Enumeration Degree Spectra of Abstract Structures

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The enumeration jump

Definition. Given $A \subseteq \omega$, set $A^+ = A \oplus (\omega \setminus A)$.

Definition.(Cooper, McEvoy) Given $A \subseteq \omega$, let $E_A = \{\langle i, x\rangle | x \in \Psi_i(A)\}$. Set $J_e(A) = E_A^+$.

The enumeration jump J_e is monotone and agrees with the Turing jump J_T in the following sense:

Theorem. *For any $A \subseteq \omega$, $J_T(A)^+ \equiv_e J_e(A^+)$.*

Definition. A set A is called *total* iff $A \equiv_e A^+$.

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The Rogers embedding. Define $\iota : \mathcal{D}_T \to \mathcal{D}_e$ by $\iota(d_T(A)) = d_e(A^+)$. Then ι is a proper embedding of $\mathcal{D}_T$ into $\mathcal{D}_e$. The enumeration degrees in the range of ι are called total.

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Degree Spectra

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k)$ *be a denumerable structure. Enumeration of* $\mathfrak{A}$ *is every total surjective mapping of* $\mathbb{N}$ *onto* $\mathbb{N}$*.*

Given an enumeration f *of* $\mathfrak{A}$ *and a subset of* A *of* $\mathbb{N}^a$*, let*

$$f^{-1}(A) = \{\langle x_1, \ldots, x_a\rangle : (f(x_1), \ldots, f(x_a)) \in A\}.$$

Set $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq)$. .

Definition.(Richter) *The Turing Degree Spectrum of* $\mathfrak{A}$ is the set

$$DS_T(\mathfrak{A}) = \{d_T(f^{-1}(\mathfrak{A})) : f \text{ is an one to one enumeration of } \mathfrak{A})\}.$$

If **a** is the least element of $DS_T(\mathfrak{A})$, then **a** is called the *degree of* $\mathfrak{A}$

Definition. *The e-Degree Spectrum of* $\mathfrak{A}$ is the set

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If **a** is the least element of $DS(\mathfrak{A})$, then **a** is called the *e-degree of* $\mathfrak{A}$

Proposition. *If* $\mathbf{a} \in DS(\mathfrak{A})$,$\mathbf{b}$ *is a total e-degree and* $\mathbf{a} \leq_e \mathbf{b}$ *then* $\mathbf{b} \in DS(\mathfrak{A})$.

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Total structures

Definition. The structure $\mathfrak{A}$ is called *total* if for every enumeration f of $\mathfrak{A}$ the set $f^{-1}(\mathfrak{A})$ is total.

Proposition. *If $\mathfrak{A}$ is a total structure then $DS(\mathfrak{A}) = \iota(DS_T(\mathfrak{A}))$.*

Given a structure $\mathfrak{A} = (\mathbb{N}, R_1, \dots, R_k)$, for every j denote by R_j^c the complement of R_j and let $\mathfrak{A}^+ = (\mathbb{N}, R_1, \dots, R_k, R_1^c, \dots, R_k^c)$.

Proposition. *The following are true:*

1. $\iota(DS_T(\mathfrak{A})) = DS(\mathfrak{A}^+)$.
2. *If $\mathfrak{A}$ is total then $DS(\mathfrak{A}) = DS(\mathfrak{A}^+)$.*

Clearly if $\mathfrak{A}$ is a total structure then $DS(\mathfrak{A})$ consists of total degrees. The vice versa is not always true.

Example. *Let K be the Kleene's set and $\mathfrak{A} = (\mathbb{N}; G_S, K)$, where G_S is the graph of the successor function. Then $DS(\mathfrak{A})$ consists of all total degrees. On the other hand if $f = \lambda x.x$, then $f^{-1}(\mathfrak{A})$ is an r.e. set. Hence $\bar{K} \not\leq_e f^{-1}(\mathfrak{A})$. Clearly $\bar{K} \leq_e (f^{-1}(\mathfrak{A}))^+$. So $f^{-1}(\mathfrak{A})$ is not total.*

Is it true that if $DS(\mathfrak{A})$ consists of total degrees then there exists a total structure $\mathfrak{B}$ s.t. $DS(\mathfrak{A}) = DS(\mathfrak{B})$?

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Jump spectra

Definition. The n-th jump spectrum of a structure $\mathfrak{A}$ is the set

$$DS_n(\mathfrak{A}) = \{\mathbf{a}^{(n)} | \mathbf{a} \in DS(\mathfrak{A})\}.$$

If $\mathbf{a}$ is the least element of $DS_n(\mathfrak{A})$ then $\mathbf{a}$ is called n-th jump degree of $\mathfrak{A}$.

Proposition. *For every $\mathfrak{A}$, $DS_1(\mathfrak{A}) \subseteq DS(\mathfrak{A})$.*

Is it true that for every $\mathfrak{A}$, $DS_1(\mathfrak{A}) \subset DS(\mathfrak{A})$? Probably the answer is "no".

