Enumeration Degree Spectra of Abstract Structures

Alexandra A. Soskova
(joint work with Ivan N. Soskov) [1]

Faculty of Mathematics and Computer Science
Sofia University

July 21, 2010

[1] Supported by BNSF Grant No. D002-258/18.12.08.

Outline

- Enumeration degrees
- Degree spectra and co-spectra
- Characterization of the co-spectra
- Representing the countable ideals as co-spectra
- Properties of upwards closed sets of degrees
- Selmans's theorem for degree spectra
- The minimal pair theorem
- Quasi-minimal degrees
- Jump spectra
- ω-degree spectra

Enumeration reducibility

Definition.(Friedberg and Rogers, 1959) We say that $\Psi : 2^\omega \to 2^\omega$ is an *enumeration operator* (or e-operator) iff for some c.e. set W_i

$$\Psi(B) = \{x | (\exists D)[\langle x, D\rangle \in W_i \,\&D \subseteq B]\}$$

for each $B \subseteq \omega$.

Definition. For any sets A and B define A is *enumeration reducible to* B, written $A \leq_e B$, by $A = \Psi(B)$ for some e-operator Ψ.

The enumeration jump

Definition. Given $A \subseteq \omega$, set $A^+ = A \oplus (\omega \setminus A)$.

Theorem. *For any $A, B \subseteq \omega$,*

1. *A is c.e. in B iff $A \leq_e B^+$.*
2. *$A \leq_T B$ iff $A^+ \leq_e B^+$.*

Definition.(Cooper, McEvoy) Given $A \subseteq \omega$, let $E_A = \{\langle i, x\rangle | x \in \Psi_i(A)\}$. Set $J_e(A) = E_A^+$.

The enumeration jump J_e is monotone and agrees with the Turing jump J_T in the following sense:

Theorem. *For any $A \subseteq \omega$, $J_T(A)^+ \equiv_e J_e(A^+)$.*

Definition. A set A is called *total* iff $A \equiv_e A^+$.

The enumeration jump

Definition. Given $A \subseteq \omega$, set $A^+ = A \oplus (\omega \setminus A)$.

Theorem. *For any* $A, B \subseteq \omega$,

1. A *is c.e. in* B *iff* $A \leq_e B^+$.
2. $A \leq_T B$ *iff* $A^+ \leq_e B^+$.

Definition.(Cooper, McEvoy) Given $A \subseteq \omega$, let $E_A = \{\langle i, x \rangle | x \in \Psi_i(A)\}$. Set $J_e(A) = E_A^+$.

The enumeration jump J_e is monotone and agrees with the Turing jump J_T in the following sense:

Theorem. *For any* $A \subseteq \omega$, $J_T(A)^+ \equiv_e J_e(A^+)$.

Definition. A set A is called *total* iff $A \equiv_e A^+$.

The enumeration degrees

Definition. Given a set A, let $d_e(A) = \{B \subseteq \omega | A \equiv_e B\}$.
Let $d_e(A) \leq_e d_e(B) \iff A \leq_e B$.

Denote by $\mathcal{D}_e$ the partial ordering of the enumeration degrees.

$\mathcal{D}_e$ is an upper semi-lattice with least element $\mathbf{0}_e$, where $d_e(A) \vee d_e(B) = d_e(A \oplus B)$ and $\mathbf{0}_e = \{W | W$ is c.e.$\}$.

The Rogers embedding. Define $\iota : \mathcal{D}_T \to \mathcal{D}_e$ by $\iota(d_T(A)) = d_e(A^+)$. Then ι is a Proper embedding of $\mathcal{D}_T$ into $\mathcal{D}_e$. The enumeration degrees in the range of ι are called total.

Let $d_e(A)' = d_e(J_e(A))$. The jump is always total and agrees with the Turing jump under the embedding ι.

The enumeration degrees

Definition. Given a set A, let $d_e(A) = \{B \subseteq \omega | A \equiv_e B\}$.
Let $d_e(A) \leq_e d_e(B) \iff A \leq_e B$.

Denote by $\mathcal{D}_e$ the partial ordering of the enumeration degrees.

$\mathcal{D}_e$ is an upper semi-lattice with least element $\mathbf{0}_e$, where $d_e(A) \vee d_e(B) = d_e(A \oplus B)$ and $\mathbf{0}_e = \{W | W$ is c.e.$\}$.

The Rogers embedding. Define $\iota : \mathcal{D}_T \to \mathcal{D}_e$ by $\iota(d_T(A)) = d_e(A^+)$. Then ι is a Proper embedding of $\mathcal{D}_T$ into $\mathcal{D}_e$. The enumeration degrees in the range of ι are called total.

Let $d_e(A)' = d_e(J_e(A))$. The jump is always total and agrees with the Turing jump under the embedding ι.

The enumeration degrees

Definition. Given a set A, let $d_e(A) = \{B \subseteq \omega | A \equiv_e B\}$. Let $d_e(A) \leq_e d_e(B) \iff A \leq_e B$.

Denote by $\mathcal{D}_e$ the partial ordering of the enumeration degrees.

$\mathcal{D}_e$ is an upper semi-lattice with least element $\mathbf{0}_e$, where $d_e(A) \vee d_e(B) = d_e(A \oplus B)$ and $\mathbf{0}_e = \{W | W \text{ is c.e.}\}$.

The Rogers embedding. Define $\iota : \mathcal{D}_T \to \mathcal{D}_e$ by $\iota(d_T(A)) = d_e(A^+)$. Then ι is a Proper embedding of $\mathcal{D}_T$ into $\mathcal{D}_e$. The enumeration degrees in the range of ι are called total.

Let $d_e(A)' = d_e(J_e(A))$. The jump is always total and agrees with the Turing jump under the embedding ι.

Degree Spectra

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k)$ *be a denumerable structure. Enumeration of* $\mathfrak{A}$ *is every total surjective mapping of* $\mathbb{N}$ *onto* $\mathbb{N}$*.*

Given an enumeration f *of* $\mathfrak{A}$ *and a subset of* A *of* $\mathbb{N}^a$*, let*

$$f^{-1}(A) = \{\langle x_1, \ldots, x_a \rangle : (f(x_1), \ldots, f(x_a)) \in A\}.$$

Set $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq)$*.*

Definition.(Richter) *The Turing Degree Spectrum of* $\mathfrak{A}$ is the set

$$DS_T(\mathfrak{A}) = \{d_T(f^{-1}(\mathfrak{A})) : f \text{ is an one to one enumeration of } \mathfrak{A})\}.$$

If $\mathbf{a}$ is the least element of $DS_T(\mathfrak{A})$, then $\mathbf{a}$ is called the *degree of* $\mathfrak{A}$

Definition. *The e-Degree Spectrum of* $\mathfrak{A}$ is the set

$$DS(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) : f \text{ is an enumeration of } \mathfrak{A})\}.$$

If $\mathbf{a}$ is the least element of $DS(\mathfrak{A})$, then $\mathbf{a}$ is called the *e-degree of* $\mathfrak{A}$

Proposition. *If* $\mathfrak{A}$ *has e-degree* $\mathbf{a}$ *then* $\mathbf{a} = d_e(f^{-1}(\mathfrak{A}))$ *for some one to one enumeration* f *of* $\mathfrak{A}$.

Proposition. *If* $\mathbf{a} \in DS(\mathfrak{A})$, $\mathbf{b}$ *is a total e-degree and* $\mathbf{a} \leq_e \mathbf{b}$ *then* $\mathbf{b} \in DS(\mathfrak{A})$.

