Relativized Degree Spectra

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Abstract. A relativized version of the notion of Degree spectrum of a structure with respect to finitely many abstract structures is presented, inspired by the notion of relatively intrinsic sets. The connection with the notion of Joint spectrum is studied. Some specific properties like Minimal Pair type theorem and the existence of Quasi-Minimal degree with respect to the Relative spectrum are shown.

1 Introduction

Let $\mathfrak{A}$ be a countable partial structure. The Degree spectrum $\mathrm{DS}(\mathfrak{A})$ of the structure $\mathfrak{A}$ is the set of all enumeration degrees generated by all enumerations of $\mathfrak{A}$. The notion is introduced by *Richter* in [?] and studied by *Knight, Ash, Jockush, Downey* and *Soskov* in [?,?,?,?]. It is a kind of a measure of complexity of the structure. The Co-spectrum $\mathrm{CS}(\mathfrak{A})$ of the structure $\mathfrak{A}$ is the set of all enumeration degrees which are lower bounds of the $\mathrm{DS}(\mathfrak{A})$. A typical example of a Degree spectrum is the cone of all total enumeration degrees, greater than or equal to some enumeration degree $\mathbf{a}$ and the respective Co-spectrum is equal to the set of all degrees less than or equal to $\mathbf{a}$. In [?] Soskov shows that the Degree spectra behave with respect to their Co-spectra very much like the cones of enumeration degrees. The Degree spectra have some general and specific properties. For example each Degree spectrum is closed upwards, i.e. if $\mathbf{a} \in \mathrm{DS}(\mathfrak{A})$ then each total enumeration degree $\mathbf{b}$ greater than or equal to $\mathbf{a}$ is in $\mathrm{DS}(\mathfrak{A})$. But not every upwards closed set of enumeration degrees is a spectrum of a structure. Some typical specific properties of the Degree spectra and their Co-spectra are the Minimal Pair type theorem and the existence of Quasi-Minimal degree. For every Degree spectrum $\mathrm{DS}(\mathfrak{A})$ there exist total enumeration degrees $\mathbf{f_0}$ and $\mathbf{f_1}$, elements of $\mathrm{DS}(\mathfrak{A})$, which determine completely the elements of the Co-spectrum $\mathrm{CS}(\mathfrak{A})$, i.e. the set of all enumeration degrees less than or equal to both $\mathbf{f_0}$ and $\mathbf{f_1}$ is exactly $\mathrm{CS}(\mathfrak{A})$. The degrees $\mathbf{f_0}$ and $\mathbf{f_1}$ are called Minimal Pair for $\mathrm{DS}(\mathfrak{A})$. For each Degree spectrum $\mathrm{DS}(\mathfrak{A})$ there is an enumeration degree $\mathbf{q} \notin \mathrm{CS}(\mathfrak{A})$, called Quasi-Minimal for $\mathrm{DS}(\mathfrak{A})$, such that for each total degree $\mathbf{a}$ if $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathrm{DS}(\mathfrak{A})$ and if $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in \mathrm{CS}(\mathfrak{A})$.

⋆ This work was partially supported by Sofia University Science Fund. The author thanks the anonymous referees for helpful comments.

In this paper we introduce and study a generalized notion of Degree spectrum of the structure $\mathfrak{A}$, relatively given structures $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$, inspired by the notion of relatively intrinsic on $\mathfrak{A}$ sets. An internal characterization of the relatively intrinsic on $\mathfrak{A}$ sets is presented in [?], [?] and in [?] with respect to the infinite sequence of sets.

The Relative spectrum $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ of $\mathfrak{A}$ with respect to the structures $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is the set of all enumeration degrees generated by all enumerations of $\mathfrak{A}$, such that the structure $\mathfrak{A}_k$ is relatively k-intrinsic on $\mathfrak{A}$, i.e. $\mathfrak{A}_k$ is admissible in the kth jump of $\mathfrak{A}$. In other words we consider the set of all enumeration degrees of the presentations of the structure $\mathfrak{A}$ in which the degrees of $\mathfrak{A}_k$ fall below the kth jump of the degrees of $\mathfrak{A}$, $k \leq n$. We will show that this generalized notion of Degree spectra posses all general and specific properties of the Degree spectra of a structure. And we will compare this notion with the notion of Joint Spectrum of $\mathfrak{A}$ with respect to the structures $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$, considered in [?], [?].

2 Preliminaries

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_s)$ be a partial structure over the set of all natural numbers $\mathbb{N}$, where each R_i is a subset of $\mathbb{N}^{r_i}$ and $=, \neq$ are among $R_1, \ldots, R_s$.

An enumeration f *of* $\mathfrak{A}$ is a total mapping from $\mathbb{N}$ onto $\mathbb{N}$.

For $A \subseteq \mathbb{N}^a$ define $f^{-1}(A) = \{\langle x_1 \ldots x_a \rangle : (f(x_1), \ldots, f(x_a)) \in A\}$. Denote by $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \ldots \oplus f^{-1}(R_s)$.

