Enumeration Degree Spectra

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Alexandra A. Soskova [1]

Faculty of Mathematics and Informatics
Sofia University

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Enumeration reducibility

Definition. We say that $\Gamma : 2^\omega \to 2^\omega$ is an *enumeration operator* iff for some c.e. set W_i for each $B \subseteq \omega$

$$\Gamma(B) = \{x | (\exists D)[\langle x, D\rangle \in W_i \,\&D \subseteq B]\}.$$

Definition. The set A is *enumeration reducible to* the set B ($A \leq_e B$), if $A = \Gamma(B)$ for some e-operator Γ.
The enumeration degree of A is $d_e(A) = \{B \subseteq \omega | A \equiv_e B\}$.
The set of all enumeration degrees is denoted by $\mathcal{D}_e$.

The enumeration jump

Definition. Given a set A, denote by $A^+ = A \oplus (\omega \setminus A)$.

Theorem. *For any sets A and B:*

1. *A is c.e. in B iff $A \leq_e B^+$.*
2. *$A \leq_T B$ iff $A^+ \leq_e B^+$.*

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Definition. For any set A let $K_A = \{\langle i, x\rangle | x \in \Gamma_i(A)\}$. Set $A' = K_A^+$.

Definition. A set A is called *total* iff $A \equiv_e A^+$.

Let $d_e(A)' = d_e(A')$. The enumeration jump is always a total degree and agrees with the Turing jump under the standard embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$ by $\iota(d_T(A)) = d_e(A^+)$.

Degree spectra

Let $\mathfrak{A} = (A; R_1, \dots, R_k)$ *be a countable structure. An enumeration of* $\mathfrak{A}$ *is every total surjective mapping of* ω *onto A.*

Given an enumeration f of $\mathfrak{A}$ *and a subset of B of* A^n*, let*

$$f^{-1}(B) = \{\langle x_1, \dots, x_n \rangle \mid (f(x_1), \dots, f(x_n)) \in B\}.$$

$f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq).$

Definition.[Richter] *The Turing degree spectrum of* $\mathfrak{A}$ is the set

$$DS_T(\mathfrak{A}) = \{d_T(f^{-1}(\mathfrak{A})) \mid f \text{ is a one-to-one enum. of } \mathfrak{A}\}.$$

If **a** is the least element of $DS_T(\mathfrak{A})$, then **a** is called the *degree of* $\mathfrak{A}$.

Enumeration degree spectra

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If $\mathbf{a}$ is the least element of $DS(\mathfrak{A})$, then $\mathbf{a}$ is called the *e-degree of* $\mathfrak{A}$.

Proposition. *The enumeration degree spectrum is closed upwards with respect to total e-degrees, i.e. if* $\mathbf{a} \in DS(\mathfrak{A})$, $\mathbf{b}$ *is a total e-degree* $\mathbf{a} \leq_e \mathbf{b}$ *then* $\mathbf{b} \in DS(\mathfrak{A})$.

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Proposition. *If* $\mathfrak{A}$ *has e-degree* $\mathbf{a}$ *then* $\mathbf{a} = d_e(f^{-1}(\mathfrak{A}))$ *for some one-to-one enumeration* f *of* $\mathfrak{A}$.

Total structures

Definition. The structure $\mathfrak{A}$ is called *total* if for every enumeration f of $\mathfrak{A}$ the set $f^{-1}(\mathfrak{A})$ is total.

Proposition. *If $\mathfrak{A}$ is a total structure then $DS(\mathfrak{A}) = \iota(DS_T(\mathfrak{A}))$.*

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Given a structure $\mathfrak{A} = (A, R_1, \ldots, R_k)$, for every j denote by R_j^c the complement of R_j and let $\mathfrak{A}^+ = (A, R_1, \ldots, R_k, R_1^c, \ldots, R_k^c)$.

Proposition.

- $\iota(DS_T(\mathfrak{A})) = DS(\mathfrak{A}^+)$.
- *If $\mathfrak{A}$ is total then $DS(\mathfrak{A}) = DS(\mathfrak{A}^+)$.*

The partial case

Definition. *The partial enumeration degree spectrum of* $\mathfrak{A}$ is the set

$$DS^p(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) \mid f \text{ is a partial enumeration of } \mathfrak{A}\}.$$

Lemma. *If* f *is a partial enumeration of* $\mathfrak{A}$ *then* $\mathrm{dom}(f) \leq_e f^{-1}(\mathfrak{A})$.