Definition. *The e-Degree Spectrum of* $\mathfrak{A}$ is the set

$$DS(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) : f \text{ is an enumeration of } \mathfrak{A})\}.$$

If $\mathbf{a}$ is the least element of $DS(\mathfrak{A})$, then $\mathbf{a}$ is called the *e-degree of* $\mathfrak{A}$

Proposition. *If* $\mathfrak{A}$ *has e-degree* $\mathbf{a}$ *then* $\mathbf{a} = d_e(f^{-1}(\mathfrak{A}))$ *for some one to one enumeration* f *of* $\mathfrak{A}$.

Proposition. *If* $\mathbf{a} \in DS(\mathfrak{A})$, $\mathbf{b}$ *is a total e-degree and* $\mathbf{a} \leq_e \mathbf{b}$ *then* $\mathbf{b} \in DS(\mathfrak{A})$.

Enumeration Degree Spectra

Definition. *The e-Degree Spectrum of* $\mathfrak{A}$ is the set

$$DS(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) : f \text{ is an enumeration of } \mathfrak{A})\}.$$

If $\mathbf{a}$ is the least element of $DS(\mathfrak{A})$, then $\mathbf{a}$ is called the *e-degree of* $\mathfrak{A}$

Proposition. *If* $\mathfrak{A}$ *has e-degree* $\mathbf{a}$ *then* $\mathbf{a} = d_e(f^{-1}(\mathfrak{A}))$ *for some one to one enumeration* f *of* $\mathfrak{A}$*.*

Proposition. *If* $\mathbf{a} \in DS(\mathfrak{A})$, $\mathbf{b}$ *is a total e-degree and* $\mathbf{a} \leq_e \mathbf{b}$ *then* $\mathbf{b} \in DS(\mathfrak{A})$*.*

Total structures

Definition. The structure $\mathfrak{A}$ is called *total* if for every enumeration f of $\mathfrak{A}$ the set $f^{-1}(\mathfrak{A})$ is total.

Proposition. *If $\mathfrak{A}$ is a total structure then $DS(\mathfrak{A}) = \iota(DS_T(\mathfrak{A}))$.*

Given a structure $\mathfrak{A} = (\mathbb{N}, R_1, \ldots, R_k)$, for every j denote by R_j^c the complement of R_j and let $\mathfrak{A}^+ = (\mathbb{N}, R_1, \ldots, R_k, R_1^c, \ldots, R_k^c)$.

Proposition. *The following are true:*

1. $\iota(DS_T(\mathfrak{A})) = DS(\mathfrak{A}^+)$.
2. *If $\mathfrak{A}$ is total then $DS(\mathfrak{A}) = DS(\mathfrak{A}^+)$.*

Clearly if $\mathfrak{A}$ is a total structure then $DS(\mathfrak{A})$ consists of total degrees. The vice versa is not always true.

Example. *Let K be the Kleene's set and $\mathfrak{A} = (\mathbb{N}; G_S, K)$, where G_S is the graph of the successor function. Then $DS(\mathfrak{A})$ consists of all total degrees. On the other hand if $f = \lambda x.x$, then $f^{-1}(\mathfrak{A})$ is an c.e. set. Hence $\bar{K} \not\leq_e f^{-1}(\mathfrak{A})$. Clearly $\bar{K} \leq_e (f^{-1}(\mathfrak{A}))^+$. So $f^{-1}(\mathfrak{A})$ is not total.*

Is it true that if $DS(\mathfrak{A})$ consists of total degrees then there exists a total structure $\mathfrak{B}$ s.t. $DS(\mathfrak{A}) = DS(\mathfrak{B})$?

Clearly if $\mathfrak{A}$ is a total structure then $DS(\mathfrak{A})$ consists of total degrees. The vice versa is not always true.

Example. *Let K be the Kleene's set and $\mathfrak{A} = (\mathbb{N}; G_S, K)$, where G_S is the graph of the successor function. Then $DS(\mathfrak{A})$ consists of all total degrees. On the other hand if $f = \lambda x.x$, then $f^{-1}(\mathfrak{A})$ is an c.e. set. Hence $\bar{K} \not\leq_e f^{-1}(\mathfrak{A})$. Clearly $\bar{K} \leq_e (f^{-1}(\mathfrak{A}))^+$. So $f^{-1}(\mathfrak{A})$ is not total.*

Is it true that if $DS(\mathfrak{A})$ consists of total degrees then there exists a total structure $\mathfrak{B}$ s.t. $DS(\mathfrak{A}) = DS(\mathfrak{B})$?

Clearly if $\mathfrak{A}$ is a total structure then $DS(\mathfrak{A})$ consists of total degrees. The vice versa is not always true.

Example. *Let K be the Kleene's set and $\mathfrak{A} = (\mathbb{N}; G_S, K)$, where G_S is the graph of the successor function. Then $DS(\mathfrak{A})$ consists of all total degrees. On the other hand if $f = \lambda x.x$, then $f^{-1}(\mathfrak{A})$ is an c.e. set. Hence $\bar{K} \not\leq_e f^{-1}(\mathfrak{A})$. Clearly $\bar{K} \leq_e (f^{-1}(\mathfrak{A}))^+$. So $f^{-1}(\mathfrak{A})$ is not total.*

Is it true that if $DS(\mathfrak{A})$ consists of total degrees then there exists a total structure $\mathfrak{B}$ s.t. $DS(\mathfrak{A}) = DS(\mathfrak{B})$?

Co-spectra

Definition. Let $\mathcal{A}$ be a nonempty set of enumeration degrees the *co-set of* $\mathcal{A}$ is the set $co(\mathcal{A})$ of all lower bounds of $\mathcal{A}$. Namely

$$co(\mathcal{A}) = \{\mathbf{b} : \mathbf{b} \in \mathcal{D}_e \ \& \ (\forall \mathbf{a} \in \mathcal{A})(\mathbf{b} \leq_e \mathbf{a})\}.$$

Example. *Fix* $\mathbf{a} \in \mathcal{D}_e$ *and set* $\mathcal{A}_\mathbf{a} = \{\mathbf{b} \in \mathcal{D}_e : \mathbf{a} \leq_e \mathbf{b}\}$. *Then* $co(\mathcal{A}_\mathbf{a}) = \{\mathbf{b} \in \mathcal{D}_e : \mathbf{b} \leq_e \mathbf{a}\}$.

Definition. Given a structure $\mathfrak{A}$, set $CS(\mathfrak{A}) = co(DS(\mathfrak{A}))$. If $\mathbf{a}$ is the greatest element of $CS(\mathfrak{A})$ then call $\mathbf{a}$ the *co-degree* of $\mathfrak{A}$.

If $\mathfrak{A}$ has a degree $\mathbf{a}$ then $\mathbf{a}$ is also the co-degree of $\mathfrak{A}$. The vice versa is not always true.

Definition. Let $\mathcal{A}$ be a nonempty set of enumeration degrees the *co-set of* $\mathcal{A}$ is the set $co(\mathcal{A})$ of all lower bounds of $\mathcal{A}$. Namely

$$co(\mathcal{A}) = \{\mathbf{b} : \mathbf{b} \in \mathcal{D}_e \ \& \ (\forall \mathbf{a} \in \mathcal{A})(\mathbf{b} \leq_e \mathbf{a})\}.$$

Example. *Fix* $\mathbf{a} \in \mathcal{D}_e$ *and set* $\mathcal{A}_{\mathbf{a}} = \{\mathbf{b} \in \mathcal{D}_e : \mathbf{a} \leq_e \mathbf{b}\}$. *Then* $co(\mathcal{A}_{\mathbf{a}}) = \{\mathbf{b} \in \mathcal{D}_e : \mathbf{b} \leq_e \mathbf{a}\}$.

Definition. Given a structure $\mathfrak{A}$, set $CS(\mathfrak{A}) = co(DS(\mathfrak{A}))$.
If $\mathbf{a}$ is the greatest element of $CS(\mathfrak{A})$ then call $\mathbf{a}$ the *co-degree* of $\mathfrak{A}$.

