Partial Degree Spectra of Abstract Structures

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[1]Supported by BNSF Grant No. D002-258/18.12.08; Sofia University Science Fund

Outline

- Enumerations
- Degree spectra of structures
- Definability on structures
- Partial degree spectra
- Relative stability

Enumerations

Definition. Let $\mathfrak{A} = (A, \omega; \theta_1, \dots, \theta_n; P_1, \dots, P_k)$ be a two sorted countable structure.
An enumeration of $\mathfrak{A}$ is $\langle f, \mathfrak{B}_f \rangle$, where f is a (partial) surjective mapping of ω onto A, $\mathfrak{B}_f = (\omega; \varphi_1, \dots, \varphi_n, \sigma_1, \dots, \sigma_k)$ and

- $\mathrm{dom}(f)$ is closed under $\varphi_1, \dots, \varphi_n$;
- $(\forall \bar{x} \in \mathrm{dom}(f))(\forall \bar{y} \in \omega)[f(\varphi_i(\bar{x}, \bar{y})) = \theta_i(f(\bar{x}), \bar{y})]$;
- $(\forall \bar{x} \in \mathrm{dom}(f))(\forall \bar{y} \in \omega)[\sigma_j(\bar{x}, \bar{y}) \iff P_j(f(\bar{x}), \bar{y})]$.

An enumeration $\langle f, \mathfrak{B}_f \rangle$ *is* total *if* $\mathrm{dom}(f) = \omega$.

Denote by $\langle \varphi \rangle = \{\langle y, x_1, \dots, x_n \rangle \mid \varphi(x_1, \dots x_n) = y\}$.

$$\langle \mathfrak{B}_f \rangle = \langle \varphi_1 \rangle \oplus \cdots \oplus \langle \varphi_n \rangle \oplus \langle \sigma_1 \rangle \oplus \cdots \oplus \langle \sigma_k \rangle$$

Definition.[Richter] *The Degree Spectrum of* $\mathfrak{A}$ is the set

$$DS(\mathfrak{A}) = \{d_e(\langle \mathfrak{B}_f \rangle) \mid \langle f, \mathfrak{B}_f \rangle \text{ is a total enumeration of } \mathfrak{A}\}.$$

If $DS(\mathfrak{A})$ has a least e-degree **a**, then **a** is called *the degree of* $\mathfrak{A}$.

Definition. *The Co-Spectrum of* $\mathfrak{A}$ is the set

$$CS(\mathfrak{A}) = \{d_e(X) \mid X \leq_e \langle \mathfrak{B}_f \rangle,\ \langle f, \mathfrak{B}_f \rangle \text{ is a tot. enum. of } \mathfrak{A}\}.$$

If $CS(\mathfrak{A})$ has a greatest e-degree **a** then **a** is called *the co-degree of* $\mathfrak{A}$.

Degrees and co-degrees

Proposition. *If a structure* $\mathfrak{A}$ *has a degree* $\mathbf{a}$ *then* $\mathbf{a}$ *is also the co-degree of* $\mathfrak{A}$.

There are examples of structures with no co-degrees and structures with co-degree but no degree.

Admissible functions in $\mathfrak{A}$

Let $\mathfrak{A} = (A, \omega; \theta_1, \ldots, \theta_n, P_1, \ldots, P_k)$ *and* $\langle f, \mathfrak{B}_f \rangle$ *is an enumeration of* $\mathfrak{A}$.
A function $\theta : \omega^r \times A^m \to A$ *is admissible in* $\langle f, \mathfrak{B}_f \rangle$ *if there is a function* φ *partial recursive in* $\mathfrak{B}_f$, *(*$\langle \varphi \rangle \leq_e \langle \mathfrak{B}_f \rangle$*) and:*

- $\mathrm{dom}(f)$ *is closed under* φ;
- $(\forall \bar{x} \in \mathrm{dom}(f))(\forall \bar{y} \in \omega)[f(\varphi(\bar{x}, \bar{y})) = \theta(f(\bar{x}), \bar{y})]$;

