

ω -Degree Spectra

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Outline

- ▶ Degree spectra and jump spectra
- ▶ ω -enumeration degrees
- ▶ ω -degree spectra
- ▶ ω -co-spectra
- ▶ A minimal pair theorem
- ▶ Quasi-minimal degrees

Enumeration of a Structure

Let $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_k, =, \neq)$ be a countable abstract structure.

- ▶ An enumeration f of \mathfrak{A} is a total mapping from \mathbb{N} onto \mathbb{N} .
- ▶ for any $A \subseteq \mathbb{N}^a$ let
$$f^{-1}(A) = \{\langle x_1 \dots x_a \rangle : (f(x_1), \dots, f(x_a)) \in A\}.$$
- ▶ $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \dots \oplus f^{-1}(R_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq).$

Definition (L. Richter, 1981)

The Turing degree spectrum of \mathfrak{A}

$$DS_T(\mathfrak{A}) = \{d_T(f^{-1}(\mathfrak{A})) \mid f \text{ is an injective enumeration of } \mathfrak{A}\}$$

- ▶ J. Knight, Ash, Jockush, Downey, Slaman.

Degree Spectra and Co-spectra

Definition (Soskov, 2004)

- ▶ The degree spectrum of \mathfrak{A}

$$DS(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) \mid f \text{ is an enumeration of } \mathfrak{A}\}.$$

- ▶ The co-spectrum of \mathfrak{A}

$$CS(\mathfrak{A}) = \{\mathbf{b} : (\forall \mathbf{a} \in DS(\mathfrak{A}))(\mathbf{b} \leq \mathbf{a})\}.$$

Degree Spectra and Co-spectra

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Degree Spectra

ω -Enumeration
Degrees

ω -Degree Spectra

Properties of the
 ω -Degree Spectra

Minimal Pair Theorem
Quasi-Minimal Degree

Definition

Let $\mathcal{A} \subseteq \mathcal{D}_e$. \mathcal{A} is *upwards closed with respect to total enumeration degrees*, if

$$\mathbf{a} \in \mathcal{A}, \mathbf{b} \text{ is total and } \mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{b} \in \mathcal{A}.$$

The degree spectra are upwards closed with respect to total enumeration degrees.

Properties of upwards closed sets

Let $\mathcal{A} \subseteq \mathcal{D}_e$ be upwards closed with respect to total enumeration degrees. Denote by

$$\text{co}(\mathcal{A}) = \{b : b \in \mathcal{D}_e \text{ \& } (\forall a \in \mathcal{A})(b \leq_e a)\}.$$

- ▶ $\mathcal{A}_t = \{\mathbf{a} : \mathbf{a} \in \mathcal{A} \text{ \& } \mathbf{a} \text{ is total}\} \implies \text{co}(\mathcal{A}) = \text{co}(\mathcal{A}_t).$
- ▶ Let $\mathbf{b} \in \mathcal{D}_e$ and $n > 0$.

$$\mathcal{A}_{\mathbf{b},n} = \{\mathbf{a} : \mathbf{a} \in \mathcal{A} \text{ \& } \mathbf{b} \leq \mathbf{a}^{(n)}\} \implies \text{co}(\mathcal{A}) = \text{co}(\mathcal{A}_{\mathbf{b},n}).$$

Properties of degree spectra and co-spectra

- ▶ Let $\mathbf{c} \in DS_n(\mathfrak{A})$ and $n > 0$. Then

$$CS(\mathfrak{A}) = co(\{\mathbf{a} \mid \mathbf{a} \in DS(\mathfrak{A}) \ \& \ \mathbf{a}^{(n)} = \mathbf{c}\}).$$

- ▶ A minimal pair theorem:
There exist \mathbf{f} and \mathbf{g} in $DS(\mathfrak{A})$:

$$(\forall \mathbf{a} \in \mathcal{D}_e)(\forall k)(\mathbf{a} \leq_e \mathbf{f}^{(k)} \ \& \ \mathbf{a} \leq_e \mathbf{g}^{(k)} \Rightarrow \mathbf{a} \in CS_k(\mathfrak{A})).$$