For any sets of natural numbers A and B the set A is enumeration reducible to B ($A \leq_{\mathrm{e}} B$) if there is an enumeration operator Γ_z such that $A = \Gamma_z(B)$. By $d_{\mathrm{e}}(A)$ we denote the enumeration degree of the set A and by $\mathcal{D}_{\mathrm{e}}$ the set of all enumeration degrees. The set A is total if $A \equiv_{\mathrm{e}} A^+$, where $A^+ = A \oplus (\mathbb{N} \setminus A)$. A degree $\mathbf{a}$ is total if $\mathbf{a}$ contains the e-degree of a total set. The jump operation "$'$" denotes here the enumeration jump introduced by *Cooper* in [?].

Let $B_0, \ldots, B_n$ be arbitrary subsets of $\mathbb{N}$. Define the set $\mathcal{P}(B_0, \ldots, B_i)$ by induction on $i \leq n$, as follows:

1. $\mathcal{P}(B_0) = B_0$;
2. If $i < n$, then $\mathcal{P}(B_0, \ldots, B_{i+1}) = (\mathcal{P}(B_0, \ldots, B_i))' \oplus B_{i+1}$.

We will use the following modification of Jump Inversion Theorem from [?]:

Theorem 1 ([?]). *Let $\{A^k_r\}_{r \in \mathbb{N}}$, $k = 0, \ldots, n-1$ be n sequences of subsets of $\mathbb{N}$, such that for every r and for all k, $0 \leq k < n$, $A^k_r \not\leq_{\mathrm{e}} \mathcal{P}(B_0, \ldots, B_k)$ and let Q be a total set, such that $\mathcal{P}(B_0, \ldots, B_n) \leq_{\mathrm{e}} Q$. Then there exists a total set F having the following properties:*

1. *$B_i \leq_{\mathrm{e}} F^{(i)}$, for all $i \leq n$;*
2. *$A^k_r \not\leq_{\mathrm{e}} F^{(k)}$, for all r and all $k < n$;*
3. *$F^{(n)} \equiv_{\mathrm{e}} Q$.*

3 Relative Spectra of Structures

Definition 2. *The Degree spectrum of* $\mathfrak{A}$ is the set

$$\mathrm{DS}(\mathfrak{A}) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) : f \text{ is an enumeration of } \mathfrak{A}\} \ .$$

Let $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ be arbitrary abstract structures on $\mathbb{N}$.

Definition 3. *The Relative spectrum* of the structure $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ is the set

$$\begin{aligned}\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) : f \text{ is an enumeration of } \mathfrak{A} \text{ such that:}\\ (\forall k \leq n)(f^{-1}(\mathfrak{A}_k) \leq_{\mathrm{e}} (f^{-1}(\mathfrak{A}))^{(k)})\} \ .\end{aligned}$$

Definition 4. Let $k \leq n$. An enumeration f of $\mathfrak{A}$ is *k-acceptable with respect to the structures* $\mathfrak{A}_1, \dots, \mathfrak{A}_k$, if $f^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} (f^{-1}(\mathfrak{A}))^{(i)}$, for each $i \leq k$.

In fact the Relative spectrum of $\mathfrak{A}$ is the set, generated by all n-acceptable enumerations of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$. First we show that the Relative spectra are closed upwards.

Lemma 5. *If F is a total set, f is a n-acceptable enumeration of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ and $f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} F$, then there exists a n-acceptable enumeration g of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$, such that*

1. $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F \oplus f^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F$;
2. $g^{-1}(B) \leq_{\mathrm{e}} F \oplus f^{-1}(B)$, *for every* $B \subseteq \mathbb{N}$.

Proof (sketch). Let $s \neq t \in \mathbb{N}$, $f(x_s) \simeq s$ and $f(x_t) \simeq t$. Define

$$g(x) \simeq \begin{cases} f(x/2) & \text{if } x \text{ is even,} \\ s & \text{if } x = 2z+1 \text{ and } z \in F, \\ t & \text{if } x = 2z+1 \text{ and } z \notin F. \end{cases}$$

It is clear that $f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} g^{-1}(\mathfrak{A})$. Since "=" and "$\neq$" are among the underlined predicates of $\mathfrak{A}$, $F \leq_e g^{-1}(\mathfrak{A})$.

Consider the predicate R_i of $\mathfrak{A}$. Let $x_1, \dots, x_{r_i}$ be arbitrary natural numbers. Define the natural numbers $y_1, \dots, y_{r_i}$ by means of the following recursive in F procedure. Let $1 \leq j \leq r_i$. If x_j is even then let $y_j = x_j/2$. If $x_j = 2z+1$ and $z \in F$, then let $y_j = x_s$. If $x_j = 2z+1$ and $z \notin F$, then let $y_j = x_t$. Clearly

$$\langle x_1, \dots, x_{r_i} \rangle \in g^{-1}(R_i) \iff \langle y_1, \dots, y_{r_i} \rangle \in f^{-1}(R_i).$$

Thus $g^{-1}(R_i) \leq_{\mathrm{e}} F \oplus f^{-1}(\mathfrak{A})$. So, we obtain that $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F \oplus f^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F$.

