)$ then A is a Σ^0_2 set in the arithmetical hierarchy. This follows from the fact that for any elementary Σ^+_1 formula $\Phi(W_1, \ldots, W_r)$, we can effectively find an elementary Σ^+_1 formula $\Psi(W_1, \ldots, W_r)$, where the predicate symbol P does not occur in Ψ such that for any fixed $t_1, \ldots, t_r \in \mathbb{N}$,

$$(\mathfrak{A}, \mathfrak{A}_1) \models \Phi(W_1/t_1, \ldots, W_r/t_r) \iff (\mathfrak{A}, \mathfrak{A}_1) \models \Psi(W_1/t_1, \ldots, W_r/t_r)\ .$$

So, we have two different generalizations of the notion of degree spectra with respect to given structures, both sharing similar properties. It is still not known what additional properties we should find to characterize the sets of enumeration degrees which are spectra of a structure relatively given structures.

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