Proposition. *The partial enumeration degree spectrum is closed upwards with respect to enumeration degrees, i.e. if* $\mathbf{a} \in DS^p(\mathfrak{A})$ *and* $\mathbf{a} \leq_e \mathbf{b}$ *then* $\mathbf{b} \in DS^p(\mathfrak{A})$.

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Proposition. *The partial enumeration degree spectrum is closed upwards with respect to enumeration degrees, i.e. if $\mathbf{a} \in DS^p(\mathfrak{A})$ and $\mathbf{a} \leq_e \mathbf{b}$ then $\mathbf{b} \in DS^p(\mathfrak{A})$.*

Theorem.[Kalimullin] *For every structure $\mathfrak{A}$ there is a structure $P(\mathfrak{A})$ with $DS^p(\mathfrak{A}^+) = DS(P(\mathfrak{A}))$.*

Kalimullin proved that there is a structure $\mathfrak{A}$ such that $DS^p(\mathfrak{A}^+) = DS(P(\mathfrak{A})) = \{d_e(\mathbf{a}) \mid \mathbf{a} \in D_e \;\&\; \mathbf{a} > 0_e\}$.

Co-spectra

Definition. Let $\mathcal{A}$ be a nonempty set of enumeration degrees. The *co-set of* $\mathcal{A}$ is the set $co(\mathcal{A})$ of all lower bounds of $\mathcal{A}$. Namely

$$co(\mathcal{A}) = \{\mathbf{b} : \mathbf{b} \in \mathcal{D}_e \;\&\; (\forall \mathbf{a} \in \mathcal{A})(\mathbf{b} \leq_e \mathbf{a})\}.$$

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Definition. Given a structure $\mathfrak{A}$, set $CS(\mathfrak{A}) = co(DS(\mathfrak{A}))$.
If $\mathbf{a}$ is the greatest element of $CS(\mathfrak{A})$ then we call $\mathbf{a}$ the *co-degree* of $\mathfrak{A}$.

If $\mathfrak{A}$ has a degree $\mathbf{a}$ then $\mathbf{a}$ is also the co-degree of $\mathfrak{A}$. The vice versa is not always true.

The admissible in $\mathfrak{A}$ sets

Definition. A set B of natural numbers is admissible in $\mathfrak{A}$ if for every enumeration f of $\mathfrak{A}$, $B \leq_e f^{-1}(\mathfrak{A})$.

Clearly $\mathbf{a} \in CS(\mathfrak{A})$ *iff* $\mathbf{a} = d_e(B)$ *for some admissible in* $\mathfrak{A}$ *set* B.

Forcing definable in $\mathfrak{A}$ sets

Every finite mapping of ω into A is called a finite part. For every finite part τ and natural numbers e, x, let

$$\tau \Vdash F_e(x) \iff x \in \Gamma_e(\tau^{-1}(\mathfrak{A})) \text{ and}$$
$$\tau \Vdash \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \nVdash F_e(x)).$$

Definition. An enumeration f of $\mathfrak{A}$ is *generic* if for every $e, x \in \omega$, there exists a $\tau \subseteq f$ s.t. $\tau \Vdash F_e(x) \vee \tau \Vdash \neg F_e(x)$.

Definition. A set B of natural numbers is *forcing definable in the structure* $\mathfrak{A}$ iff there exist a finite part δ and a natural number e s.t.

$$B = \{x | (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}.$$

Forcing definable in $\mathfrak{A}$ sets

Theorem. *Let $B \subseteq \omega$ and $d_e(C) \in DS(\mathfrak{A})$. Then the following are equivalent:*

1. *B is admissible in $\mathfrak{A}$.*
2. *$B \leq_e f^{-1}(\mathfrak{A})$ for all generic enumerations f of $\mathfrak{A}$ s.t. $(f^{-1}(\mathfrak{A}))' \equiv_e C'$.*
3. *B is forcing definable on $\mathfrak{A}$.*

The formally definable sets on $\mathfrak{A}$

Definition. A Σ_1^+ formula with free variables among $X_1, \ldots, X_r$ is a c.e. disjunction of existential formulae of the form $\exists Y_1 \ldots \exists Y_k \theta(\bar{Y}, \bar{X})$, where θ is a finite conjunction of atomic formulae.