Every jump spectrum is spectrum of a total structure

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_n)$.
Let $\bar{0} \notin \mathbb{N}$. Set $\mathbb{N}_0 = \mathbb{N} \cup \{\bar{0}\}$. Let $\langle ., . \rangle$ be a pairing function s.t. none of the elements of $\mathbb{N}_0$ is a pair and N^* be the least set containing $\mathbb{N}_0$ and closed under $\langle ., . \rangle$.

Definition. *Moschovakis' extension* of $\mathfrak{A}$ is the structure

$$\mathfrak{A}^* = (\mathbb{N}^*, R_1, \ldots, R_n, \mathbb{N}_0, G_{\langle ., . \rangle}).$$

Proposition. $DS(\mathfrak{A}) = DS(\mathfrak{A}^*)$

Let $K_{\mathfrak{A}} = \{\langle \delta, e, x \rangle : (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}$.
Set $\mathfrak{A}' = (\mathfrak{A}^*, K_{\mathfrak{A}}, \mathbb{N}^* \setminus K_{\mathfrak{A}})$.

Theorem.

1. *The structure $\mathfrak{A}'$ is total.*
2. $DS_1(\mathfrak{A}) = DS(\mathfrak{A}')$.

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The Jump Inversion Theorem

Consider two structures $\mathfrak{A}$ and $\mathfrak{B}$. Suppose that

$$DS(\mathfrak{B})_t = \{\mathbf{a} | \mathbf{a} \in DS(\mathfrak{B}) \text{ and } \mathbf{a} \text{ is total}\} \subseteq DS_1(\mathfrak{A}).$$

Theorem. *There exists a structure $\mathfrak{C}$ s.t. $DS(\mathfrak{C}) \subseteq DS(\mathfrak{A})$ and $DS_1(\mathfrak{C}) = DS(\mathfrak{B})_t$.*

Corollary. *Let $DS(\mathfrak{B}) \subseteq DS_1(\mathfrak{A})$. Then there exists a structure $\mathfrak{C}$ s.t. $DS(\mathfrak{C}) \subseteq DS(\mathfrak{A})$ and $DS(\mathfrak{B}) = DS_1(\mathfrak{C})$.*

Corollary. *Suppose that $DS(\mathfrak{B})$ consists of total degrees greater than or equal to $\mathbf{0}'$. Then there exists a total structure $\mathfrak{C}'$ such that $DS(\mathfrak{B}) = DS(\mathfrak{C}')$.*

Theorem. *Let $n \geq 1$. Suppose that $DS(\mathfrak{B}) \subseteq DS_n(\mathfrak{A})$. There exists a structure $\mathfrak{C}$ s.t. $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.*

Corollary. *Suppose that $DS(\mathfrak{B})$ consists of total degrees greater than or equal to $\mathbf{0}^{(n)}$. Then there exists a total structure $\mathfrak{C}$ s.t. $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.*

Example. *Let $n \geq 0$. There exists a total structure $\mathfrak{C}$ s.t. $\mathfrak{C}$ has a $n+1$-th jump degree $\mathbf{0}^{(n+1)}$ but has no k-th jump degree for $k \leq n$.*

It is sufficient to construct a structure $\mathfrak{B}$ satisfying:

1. *$DS(\mathfrak{B})$ has not least element.*
2. *$\mathbf{0}^{(n+1)}$ is the least element of $DS_1(\mathfrak{B})$.*
3. *All elements of $DS(\mathfrak{B})$ are total and above $\mathbf{0}^{(n)}$.*

Consider a set B satisfying:

1. *B is quasi-minimal above $\mathbf{0}^{(n)}$.*
2. *$B' \equiv_e \mathbf{0}^{(n+1)}$.*

Let G be a subgroup of the additive group of the rationales s.t. $S_G \equiv_e B$. Recall that $DS(G) = \{\mathbf{a} | d_e(S_G) \leq_e \mathbf{a}$ and $\mathbf{a}$ is total$\}$ and $d_e(S_G)'$ is the least element of $DS_1(G)$.

Applications

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Let $n \geq 0$. There exists a total structure $\mathfrak{C}$ such that $DS_n(\mathfrak{C}) = \{\mathbf{a} | \mathbf{0}^{(n)} <_e \mathbf{a}\}$.
It is sufficient to construct a structure $\mathfrak{B}$ such that the elements of $DS(\mathfrak{B})$ are exactly the total e-degrees greater than $\mathbf{0}^{(n)}$.
This is done by Whener's construction using a special family of sets:

Theorem. *Let $n \geq 0$. There exists a family $\mathcal{F}$ of sets of natural number s.t. for every X strictly above $\mathbf{0}^{(n)}$ there exists a recursive in X set U satisfying the equivalence:*

$$F \in \mathcal{F} \iff (\exists a)(F = \{x | (a, x) \in U\}).$$

But there is no r.e. in $\mathbf{0}^{(n)}$ such U.

Thank you!