If $\mathfrak{A}$ has a degree $\mathbf{a}$ then $\mathbf{a}$ is also the co-degree of $\mathfrak{A}$. The vice versa is not always true.

Definition. A set A of natural numbers is admissible in $\mathfrak{A}$ if for every enumeration f of $\mathfrak{A}$, $A \leq_e f^{-1}(\mathfrak{A})$.

Clearly $\mathbf{a} \in CS(\mathfrak{A})$ iff $\mathbf{a} = d_e(A)$ for some admissible in $\mathfrak{A}$ set A.
Every finite mapping of $\mathbb{N}$ into $\mathbb{N}$ is called *finite part*.
For every finite part τ and natural numbers e, x, let

$$\tau \Vdash F_e(x) \iff x \in \Psi_e(\tau^{-1}(\mathfrak{A})) \text{ and}$$
$$\tau \Vdash \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \nVdash F_e(x)).$$

Definition. An enumeration f is *generic* if for every $e, x \in \mathbb{N}$, there exists a $\tau \subseteq f$ s.t. $\tau \Vdash F_e(x) \vee \tau \Vdash \neg F_e(x)$.

The admissible sets

Definition. A set A of natural numbers is admissible in $\mathfrak{A}$ if for every enumeration f of $\mathfrak{A}$, $A \leq_e f^{-1}(\mathfrak{A})$.

Clearly $\mathbf{a} \in CS(\mathfrak{A})$ iff $\mathbf{a} = d_e(A)$ for some admissible in $\mathfrak{A}$ set A.
Every finite mapping of $\mathbb{N}$ into $\mathbb{N}$ is called *finite part*.
For every finite part τ and natural numbers e, x, let

$$\tau \Vdash F_e(x) \iff x \in \Psi_e(\tau^{-1}(\mathfrak{A})) \text{ and}$$
$$\tau \Vdash \neg F_e(x) \iff (\forall \rho \supseteq \tau)(\rho \nVdash F_e(x)).$$

Definition. An enumeration f is *generic* if for every $e, x \in \mathbb{N}$, there exists a $\tau \subseteq f$ s.t. $\tau \Vdash F_e(x) \vee \tau \Vdash \neg F_e(x)$.

Definition. A set A of natural numbers is *forcing definable in the structure* $\mathfrak{A}$ iff there exist finite part δ and natural number e s.t.

$$A = \{x | (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}.$$

Theorem. *Let $A \subseteq \mathbb{N}$ and $d_e(B) \in DS(\mathfrak{A})$. Then the following are equivalent:*

1. *A is admissible in $\mathfrak{A}$.*
2. *$A \leq_e f^{-1}(\mathfrak{A})$ for all generic enumerations f of $\mathfrak{A}$ s.t. $(f^{-1}(\mathfrak{A}))' \equiv_e B'$.*
3. *A is forcing definable.*

Normal form of the admissible sets

Definition. A set A of natural numbers is *forcing definable in the structure* $\mathfrak{A}$ iff there exist finite part δ and natural number e s.t.

$$A = \{x | (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}.$$

Theorem. *Let $A \subseteq \mathbb{N}$ and $d_e(B) \in DS(\mathfrak{A})$. Then the following are equivalent:*

1. *A is admissible in $\mathfrak{A}$.*
2. *$A \leq_e f^{-1}(\mathfrak{A})$ for all generic enumerations f of $\mathfrak{A}$ s.t. $(f^{-1}(\mathfrak{A}))' \equiv_e B'$.*
3. *A is forcing definable.*

Some examples

Example. *(Richter 1981) Let* $\mathfrak{A} = (\mathbb{N}; <)$ *be a linear ordering. Then* $DS(\mathfrak{A})$ *contains a minimal pair of degrees and hence* $CS(\mathfrak{A}) = \{\mathbf{0}_e\}$. *Clearly* $\mathbf{0}_e$ *is the co-degree of* $\mathfrak{A}$. *Therefore if* $\mathfrak{A}$ *has a degree* $\mathbf{a}$, *then* $\mathbf{a} = \mathbf{0}_e$.

Definition. Let $n \geq 0$. The n-th jump spectrum of a structure $\mathfrak{A}$ is defined by $DS_n(\mathfrak{A}) = \{\mathbf{a}^{(n)} | \mathbf{a} \in DS(\mathfrak{A})\}$. Set $CS_n(\mathfrak{A}) = co(DS_n(\mathfrak{A}))$.

Example. *(Knight 1986) Consider again a linear ordering* $\mathfrak{A}$. *Then* $CS_1(\mathfrak{A})$ *consists of all* Σ^0_2 *sets. The first jump co-degree of* $\mathfrak{A}$ *is* $\mathbf{0}'_e$.

Example. *(Slaman 1998,Whener 1998) There exists an* $\mathfrak{A}$ *s.t.*

$$DS(\mathfrak{A}) = \{\mathbf{a} : \mathbf{a} \text{ is total and } \mathbf{0}_e < \mathbf{a}\}.$$

Clearly the structure $\mathfrak{A}$ *has co-degree* $\mathbf{0}_e$ *but has not a degree.*

Some examples

Example. *(Richter 1981) Let $\mathfrak{A} = (\mathbb{N}; <)$ be a linear ordering. Then $DS(\mathfrak{A})$ contains a minimal pair of degrees and hence $CS(\mathfrak{A}) = \{\mathbf{0}_e\}$. Clearly $\mathbf{0}_e$ is the co-degree of $\mathfrak{A}$. Therefore if $\mathfrak{A}$ has a degree $\mathbf{a}$, then $\mathbf{a} = \mathbf{0}_e$.*

Definition. Let $n \geq 0$. The n-th jump spectrum of a structure $\mathfrak{A}$ is defined by $DS_n(\mathfrak{A}) = \{\mathbf{a}^{(n)} | \mathbf{a} \in DS(\mathfrak{A})\}$. Set $CS_n(\mathfrak{A}) = co(DS_n(\mathfrak{A}))$.

Example. *(Knight 1986) Consider again a linear ordering $\mathfrak{A}$. Then $CS_1(\mathfrak{A})$ consists of all Σ^0_2 sets. The first jump co-degree of $\mathfrak{A}$ is $\mathbf{0}'_e$.*

Example. *(Slaman 1998, Whener 1998) There exists an $\mathfrak{A}$ s.t.*

$$DS(\mathfrak{A}) = \{\mathbf{a} : \mathbf{a} \text{ is total and } \mathbf{0}_e < \mathbf{a}\}.$$

Clearly the structure $\mathfrak{A}$ has co-degree $\mathbf{0}_e$ but has not a degree.

Some examples

Example. *(Richter 1981) Let* $\mathfrak{A} = (\mathbb{N}; <)$ *be a linear ordering. Then* $DS(\mathfrak{A})$ *contains a minimal pair of degrees and hence* $CS(\mathfrak{A}) = \{\mathbf{0}_e\}$. *Clearly* $\mathbf{0}_e$ *is the co-degree of* $\mathfrak{A}$. *Therefore if* $\mathfrak{A}$ *has a degree* $\mathbf{a}$, *then* $\mathbf{a} = \mathbf{0}_e$.

Definition. Let $n \geq 0$. The n-th jump spectrum of a structure $\mathfrak{A}$ is defined by $DS_n(\mathfrak{A}) = \{\mathbf{a}^{(n)} | \mathbf{a} \in DS(\mathfrak{A})\}$. Set $CS_n(\mathfrak{A}) = co(DS_n(\mathfrak{A}))$.

Example. *(Knight 1986) Consider again a linear ordering* $\mathfrak{A}$. *Then* $CS_1(\mathfrak{A})$ *consists of all* Σ^0_2 *sets. The first jump co-degree of* $\mathfrak{A}$ *is* $\mathbf{0}'_e$.