Definition. A set $B \subseteq \omega$ is *formally definable* on $\mathfrak{A}$ if there exists a recursive function $\gamma(x)$, such that $\bigvee_{x\in\omega} \Phi_{\gamma(x)}$ is a Σ_1^+ formula with free variables among $X_1, \ldots, X_r$ and elements $t_1, \ldots, t_r$ of A such that the following equivalence holds:

$$x \in B \iff \mathfrak{A} \models \Phi_{\gamma(x)}(X_1/t_1, \ldots, X_r/t_r)\ .$$

Theorem. *Let $B \subseteq \omega$. Then*

1. *B is admissible in $\mathfrak{A}$ ($d_e(B) \in CS(\mathfrak{A})$) iff*
2. *B is forcing definable on $\mathfrak{A}$ iff*
3. *B is formally definable on $\mathfrak{A}$.*

Jump spectra and jump co-spectra

Definition. The *n*th jump spectrum of $\mathfrak{A}$ is the set

$$DS_n(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})^{(n)}) : f \text{ is an enumeration of } \mathfrak{A}\}.$$

If **a** is the least element of $DS_n(\mathfrak{A})$, then **a** is called the *nth jump degree of* $\mathfrak{A}$.

Definition. The co-set $CS_n(\mathfrak{A})$ of the *n*th jump spectrum of $\mathfrak{A}$ is called *n*th jump co-spectrum of $\mathfrak{A}$.
If $CS_n(\mathfrak{A})$ has a greatest element then it is called the *nth jump co-degree of* $\mathfrak{A}$.

Some examples

Example.*[Richter] Let* $\mathfrak{A} = (A; <)$ *be a linear ordering.* $DS(\mathfrak{A})$ *contains a minimal pair of degrees and hence* $CS(\mathfrak{A}) = \{\mathbf{0}_e\}$*.* $\mathbf{0}_e$ *is the co-degree of* $\mathfrak{A}$*. So, if* $\mathfrak{A}$ *has a degree* $\mathbf{a}$*, then* $\mathbf{a} = \mathbf{0}_e$*.*

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Example.*[Knight] For a linear ordering* $\mathfrak{A}$, $CS_1(\mathfrak{A})$ *consists of all e-degrees of* Σ^0_2 *sets. The first jump co-degree of* $\mathfrak{A}$ *is* $\mathbf{0}'_e$.

A special kind of co-degree

Definition. [Knight 98] A structure $\mathfrak{A}$ has "enumeration degree X" if every enumeration of X computes a copy of $\mathfrak{A}$, and every copy of $\mathfrak{A}$ computes an enumeration of X.

In our terms this can be formulated as $\mathfrak{A}^+$ has a co-degree $d_e(X)$ and $DS(\mathfrak{A}) = \{\mathbf{a} \mid \mathbf{a} \text{ is total and } d_e(X) \leq \mathbf{a}\}$.

Example. *Given $X \subseteq \omega$, consider the group $G_X = \bigoplus_{i \in X} \mathbb{Z}_{p_i}$, where p_i is the ith prime number. Then G_X has "enumeration degree X": We can easily build G_X out of an enumeration of X, and for the other direction, we have that $n \in X$ if and only if there exists $g \in G_X$ of order p_n.*

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Theorem. *[A. Montalban] Let K be Π^c_2 class of $\exists$-atomic structures, i.e. K is the class of structures axiomatized by some Π^c_2 sentence and for every structure $\mathfrak{A}$ in K and every tuple $\bar{a} \in |\mathfrak{A}|$ the orbit of $\bar{a}$ is existentially definable (with parameters $\bar{a}$). Then every structure in K has "enumeration degree" given by its $\exists$-theory.*

Representing the principle countable ideals as co-spectra

Example. *[Coles, Downey, Slaman; Soskov] Let G be a torsion free abelian group of rank 1, i.e. G is a subgroup of Q.*
There exists an enumeration degree $\mathbf{s}_G$ *such that*

- $DS(G) = \{\mathbf{b} : \mathbf{b}$ *is total and* $\mathbf{s}_G \leq_e \mathbf{b}\}$.
- *The co-degree of G is* $\mathbf{s}_G$.
- *G has a degree iff* $\mathbf{s}_G$ *is a total e-degree.*
- *If* $1 \leq n$, *then* $\mathbf{s}_G^{(n)}$ *is the n-th jump degree of G.*

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- *G has a degree iff* $\mathbf{s}_G$ *is a total e-degree.*
- *If* $1 \leq n$, *then* $\mathbf{s}_G^{(n)}$ *is the n-th jump degree of G.*

For every $\mathbf{d} \in \mathcal{D}_e$ *there exists a G, s.t.* $\mathbf{s}_G = \mathbf{d}$.

