Structural properties of spectra and co-spectra

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Degree spectra

Let $\mathfrak{A} = (A; R_1, \ldots, R_k)$ *be a countable structure. An enumeration of* $\mathfrak{A}$ *is every total surjective mapping of* ω *onto* A*.*

Given an enumeration f *of* $\mathfrak{A}$ *and a subset of* B *of* A^n*, let*

$$f^{-1}(B) = \{\langle x_1, \ldots, x_n\rangle \mid (f(x_1), \ldots, f(x_n)) \in B\}.$$

$$f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq).$$

Definition.[Richter, Jockusch] *The Turing degree spectrum of* $\mathfrak{A}$ is the set

$$DS_T(\mathfrak{A}) = \{d_T(f^{-1}(\mathfrak{A})) \mid f \text{ is a one-to-one enum. of } \mathfrak{A}\}.$$

If **a** is the least element of $DS_T(\mathfrak{A})$, then **a** is called the *degree of* $\mathfrak{A}$.

Enumeration degree spectra

Definition.[Soskov] *The enumeration degree spectrum of* $\mathfrak{A}$ is the set

$$DS(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) \mid f \text{ is an enumeration of } \mathfrak{A}\}.$$

If $\mathbf{a}$ is the least element of $DS(\mathfrak{A})$, then $\mathbf{a}$ is called the *e-degree of* $\mathfrak{A}$.

Proposition. *The enumeration degree spectrum is closed upwards with respect to total e-degrees, i.e. if* $\mathbf{a} \in DS(\mathfrak{A})$, $\mathbf{b}$ *is a total e-degree* $\mathbf{a} \leq_e \mathbf{b}$ *then* $\mathbf{b} \in DS(\mathfrak{A})$.

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Proposition. *If* $\mathfrak{A}$ *has e-degree* **a** *then* $\mathbf{a} = d_e(f^{-1}(\mathfrak{A}))$ *for some one-to-one enumeration* f *of* $\mathfrak{A}$.

Total structures

Given a structure $\mathfrak{A} = (A, R_1, \ldots, R_k)$*, for every* j *denote by* R_j^c *the complement of* R_j *and let* $\mathfrak{A}^+ = (A, R_1, \ldots, R_k, R_1^c, \ldots, R_k^c)$*.*

Proposition.

- $\iota(DS_T(\mathfrak{A})) = DS(\mathfrak{A}^+)$.
- *If* $\mathfrak{A}$ *is total then* $DS(\mathfrak{A}) = DS(\mathfrak{A}^+)$*.*

Co-spectra

Definition. Let $\mathcal{A}$ be a nonempty set of enumeration degrees. The *co-set of* $\mathcal{A}$ is the set $co(\mathcal{A})$ of all lower bounds of $\mathcal{A}$. Namely

$$co(\mathcal{A}) = \{\mathbf{b} : \mathbf{b} \in \mathcal{D}_e \ \& \ (\forall \mathbf{a} \in \mathcal{A})(\mathbf{b} \leq_e \mathbf{a})\}.$$

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Definition. Given a structure $\mathfrak{A}$, set $CS(\mathfrak{A}) = co(DS(\mathfrak{A}))$.
If $\mathbf{a}$ is the greatest element of $CS(\mathfrak{A})$ then we call $\mathbf{a}$ the *co-degree* of $\mathfrak{A}$.

If $\mathfrak{A}$ has a degree $\mathbf{a}$ then $\mathbf{a}$ is also the co-degree of $\mathfrak{A}$. The vice versa is not always true.

The admissible in $\mathfrak{A}$ sets

Definition. A set B of natural numbers is admissible in $\mathfrak{A}$ if for every enumeration f of $\mathfrak{A}$, $B \leq_e f^{-1}(\mathfrak{A})$.

Clearly $\mathbf{a} \in CS(\mathfrak{A})$ *iff* $\mathbf{a} = d_e(B)$ *for some admissible in* $\mathfrak{A}$ *set* B.

Forcing definable in $\mathfrak{A}$ sets

Every finite mapping of ω into A is called a finite part. For every finite part τ and natural numbers e, x, let

$$\tau \Vdash F_e(x) \iff x \in \Gamma_e(\tau^{-1}(\mathfrak{A})) \text{ and}$$
$$\tau \Vdash \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \nVdash F_e(x)).$$

Definition. An enumeration f of $\mathfrak{A}$ is *generic* if for every $e, x \in \omega$, there exists a $\tau \subseteq f$ s.t. $\tau \Vdash F_e(x) \vee \tau \Vdash \neg F_e(x)$.