Example. *(Slaman 1998,Whener 1998) There exists an* $\mathfrak{A}$ *s.t.*

$$DS(\mathfrak{A}) = \{\mathbf{a} : \mathbf{a} \text{ is total and } \mathbf{0}_e < \mathbf{a}\}.$$

Clearly the structure $\mathfrak{A}$ *has co-degree* $\mathbf{0}_e$ *but has not a degree.*

Representing countable ideals as co-spectra

Example. *(based on Coles, Dawney, Slaman - 1998) Let G be a torsion free Abelian group of rank 1, i.e. G is a subgroup of Q. There exists an enumeration degree $\mathbf{s}_G$ such that*

- $DS(G) = \{\mathbf{b} : \mathbf{b}$ *is total and* $\mathbf{s}_G \leq_e \mathbf{b}\}$.
- *The co-degree of G is* $\mathbf{s}_G$.
- *G has a degree iff* $\mathbf{s}_G$ *is total*
- *If* $1 \leq n$, *then* $\mathbf{s}_G^{(n)}$ *is the n-th jump degree of G.*

For every $\mathbf{d} \in \mathcal{D}_e$ *there exists a G, s.t.* $\mathbf{s}_G = \mathbf{d}$. *Hence every principle ideal of enumeration degrees is* $CS(G)$ *for some G.*

Similar results on algebraic fields:
W. Calvert, V. Harizanov and A. Shlapentokh (2007)
A. Frolov, I. Kalimullin and R. Miller(2009)

I. Kalimullin: $\exists\mathfrak{A}[DS_T(\mathfrak{A}) = \{\mathbf{x} \mid \mathbf{x} \not\leq_T \mathbf{b}\}]$ *for low* $\mathbf{b}$.

Representing countable ideals as co-spectra

Example. *(based on Coles, Dawney, Slaman - 1998) Let G be a torsion free Abelian group of rank 1, i.e. G is a subgroup of Q. There exists an enumeration degree $\mathbf{s}_G$ such that*

- $DS(G) = \{\mathbf{b} : \mathbf{b}$ *is total and* $\mathbf{s}_G \leq_e \mathbf{b}\}$.
- *The co-degree of G is $\mathbf{s}_G$.*
- *G has a degree iff $\mathbf{s}_G$ is total*
- *If $1 \leq n$, then $\mathbf{s}_G^{(n)}$ is the n-th jump degree of G.*

For every $\mathbf{d} \in \mathcal{D}_e$ there exists a G, s.t. $\mathbf{s}_G = \mathbf{d}$. Hence every principle ideal of enumeration degrees is $CS(G)$ for some G.

Similar results on algebraic fields:
W. Calvert, V. Harizanov and A. Shlapentokh (2007)
A. Frolov, I. Kalimullin and R. Miller(2009)

I. Kalimullin: $\exists\mathfrak{A}[DS_T(\mathfrak{A}) = \{\mathbf{x} \mid \mathbf{x} \not\leq_T \mathbf{b}\}]$ *for low* $\mathbf{b}$.

Representing countable ideals as co-spectra

Example. *(based on Coles, Dawney, Slaman - 1998) Let G be a torsion free Abelian group of rank 1, i.e. G is a subgroup of Q. There exists an enumeration degree $\mathbf{s}_G$ such that*

- $DS(G) = \{\mathbf{b} : \mathbf{b}$ *is total and* $\mathbf{s}_G \leq_e \mathbf{b}\}$.
- *The co-degree of G is $\mathbf{s}_G$.*
- *G has a degree iff $\mathbf{s}_G$ is total*
- *If $1 \leq n$, then $\mathbf{s}_G^{(n)}$ is the n-th jump degree of G.*

For every $\mathbf{d} \in \mathcal{D}_e$ there exists a G, s.t. $\mathbf{s}_G = \mathbf{d}$. Hence every principle ideal of enumeration degrees is $CS(G)$ for some G.

Similar results on algebraic fields:
W. Calvert, V. Harizanov and A. Shlapentokh (2007)
A. Frolov, I. Kalimullin and R. Miller(2009)

I. Kalimullin: $\exists \mathfrak{A}[DS_T(\mathfrak{A}) = \{\mathbf{x} \mid \mathbf{x} \not\leq_T \mathbf{b}\}]$ *for low* $\mathbf{b}$.

Representing non-principle countable ideals as co-spectra

Example. *Let* $B_0, \dots, B_n, \dots$ *be a sequence of sets of natural numbers. Set* $\mathfrak{A} = (\mathbb{N}; f; \sigma)$,

$$f(\langle i, n\rangle) = \langle i+1, n\rangle;$$
$$\sigma = \{\langle i, n\rangle : n = 2k+1 \vee n = 2k \;\&\; i \in B_k\}.$$

Then $CS(\mathfrak{A}) = I(d_e(B_0), \dots, d_e(B_n), \dots)$

General Properties of Upwards Closed Sets

Definition. Consider a subset $\mathcal{A}$ of $\mathcal{D}_e$. Say that $\mathcal{A}$ is *upwards closed* if for every $\mathbf{a} \in \mathcal{A}$ all total degrees greater than $\mathbf{a}$ are contained in $\mathcal{A}$.

Let $\mathcal{A}$ be an upwards closed set of degrees.
Note that if $\mathcal{B} \subseteq \mathcal{A}$, then $co(\mathcal{A}) \subseteq co(\mathcal{B})$.

Proposition. *(Selman) Let $\mathcal{A}_t = \{\mathbf{a} : \mathbf{a} \in \mathcal{A}\ \&\ \mathbf{a}$ is total$\}$. Then $co(\mathcal{A}) = co(\mathcal{A}_t)$.*

Proposition. *Let $\mathbf{b}$ be an arbitrary enumeration degree and $n > 0$. Set $\mathcal{A}_{\mathbf{b},n} = \{\mathbf{a} : \mathbf{a} \in \mathcal{A}\ \&\ \mathbf{b} \leq_e \mathbf{a}^{(n)}\}$. Then $co(\mathcal{A}) = co(\mathcal{A}_{\mathbf{b},n})$.*

Theorem. *Let $\mathfrak{A}$ be a structure, $1 \leq n$ and $\mathbf{c} \in DS_n(\mathfrak{A})$. Then*

$$CS(\mathfrak{A}) = co(\{\mathbf{b} \in DS(\mathfrak{A}) : \mathbf{b}^{(n)} = \mathbf{c}\}).$$

Example. *(Upwards closed set for which the Theorem is not true)*

Let $B \not\leq_e A$ and $A \not\leq_e B'$. Let

$$\mathcal{D} = \{\mathbf{a} : d_e(A) \leq_e \mathbf{a}\} \cup \{\mathbf{a} : d_e(B) \leq_e \mathbf{a}\}.$$

Set $\mathcal{A} = \{\mathbf{a} : \mathbf{a} \in \mathcal{D} \ \& \ \mathbf{a}' = d_e(B)'\}$.

- *$d_e(B)$ is the least element of $\mathcal{A}$ and hence $d_e(B) \in co(\mathcal{A})$.*
- *$d_e(B) \not\leq d_e(A)$ and hence $d_e(B) \notin co(\mathcal{D})$.*

Specific Properties of Degree Spectra

Theorem. *Let $\mathfrak{A}$ be a structure, $1 \leq n$ and $\mathbf{c} \in DS_n(\mathfrak{A})$. Then*

$$CS(\mathfrak{A}) = co(\{\mathbf{b} \in DS(\mathfrak{A}) : \mathbf{b}^{(n)} = \mathbf{c}\}).$$

Example. *(Upwards closed set for which the Theorem is not true)*

Let $B \not\leq_e A$ and $A \not\leq_e B'$. Let

$$\mathcal{D} = \{\mathbf{a} : d_e(A) \leq_e \mathbf{a}\} \cup \{\mathbf{a} : d_e(B) \leq_e \mathbf{a}\}.$$

Set $\mathcal{A} = \{\mathbf{a} : \mathbf{a} \in \mathcal{D} \ \& \ \mathbf{a}' = d_e(B)'\}$.