Corollary. *Every principle ideal of enumeration degrees is* $CS(G)$ *for some G.*

Representing non-principle countable ideals as co-spectra

Theorem.[Soskov] *Every countable ideal is the co-spectrum of a structure.*

Proof.

Let $B_0, \dots, B_n, \dots$ be a sequence of sets of natural numbers. Set $\mathfrak{A} = (\mathbb{N}; G_f; \sigma)$,

$$f(\langle i, n\rangle) = \langle i+1, n\rangle;$$
$$\sigma = \{\langle i, n\rangle : n = 2k+1 \vee n = 2k \ \&\ i \in B_k\}.$$

Then $CS(\mathfrak{A}) = I(d_e(B_0), \dots, d_e(B_n), \dots)$

□

Spectra with a countable base

Definition. Let $\mathcal{B} \subseteq \mathcal{A}$ be sets of degrees. Then $\mathcal{B}$ is a base of $\mathcal{A}$ if

$$(\forall \mathbf{a} \in \mathcal{A})(\exists \mathbf{b} \in \mathcal{B})(\mathbf{b} \leq \mathbf{a}).$$

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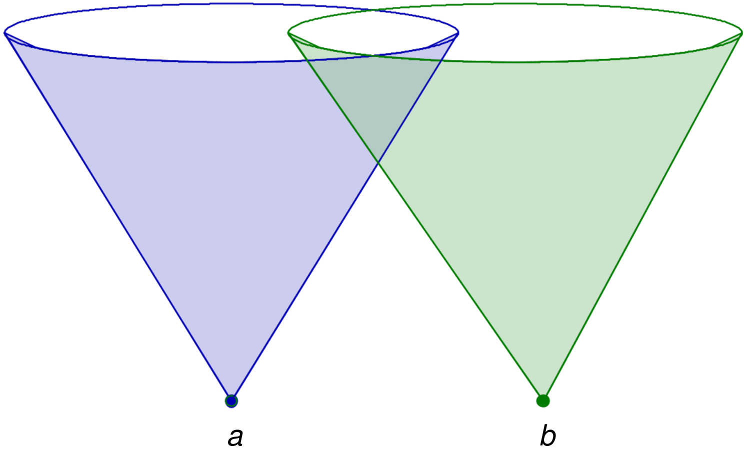
Theorem. *A structure $\mathfrak{A}$ has e-degree if and only if $DS(\mathfrak{A})$ has a countable base.*

Suppose that the sequence of e-degrees $\{\mathbf{b}_i\}_i$ is a base for $DS(\mathfrak{A})$.
Assume that no $\mathbf{b}_i$ is an e-degree of $\mathfrak{A}$. Then for every i, $\mathbf{b}_i \notin CS(\mathfrak{A})$.
Let $B_i \in \mathbf{b}_i$ for every $i \in \omega$. Then all the sets B_i have no forcing normal form.
We can construct a generic enumeration f of $\mathfrak{A}$, omitting all B_i, i.e. $B_i \not\leq_e f^{-1}(\mathfrak{A})$.
This contradicts with fact that $\{\mathbf{b}_i\}_i$ is a base for $DS(\mathfrak{A})$.

An upwards closed set of degrees which is not a degree spectra of a structure



The minimal pair theorem

Theorem. *Let* $\mathbf{c} \in DS_2(\mathfrak{A})$. *There exist* $\mathbf{f},\mathbf{g} \in DS(\mathfrak{A})$ *s.t.* $\mathbf{f},\mathbf{g}$ *are total,* $\mathbf{f}'' = \mathbf{g}'' = \mathbf{c}$ *and* $CS(\mathfrak{A}) = co(\{\mathbf{f},\mathbf{g}\})$.