- ▶ Quasi-minimal degree:
There exists \mathbf{q}_0 quasi-minimal for $DS(\mathfrak{A})$
 - ▶ $\mathbf{q}_0 \notin CS(\mathfrak{A})$;
 - ▶ for every total e -degree \mathbf{a} : $\mathbf{a} \geq_e \mathbf{q}_0 \Rightarrow \mathbf{a} \in DS(\mathfrak{A})$ and $\mathbf{a} \leq_e \mathbf{q}_0 \Rightarrow \mathbf{a} \in CS(\mathfrak{A})$.
- ▶ Every countable ideal can be represented as a co-spectrum of some structure \mathfrak{A} .

ω -Enumeration Degrees

- ▶ Uniform reducibility on sequences of sets
- ▶ S the set of all sequences of sets of natural numbers
- ▶ For $\mathcal{B} = \{B_n\}_{n < \omega} \in S$ call *the jump class of \mathcal{B}* the set

$$J_{\mathcal{B}} = \{d_T(X) \mid (\forall n)(B_n \text{ is c.e. in } X^{(n)} \text{ uniformly in } n)\} .$$

- ▶ $\mathcal{A} \leq_{\omega} \mathcal{B}$ (\mathcal{A} is ω -enumeration reducible to \mathcal{B}) if $J_{\mathcal{B}} \subseteq J_{\mathcal{A}}$
- ▶ $\mathcal{A} \equiv_{\omega} \mathcal{B}$ if $J_{\mathcal{A}} = J_{\mathcal{B}}$.

- ▶ \equiv_ω is an equivalence relation on \mathcal{S} .
- ▶ $d_\omega(\mathcal{B}) = \{\mathcal{A} \mid \mathcal{A} \equiv_\omega \mathcal{B}\}$
- ▶ $\mathcal{D}_\omega = \{d_\omega(\mathcal{B}) \mid \mathcal{B} \in \mathcal{S}\}$.
- ▶ If $A \subseteq \mathbb{N}$ denote by $A \uparrow \omega = \{A, \emptyset, \emptyset, \dots\}$.
- ▶ For every $A, B \subseteq \mathbb{N}$:

$$A \leq_e B \iff A \uparrow \omega \leq_\omega B \uparrow \omega.$$

- ▶ The mapping $\kappa(d_e(A)) = d_\omega(A \uparrow \omega)$ gives an isomorphic embedding of \mathcal{D}_e to \mathcal{D}_ω .

ω -Enumeration Degrees

Let $\mathcal{B} = \{B_n\}_{n < \omega} \in \mathcal{S}$.

A jump sequence $\mathcal{P}(\mathcal{B}) = \{\mathcal{P}_n(\mathcal{B})\}_{n < \omega}$:

1 $\mathcal{P}_0(\mathcal{B}) = B_0$

2 $\mathcal{P}_{n+1}(\mathcal{B}) = (\mathcal{P}_n(\mathcal{B}))' \oplus B_{n+1}$

Theorem (Soskov, Kovachev)

$A \leq_{\omega} B$, if $A_n \leq_e \mathcal{P}_n(\mathcal{B})$ uniformly in n .

ω -Enumeration Jump

- ▶ For every $\mathcal{A} \in \mathcal{S}$ the ω -enumeration jump of \mathcal{A} is $\mathcal{A}' = \{\mathcal{P}_{n+1}(\mathcal{A})\}_{n < \omega}$
- ▶ $d_\omega(\mathcal{A})' = d_\omega(\mathcal{A}')$
- ▶ $\mathcal{A}^{(k+1)} = (\mathcal{A}^{(k)})'$
- ▶ $d_\omega(\mathcal{A})^{(k+1)} = d_\omega(\mathcal{A}^{(k+1)})$
- ▶ $\mathcal{A}^{(k)} = \{\mathcal{P}_{n+k}(\mathcal{A})\}_{n < \omega}$ for each k .