From the definition of g it follows that $g^{-1}(B) \leq_{\mathrm{e}} F \oplus f^{-1}(B)$, for any $B \subseteq \mathbb{N}$. Then, for each $i \leq n$, $g^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} F \oplus f^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} F \oplus (f^{-1}(\mathfrak{A}))^{(i)} \leq_{\mathrm{e}} F \oplus F^{(i)} \equiv_{\mathrm{e}} F^{(i)} \equiv_{\mathrm{e}} (g^{-1}(\mathfrak{A}))^{(i)}$.

Corollary 6. *If* $\mathbf{b}$ *is a total e-degree,* $\mathbf{a} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$*, and* $\mathbf{a} \leq \mathbf{b}$*, then* $\mathbf{b} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.

Denote by $\mathcal{P}_k^f = \mathcal{P}(f^{-1}(\mathfrak{A}), f^{-1}(\mathfrak{A}_1), \dots, f^{-1}(\mathfrak{A}_k))$, for every enumeration f of $\mathfrak{A}$ and $k \leq n$.

Lemma 7. *Let f be an arbitrary enumeration of $\mathfrak{A}$, then there exists a n-acceptable enumeration g of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$, such that $f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} g^{-1}(\mathfrak{A})$ and $g^{-1}(\mathfrak{A})$ is a total set.*

Let Q be a total set such that $\mathcal{P}_n^f \leq_{\mathrm{e}} Q$. Apply Theorem **??** and the construction of Lemma **??**.

Definition 8. Let $k \leq n$. *The kth Jump Relative spectrum of $\mathfrak{A}$ with respect to* $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ is the set

$$\mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \{\mathbf{a}^{(k)} : \mathbf{a} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)\} \ .$$

Proposition 9. *Let $k \leq n$. $\mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots \mathfrak{A}_n)$ is closed upwards, i.e. if $\mathbf{b}$ is a total e-degree, $\mathbf{a} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots \mathfrak{A}_n)$ and $\mathbf{a}^{(k)} \leq \mathbf{b}$, then $\mathbf{b} \in \mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.*

Proof. Let G be a total set, $G \in \mathbf{b}$, and $(f^{-1}(\mathfrak{A}))^{(k)} \leq_{\mathrm{e}} G$, for some n-acceptable enumeration f of $\mathfrak{A}$, with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$. Then $\mathcal{P}_k^f \leq_{\mathrm{e}} (f^{-1}(\mathfrak{A}))^{(k)} \leq_{\mathrm{e}} G$. By Theorem **??** there exists a total set F, such that $f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} F$, $f^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} F^{(i)}$, for $i \leq k$ and $F^{(k)} \equiv_{\mathrm{e}} G$. As in Lemma **??**, we construct a k-acceptable enumeration g of $\mathfrak{A}$, with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_k$, so that $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F$. So, $g^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} (g^{-1}(\mathfrak{A}))^{(i)}$, for $i \leq k$. But for $k \leq j \leq n$ we have $g^{-1}(\mathfrak{A}_j) \leq_{\mathrm{e}} F \oplus f^{-1}(\mathfrak{A}_j) \leq_{\mathrm{e}} F \oplus (f^{-1}(\mathfrak{A}))^{(j)} \leq_{\mathrm{e}} F \oplus F^{(j)} \equiv_{\mathrm{e}} F^{(j)} \equiv_{\mathrm{e}} (g^{-1}(\mathfrak{A}))^{(j)}$. Thus $G \equiv_{\mathrm{e}} (g^{-1}(\mathfrak{A}))^{(k)}$, $d_{\mathrm{e}}(g^{-1}(\mathfrak{A})) \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$ and hence $d_{\mathrm{e}}(G) \in \mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.

4 Relative Co-spectra of Structures

Let $\mathcal{A}$ be a set of enumeration degrees. The co-set of $\mathcal{A}$ is the set of all lower bounds of $\mathcal{A}$.

Definition 10. *The Relative co-spectrum of $\mathfrak{A}$ with respect to* $\mathfrak{A}_1, \dots, \mathfrak{A}_n$, is the co-set of $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$, i.e.

$$\mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \{\mathbf{b} : \mathbf{b} \in \mathcal{D}_{\mathrm{e}} \& (\forall \mathbf{a} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n))(\mathbf{b} \leq \mathbf{a})\} \ .$$

Definition 11. Let $k \leq n$. *The Relative kth co-spectrum of $\mathfrak{A}$ with respect to* $\mathfrak{A}_1, \dots, \mathfrak{A}_n$, is the co-set of $\mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$, i.e.

$$\mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \{\mathbf{b} : \mathbf{b} \in \mathcal{D}_{\mathrm{e}} \& (\forall \mathbf{a} \in \mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n))(\mathbf{b} \leq \mathbf{a})\} \ .$$

Proposition 12. $\mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_k, \dots, \mathfrak{A}_n) = \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_k)$.

Proof. It is clear that $\mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_k, \dots, \mathfrak{A}_n) \subseteq \mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_k)$. Thus $\mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_k) \subseteq \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_k, \dots, \mathfrak{A}_n)$.