Definition. A set B of natural numbers is *forcing definable in the structure* $\mathfrak{A}$ iff there exist a finite part δ and a natural number e s.t.

$$B = \{x | (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}.$$

Forcing definable in $\mathfrak{A}$ sets

Theorem. *Let $B \subseteq \omega$ and $d_e(C) \in DS(\mathfrak{A})$. Then the following are equivalent:*

1. *B is admissible in $\mathfrak{A}$.*
2. *$B \leq_e f^{-1}(\mathfrak{A})$ for all generic enumerations f of $\mathfrak{A}$ s.t. $(f^{-1}(\mathfrak{A}))' \equiv_e C'$.*
3. *B is forcing definable on $\mathfrak{A}$.*

The formally definable sets on $\mathfrak{A}$

Definition. A Σ_1^+ formula with free variables among $X_1, \dots, X_r$ is a c.e. disjunction of existential formulae of the form $\exists Y_1 \dots \exists Y_k \theta(\bar{Y}, \bar{X})$, where θ is a finite conjunction of atomic formulae.

Definition. A set $B \subseteq \omega$ is *formally definable* on $\mathfrak{A}$ if there exists a recursive function $\gamma(x)$, such that $\bigvee_{x \in \omega} \Phi_{\gamma(x)}$ is a Σ_1^+ formula with free variables among $X_1, \dots, X_r$ and elements $t_1, \dots, t_r$ of A such that the following equivalence holds:

$$x \in B \iff \mathfrak{A} \models \Phi_{\gamma(x)}(X_1/t_1, \dots, X_r/t_r) \ .$$

Theorem. *Let $B \subseteq \omega$. Then*

1. *B is admissible in $\mathfrak{A}$ ($d_e(B) \in CS(\mathfrak{A})$) iff*
2. *B is forcing definable on $\mathfrak{A}$ iff*
3. *B is formally definable on $\mathfrak{A}$.*

Jump spectra and jump co-spectra

Definition. The nth jump spectrum of $\mathfrak{A}$ is the set

$$DS_n(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})^{(n)}) : f \text{ is an enumeration of } \mathfrak{A}\}.$$

If **a** is the least element of $DS_n(\mathfrak{A})$, then **a** is called the *nth jump degree of* $\mathfrak{A}$.

Definition. The co-set $CS_n(\mathfrak{A})$ of the nth jump spectrum of $\mathfrak{A}$ is called nth jump co-spectrum of $\mathfrak{A}$.
If $CS_n(\mathfrak{A})$ has a greatest element then it is called the *nth jump co-degree of* $\mathfrak{A}$.

Some examples

Example. *[Richter] Let* $\mathfrak{A} = (A; <)$ *be a linear ordering.* $DS(\mathfrak{A})$ *contains a minimal pair of degrees and hence* $CS(\mathfrak{A}) = \{\mathbf{0}_e\}$*.* $\mathbf{0}_e$ *is the co-degree of* $\mathfrak{A}$*. So, if* $\mathfrak{A}$ *has a degree* $\mathbf{a}$*, then* $\mathbf{a} = \mathbf{0}_e$*.*

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Example.*[Knight] For a linear ordering* $\mathfrak{A}$, $CS_1(\mathfrak{A})$ *consists of all e-degrees of* Σ^0_2 *sets. The first jump co-degree of* $\mathfrak{A}$ *is* $\mathbf{0}'_e$.

A special kind of co-degree

Definition. [Knight, Motalbán] A structure $\mathfrak{A}$ has "enumeration degree X" if every enumeration of X computes a copy of $\mathfrak{A}$, and every copy of $\mathfrak{A}$ computes an enumeration of X.

In our terms this can be formulated as $\mathfrak{A}^+$ has a co-degree $d_e(X)$ and $DS(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a}$ is total and $d_e(X) \leq \mathbf{a}\}$.

Example. *Given $X \subseteq \omega$, consider the group $G_X = \bigoplus_{i \in X} \mathbb{Z}_{p_i}$, where p_i is the ith prime number. Then G_X has "enumeration degree X": We can easily build G_X out of an enumeration of X, and for the other direction, we have that $n \in X$ if and only if there exists $g \in G_X$ of order p_n.*

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Theorem. *[A. Montalbán] Let K be Π^c_2 class of $\exists$-atomic structures, i.e. K is the class of structures axiomatized by some Π^c_2 sentence and for every structure $\mathfrak{A}$ in K and every tuple $\bar{a} \in |\mathfrak{A}|$ the orbit of $\bar{a}$ is existentially definable (with parameters $\bar{a}$). Then every structure in K has "enumeration degree" given by its $\exists$-theory.*

Representing the principle countable ideals as co-spectra

Example.*[Coles, Downey, Slaman; Soskov] Let G be a torsion free abelian group of rank 1, i.e. G is a subgroup of Q.*
There exists an enumeration degree $\mathbf{s}_G$ *such that*