And $\theta : \omega^r \times A^m \to \omega$ *is admissible in* $\langle f, \mathfrak{B}_f \rangle$ *if there is a function* φ *partial recursive in* $\mathfrak{B}_f$

- $\mathrm{dom}(f)$ *is closed under* φ;
- $(\forall \bar{x} \in \mathrm{dom}(f))(\forall \bar{y} \in \omega)[\varphi(\bar{x}, \bar{y}) = \theta(f(\bar{x}), \bar{y})]$.

Definition.

- A function θ is (search) computable in $\mathfrak{A}$ iff θ is admissible in all total enumerations of $\mathfrak{A}$.
- A function θ is (REDS) partially computable in $\mathfrak{A}$ iff θ is admissible in all (partial) enumerations of $\mathfrak{A}$.

- *Search computability by Moschkovakis (Fraissé, Lacombe, Montague);*
- *Computability by means of Recursively Enumerable Definitional Schemes (REDS) by Shepherdson (Friedman EDS).*

The domains of the computable functions in $\mathfrak{A}$ *we call the* **computably enumerable (c.e.) on $\mathfrak{A}$ sets**.

Let L be the language of $\mathfrak{A}$. We add a unary predicate symbol T_0 to L to represent a predicate which is true everywhere.

Proposition. *A set $X \subseteq \omega^r \times A^m$ is c.e. on $\mathfrak{A}$ iff there is a recursive function $\gamma : \omega^{r+1} \to \omega$, such that for any n, $E^{\gamma(n,\bar{y})}(\bar{X}, \bar{W})$ is an elementary Σ_1 formula in L and there exist parameters $t_1, \ldots, t_l$ of A such that:*

$$(\bar{y}, \bar{x}) \in X \iff (\exists n \in \omega)[\mathfrak{A} \models E^{\gamma(n,\bar{y})}(\bar{X}/\bar{x}, \bar{W}/\bar{t})].$$

These sets are exactly *the relative intrinsically sets* on $\mathfrak{A}$.

The Partially Computably enumerable Sets on $\mathfrak{A}$

The domains of the partially computable functions in $\mathfrak{A}$ we call **partially c.e. on $\mathfrak{A}$ sets**.

Proposition. *A set $X \subseteq \omega^r \times A^m$ is p.c.e. in $\mathfrak{A}$ if there is a recursive function $\gamma : \omega^{r+1} \to \omega$, such that for any n, $P^{\gamma(n,\bar{y})}(\bar{X}, \bar{W})$ is a finite conjunctions of atoms or negated atoms in L and there exist parameters $t_1, \ldots, t_l$ of A such that:*

$$(\bar{y}, \bar{x}) \in X \iff (\exists n \in \omega)[\mathfrak{A} \models P^{\gamma(n,\bar{y})}(\bar{X}/\bar{x}, \bar{W}/\bar{t})].$$

Example of a structure with no co-degree

Consider $\mathfrak{A} = (\mathbb{N}, \omega; \Psi; P)$, *where* $\Psi : \mathbb{N} \to \mathbb{N}$ *and* $\Psi(\langle n, x\rangle) = \langle n, x+1\rangle$ *and the predicate* $P \subseteq \mathbb{N}$*:*

$$P(x) = \begin{cases} 0 & \exists t(x = \langle 0, t\rangle), \\ 0 & \exists n \exists t(x = \langle n+1, t\rangle \ \& \ t \in \emptyset^{(n+1)}), \\ \perp & \text{otherwise.} \end{cases}$$

For every $X \subseteq \omega$*:* X *is c.e. in* $\mathfrak{A}$ *iff* $\exists n(X \leq_e \emptyset^{(n)})$*.*
Consider the sequence $\emptyset <_e \emptyset' <_e \cdots < \emptyset^{(n)} <_e \ldots$. *There is no set* W *so that:*

$$(\forall X \subseteq \omega)(X \leq_e W \iff \exists n(X \leq_e \emptyset^{(n)})).$$