Let $\mathbf{a} \in \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1 \dots \mathfrak{A}_k \dots, \mathfrak{A}_n)$, $A \in \mathbf{a}$ and assume that $A \not\leq_{\mathrm{e}} (f^{-1}(\mathfrak{A}))^{(k)}$ for some k-acceptable enumeration f of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_k$. Then $A \not\leq_{\mathrm{e}}$

$\mathcal{P}_k^f$. Hence by Theorem **??** for $B_0 = f^{-1}(\mathfrak{A}), B_1 = f^{-1}(\mathfrak{A}_1), \dots, B_n = f^{-1}(\mathfrak{A}_n)$, $B_{n+1} = \mathbb{N}$, there exists a total set F, such that $f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} F$, for each $i \leq n$ $f^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} F^{(i)}$, and $A \not\leq_{\mathrm{e}} F^{(k)}$. As in Lemma **??**, we construct a k-acceptable enumeration g of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_k$, such that $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F$. Then $A \not\leq_{\mathrm{e}} (g^{-1}(\mathfrak{A}))^{(k)}$ and $g^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} (g^{-1}(\mathfrak{A}))^{(i)}$, for $i \leq k$. But for $k \leq j \leq n$, $g^{-1}(\mathfrak{A}_j) \leq_{\mathrm{e}} F \oplus f^{-1}(\mathfrak{A}_j) \leq_{\mathrm{e}} F \oplus F^{(j)} \equiv_{\mathrm{e}} F^{(j)} \equiv_{\mathrm{e}} (g^{-1}(\mathfrak{A}))^{(j)}$, i.e. g is a n-acceptable enumeration of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ and $A \not\leq_{\mathrm{e}} (g^{-1}(\mathfrak{A}))^{(k)}$, which contradicts with the choice of A.

In order to obtain a forcing normal form of the sets with enumeration degrees in $\mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$ we shall define the notion of forcing relation $\tau \Vdash_k F_e(x)$ and the relations $f \models_k F_e(x)$, for $k \leq n$, as in [**?**].

Let $W_0, \dots, W_z, \dots$ be a Gödel's enumeration of the c.e. sets and D_v be the finite set having the canonical code v. Let f be an enumeration of $\mathfrak{A}$.

For every $i \leq n$, e and x in $\mathbb{N}$ define the relations $f \models_i F_e(x)$ and $f \models_i \neg F_e(x)$ by induction on i:

1. $f \models_0 F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ D_v \subseteq f^{-1}(\mathfrak{A}))$;
2. $f \models_{i+1} F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(u = \langle 0, e_u, x_u\rangle \ \&$ $f \models_i F_{e_u}(x_u) \ \vee u = \langle 1, e_u, x_u\rangle \ \&\ f \models_i \neg F_{e_u}(x_u) \ \vee\ u = \langle 2, x_u\rangle \ \&$ $x_u \in f^{-1}(\mathfrak{A}_{i+1})))$;
3. $f \models_i \neg F_e(x) \iff f \not\models_i F_e(x)$.

From the definition it follows that for any $A \subseteq \mathbb{N}$ and $k \leq n$

$$A \leq_{\mathrm{e}} \mathcal{P}_k^f \iff (\exists e)(A = \{x : f \models_k F_e(x)\}) \ .$$

The forcing conditions, called *finite parts*, are finite mappings τ of $\mathbb{N}$ in $\mathbb{N}$.

For any $i \leq n$, e and x in $\mathbb{N}$ and every finite part τ define the forcing relations $\tau \Vdash_i F_e(x)$ and $\tau \Vdash_i \neg F_e(x)$ following the definition of relation "$\models_i$".

1. $\tau \Vdash_0 F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ D_v \subseteq \tau^{-1}(\mathfrak{A}))$;
2. $\tau \Vdash_{i+1} F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(u = \langle 0, e_u, x_u\rangle \ \&$ $\tau \Vdash_i F_{e_u}(x_u) \ \vee u = \langle 1, e_u, x_u\rangle \ \&\ \tau \Vdash_i \neg F_{e_u}(x_u) \ \vee\ u = \langle 2, x_u\rangle \ \&$ $x_u \in \tau^{-1}(\mathfrak{A}_{i+1})))$;
3. $\tau \Vdash_i \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \nVdash_i F_e(x))$.

For any $i \leq n, e, x \in \mathbb{N}$ denote by $X^i_{\langle e,x\rangle} = \{\rho : \rho \Vdash_i F_e(x)\}$.

Definition 13. Let $k \leq n+1$. An enumeration f of $\mathfrak{A}$ is k-*generic* with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$, if for every $j < k$, $e, x \in \mathbb{N}$

$$(\forall \tau \subseteq f)(\exists \rho \in X^j_{\langle e,x\rangle})(\tau \subseteq \rho) \Longrightarrow (\exists \tau \subseteq f)(\tau \in X^j_{\langle e,x\rangle}) \ .$$

In [**?**] the following properties of the k-generic enumerations are shown:

1. The forcing relation is monotone.
2. If f is a $(k+1)$-generic enumeration of $\mathfrak{A}$, with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$, then

$$f \models_k (\neg)F_e(x) \iff (\exists \tau \subseteq f)(\tau \Vdash_k (\neg)F_e(x)) \ .$$