- *$d_e(B)$ is the least element of $\mathcal{A}$ and hence $d_e(B) \in co(\mathcal{A})$.*
- *$d_e(B) \not\leq d_e(A)$ and hence $d_e(B) \notin co(\mathcal{D})$.*

The minimal pair theorem

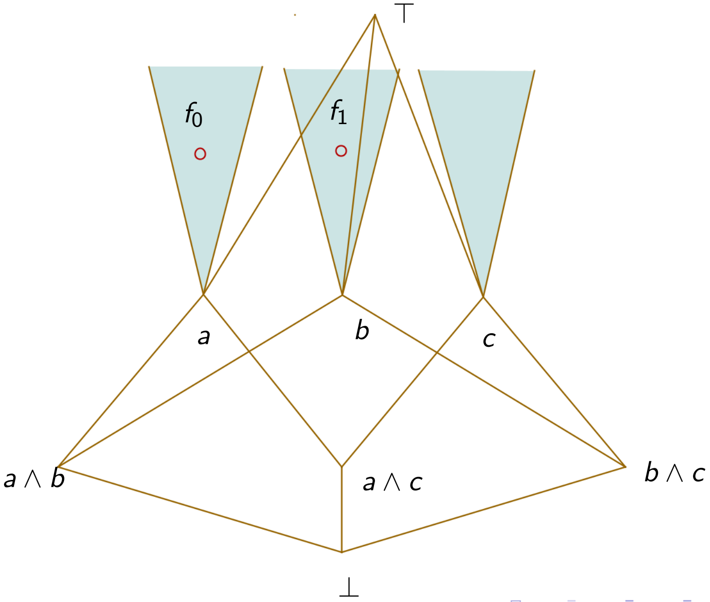
Theorem. *Let* $\mathbf{c} \in DS_2(\mathfrak{A})$. *There exist* $\mathbf{f},\mathbf{g} \in DS(\mathfrak{A})$ *s.t.* $\mathbf{f},\mathbf{g}$ *are total,* $\mathbf{f}'' = \mathbf{g}'' = \mathbf{c}$ *and* $CS(\mathfrak{A}) = co(\{\mathbf{f},\mathbf{g}\})$.

Notice that for every enumeration degree **a** there exists a structure $\mathfrak{A}_{\mathbf{a}}$ s. t. $DS(\mathfrak{A}_{\mathbf{a}}) = \{\mathbf{x} \in \mathcal{D}_T | \mathbf{a} <_e \mathbf{x}\}$. Hence

Corollary. *(Rozinas) For every* $\mathbf{b} \in \mathcal{D}_e$ *there exist total* $\mathbf{f}, \mathbf{g}$ *below* $\mathbf{b}''$ *which are a minimal pair over* $\mathbf{b}$.

Not every upwards closed set of enumeration degrees has a minimal pair:

An upwards closed set with no minimal pair



The Quasi-minimal degree

Definition. Let $\mathcal{A}$ be a set of enumeration degrees. The degree $\mathbf{q}$ is quasi-minimal with respect to $\mathcal{A}$ if:

- $\mathbf{q} \notin co(\mathcal{A})$.
- If $\mathbf{a}$ is total and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathcal{A}$.
- If $\mathbf{a}$ is total and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in co(\mathcal{A})$.

Theorem. *If $\mathbf{q}$ is quasi-minimal with respect to $\mathcal{A}$, then $\mathbf{q}$ is an upper bound of $co(\mathcal{A})$.*

Theorem. *For every structure $\mathfrak{A}$ there exists a quasi-minimal with respect to $DS(\mathfrak{A})$ degree.*

Corollary. *(Slaman and Sorbi) Let I be a countable ideal of enumeration degrees. There exist an enumeration degree* $\mathbf{q}$ *s.t.*

1. *If* $\mathbf{a} \in I$ *then* $\mathbf{a} <_e \mathbf{q}$.
2. *If* $\mathbf{a}$ *is total and* $\mathbf{a} \leq_e \mathbf{q}$ *then* $\mathbf{a} \in I$.

Definition. Let $\mathcal{B} \subseteq \mathcal{A}$ be sets of degrees. Then $\mathcal{B}$ is a base of $\mathcal{A}$ if

$$(\forall \mathbf{a} \in \mathcal{A})(\exists \mathbf{b} \in \mathcal{B})(\mathbf{b} \leq \mathbf{a}).$$

Theorem. *Let $\mathcal{A}$ be an upwards closed set of degrees possessing a quasi-minimal degree. Suppose that there exists a countable base $\mathcal{B}$ of $\mathcal{A}$ such that all elements of $\mathcal{B}$ are total. Then $\mathcal{A}$ has a least element.*

Corollary. *A total structure $\mathfrak{A}$ has a degree if and only if $DS(\mathfrak{A})$ has a countable base.*

Corollary. *(Slaman and Sorbi) Let I be a countable ideal of enumeration degrees. There exist an enumeration degree* $\mathbf{q}$ *s.t.*

1. *If* $\mathbf{a} \in I$ *then* $\mathbf{a} <_e \mathbf{q}$.
2. *If* $\mathbf{a}$ *is total and* $\mathbf{a} \leq_e \mathbf{q}$ *then* $\mathbf{a} \in I$.

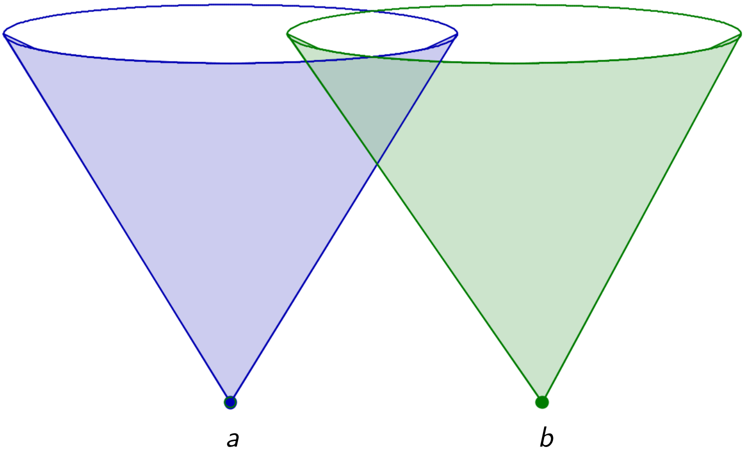
Definition. Let $\mathcal{B} \subseteq \mathcal{A}$ be sets of degrees. Then $\mathcal{B}$ is a base of $\mathcal{A}$ if

$$(\forall \mathbf{a} \in \mathcal{A})(\exists \mathbf{b} \in \mathcal{B})(\mathbf{b} \leq \mathbf{a}).$$

Theorem. *Let $\mathcal{A}$ be an upwards closed set of degrees possessing a quasi-minimal degree. Suppose that there exists a countable base $\mathcal{B}$ of $\mathcal{A}$ such that all elements of $\mathcal{B}$ are total. Then $\mathcal{A}$ has a least element.*

Corollary. *A total structure $\mathfrak{A}$ has a degree if and only if $DS(\mathfrak{A})$ has a countable base.*

An upwards closed set with no quasi-minimal degree



Jump spectra

Definition. The *n*-th jump spectrum of a structure $\mathfrak{A}$ is the set

$$DS_n(\mathfrak{A}) = \{\mathbf{a}^{(n)} | \mathbf{a} \in DS(\mathfrak{A})\}.$$

If $\mathbf{a}$ is the least element of $DS_n(\mathfrak{A})$ then $\mathbf{a}$ is called *n*-th jump degree of $\mathfrak{A}$.