Let $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ be given structures.

Definition

The *relative spectrum* $RS(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$ of the structure \mathfrak{A} with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ is the set

$$\{d_e(f^{-1}(\mathfrak{A})) \mid f \text{ is an enumeration of } \mathfrak{A} \text{ \& } (\forall k \leq n)(f^{-1}(\mathfrak{A}_k) \leq_e f^{-1}(\mathfrak{A})^{(k)})\}$$

It turns out that all properties of the degree spectra remain true for the relative spectra.

Let $\mathcal{B} = \{B_n\}_{n < \omega}$ be a fixed sequence of sets.

The enumeration f of the structure \mathfrak{A} is *acceptable with respect to \mathcal{B}* , if for every n ,

$$f^{-1}(B_n) \leq_e f^{-1}(\mathfrak{A})^{(n)} \text{ uniformly in } n.$$

Denote by $\mathcal{E}(\mathfrak{A}, \mathcal{B})$ - the class of all acceptable enumerations.

Definition

The ω -degree spectrum of \mathfrak{A} with respect to $\mathcal{B} = \{B_n\}_{n < \omega}$ is the set

$$\text{DS}(\mathfrak{A}, \mathcal{B}) = \{d_e(f^{-1}(\mathfrak{A})) \mid f \in \mathcal{E}(\mathfrak{A}, \mathcal{B})\}$$

- ▶ It is easy to find a structure \mathfrak{A} and a sequence \mathcal{B} such that $DS(\mathfrak{A}, \mathcal{B}) \neq DS(\mathfrak{A})$.
- ▶ The notion of the ω -degree spectrum is a generalization of the relative spectrum:
 $RS(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = DS(\mathfrak{A}, \mathcal{B})$, where $\mathcal{B} = \{B_k\}_{k < \omega}$,
 - ▶ $B_0 = \emptyset$,
 - ▶ B_k is the positive diagram of the structure \mathfrak{A}_k , $k \leq n$
 - ▶ $B_k = \emptyset$ for all $k > n$.

ω -Degree Spectra and Jump Spectra

Proposition

$DS(\mathfrak{A}, \mathcal{B})$ is upwards closed with respect to total e -degrees.

Definition

The k th ω -jump spectrum of \mathfrak{A} with respect to \mathcal{B} is the set

$$DS_k(\mathfrak{A}, \mathcal{B}) = \{\mathbf{a}^{(k)} \mid \mathbf{a} \in DS(\mathfrak{A}, \mathcal{B})\}.$$

Proposition

$DS_k(\mathfrak{A}, \mathcal{B})$ is upwards closed with respect to total e -degrees.

For every $\mathcal{A} \subseteq \mathcal{D}_\omega$ let

$$\text{co}(\mathcal{A}) = \{\mathbf{b} \mid \mathbf{b} \in \mathcal{D}_\omega \ \& \ (\forall \mathbf{a} \in \mathcal{A})(\mathbf{b} \leq_\omega \mathbf{a})\}.$$

Definition

The ω -co-spectrum of \mathfrak{A} with respect to \mathcal{B} is the set

$$\text{CS}(\mathfrak{A}, \mathcal{B}) = \text{co}(\text{DS}(\mathfrak{A}, \mathcal{B})).$$

Definition

The k th ω -co-spectrum of \mathfrak{A} with respect to \mathcal{B} is the set

$$\text{CS}_k(\mathfrak{A}, \mathcal{B}) = \text{co}(\text{DS}_k(\mathfrak{A}, \mathcal{B})).$$

Normal Form Theorem

Let \mathcal{L} be the language of the structure \mathfrak{A} . For each n let P_n be a new unary predicate representing the set B_n .