And hence $\mathfrak{A}$ *has no co-degree.*

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Consider the sequence $\emptyset <_e \emptyset' <_e \cdots < \emptyset^{(n)} <_e \ldots$*. There is no set* W *so that:*

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And hence $\mathfrak{A}$ *has no co-degree.*

Example of a structure with a co-degree but no degree

Proposition. *Let* $\mathfrak{A} = (A, \omega; R, =_A)$, *where* A *is countable set and* $R \subseteq A$ *is a linear order. Then* $d_e(\emptyset)$ *is a co-degree of* $\mathfrak{A}$.

For every $X \subseteq \omega$, *if* X *is c.e. in* $\mathfrak{A}$ *then there is a recursive function* γ *and there exist parameters* $t_1, \ldots, t_l$ *of* A *such that:*

$$y \in X \iff (\exists n \in \omega)[\mathfrak{A} \models E^{\gamma(n,y)}(\bar{W}/\bar{t})].$$

And then $X \leq_e \emptyset$.
Hence $d_e(\emptyset)$ *is a co-degree of* $\mathfrak{A}$.

Corollary. *[Richter] If* $\mathfrak{A}$ *is a countable linear ordering with a degree, then this degree is* $\mathbf{0}_e = d_e(\emptyset)$.

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For every $X \subseteq \omega$, if X is c.e. in $\mathfrak{A}$ then there is a recursive function γ and there exist parameters $t_1, \ldots, t_l$ of A such that:

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Example of a structure a co-degree but no degree

An ordinal ξ is constructive if the structure $\xi = (\xi, \omega; \in, =)$ is isomorphic to a computable well ordering.

Proposition. *Let ξ be a countable ordinal. Then the structure $\xi = (\xi, \omega; \in, =)$ has a degree if and only if ξ is a constructive ordinal.*

Corollary. *If ξ is a countable $\xi \geq \omega_1^{CK}$ then ξ has a co-degree and no degree.*

Definition. *The Partial Degree Spectrum of* $\mathfrak{A}$ is the set

$$PDS(\mathfrak{A}) = \{d_e(\langle \mathfrak{B}_f \rangle) \mid \ \langle f, \mathfrak{B}_f \rangle \text{ is a partial enumeration of } \mathfrak{A}\}.$$

The least element of $\mathfrak{A}$ (if it exists) is called *a partial degree of* $\mathfrak{A}$.

Definition. *The Partial Co-Spectrum of* $\mathfrak{A}$ is the set

$$PCS(\mathfrak{A}) = \{d_e(X) \mid X \leq_e \langle \mathfrak{B}_f \rangle, \ \langle f, \mathfrak{B}_f \rangle \text{ is an enumeration of } \mathfrak{A}\}.$$

If $PCS(\mathfrak{A})$ has a greatest e-degree **a** then **a** is called *a partial co-degree of* $\mathfrak{A}$.

Proposition. *If* **a** *is a partial degree of* $\mathfrak{A}$ *then* **a** *is a partial co-degree of* $\mathfrak{A}$.

Partial Degrees and Co-degrees

If **a** *is a degree of* $\mathfrak{A}$ *and* **b** *is a partial degree of* $\mathfrak{A}$ *then* $\mathbf{b} \leq \mathbf{a}$*. There are structures (e.g. that from Example 1) with no partial degree.*

Definition. A set $W \subseteq \mathbb{N}$ is *total* if $(\omega \setminus W) \leq_e W$. An e-degree is *total* if it contains a total set.