Proposition. *For every* $\mathfrak{A}$, $DS_1(\mathfrak{A}) \subseteq DS(\mathfrak{A})$.

Is it true that for every $\mathfrak{A}$, $DS_1(\mathfrak{A}) \subset DS(\mathfrak{A})$*? Probably the answer is "no".*

Every jump spectrum is spectrum of a total structure

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_n)$.
Let $\bar{0} \notin \mathbb{N}$. Set $\mathbb{N}_0 = \mathbb{N} \cup \{\bar{0}\}$. Let $\langle ., . \rangle$ be a pairing function s.t. none of the elements of $\mathbb{N}_0$ is a pair and N^* be the least set containing $\mathbb{N}_0$ and closed under $\langle ., . \rangle$.

Definition. *Moschovakis' extension* of $\mathfrak{A}$ is the structure

$$\mathfrak{A}^* = (\mathbb{N}^*, R_1, \ldots, R_n, \mathbb{N}_0, G_{\langle .,. \rangle}).$$

Proposition. $DS(\mathfrak{A}) = DS(\mathfrak{A}^*)$

Let $K_{\mathfrak{A}} = \{\langle \delta, e, x \rangle : (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}$.
Set $\mathfrak{A}' = (\mathfrak{A}^*, K_{\mathfrak{A}}, \mathbb{N}^* \setminus K_{\mathfrak{A}})$.

Theorem.

1. *The structure $\mathfrak{A}'$ is total.*
2. $DS_1(\mathfrak{A}) = DS(\mathfrak{A}')$.

Every jump spectrum is spectrum of a total structure

Let $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_n)$.
Let $\bar{0} \notin \mathbb{N}$. Set $\mathbb{N}_0 = \mathbb{N} \cup \{\bar{0}\}$. Let $\langle .,. \rangle$ be a pairing function s.t. none of the elements of $\mathbb{N}_0$ is a pair and N^* be the least set containing $\mathbb{N}_0$ and closed under $\langle .,. \rangle$.

Definition. *Moschovakis' extension* of $\mathfrak{A}$ is the structure

$$\mathfrak{A}^* = (\mathbb{N}^*, R_1, \dots, R_n, \mathbb{N}_0, G_{\langle .,. \rangle}).$$

Proposition. $DS(\mathfrak{A}) = DS(\mathfrak{A}^*)$

Let $K_{\mathfrak{A}} = \{\langle \delta, e, x \rangle : (\exists \tau \supseteq \delta)(\tau \Vdash F_e(x))\}$.
Set $\mathfrak{A}' = (\mathfrak{A}^*, K_{\mathfrak{A}}, \mathbb{N}^* \setminus K_{\mathfrak{A}})$.

Theorem.

1. *The structure $\mathfrak{A}'$ is total.*
2. $DS_1(\mathfrak{A}) = DS(\mathfrak{A}')$.

The Jump Inversion Theorem

Consider two structures $\mathfrak{A}$ and $\mathfrak{B}$. Suppose that

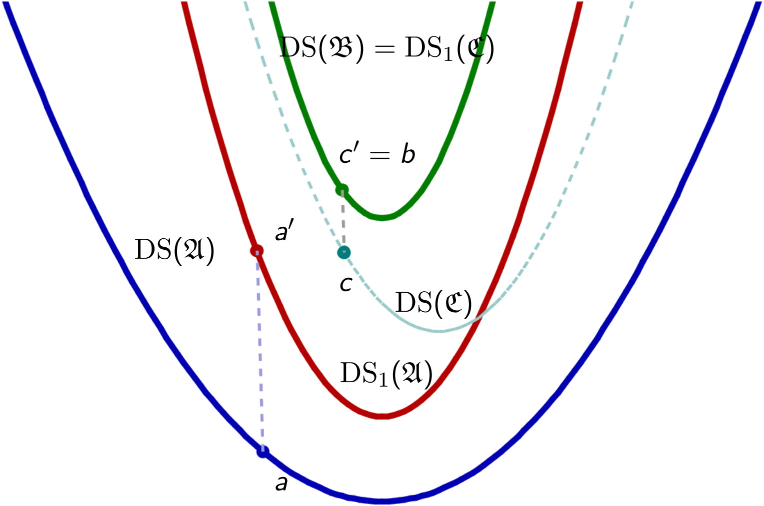
$$DS(\mathfrak{B})_t = \{\mathbf{a} | \mathbf{a} \in DS(\mathfrak{B}) \text{ and } \mathbf{a} \text{ is total}\} \subseteq DS_1(\mathfrak{A}).$$

Theorem. *There exists a structure $\mathfrak{C}$ s.t. $DS(\mathfrak{C}) \subseteq DS(\mathfrak{A})$ and $DS_1(\mathfrak{C}) = DS(\mathfrak{B})_t$.*

Method: Marker's extentions.

Corollary. *Let $DS(\mathfrak{B}) \subseteq DS_1(\mathfrak{A})$. Then there exists a structure $\mathfrak{C}$ s.t. $DS(\mathfrak{C}) \subseteq DS(\mathfrak{A})$ and $DS(\mathfrak{B}) = DS_1(\mathfrak{C})$.*

Corollary. *Suppose that $DS(\mathfrak{B})$ consists of total degrees greater than or equal to $\mathbf{0}'$. Then there exists a total structure $\mathfrak{C}'$ such that $DS(\mathfrak{B}) = DS(\mathfrak{C}')$.*


$\mathrm{DS}(\mathfrak{B}) = \mathrm{DS}_1(\mathfrak{C})$
$c' = b$
a'
$\mathrm{DS}(\mathfrak{A})$
c
$\mathrm{DS}(\mathfrak{C})$
$\mathrm{DS}_1(\mathfrak{A})$
a

Theorem. *Let $n \geq 1$. Suppose that $DS(\mathfrak{B}) \subseteq DS_n(\mathfrak{A})$. There exists a structure $\mathfrak{C}$ s.t. $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.*

Corollary. *Suppose that $DS(\mathfrak{B})$ consists of total degrees greater than or equal to $\mathbf{0}^{(n)}$. Then there exists a total structure $\mathfrak{C}$ s.t. $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.*

Remark. *Similar results*
A. Montalban (2009) different approach with complete set of Π^c_n formulas.
A. Stukachev (2009) for Σ reducibility with Marker's extentions.

Theorem. *Let $n \geq 1$. Suppose that $DS(\mathfrak{B}) \subseteq DS_n(\mathfrak{A})$. There exists a structure $\mathfrak{C}$ s.t. $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.*

Corollary. *Suppose that $DS(\mathfrak{B})$ consists of total degrees greater than or equal to $\mathbf{0}^{(n)}$. Then there exists a total structure $\mathfrak{C}$ s.t. $DS_n(\mathfrak{C}) = DS(\mathfrak{B})$.*

Remark. *Similar results*
A. Montalban (2009) different approach with complete set of Π^c_n formulas.
A. Stukachev (2009) for Σ reducibility with Marker's extentions.

Example. *(Ash, Jockush, Knight and Downey) Let $n \geq 0$. There exists a total structure $\mathfrak{C}$ s.t. $\mathfrak{C}$ has a $n+1$-th jump degree $\mathbf{0}^{(n+1)}$ but has no k-th jump degree for $k \leq n$.*
It is sufficient to construct a structure $\mathfrak{B}$ satisfying:

1. *$DS(\mathfrak{B})$ has not least element.*
2. *$\mathbf{0}^{(n+1)}$ is the least element of $DS_1(\mathfrak{B})$.*
3. *All elements of $DS(\mathfrak{B})$ are total and above $\mathbf{0}^{(n)}$.*

Consider a set B satisfying:

1. *B is quasi-minimal above $\mathbf{0}^{(n)}$.*
2. *$B' \equiv_e \mathbf{0}^{(n+1)}$.*

Let G be a subgroup of the additive group of the rationales s.t. $S_G \equiv_e B$. Recall that $DS(G) = \{\mathbf{a} | d_e(S_G) \leq_e \mathbf{a}$ and $\mathbf{a}$ is total$\}$ and $d_e(S_G)'$ is the least element of $DS_1(G)$.