- ▶ An elementary Σ_0^+ formula is an existential formula of the form $\exists Y_1 \dots \exists Y_m \Phi(W_1, \dots, W_r, Y_1, \dots, Y_m)$, where Φ is a finite conjunction of atomic formulae in $\mathcal{L} \cup \{P_0\}$;
- ▶ A Σ_n^+ formula is a c.e. disjunction of elementary Σ_n^+ formulae;
- ▶ An elementary Σ_{n+1}^+ formula is a formula of the form $\exists Y_1 \dots \exists Y_m \Phi(W_1, \dots, W_r, Y_1, \dots, Y_m)$, where Φ is a finite conjunction of atoms of the form $P_{n+1}(Y_j)$ or $P_{n+1}(W_i)$ and Σ_n^+ formulae or negations of Σ_n^+ formulae in $\mathcal{L} \cup \{P_0\} \cup \dots \cup \{P_n\}$.

Definition

The sequence $\mathcal{A} = \{A_n\}_{n < \omega}$ is *formally k -definable* on \mathfrak{A} with respect to \mathcal{B} if there exists a computable sequence $\{\Phi^{\gamma(n,x)}(W_1, \dots, W_r)\}_{n,x < \omega}$ of Σ_{n+k}^+ formulae and elements t_1, \dots, t_r of \mathbb{N} such that for every $x \in \mathbb{N}$, the following equivalence holds:

$$x \in A_n \iff (\mathfrak{A}, \mathcal{B}) \models \Phi^{\gamma(n,x)}(W_1/t_1, \dots, W_r/t_r).$$

Theorem

The sequence \mathcal{A} is formally k -definable on \mathfrak{A} with respect to \mathcal{B} iff $d_\omega(\mathcal{A}) \in \text{CS}_k(\mathfrak{A}, \mathcal{B})$.

Properties of upwards closed sets

Let $\mathcal{A} \subseteq \mathcal{D}_e$ be an upwards closed set with respect to total e-degrees.

Proposition

$co(\mathcal{A}) = co(\{\mathbf{a} : \mathbf{a} \in \mathcal{A} \ \& \ \mathbf{a} \text{ is total}\})$.

Corrolary

$CS(\mathcal{A}, \mathcal{B}) = co(\{\mathbf{a} \mid \mathbf{a} \in DS(\mathcal{A}, \mathcal{B}) \ \& \ \mathbf{a} \text{ is a total e-degree}\})$.

Negative results (Vatev)

Let $\mathcal{A} \subseteq \mathcal{D}_e$ be an upwards closed set with respect to total e-degrees and $k > 0$.

- ▶ There exists $\mathbf{b} \in \mathcal{D}_e$ such that

$$\text{co}(\mathcal{A}) \neq \text{co}(\{\mathbf{a} : \mathbf{a} \in \mathcal{A} \ \& \ \mathbf{b} \leq \mathbf{a}^{(k)}\}).$$

- ▶ Let $n > 0$. There is a structure \mathfrak{A} , a sequence \mathcal{B} and $\mathbf{c} \in \text{DS}_n(\mathfrak{A}, \mathcal{B})$ such that

$$\text{CS}(\mathfrak{A}, \mathcal{B}) \neq \text{co}(\{\mathbf{a} \in \text{DS}(\mathfrak{A}, \mathcal{B}) \mid \mathbf{a}^{(n)} = \mathbf{c}\}).$$

Theorem

For every structure \mathfrak{A} and every sequence $\mathcal{B} \in \mathcal{S}$ there exist total enumeration degrees \mathbf{f} and \mathbf{g} in $\text{DS}(\mathfrak{A}, \mathcal{B})$ such that for every ω -enumeration degree \mathbf{a} and $k \in \mathbb{N}$:

$$\mathbf{a} \leq_{\omega} \mathbf{f}^{(k)} \ \& \ \mathbf{a} \leq_{\omega} \mathbf{g}^{(k)} \Rightarrow \mathbf{a} \in \text{CS}_k(\mathfrak{A}, \mathcal{B}) .$$

Corrolary

$CS_k(\mathfrak{A}, \mathfrak{B})$ is the least ideal containing all k th ω -jumps of the elements of $CS(\mathfrak{A}, \mathfrak{B})$.