Proposition. *Let* $\mathfrak{A}$ *be a total countable structure with a partial co-degree* **a**. *Then* **a** *is a total e-degree.*

Consider a set $W \in \mathbf{a}$*. Then* W *is p.c.e. in* $\mathfrak{A}$*, i.e. there is a recursive function* γ *and parameters* $t_1, \ldots, t_l$ *of* A *such that:*

$$y \in W \iff (\exists n \in \omega)[\mathfrak{A} \models P^{\gamma(n,y)}(\bar{Z}/\bar{t})].$$

The set $\{\hat{L} \mid L(\bar{Z}/\bar{t}) = 0\}$ *is total and e-equivalent to* W*.*

Partial Degrees and Co-degrees

Theorem. *If the structure $\mathfrak{A}$ has a p. co-degree which is a total e-degree then $\mathfrak{A}$ has a p. degree too.*

Let $\mathbf{a}$ *be e p.co-degree of* $\mathfrak{A}$ *and* $W \in \mathbf{a}$ *be a total set.*
We construct a standard enumeration $\langle f, \mathfrak{B}_f \rangle$ *of* $\mathfrak{A}$ *such that* $\langle \mathfrak{B}_f \rangle \leq_e W$.
Fact: Since W *is a total set then* W *is e-equivalent to its characteristic function.*
Hence for each r *there is a p.r in* W *universal function* Φ_r *for the p.r. in* W *functions of* r *arguments.*

If W *is not total, then we can construct an enumeration* $\langle f, \mathfrak{B}_f \rangle$ *of* $\mathfrak{A}$, $W \equiv_e \langle \mathfrak{B}_f \rangle$, *but the functions in* $\mathfrak{B}_f$ *are not single valued outside the domain of* f.

Partial Degrees and Co-degrees

Corollary. *Every total structure $\mathfrak{A}$ with a partial co-degree has a partial degree.*

Proposition. *Let $\mathfrak{A} = (A, \omega; R_1, \ldots, R_k)$, where all the predicates $R_j \subseteq A^{m_j}$. Then $\mathfrak{A}$ has a partial co-degree $\mathbf{0_e}$.*

Corollary. *Every countable linear ordering has a partial degree $\mathbf{0_e}$. And hence if ξ is not constructive ordinal, then the structure $(\xi, \omega; \in, =)$ has a partial degree $\mathbf{0_e}$ and has no degree.*

Relative Stability

Let $\mathfrak{A} = (\mathbb{N}, \omega; \theta_1, \ldots, \theta_n; P_1, \ldots, P_k)$.

Definition. The structure $\mathfrak{A}$ is *relatively stable* if for every total enumeration $\langle f, \mathfrak{B}_f \rangle$ of $\mathfrak{A}$ the mapping f is partially recursive in $\mathfrak{B}_f$.

Definition. The structure $\mathfrak{A}$ is *algorithmic complete* if all the p.r. functions on $\mathbb{N}$ are computable in $\mathfrak{A}$ considered as functions on $\mathbb{N}$ and on ω.

Proposition. *The following conditions are equivalent:*

- $\mathfrak{A}$ *is relatively stable;*
- *the converting function* $\alpha : \mathbb{N} \to \omega$, $\lambda n.\alpha(n) = n$ *is computable;*
- $\mathfrak{A}$ *is algorithmic complete.*

- *Ash, Knight, Manasse, Slaman, Chisholm*

Theorem. $\mathfrak{A}$ *is algorithmic complete if there exists a recursive function* $\gamma(n,x)$ *and parameters* $t_1, \ldots, t_l \in \mathbb{N}$ *such that*

$$(\forall x \in \mathbb{N})(\forall y \in \omega)(x = y \iff (\exists n \in \omega)(\mathfrak{A} \models E^{\gamma(n,y)}(\bar{Z}/\bar{t}, X/x))).$$

Proposition. *The structure* $\mathfrak{A} = (\mathbb{N}, \omega; S, =_{\mathbb{N}})$, *where* $S : \mathbb{N} \to \mathbb{N}$ *is the successor function on* $\mathbb{N}$ *is algorithmic complete..*

If $E^y = T(F^y(Z), X)$ *then* $\mathfrak{A} \models E^y(Z/0, X/x) \iff x = y$.