Example. *(Ash, Jockush, Knight and Downey) Let $n \geq 0$. There exists a total structure $\mathfrak{C}$ s.t. $\mathfrak{C}$ has a $n+1$-th jump degree $\mathbf{0}^{(n+1)}$ but has no k-th jump degree for $k \leq n$.*
It is sufficient to construct a structure $\mathfrak{B}$ satisfying:

1. *$DS(\mathfrak{B})$ has not least element.*
2. *$\mathbf{0}^{(n+1)}$ is the least element of $DS_1(\mathfrak{B})$.*
3. *All elements of $DS(\mathfrak{B})$ are total and above $\mathbf{0}^{(n)}$.*

Consider a set B satisfying:

1. *B is quasi-minimal above $\mathbf{0}^{(n)}$.*
2. *$B' \equiv_e \mathbf{0}^{(n+1)}$.*

Let G be a subgroup of the additive group of the rationales s.t. $S_G \equiv_e B$. Recall that $DS(G) = \{\mathbf{a} | d_e(S_G) \leq_e \mathbf{a}$ and $\mathbf{a}$ is total$\}$ and $d_e(S_G)'$ is the least element of $DS_1(G)$.

Let $n \geq 0$. There exists a total structure $\mathfrak{C}$ such that $DS_n(\mathfrak{C}) = \{\mathbf{a} | \mathbf{0}^{(n)} <_e \mathbf{a}\}$.
It is sufficient to construct a structure $\mathfrak{B}$ such that the elements of $DS(\mathfrak{B})$ are exactly the total e-degrees greater than $\mathbf{0}^{(n)}$.
This is done by Whener's construction using a special family of sets:

Theorem. *Let $n \geq 0$. There exists a family $\mathcal{F}$ of sets of natural numbers s.t. for every X strictly above $\mathbf{0}^{(n)}$ there exists a recursive in X set U satisfying the equivalence:*

$$F \in \mathcal{F} \iff (\exists a)(F = \{x | (a, x) \in U\}).$$

But there is no c.e. in $\mathbf{0}^{(n)}$ such U.

Relative Spectra

Let $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ be given structures.

Definition. *The relative spectrum* $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1 \dots, \mathfrak{A}_n)$ of the structure $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ is the set

$$\{d_e(f^{-1}(\mathfrak{A})) \mid \begin{array}{l} f \text{ is an enumeration of } \mathfrak{A} \ \& \\ (\forall k \le n)(f^{-1}(\mathfrak{A}_k) \le_e f^{-1}(\mathfrak{A})^{(k)}).\} \end{array}$$

It turns out that almost all properties of the degree spectra remain true for the relative spectra.

Uniform reducibility on sequences of sets

Let $\mathcal{S}$ be the set of all sequences of sets of natural numbers.
For $\mathcal{B} = \{B_n\}_{n<\omega} \in \mathcal{S}$ call the jump class of $\mathcal{B}$ *the set*

$$J_{\mathcal{B}} = \{d_{\mathrm{T}}(X) \mid (\forall n)(B_n \textit{ is c.e. in } X^{(n)} \textit{ uniformly in } n)\} \ .$$

$\mathcal{A}$ *is* ω-enumeration reducible to $\mathcal{B}$ *(*$\mathcal{A} \leq_\omega \mathcal{B}$ *) if* $J_{\mathcal{B}} \subseteq J_{\mathcal{A}}$
$\mathcal{A} \equiv_\omega \mathcal{B}$ *if* $J_{\mathcal{A}} = J_{\mathcal{B}}$.

ω-Enumeration Degrees

Let $\mathcal{B} = \{B_n\}_{n<\omega} \in \mathcal{S}$.

Definition. *A jump sequence* $\mathcal{P}(\mathcal{B}) = \{\mathcal{P}_n(\mathcal{B})\}_{n<\omega}$:

1. $\mathcal{P}_0(\mathcal{B}) = B_0$
2. $\mathcal{P}_{n+1}(\mathcal{B}) = (\mathcal{P}_n(\mathcal{B}))' \oplus B_{n+1}$

Theorem. *[Soskov, Kovachev] $\mathcal{A} \leq_\omega \mathcal{B}$, if $A_n \leq_{\mathrm{e}} \mathcal{P}_n(\mathcal{B})$ uniformly in n.*

ω-Enumeration Degrees

$d_\omega(\mathcal{B}) = \{\mathcal{A} \mid \mathcal{A} \equiv_\omega \mathcal{B}\}$
$\mathcal{D}_\omega = \{d_\omega(\mathcal{B}) \mid \mathcal{B} \in \mathcal{S}\}$.
If $A \subseteq \mathbb{N}$ *denote by* $A \uparrow \omega = \{A, \emptyset, \emptyset, \dots\}$.
For every $A, B \subseteq \mathbb{N}$*:*

$$A \leq_{\mathrm{e}} B \iff A \uparrow \omega \leq_\omega B \uparrow \omega.$$

The mapping $\kappa(d_{\mathrm{e}}(A)) = d_\omega(A \uparrow \omega)$ *gives an isomorphic embedding of* $\mathcal{D}_{\mathrm{e}}$ *to* $\mathcal{D}_\omega$.

ω-Enumeration Jump

Definition. For every $\mathcal{A} \in \mathcal{S}$ *the ω-enumeration jump of $\mathcal{A}$* is $\mathcal{A}' = \{\mathcal{P}_{n+1}(\mathcal{A})\}_{n<\omega}$

Let $d_\omega(\mathcal{A})' = d_\omega(\mathcal{A}')$.

$\mathcal{A}^{(k)} = \{\mathcal{P}_{n+k}(\mathcal{A})\}_{n<\omega}$ *for each k.*
$d_\omega(\mathcal{A})^{(k)} = d_\omega(\mathcal{A}^{(k)})$.

Let $\mathcal{B} = \{B_n\}_{n<\omega}$ be a fixed sequence of sets.

Definition. The enumeration f of the structure $\mathfrak{A}$ is *acceptable with respect to* $\mathcal{B}$, if for every n,

$$f^{-1}(B_n) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(n)} \text{ uniformly in } n.$$

Denote by $\mathcal{E}(\mathfrak{A}, \mathcal{B})$ - the class of all acceptable enumerations.

Definition. *The* ω- *degree spectrum of* $\mathfrak{A}$ *with respect to* $\mathcal{B} = \{B_n\}_{n<\omega}$ is the set

$$\mathrm{DS}(\mathfrak{A}, \mathcal{B}) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) \mid f \in \mathcal{E}(\mathfrak{A}, \mathcal{B}).\}$$

ω- Degree Spectra

It is easy to find a structure $\mathfrak{A}$ and a sequence $\mathcal{B}$ such that $\mathrm{DS}(\mathfrak{A}, \mathcal{B}) \neq \mathrm{DS}(\mathfrak{A})$.

The notion of the ω-degree spectrum is a generalization of the relative spectrum: $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \mathrm{DS}(\mathfrak{A}, \mathcal{B})$, where $\mathcal{B} = \{B_k\}_{k<\omega}$,

- $B_0 = \emptyset$,
- B_k is the positive diagram of the structure $\mathfrak{A}_k$, $k \leq n$
- $B_k = \emptyset$ for all $k > n$.