Notice that for every enumeration degree $\mathbf{b}$ *there exists a structure* $\mathfrak{A}_{\mathbf{b}}$ *s. t.* $DS(\mathfrak{A}_{\mathbf{b}}) = \{\mathbf{x} \in \mathcal{D}_T | \mathbf{b} <_e \mathbf{x}\}$. *Hence*

Corollary.*[Rozinas] For every* $\mathbf{b} \in \mathcal{D}_e$ *there exist total* $\mathbf{f}, \mathbf{g}$ *below* $\mathbf{b}''$ *which are a minimal pair over* $\mathbf{b}$.

The quasi-minimal degree

Definition. Let $\mathcal{A}$ be a set of enumeration degrees. The degree **q** is quasi-minimal with respect to $\mathcal{A}$ if:

- $\mathbf{q} \notin co(\mathcal{A})$.
- If **a** is total and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathcal{A}$.
- If **a** is total and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in co(\mathcal{A})$.

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Theorem. *For every structure $\mathfrak{A}$ there exists a quasi-minimal with respect to $DS(\mathfrak{A})$ degree.*

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Theorem. *For every structure $\mathfrak{A}$ there exists a quasi-minimal with respect to $DS(\mathfrak{A})$ degree.*

Corollary.*[Slaman and Sorbi] Let I be a countable ideal of enumeration degrees. There exists an enumeration degree $\mathbf{q}$ s.t.*

1. *If $\mathbf{a} \in I$ then $\mathbf{a} <_e \mathbf{q}$.*
2. *If $\mathbf{a}$ is total and $\mathbf{a} \leq_e \mathbf{q}$ then $\mathbf{a} \in I$.*

Properties of the quasi-minimal degrees

Proposition. *For every countable structure $\mathfrak{A}$ there exist uncountably many quasi-minimal degrees with respect to $DS(\mathfrak{A})$.*

Proof.

Suppose that all quasi-minimal degrees with respect to $DS(\mathfrak{A})$ are $\mathbf{q}_0, \mathbf{q}_1, \ldots, \mathbf{q}_n, \ldots$ and let $X_i \in \mathbf{q}_i$, for all $i \in \omega$. Then all $\mathbf{q}_i$ are not in $CS(\mathfrak{A})$ and hence every X_i is not forcing definable on $\mathfrak{A}$.
We could build a partial generic enumeration f of $\mathfrak{A}$ such that $X_i \not\leq_e f^{-1}(\mathfrak{A})$.
Thus $d_e(f^{-1}(\mathfrak{A}))$ is quasi-minimal with respect to $DS(\mathfrak{A})$ and not in $\{\mathbf{q}_i\}$. □

Jumps of quasi-minimal degrees

Lemma. *Let* $\mathbf{a} \in DS_1(\mathfrak{A})$ *and* g *be an enumeration of* $\mathfrak{A}$ *such that* $g^{-1}(\mathfrak{A})' \in \mathbf{a}$. *There exists a partial generic enumeration* f *such that* $f^{-1}(\mathfrak{A})' \equiv_e g^{-1}(\mathfrak{A})'$.

Proposition. *The first jump spectrum of every structure* $\mathfrak{A}$ *consists exactly of the enumeration jumps of the quasi-minimal degrees.*

Corollary.*[McEvoy] For every total e-degree* $\mathbf{a} \geq_e \mathbf{0}'_e$ *there is a quasi-minimal degree* $\mathbf{q}$ *with* $\mathbf{q}' = \mathbf{a}$.

Splitting a total set

Proposition.[Jockusch] *For every total e-degree* $\mathbf{a}$ *there are quasi-minimal degrees* $\mathbf{p}$ *and* $\mathbf{q}$ *such that* $\mathbf{a} = \mathbf{p} \vee \mathbf{q}$.

Proposition. *For every element* $\mathbf{a}$ *of the jump spectrum of a structure* $\mathfrak{A}$ *there exists quasi-minimal with respect to* $\mathfrak{A}$ *degrees* $\mathbf{p}$ *and* $\mathbf{q}$ *such that* $\mathbf{a} = \mathbf{p} \vee \mathbf{q}$.