Definition. The structure $\mathfrak{A}$ is *super relatively stable* if for every enumeration $\langle f, \mathfrak{B}_f \rangle$ of $\mathfrak{A}$ the mapping f has a p.r. in $\mathfrak{B}_f$ function $g \supseteq f$, i.e. for every n if $f(n)$ is defined then $g(n)$ is defined and $f(n) = g(n)$.

Let $\langle f, \mathfrak{B}_f \rangle$ be an enumeration of $\mathfrak{A}$. Then for every function φ with the property $\varphi(x) = \alpha(f(x))$ for $x \in \mathrm{dom}(\alpha)$, $\varphi \supseteq f$.

Proposition. *The following conditions are equivalent:*

- $\mathfrak{A}$ *is super relatively stable;*
- *The converting function* $\alpha : \mathbb{N} \to \omega$, $\lambda n.\alpha(n) = n$ *is partially computable in* $\mathfrak{A}$*;*
- *Every c.e subset of* ω^{r+m}*, considered as a subset of* $\omega^r \times \mathbb{N}^m$*, is c.e. in* $\mathfrak{A}$*.*
- *There exists a recursive function* $\gamma(n, x)$ *and parameters* $t_1, \ldots, t_l \in \mathbb{N}$ *such that*

$$(\forall x \in \mathbb{N})(\forall y \in \omega)(x = y \Leftrightarrow (\exists n \in \omega)(\mathfrak{A} \models P^{\gamma(n,y)}(\bar{Z}/\bar{t}, X/x))).$$

Partially algorithmic completeness

Definition. The structure $\mathfrak{A}$ is *partially algorithmic complete* if all the p.r. functions on $\mathbb{N}$ are partially computable in $\mathfrak{A}$ considered as functions on $\mathbb{N}$ and on ω.

Definition. A structure $\mathfrak{A}$ is finitely generated if there are finitely many elements $t_1, \ldots, t_l$ and variables $W_1, \ldots, W_l$, such that

$$A = \{\lambda(\bar{W}/\bar{t}) \mid \lambda \text{ is a term on } \bar{W}\}.$$

Proposition. *If a structure $\mathfrak{A}$ is partially algorithmic complete then it is finitely generated and hence the computable functions in $\mathfrak{A}$ and the partially computable functions coincide.*

Theorem. *A structure $\mathfrak{A}$ is partially algorithmic complete if and only if $\mathfrak{A}$ is super relatively stable and finitely generated.*

Example of algorithmic complete structures

Consider the structure $\mathfrak{A} = (\mathbb{N}, \omega; P; Z)$*,*
where $P : \mathbb{N} \to \mathbb{N}$, $P(x) = x - 1$ *for* $x > 0$ *and* $P(0) = 0$*,*
and $Z(x) = 0$ *if* $x = 0$*, and* $Z(x) = 1$ *if* $x > 0$*.*
It is clear that $\mathfrak{A}$ *is not finitely generated. Thus it is not partially algorithmic complete.*
Let $L = (F, T)$ *be the language of* $\mathfrak{A}$ *and* $x \in \mathbb{N}, y \in \omega$*.*

$$x = y \iff \mathfrak{A} \models \neg T(X/x) \ \& \ \cdots \neg T(F^{y-1}(X/x) \ \& \ T(F^{y}(X/x)).$$

Since it is super relative stable and hence relatively stable. Then it is algorithmic complete.

An example of partially algorithmic complete structure is
$\mathfrak{A} = (\mathbb{N}, \omega; S, P; Z)$*, where*
$S(x) = x + 1$,
$P(x) = x - 1$ *for* $x > 0$ *and not defined if* $x = 0$*,*
$Z(x) = 0$ *if* $x = 0$ *and not defined if* $x > 0$*.*

Thank you!