- ▶ $I = CS(\mathfrak{A}, \mathfrak{B})$ is a countable ideal;
- ▶ $CS(\mathfrak{A}, \mathfrak{B}) = I(\mathbf{f}) \cap I(\mathbf{g})$;
- ▶ $I^{(k)}$ - the least ideal, containing all k th ω -jumps of the elements of I ;
- ▶ (Ganchev)
 $I = I(\mathbf{f}) \cap I(\mathbf{g}) \implies I^{(k)} = I(\mathbf{f}^{(k)}) \cap I(\mathbf{g}^{(k)})$ for every k ;
- ▶ $I(\mathbf{f}^{(k)}) \cap I(\mathbf{g}^{(k)}) = CS_k(\mathfrak{A}, \mathfrak{B})$ for each k
- ▶ Thus $I^{(k)} = CS_k(\mathfrak{A}, \mathfrak{B})$.

Countable ideals of ω -enumeration degrees

There is a countable ideal I of ω -enumeration degrees for which there is no structure \mathfrak{A} and sequence \mathcal{B} such that $I = \text{CS}(\mathfrak{A}, \mathcal{B})$.

- ▶ $\mathcal{A} = \{\mathbf{0}, \mathbf{0}', \mathbf{0}'', \dots, \mathbf{0}^{(n)}, \dots\}$;
- ▶ $I = I(\mathcal{A}) = \{\mathbf{a} \mid \mathbf{a} \in \mathcal{D}_\omega \ \& \ (\exists n)(\mathbf{a} \leq_\omega \mathbf{0}^{(n)})\}$ - a countable ideal generated by \mathcal{A} .
- ▶ Assume that there is a structure \mathfrak{A} and a sequence \mathcal{B} such that $I = \text{CS}(\mathfrak{A}, \mathcal{B})$
- ▶ Then there is a minimal pair \mathbf{f} and \mathbf{g} for $\text{DS}(\mathfrak{A}, \mathcal{B})$, so $I^{(n)} = I(\mathbf{f}^{(n)}) \cap I(\mathbf{g}^{(n)})$ for each n .
- ▶ $\mathbf{f} \geq \mathbf{0}^{(n)}$ and $\mathbf{g} \geq \mathbf{0}^{(n)}$ for each n .
- ▶ Then by Enderton and Putnam [1970], Sacks [1971]: $\mathbf{f}'' \geq \mathbf{0}^{(\omega)}$ and $\mathbf{g}'' \geq \mathbf{0}^{(\omega)}$.
- ▶ Hence $I'' \neq I(\mathbf{f}'') \cap I(\mathbf{g}'')$. A contradiction.







Theorem

For every structure \mathfrak{A} and every sequence \mathcal{B} , there exists $F \subseteq \mathbb{N}$, such that $\mathbf{q} = d_\omega(F \uparrow \omega)$ and:

- 1. $\mathbf{q} \notin \text{CS}(\mathfrak{A}, \mathcal{B})$;*
- 2. If \mathbf{a} is a total e-degree and $\mathbf{a} \geq_\omega \mathbf{q}$ then $\mathbf{a} \in \text{DS}(\mathfrak{A}, \mathcal{B})$*
- 3. If \mathbf{a} is a total e-degree and $\mathbf{a} \leq_\omega \mathbf{q}$ then $\mathbf{a} \in \text{CS}(\mathfrak{A}, \mathcal{B})$.*

► Questions:

- Is it true that for every structure \mathfrak{A} and every sequence \mathcal{B} there exists a structure \mathfrak{B} such that $DS(\mathfrak{B}) = DS(\mathfrak{A}, \mathcal{B})$?
- If for a countable ideal $I \subseteq \mathcal{D}_\omega$ there is an exact pair then are there a structure \mathfrak{A} and a sequence \mathcal{B} so that $CS(\mathfrak{A}, \mathcal{B}) = I$?

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