- $DS(G) = \{\mathbf{b} : \mathbf{b}$ *is total and* $\mathbf{s}_G \leq_e \mathbf{b}\}$.
- *The co-degree of G is* $\mathbf{s}_G$.
- *G has a degree iff* $\mathbf{s}_G$ *is a total e-degree.*
- *If* $1 \leq n$*, then* $\mathbf{s}_G^{(n)}$ *is the n-th jump degree of G.*

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- *The co-degree of G is $\mathbf{s}_G$.*
- *G has a degree iff $\mathbf{s}_G$ is a total e-degree.*
- *If $1 \leq n$, then $\mathbf{s}_G^{(n)}$ is the n-th jump degree of G.*

For every $\mathbf{d} \in \mathcal{D}_e$ there exists a G, s.t. $\mathbf{s}_G = \mathbf{d}$.

Corollary. *Every principle ideal of enumeration degrees is $CS(G)$ for some G.*

Representing non-principle countable ideals as co-spectra

Theorem.*[Soskov] Every countable ideal is the co-spectrum of a structure.*

Spectra with a countable base

Definition. Let $\mathcal{B} \subseteq \mathcal{A}$ be sets of degrees. Then $\mathcal{B}$ is a base of $\mathcal{A}$ if

$$(\forall \mathbf{a} \in \mathcal{A})(\exists \mathbf{b} \in \mathcal{B})(\mathbf{b} \leq \mathbf{a}).$$

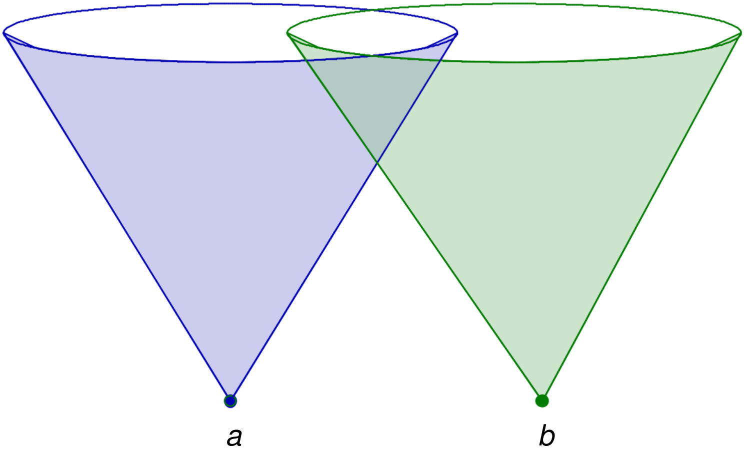
Spectra with a countable base

Definition. Let $\mathcal{B} \subseteq \mathcal{A}$ be sets of degrees. Then $\mathcal{B}$ is a base of $\mathcal{A}$ if

$$(\forall \mathbf{a} \in \mathcal{A})(\exists \mathbf{b} \in \mathcal{B})(\mathbf{b} \leq \mathbf{a}).$$

Theorem. *A structure $\mathfrak{A}$ has e-degree if and only if $DS(\mathfrak{A})$ has a countable base.*

An upwards closed set of degrees which is not a degree spectra of a structure



The minimal pair theorem

Theorem. *Let* $\mathbf{c} \in DS_2(\mathfrak{A})$. *There exist* $\mathbf{f}$,$\mathbf{g} \in DS(\mathfrak{A})$ *s.t.* $\mathbf{f}$,$\mathbf{g}$ *are total,* $\mathbf{f}'' = \mathbf{g}'' = \mathbf{c}$ *and* $CS(\mathfrak{A}) = co(\{\mathbf{f},\mathbf{g}\})$.

Notice that for every enumeration degree $\mathbf{b}$ *there exists a structure* $\mathfrak{A}_{\mathbf{b}}$ *s. t.* $DS(\mathfrak{A}_{\mathbf{b}}) = \{\mathbf{x} \in \mathcal{D}_T | \mathbf{b} <_e \mathbf{x}\}$. *Hence*

Corollary.*[Rozinas] For every* $\mathbf{b} \in \mathcal{D}_e$ *there exist total* $\mathbf{f}, \mathbf{g}$ *below* $\mathbf{b}''$ *which are a minimal pair over* $\mathbf{b}$.

The quasi-minimal degree

Definition. Let $\mathcal{A}$ be a set of enumeration degrees. The degree **q** is quasi-minimal with respect to $\mathcal{A}$ if:

- $\mathbf{q} \notin co(\mathcal{A})$.
- If **a** is total and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathcal{A}$.
- If **a** is total and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in co(\mathcal{A})$.