Definition 14. Let $A \subseteq \mathbb{N}$ and $k \leq n$. The set A is *forcing k-definable* on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ if there exist a finite part δ and $e \in \mathbb{N}$ such that

$$x \in A \iff (\exists \tau \supseteq \delta)(\tau \Vdash_k F_e(x)) \ .$$

Proposition 15. *Let $\{A_r^k\}_{r \in \mathbb{N}}$, $k = 0, \ldots, n$ be $n+1$ sequences of subsets of $\mathbb{N}$, such that for every r and for all k, $0 \leq k \leq n$, the set A_r^k be not forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. Then there exists a $(n+1)$-generic enumeration f of $\mathfrak{A}$ such that $A_r^k \not\leq_{\mathrm{e}} \mathcal{P}_k^f$ for all r and $k \leq n$.*

Corollary 16. *Let $\{A_r^k\}_{r \in \mathbb{N}}$, $k = 0, \ldots, n$ be $n+1$ sequences of subsets of $\mathbb{N}$, such that for every r and for all k, $0 \leq k \leq n$, the set A_r^k be not forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. Then there exists a n-acceptable enumeration f of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$, such that the enumeration degree of $f^{-1}(\mathfrak{A})$ is total and $A_r^k \not\leq_{\mathrm{e}} (f^{-1}(\mathfrak{A}))^{(k)}$ for all r and $k \leq n$.*

This follows from the previous proposition, Theorem **??** and Lemma **??**.

Theorem 17. *For every $A \subseteq \mathbb{N}$ and $k \leq n$, the following are equivalent:*

1. *$d_{\mathrm{e}}(A) \in \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.*
2. *$A \leq_{\mathrm{e}} \mathcal{P}_k^f$, for every k-acceptable enumeration f of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$.*
3. *A is forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$.*

5 Normal Form Theorem

In this section a normal form of the forcing k-definable sets on the structure $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is presented. According to [?], these sets coincide with the sets which are definable on $\mathfrak{A}$ by means of *positive* recursive Σ_k^0 formulae [?].

Let $\mathcal{L} = \{T_1, \ldots, T_s\}$ be the first order language corresponding to the structure $\mathfrak{A}$. Let $\mathcal{L}_1, \ldots, \mathcal{L}_n$ be the languages of $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. Assume that the languages $\mathcal{L}, \mathcal{L}_1, \ldots, \mathcal{L}_n$ are disjoined.

For each $i \leq n$, define the elementary Σ_i^+ formulae and the Σ_i^+ formulae by induction on i, as follows.

Definition 18. (1) The elementary Σ_0^+ formulae are formulae in prenex normal form with a finite number of existential quantifiers and a matrix which is a finite conjunction of atomic predicates built up from the variables and the predicate symbols $T_1, \ldots, T_s$.

(2) An elementary Σ_{i+1}^+ formula is in the form

$$\exists Y_1 \ldots \exists Y_m \Phi(X_1, \ldots, X_l, Y_1, \ldots, Y_m),$$

where Φ is a finite conjunction of atoms built up from the variables $X_1, \ldots, X_l, Y_1, \ldots, Y_m$ and the predicate symbols from $\mathcal{L}_{i+1}$, Σ_i^+ formulae and negations of Σ_i^+ formulae with free variables among $X_1, \ldots, X_l$, $Y_1, \ldots, Y_m$.

(3) A Σ_i^+ formula with free variables among $X_1, \ldots, X_l$ is an c.e. infinitary disjunction of elementary Σ_i^+ formulae with free variables among $X_1, \ldots, X_l$.

Let Φ be a Σ_i^+ formula with free variables among $W_1, \ldots, W_r$ and let $t_1, \ldots, t_r$ be elements of $\mathbb{N}$. Then by $(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) \models \Phi(W_1/t_1, \ldots, W_r/t_r)$ we denote that Φ is true on a structure, obtained from $\mathfrak{A}$ by adding the predicates from $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$, under the variable assignment v such that $v(W_1) = t_1, \ldots, v(W_n) = t_n$.

Definition 19. Let $A \subseteq \mathbb{N}$ and let $k \leq n$. The set A is *formally k-definable* on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ if there exists a recursive sequence $\{\Phi^{\gamma(x)}\}$ of Σ_k^+ formulae with free variables among $W_1, \ldots, W_r$ and elements $t_1, \ldots, t_r$ of $\mathbb{N}$ such that for every $x \in \mathbb{N}$ the following equivalence holds:

$$x \in A \iff (\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) \models \Phi^{\gamma(x)}(W_1/t_1, \ldots, W_r/t_r).$$

The next theorem is proved, following the construction from [**?**].

Theorem 20. *A set $A \subseteq \mathbb{N}$ is forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots,$ $\mathfrak{A}_n$ if and only if A is formally k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$.*

6 The connection with the Joint Spectra

In [**?**] another generalization of the notion of Degree spectra is considered.

Definition 21. *The Joint spectrum of* $\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is the set

$$\mathrm{DS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \{\mathbf{a} : \mathbf{a} \in \mathrm{DS}(\mathfrak{A}), \mathbf{a}' \in \mathrm{DS}(\mathfrak{A}_1), \ldots, \mathbf{a}^{(n)} \in \mathrm{DS}(\mathfrak{A}_n)\} \ .$$

The co-set of $\mathrm{DS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ is denoted by $\mathrm{CS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$. *The kth Jump spectrum of* $\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is the set $\mathrm{DS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ of all kth jumps of the elements of the Joint spectrum $\mathrm{DS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$. The co-set of $\mathrm{DS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots,$ $\mathfrak{A}_n)$ is denoted by $\mathrm{CS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

The properties of both notions of spectra are very similar, for example the Joint Spectra are closed upwards, the kth Co-spectrum depends only on the first k structures.