Every jump spectrum is the spectrum of a total structure

Let $\mathfrak{A} = (A; R_1, \ldots, R_n)$.
Let $\bar{0} \notin A$. *Set* $A_0 = A \cup \{\bar{0}\}$. *Let* $\langle ., . \rangle$ *be a pairing function s.t. none of the elements of* A_0 *is a pair and* A^* *be the least set containing* A_0 *and closed under* $\langle ., . \rangle$. *Let L and R be the decoding functions.*

Definition. *Moschovakis' extension* of $\mathfrak{A}$ is the structure

$$\mathfrak{A}^* = (A^*, R_1, \ldots, R_n, A_0, G_{\langle ., . \rangle}, G_L, G_R).$$

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Let $K_{\mathfrak{A}} = \{\langle \delta, e, x \rangle : (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}$*.*
Set $\mathfrak{A}' = (\mathfrak{A}^*, K_{\mathfrak{A}}, A^* \setminus K_{\mathfrak{A}})$*.*

Theorem.

1. *The structure* $\mathfrak{A}'$ *is total.*
2. $DS_1(\mathfrak{A}) = DS(\mathfrak{A}')$*.*

The jump inversion theorem

Let $\alpha < \omega_1^{CK}$ and $\mathfrak{A}$ be a countable structure such that all elements of $DS(\mathfrak{A})$ are above $\mathbf{0}^{(\alpha)}$.

The jump inversion theorem

Let $\alpha < \omega_1^{CK}$ and $\mathfrak{A}$ be a countable structure such that all elements of $DS(\mathfrak{A})$ are above $\mathbf{0}^{(\alpha)}$.

Does there exist a structure $\mathfrak{M}$ such that $DS_\alpha(\mathfrak{M}) = DS(\mathfrak{A})$?

The Jump Inversion Theorem

Consider two structures $\mathfrak{A}$ and $\mathfrak{B}$. Suppose that

$$DS(\mathfrak{B})_t = \{\mathbf{a} | \mathbf{a} \in DS(\mathfrak{B}) \text{ and } \mathbf{a} \text{ is total}\} \subseteq DS_1(\mathfrak{A}).$$

Theorem. *There exists a structure $\mathfrak{C}$ s.t. $DS(\mathfrak{C}) \subseteq DS(\mathfrak{A})$ and $DS_1(\mathfrak{C}) = DS(\mathfrak{B})_t$.*

Method: Marker's extensions.

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Method: Marker's extensions.

Corollary. *Let $DS(\mathfrak{B}) \subseteq DS_1(\mathfrak{A})$. Then there exists a structure $\mathfrak{C}$ s.t. $DS(\mathfrak{C}) \subseteq DS(\mathfrak{A})$ and $DS(\mathfrak{B}) = DS_1(\mathfrak{C})$.*

Corollary. *Suppose that $DS(\mathfrak{B})$ consists of total degrees greater than or equal to $\mathbf{0}'$. Then there exists a total structure $\mathfrak{C}$ such that $DS(\mathfrak{B}) = DS(\mathfrak{C})$.*

The jump inversion theorem

Theorem. *Let $n \geq 1$. Suppose that $DS(\mathfrak{B}) \subseteq DS_n(\mathfrak{A})$. There exists a structure $\mathfrak{C}$ s.t. $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.*

Corollary. *Suppose that $DS(\mathfrak{B})$ consists of total degrees greater than or equal to $\mathbf{0}^{(n)}$. Then there exists a total structure $\mathfrak{C}$ s.t. $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.*

The jump inversion theorem

Theorem. *Let $n \geq 1$. Suppose that $DS(\mathfrak{B}) \subseteq DS_n(\mathfrak{A})$. There exists a structure $\mathfrak{C}$ s.t. $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.*

Corollary. *Suppose that $DS(\mathfrak{B})$ consists of total degrees greater than or equal to $\mathbf{0}^{(n)}$. Then there exists a total structure $\mathfrak{C}$ s.t. $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.*

Remark.