ω- Degree Spectra and Jump Spectra

Proposition. $\mathrm{DS}(\mathfrak{A}, \mathcal{B})$ *is upwards closed with respect to total e-degrees.*

Definition. *The* k*th* ω*-jump spectrum of* $\mathfrak{A}$ *with respect to* $\mathcal{B}$ is the set

$$\mathrm{DS}_k(\mathfrak{A}, \mathcal{B}) = \{\mathbf{a}^{(\mathbf{k})} \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A}, \mathcal{B})\}.$$

Proposition. $\mathrm{DS}_k(\mathfrak{A}, \mathcal{B})$ *is upwards closed with respect to total e-degrees.*

ω-Co-Spectra

For every $\mathcal{A} \subseteq \mathcal{D}_\omega$ let $co(\mathcal{A}) = \{\mathbf{b} \mid \mathbf{b} \in \mathcal{D}_\omega \ \& \ (\forall \mathbf{a} \in \mathcal{A})(\mathbf{b} \leq_\omega \mathbf{a})\}$.

Definition. *The ω-co-spectrum of $\mathfrak{A}$ with respect to $\mathcal{B}$* is the set

$$\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = co(\mathrm{DS}(\mathfrak{A}, \mathcal{B})).$$

Definition. *The kth ω-co-spectrum of $\mathfrak{A}$ with respect to $\mathcal{B}$* is the set

$$\mathrm{CS}_k(\mathfrak{A}, \mathcal{B}) = co(\mathrm{DS}_k(\mathfrak{A}, \mathcal{B})).$$

Properties of the co-sets of omega degrees of upwards closed sets

Let $\mathcal{A} \subseteq \mathcal{D}_e$ be an upwards closed set with respect to total e-degrees.

Proposition. $co(\mathcal{A}) = co(\{\mathbf{a} : \mathbf{a} \in \mathcal{A}\ \&\ \mathbf{a}\ \textit{is total}\})$.

Corollary.
$\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = co(\{\mathbf{a} \mid \mathbf{a} \in \mathrm{DS}(\mathfrak{A}, \mathcal{B})\ \&\ \mathbf{a}\ \textit{is a total e-degree}\})$.

Negative results (Stefan Vatev)

Let $\mathcal{A} \subseteq \mathcal{D}_e$ be an upwards closed set with respect to total e-degrees and $k > 0$.

There exists $\mathbf{b} \in \mathcal{D}_e$ *such that*

$$co(\mathcal{A}) \neq co(\{\mathbf{a} : \mathbf{a} \in \mathcal{A} \ \& \ \mathbf{b} \leq \mathbf{a}^{(k)}\}).$$

Let $n > 0$. *There is a structure* $\mathfrak{A}$, *a sequence* $\mathcal{B}$ *and* $\mathbf{c} \in \mathrm{DS}_n(\mathfrak{A}, \mathcal{B})$ *such that*

$$\mathrm{CS}(\mathfrak{A}, \mathcal{B}) \neq co(\{\mathbf{a} \in \mathrm{DS}(\mathfrak{A}, \mathcal{B}) \mid \mathbf{a}^{(n)} = \mathbf{c}\}).$$

Minimal pair theorem

Theorem. *For every structure $\mathfrak{A}$ and every sequence $\mathcal{B} \in \mathcal{S}$ there exist total enumeration degrees* $\mathbf{f}$ *and* $\mathbf{g}$ *in* $\mathrm{DS}(\mathfrak{A}, \mathcal{B})$ *such that for every ω-enumeration degree* $\mathbf{a}$ *and $k \in \mathbb{N}$:*

$$\mathbf{a} \leq_\omega \mathbf{f}^{(k)} \ \& \ \mathbf{a} \leq_\omega \mathbf{g}^{(k)} \Rightarrow \mathbf{a} \in \mathrm{CS}_k(\mathfrak{A}, \mathcal{B}) \ .$$

Countable ideals of ω-enumeration degrees

Corollary. *$\mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$ is the least ideal containing all kth ω-jumps of the elements of $\mathrm{CS}(\mathfrak{A}, \mathcal{B})$.*

- $I = \mathrm{CS}(\mathfrak{A}, \mathcal{B})$ is a countable ideal;
- $\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = I(\mathbf{f}) \cap I(\mathbf{g})$;
- $I^{(k)}$ - the least ideal, containing all kth ω-jumps of the elements of I;
- (Hristo Ganchev)
  $I = I(\mathbf{f}) \cap I(\mathbf{g}) \Longrightarrow I^{(k)} = I(\mathbf{f}^{(k)}) \cap I(\mathbf{g}^{(k)})$ for every k;
- $I(\mathbf{f}^{(k)}) \cap I(\mathbf{g}^{(k)}) = \mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$ for each k
- Thus $I^{(k)} = \mathrm{CS}_k(\mathfrak{A}, \mathcal{B})$.

Countable ideals of ω-enumeration degrees

There is a countable ideal I of ω-enumeration degrees for which there is no structure $\mathfrak{A}$ and sequence $\mathcal{B}$ such that $I = \mathrm{CS}(\mathfrak{A}, \mathcal{B})$.

- $\mathcal{A} = \{\mathbf{0}, \mathbf{0}', \mathbf{0}'', \dots, \mathbf{0}^{(n)}, \dots\}$;
- $I = I(\mathcal{A}) = \{\mathbf{a} \mid \mathbf{a} \in \mathcal{D}_\omega \ \& \ (\exists n)(\mathbf{a} \leq_\omega \mathbf{0}^{(n)})\}$ - a countable ideal generated by $\mathcal{A}$.
- Assume that there is a structure $\mathfrak{A}$ and a sequence $\mathcal{B}$ such that $I = \mathrm{CS}(\mathfrak{A}, \mathcal{B})$
- Then there is a minimal pair $\mathbf{f}$ and $\mathbf{g}$ for $\mathrm{DS}(\mathfrak{A}, \mathcal{B})$, so $I^{(n)} = I(\mathbf{f}^{(n)}) \cap I(\mathbf{g}^{(n)})$ for each n.
- $\mathbf{f} \geq \mathbf{0}^{(n)}$ and $\mathbf{g} \geq \mathbf{0}^{(n)}$ for each n.
- Then by Enderton and Putnam [1970], Sacks [1971]: $\mathbf{f}'' \geq \mathbf{0}^{(\omega)}$ and $\mathbf{g}'' \geq \mathbf{0}^{(\omega)}$.
- Hence $I'' \neq I(\mathbf{f}'') \cap I(\mathbf{g}'')$. A contradiction.

Quasi-Minimal Degree

Theorem. *For every structure* $\mathfrak{A}$ *and every sequence* $\mathcal{B}$, *there exists* $F \subseteq \mathbb{N}$, *such that* $\mathbf{q} = d_\omega(F \uparrow \omega)$ *and:*

1. $\mathbf{q} \notin \mathrm{CS}(\mathfrak{A}, \mathcal{B})$;
2. *If* $\mathbf{a}$ *is a total e-degree and* $\mathbf{a} \geq_\omega \mathbf{q}$ *then* $\mathbf{a} \in \mathrm{DS}(\mathfrak{A}, \mathcal{B})$
3. *If* $\mathbf{a}$ *is a total e-degree and* $\mathbf{a} \leq_\omega \mathbf{q}$ *then* $\mathbf{a} \in \mathrm{CS}(\mathfrak{A}, \mathcal{B})$.

- Questions:
  - Is it true that for every structure $\mathfrak{A}$ and every sequence $\mathcal{B}$ there exists a structure $\mathfrak{B}$ such that $\mathrm{DS}(\mathfrak{B}) = \mathrm{DS}(\mathfrak{A}, \mathcal{B})$?
  - If for a countable ideal $I \subseteq \mathcal{D}_\omega$ there is an exact pair then are there a structure $\mathfrak{A}$ and a sequence $\mathcal{B}$ so that $\mathrm{CS}(\mathfrak{A}, \mathcal{B}) = I$?

Thank you!