Proposition 22. $\mathrm{CS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

This follows from the fact that $\mathrm{CS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \mathrm{CS}(\mathfrak{A})$ by [**?**], and $\mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \mathrm{CSR}(\mathfrak{A}) = \mathrm{CS}(\mathfrak{A})$ by Proposition **??**.

The difference between the co-sets of these spectra we can see first from the forcing normal form of both sets. In [**?**] is shown that for any set $A \subseteq \mathbb{N}$:

$$d_{\mathrm{e}}(A) \in \mathrm{CS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) \iff A \leq_{\mathrm{e}} \mathcal{P}(f^{-1}(\mathfrak{A}), f_1^{-1}(\mathfrak{A}_1) \ldots, f_k^{-1}(\mathfrak{A}_k)),$$

for every enumerations f of $\mathfrak{A}$, f_1 of $\mathfrak{A}_1$, ..., f_k of $\mathfrak{A}_k$. While by Theorem **??**:

$$d_{\mathrm{e}}(A) \in \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) \iff A \leq_{\mathrm{e}} \mathcal{P}(f^{-1}(\mathfrak{A}), f^{-1}(\mathfrak{A}_1) \ldots, f^{-1}(\mathfrak{A}_k)),$$

for any k-acceptable enumeration f of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_k$.

Second, from the normal form of forcing k-definable sets from [**?**] we know that these sets are definable on $\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n$ by a recursive sequence of Σ_k^+ formulae, which differ from these considered here only by the induction step 2, where the existential quantifiers for the structure $\mathfrak{A}_{i+1}$ are different from the others. More precisely, in [**?**]:

(2) *An elementary Σ_{i+1}^+ formula* with free variables among $\bar{X}^0 \dots \bar{X}^{i+1}$ is in the form

$$\exists \bar{Y}^0 \dots \exists \bar{Y}^{i+1} \Phi(\bar{X}^0 \dots \bar{X}^{i+1}, \bar{Y}^0, \dots, \bar{Y}^{i+1})$$

where Φ is a finite conjunction of Σ_i^+ formulae and negations of Σ_i^+ formulae with free variables among $\bar{Y}^0 \dots \bar{Y}^i, \bar{X}^0 \dots \bar{X}^i$ and atoms of $\mathcal{L}_{i+1}$ with variables among $\bar{X}^{i+1}, \bar{Y}^{i+1}$;

Notice that, the variables for each structure are different. Moreover, when we get the value of a Σ_i^+ formula in $(\mathfrak{A}, \mathfrak{A}_1 \dots, \mathfrak{A}_n)$ under an assignment then we treat the structure $(\mathfrak{A}, \mathfrak{A}_1 \dots, \mathfrak{A}_n)$ as a many-sorted structure with separated sorts.

From this point we will prove that there are structures $\mathfrak{A}$ and $\mathfrak{A}_1$, for which $\mathrm{CS}_1(\mathfrak{A}, \mathfrak{A}_1) \neq \mathrm{CRS}_1(\mathfrak{A}, \mathfrak{A}_1)$.

Example 23. Fix an effective bijective coding of the pairs of natural numbers. Denote by $\langle i, j \rangle$ the code of the ordered pair (i, j). Let R and S be binary predicates defined as follows: for every $i, j \in \mathbb{N}$, $R(\langle i, j \rangle, \langle i+1, j \rangle)$, i.e. R is the graph of the successor function for the first coordinate. For every $i, j \in \mathbb{N}$, $S(\langle i, j \rangle, \langle i, j+1 \rangle)$, i.e. S is the graph of the successor function for the second coordinate. Let $\mathfrak{A} = (\mathbb{N}, R, S, =, \neq)$ and let the language of $\mathfrak{A}$ be $\mathcal{L} = (R, S, =, \neq)$.

Consider a set M which is Σ_3^0, but not Σ_2^0 in the arithmetical hierarchy, and let $M = \{j_0, \dots, j_i, \dots\}$ be a fixed enumeration of the elements of M.

Define $\mathfrak{A}_1 = (\mathbb{N}, P, =, \neq)$, where $P(\langle i, j_i \rangle) \iff j_i \in M$. Let $\mathcal{L}_1 = (P, =, \neq)$.

Claim: $d_{\mathrm{e}}(M) \in \mathrm{CRS}_1(\mathfrak{A}, \mathfrak{A}_1)$ and $d_{\mathrm{e}}(M) \notin \mathrm{CS}_1(\mathfrak{A}, \mathfrak{A}_1)$.