2009 *Montalban, Notes on the jump of a structure, Mathematical Theory and Computational Practice, 372–378.*

2009 *Stukachev, A jump inversion theorem for the semilattices of Sigma-degrees, Siberian Electronic Mathematical Reports, v. 6, 182 – 190*

Jump inversion for a successor ordinal

Theorem.*[Goncharov-Harizanov-Knight-McCoy-Miller-Solomon, 2006]* *Let α be a computable successor ordinal and $\mathfrak{B}_1$ and $\mathfrak{B}_2$ in $\mathcal{L}$ are computable and α-friendly structures and such that*

- *$\mathfrak{B}_1$ and $\mathfrak{B}_2$ satisfy the same Σ_β sentences of $\mathcal{L}$ for each $\beta < \alpha$,*
- *each $\mathfrak{B}_i$ satisfies some Σ^c_α sentence that is not true in the other.*

Then there is a graph $\mathfrak{N}$ built from the sequences which strongly encodes the initial predicates of $\mathfrak{A}$ and
$\mathfrak{N}$ has an X computable copy iff $\mathfrak{A}$ has a $\Delta^0_\alpha(X)$ computable copy.

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S. Vatev using this idea proved:

Theorem.[S. Vatev,2013] *For every computable successor ordinal $\alpha \geq 2$ and a countable structure $\mathfrak{A}$ such that $DS(\mathfrak{A}) \subseteq \{\mathbf{a} \mid \mathbf{0}^{(\alpha)} \leq_T \mathbf{a}\}$ there is a structure $\mathfrak{N}$ such that:*

- $DS_\alpha(\mathfrak{N}) = DS(\mathfrak{A})$;
- $(\forall X \subseteq A)[X \in \Sigma^c_{\alpha+1}(\mathfrak{N}) \iff X \in \Sigma^c_1(\mathfrak{A})]$.

The jump inversion theorem - a negative solution

Theorem.*[Soskov 2013] There is a structure* $\mathfrak{A}$ *with* $DS(\mathfrak{A}) \subseteq \{\mathbf{b} \mid \mathbf{0}^{(\omega)} \leq \mathbf{b}\}$ *for which there is no structure* $\mathfrak{M}$ *with* $DS_\omega(\mathfrak{M}) = DS(\mathfrak{A})$.

Applications

Example. *[Ash, Jockusch, Knight and Downey] Let $n \geq 0$. There exists a total structure $\mathfrak{C}$ s.t. $\mathfrak{C}$ has a $n+1$-th jump degree $\mathbf{0}^{(n+1)}$ but has no k-th jump degree for $k \leq n$.*

It is sufficient to construct a structure $\mathfrak{B}$ satisfying:

1. *$DS(\mathfrak{B})$ has not a least element.*
2. *$\mathbf{0}^{(n+1)}$ is the least element of $DS_1(\mathfrak{B})$.*
3. *All elements of $DS(\mathfrak{B})$ are total and above $\mathbf{0}^{(n)}$.*

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Consider a set B satisfying:

1. *B is quasi-minimal above $\mathbf{0}^{(n)}$.*
2. *$B' \equiv_e \mathbf{0}^{(n+1)}$.*

Let G be a subgroup of the additive group of the rationals s.t. $S_G \equiv_e B$. Recall that $DS(G) = \{\mathbf{a} \mid d_e(S_G) \leq_e \mathbf{a}$ and $\mathbf{a}$ is total$\}$ and $d_e(S_G)'$ is the least element of $DS_1(G)$.

Applications

Let $n \geq 0$. There exists a total structure $\mathfrak{C}$ such that $DS_n(\mathfrak{C}) = \{\mathbf{a} \mid \mathbf{0}^{(n)} <_e \mathbf{a}\}$.
It is sufficient to construct a structure $\mathfrak{B}$ such that the elements of $DS(\mathfrak{B})$ are exactly the total e-degrees greater than $\mathbf{0}^{(n)}$.
This could be done by Whener's construction using a special family of sets:

Theorem. *Let $n \geq 0$. There exists a family $\mathcal{F}$ of sets of natural numbers s.t. for every X strictly above $\mathbf{0}^{(n)}$ there exists a computable in X set U satisfying the equivalence:*

$$F \in \mathcal{F} \iff (\exists a)(F = \{x | (a, x) \in U\}).$$

But there is no such U c.e. in $\mathbf{0}^{(n)}$.

Degree spectra

- Questions:
  - Describe the sets of Turing degrees (enumeration degrees) which are equal to $DS(\mathfrak{A})$ for some structure $\mathfrak{A}$.
  - Is the set of all Muchnik degrees containing some degree spectra definable in the lattice of the Muchnik degrees?