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Theorem. *For every structure $\mathfrak{A}$ there exists a quasi-minimal with respect to $DS(\mathfrak{A})$ degree.*

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Theorem. *For every structure $\mathfrak{A}$ there exists a quasi-minimal with respect to $DS(\mathfrak{A})$ degree.*

Corollary.*[Slaman and Sorbi] Let I be a countable ideal of enumeration degrees. There exists an enumeration degree $\mathbf{q}$ s.t.*

1. *If $\mathbf{a} \in I$ then $\mathbf{a} <_e \mathbf{q}$.*
2. *If $\mathbf{a}$ is total and $\mathbf{a} \leq_e \mathbf{q}$ then $\mathbf{a} \in I$.*

Properties of the quasi-minimal degrees

Proposition. *For every countable structure $\mathfrak{A}$ there exist uncountably many quasi-minimal degrees with respect to $DS(\mathfrak{A})$.*

Proposition. *The first jump spectrum of every structure $\mathfrak{A}$ consists exactly of the enumeration jumps of the quasi-minimal degrees.*

Corollary.*[McEvoy] For every total e-degree $\mathbf{a} \geq_e \mathbf{0}'_e$ there is a quasi-minimal degree $\mathbf{q}$ with $\mathbf{q}' = \mathbf{a}$.*

Splitting a total set

Proposition.[Jockusch] *For every total e-degree* $\mathbf{a}$ *there are quasi-minimal degrees* $\mathbf{p}$ *and* $\mathbf{q}$ *such that* $\mathbf{a} = \mathbf{p} \vee \mathbf{q}$.

Proposition. *For every element* $\mathbf{a}$ *of the jump spectrum of a structure* $\mathfrak{A}$ *there exists quasi-minimal with respect to* $\mathfrak{A}$ *degrees* $\mathbf{p}$ *and* $\mathbf{q}$ *such that* $\mathbf{a} = \mathbf{p} \vee \mathbf{q}$.

Every jump spectrum is the spectrum of a total structure

Let $\mathfrak{A} = (A; R_1, \dots, R_n)$.
Let $\bar{0} \notin A$. *Set* $A_0 = A \cup \{\bar{0}\}$. *Let* $\langle ., . \rangle$ *be a pairing function s.t. none of the elements of* A_0 *is a pair and* A^* *be the least set containing* A_0 *and closed under* $\langle ., . \rangle$. *Let L and R be the decoding functions.*

Definition. *Moschovakis' extension* of $\mathfrak{A}$ is the structure

$$\mathfrak{A}^* = (A^*, R_1, \dots, R_n, A_0, G_{\langle .,. \rangle}, G_L, G_R).$$

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Let $K_{\mathfrak{A}} = \{\langle \delta, e, x \rangle : (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}$.
Set $\mathfrak{A}' = (\mathfrak{A}^*, K_{\mathfrak{A}}, A^* \setminus K_{\mathfrak{A}})$.

Theorem. *[Soskov, A. Soskova]* $DS_1(\mathfrak{A}) = DS(\mathfrak{A}')$.

The jump inversion theorem

Let $\alpha < \omega_1^{CK}$ *and* $\mathfrak{A}$ *be a countable structure such that all elements of* $DS(\mathfrak{A})$ *are above* $\mathbf{0}^{(\alpha)}$.

Does there exist a structure $\mathfrak{M}$ *such that* $DS_\alpha(\mathfrak{M}) = DS(\mathfrak{A})$*?*

The jump inversion theorem

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Does there exist a structure $\mathfrak{M}$ such that $DS_\alpha(\mathfrak{M}) = DS(\mathfrak{A})$?

Theorem.*[Soskov, A. Soskova] Let $DS(\mathfrak{B}) \subseteq DS_1(\mathfrak{A})$. Then there exists a structure $\mathfrak{C}$ s.t. $DS(\mathfrak{C}) \subseteq DS(\mathfrak{A})$ and $DS(\mathfrak{B}) = DS_1(\mathfrak{C})$.*

The jump inversion theorem

Remark.

2009 *Montalbán, Notes on the jump of a structure.*

2009 *Stukachev, A jump inversion theorem for the semilattices of Sigma-degrees.*

2006 *Goncharov-Harizanov-Knight-McCoy-Miller-Solomon, Enumerations in computable structure theory.*

2013 *Vatev, Another Jump Inversion Theorem for Structures*

The jump inversion theorem - a negative solution

Theorem.*[Soskov 2013] There is a structure $\mathfrak{A}$ with $DS(\mathfrak{A}) \subseteq \{\mathbf{b} \mid \mathbf{0}^{(\omega)} \leq \mathbf{b}\}$ for which there is no structure $\mathfrak{M}$ with $DS_\omega(\mathfrak{M}) = DS(\mathfrak{A})$.*

Every member of $\mathbf{a} \in CS_\omega(\mathfrak{M})$ is bounded by a total degree $\mathbf{b}$, which is also a member of $CS_\omega(\mathfrak{M})$.