Let $t_0 = \langle 0, 0 \rangle$. Then $d_{\mathrm{e}}(M) \in \mathrm{CRS}_1(\mathfrak{A}, \mathfrak{A}_1)$, since

$$\begin{aligned} j \in M \iff & \exists Y_0 \dots \exists Y_i \exists Z_0 \dots \exists Z_j (Y_0 = t_0 \mathbin{\&} R(Y_0, Y_1) \mathbin{\&} \dots \mathbin{\&} R(Y_{i-1}, Y_i) \\ & \mathbin{\&} Y_i = Z_0 \mathbin{\&} S(Z_0, Z_1) \mathbin{\&} \dots \mathbin{\&} S(Z_{j-1}, Z_j) \mathbin{\&} P(Z_j)) \ . \end{aligned}$$

On the other hand if $A \subseteq \mathbb{N}$ and $d_{\mathrm{e}}(A) \in \mathrm{CS}_1(\mathfrak{A}, \mathfrak{A}_1)$, then A is Σ_2^0 set in the arithmetical hierarchy. This follows from the fact that for any elementary Σ_1^+ formula $\Phi(W_1, \dots, W_r)$ we can effectively find an elementary Σ_1^+ formula $\Psi(W_1, \dots, W_r)$, where the predicate symbol P does not occur in Ψ, such that for any fixed $t_1, \dots, t_r \in \mathbb{N}$

$$(\mathfrak{A}, \mathfrak{A}_1) \models \Phi(W_1/t_1, \dots, W_r/t_r) \iff (\mathfrak{A}, \mathfrak{A}_1) \models \Psi(W_1/t_1, \dots, W_r/t_r) \ .$$

7 Minimal Pair Theorem

In [**?**] a Minimal Pair Theorem for Degree spectrum of a structure $\mathfrak{A}$ is presented. There it is proved that for each constructive ordinal α there exist elements $\mathbf{f}$ and

$\mathbf{g}$ of $\mathrm{DS}(\mathfrak{A})$ such that for any enumeration degree $\mathbf{a}$ and any $\beta + 1 < \alpha$

$$\mathbf{a} \leq \mathbf{f}^{(\beta)} \ \& \ \mathbf{a} \leq \mathbf{g}^{(\beta)} \Rightarrow \mathbf{a} \in \mathrm{CS}_\beta(\mathfrak{A}) \ .$$

We shall prove an analogue of the Minimal Pair Theorem for the Relative spectrum.

Theorem 24. *For any structures $\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$, there exist enumeration degrees $\mathbf{f}$ and $\mathbf{g}$ in $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$, such that for any enumeration degree $\mathbf{a}$ and $k \leq n$:*

$$\mathbf{a} \leq \mathbf{f}^{(k)} \ \& \ \mathbf{a} \leq \mathbf{g}^{(k)} \Rightarrow \mathbf{a} \in \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) \ .$$

Proof. Let h be an arbitrary enumeration of $\mathfrak{A}$. By Lemma **??** there exists a n-acceptable enumeration f of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$, such that $h^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})$ and $F = f^{-1}(\mathfrak{A})$ is a total set. Hence $d_{\mathrm{e}}(F) \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ and since f is n-acceptable enumeration of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$, $F^{(k)} \equiv_{\mathrm{e}} \mathcal{P}^f_k$. For each $k \leq n$, denote by $\{X^k_r\}_{r \in \mathbb{N}}$ the sequence of all sets enumeration reducible to $\mathcal{P}^f_k$.

For each $k \leq n$ consider the sequence $\{A^k_r\}_{r \in \mathbb{N}}$ of these sets among the sets $\{X^k_r\}_{r \in \mathbb{N}}$, which are not forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. By Corollary **??** there is a n-acceptable enumeration g such that for all r, and all $k = 0, \ldots, n$, $A^k_r \not\leq_{\mathrm{e}} (g^{-1}(\mathfrak{A}))^{(k)}$ and $g^{-1}(\mathfrak{A})$ is a total set. Let $G = g^{-1}(\mathfrak{A})$. It is clear that $d_{\mathrm{e}}(G) \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

Suppose now, that $k \leq n$ and a set X, $X \leq_{\mathrm{e}} F^{(k)}$ and $X \leq_{\mathrm{e}} G^{(k)}$. From $X \leq_{\mathrm{e}} F^{(k)}$ and $F^{(k)} \equiv_{\mathrm{e}} \mathcal{P}^f_k$, it follows that $X = X^k_r$ for some r. Assume for contradiction that X is not forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. Then $X = A^k_l$ for some l and then $X \not\leq_{\mathrm{e}} G^{(k)}$. Hence X is forcing k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. By Theorem **??**, $d_{\mathrm{e}}(X) \in \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$. Let $\mathbf{f} = d_{\mathrm{e}}(F)$ and $\mathbf{g} = d_{\mathrm{e}}(G)$.

8 Quasi-Minimal Degree

Let $\mathcal{A}$ be a set of enumeration degrees and $co(\mathcal{A})$ be the co-set of $\mathcal{A}$. The degree $\mathbf{q}$ is *quasi-minimal with respect to* $\mathcal{A}$ if the following conditions hold ([**?**]):

1. $\mathbf{q} \notin co(\mathcal{A})$.
2. If $\mathbf{a}$ is a total degree and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathcal{A}$.
3. If $\mathbf{a}$ is a total degree and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in co(\mathcal{A})$.

It is shown in [**?**] that for any structure $\mathfrak{A}$, there is a quasi-minimal degree $\mathbf{q}$ with respect to $\mathrm{DS}(\mathfrak{A})$, i.e. $\mathbf{q} \notin \mathrm{CS}(\mathfrak{A})$ and for every total degree $\mathbf{a}$: if $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathrm{DS}(\mathfrak{A})$ and if $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in \mathrm{CS}(\mathfrak{A})$.

Theorem 25. *For any structures $\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$ there exists an enumeration degree $\mathbf{q}$ such that:*

1. $\mathbf{q} \notin \mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$*;*
2. *If $\mathbf{a}$ is a total degree and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$;*

3. If $\mathbf{a}$ *is a total degree and* $\mathbf{a} \leq \mathbf{q}$*, then* $\mathbf{a} \in \mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$*.*

Proof (sketch). Let f be a partial generic enumeration of $\mathfrak{A}$ constructed as in [**?**]. Then by [**?**], $d_{\mathrm{e}}(f^{-1}(\mathfrak{A}))$ is quasi-minimal with respect to $\mathrm{DS}(\mathfrak{A})$. By Theorem 4. from [**?**] there is a quasi-minimal over $f^{-1}(\mathfrak{A})$ set F, such that $f^{-1}(\mathfrak{A}) <_{\mathrm{e}} F$, $f^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} F^{(i)}$, for $i \leq n$, and for any total set A, if $A \leq_{\mathrm{e}} F$, then $A \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})$. The set F is constructed as a partial regular enumeration which is quasi-minimal over $f^{-1}(\mathfrak{A})$ with respect to $f^{-1}(\mathfrak{A}_i)$, $i \leq n$. Take $\mathbf{q} = d_{\mathrm{e}}(F)$.

Since $d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) \notin \mathrm{CS}(\mathfrak{A})$ and $d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) < \mathbf{q}$ then $\mathbf{q} \notin \mathrm{CS}(\mathfrak{A})$. But $\mathrm{CS}(\mathfrak{A}) = \mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.

Let X be a total set.

If $X \leq_{\mathrm{e}} F$, then by the choice of F, $X \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})$. Thus $d_{\mathrm{e}}(X) \in \mathrm{CS}(\mathfrak{A}) = \mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$ by the choice of $f^{-1}(\mathfrak{A})$.

If $X \geq_{\mathrm{e}} F$, then $X \geq_{\mathrm{e}} f^{-1}(\mathfrak{A})$. Since "=" is in $\mathfrak{A}$, $\mathrm{dom}(f) \leq_{\mathrm{e}} X$ and since X is a total set, $\mathrm{dom}(f)$ is r.e. in X. Let ρ be a recursive in X enumeration of $\mathrm{dom}(f)$. Set $h = \lambda n.f(\rho(n))$. Thus $h^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} X$ and $h^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} X^{(i)}$, for $i \leq n$. Construct an enumeration g as in Lemma **??**, $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} X$, and for each $i \leq n$, $g^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} X \oplus h^{-1}(\mathfrak{A}_i) \leq_{\mathrm{e}} X \oplus X^{(i)} \equiv_{\mathrm{e}} (g^{-1}(\mathfrak{A}))^{(i)}$. And then $d_{\mathrm{e}}(X) \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.

References

1. Ash, C. J. : Generalizations of enumeration reducibility using recursive infinitary propositional senetences. Ann. Pure Appl. Logic **58** (1992) 173–184.
2. Ash, C. J., Knight, J. F., Manasse, M., Slaman, T. : Generic copies of countable structures. Ann. Pure Appl. Logic **42** (1989) 195–205.
3. Ash, C. J., Jockush, C., Knight, J. F. : Jumps of orderings. Trans. Amer. Math. Soc. **319** (1990) 573–599.
4. Chisholm, J. : Effective model theory vs. recursive model theory. J. Symbolic Logic **55** (1990) 1168–1191.
5. Cooper, S. B. : Partial degrees and the density problem. Part 2: The enumeration degrees of the Σ_2 sets are dense. J. Symbolic Logic **49** (1984) 503–513.
6. Downey, R. G., Knight, J. F. : Orderings with αth jump degree $\mathbf{0}^{(\alpha)}$. Proc. Amer. Math. Soc. **114** (1992) 545–552.
7. Knight, J. F. : Degrees coded in jumps of orderings. J. Symbolic Logic **51** (1986) 1034–1042.
8. Richter, L. J. : Degrees of structures. J. Symbolic Logic **46** (1981) 723–731.
9. Soskov, I. N. : A jump inversion theorem for the enumeration jump. Arch. Math. Logic **39** (2000), 417–437.
10. Soskov, I. N. : Degree spectra and co-spectra of structures. Ann. Univ. Sofia **96** (2004), 45–68.
11. Soskov, I. N., Baleva, V. : Ash's theorem for abstract structures. Proceedings of Logic Colloquium'2002 (to appear).
12. Soskova, A. A., Soskov, I. N. : Co-spectra of joint spectra of structures. Ann. Univ. Sofia **96** (2004) 35–44.
13. Soskova, A. A. : Minimal Pairs and Quasi-minimal degrees for the Joint Spectra of Structures. New Computational Pradigms (S. B. Cooper; B. Loewe, eds.) Lecture Notes in Comp. Sci **3526** (2005) Springer-Verlag 451–460.