Relativized Degree Spectra

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Outline

- Enumeration of a structure
- Degree spectra and co-spectra
- Relatively α-intrinsic sets
- Relative spectra of structures
- Normal Form Theorem
- The connection with the Joint Spectra
- The Minimal pair theorem
- Quasi-minimal degrees

Enumeration of a structure

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k, =, \neq)$ be a countable abstract structure.

- An enumeration f of $\mathfrak{A}$ is a total mapping from $\mathbb{N}$ onto $\mathbb{N}$.
- for any $A \subseteq \mathbb{N}^a$ let
  $f^{-1}(A) = \{\langle x_1 \ldots x_a \rangle : (f(x_1), \ldots, f(x_a)) \in A\}$.
- $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k) \oplus f^{-1}(=) \oplus f^{-1}(\neq)$.

Degree spectra of structures

Definition

- **The Degree spectrum of** $\mathfrak{A}$ is the set

$$\mathrm{DS}(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) \mid f \text{ is an enumeration of } \mathfrak{A}\}.$$

- L. Richter [1981] — degrees of structures.
- J. Knight [1986] — jump degrees of structures.
- Ash, Jockush, Downey, Slaman, Soskov.

Definition

- **The Co-spectrum of** $\mathfrak{A}$ is the set

$$\mathrm{CS}(\mathfrak{A}) = \{\mathbf{b} : (\forall \mathbf{a} \in \mathrm{DS}(\mathfrak{A}))(\mathbf{b} \leq \mathbf{a})\}.$$

Examples

1981 (Richter) Let $\mathfrak{A} = (\mathbb{N}; <, =, \neq)$ be a linear ordering.

- $\mathrm{DS}(\mathfrak{A})$ contains a minimal pair of degrees, $\mathrm{CS}(\mathfrak{A}) = \{\mathbf{0}_e\}$.
- If $\mathrm{DS}(\mathfrak{A})$ has a least element $\mathbf{a}$, then $\mathbf{a} = \mathbf{0}_e$.

1998 (Slaman) $\mathrm{DS}(\mathfrak{A}) = \{\mathbf{a} : \mathbf{a}$ is total and $\mathbf{0}_e < \mathbf{a}\}$, $\mathrm{CS}(\mathfrak{A}) = \{\mathbf{0}_e\}$.

- $\mathrm{DS}(\mathfrak{A})$ has not a least element.

1998 (Coles, Downey, Slaman) Every principle ideal of enumeration degrees is a $\mathrm{CS}(\mathfrak{A})$ for some torsion free abelian group $\mathfrak{A}$.

2002 (Soskov) Every countable ideal is a $\mathrm{CS}(\mathfrak{A})$ for some $\mathfrak{A}$.

Definition

Let $\mathcal{A} \subseteq \mathcal{D}_e$. Then $\mathcal{A}$ is *upwards closed* if

$$\mathbf{a} \in \mathcal{A}, \mathbf{b} \text{ is total and } \mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{b} \in \mathcal{A}.$$

The Degree spectra are upwards closed.

- General properties of upwards closed sets of degrees.
- Specific properties:
  - the Minimal pair type theorem;
  - the existence of Quasi-minimal degree.

1989 (Ash, Knight, Manasse, Slaman, Chisholm).

- The set A is *relatively α-intrinsic on $\mathfrak{A}$* if for every enumeration f of $\mathfrak{A}$ the set $f^{-1}(A) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(\alpha)}$, $\alpha < \omega_1^{CK}$.

2002 (Soskov, Baleva)

- Let $\{B_\alpha\}_{\alpha \leq \zeta}$ be a sequence of subset of $\mathbb{N}$ and $\zeta < \omega_1^{CK}$.
- Add each set B_α to the structure $\mathfrak{A}$ as a new predicate which is relatively α-intrinsic on $\mathfrak{A}$.
- Restrict the class of all enumerations of $\mathfrak{A}$ to the class of those enumerations f of $\mathfrak{A}$ for which $f^{-1}(B_\alpha) \leq_{\mathrm{e}} f^{-1}(\mathfrak{A})^{(\alpha)}$.

Relative Spectra of Structures

Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ be arbitrary abstract structures on $\mathbb{N}$, $k \le n$. An enumeration f of $\mathfrak{A}$ is **k-acceptable** with respect to the structures $\mathfrak{A}_1, \ldots, \mathfrak{A}_k$, if

$$f^{-1}(\mathfrak{A}_1) \le_{\mathrm{e}} (f^{-1}(\mathfrak{A}))', \ldots, f^{-1}(\mathfrak{A}_k) \le_{\mathrm{e}} (f^{-1}(\mathfrak{A}))^{(k)}.$$

Denote by $\mathcal{E}_k$ the class of all k-acceptable enumerations.

Definition

The Relative spectrum of the structure $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is the set

$$\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) \mid f \in \mathcal{E}_n\}$$

Lemma

If F is a total set, $f \in \mathcal{E}_n$ and $f^{-1}(\mathfrak{A}) \leq_{\mathrm{e}} F$, then there exists an enumeration $g \in \mathcal{E}_n$, such that

1. $g^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F \oplus f^{-1}(\mathfrak{A}) \equiv_{\mathrm{e}} F$;
2. $g^{-1}(B) \leq_{\mathrm{e}} F \oplus f^{-1}(B)$, *for every* $B \subseteq \mathbb{N}$.

Corollary

The Relative spectrum $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$ *is upwards closed.*

Let $k \leq n$. **The kth Jump Relative spectrum** of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is the set

$$\mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \{\mathbf{a}^{(\mathbf{k})} \mid \mathbf{a} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)\}.$$

Proposition

The kth Jump Relative spectrum $\mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots \mathfrak{A}_n)$ is upwards closed.

Relative Co-spectra of Structures

Definition

The Relative co-spectrum of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$, is the co-set of $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$, i.e.

$$\mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \{\mathbf{b} \mid (\forall \mathbf{a} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n))(\mathbf{b} \le \mathbf{a})\}.$$

Let $k \le n$. **The Relative *k*th co-spectrum** of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$, is the co-set of $\mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$, i.e.

$$\mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n) = \{\mathbf{b} \mid (\forall \mathbf{a} \in \mathrm{RS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n))(\mathbf{b} \le \mathbf{a})\}.$$

The jump set

The jump set $\mathcal{P}^f_k$ of $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \dots, \mathfrak{A}_n$:

1. $\mathcal{P}^f_0 = f^{-1}(\mathfrak{A})$.
2. $\mathcal{P}^f_{k+1} = (\mathcal{P}^f_k)' \oplus f^{-1}(\mathfrak{A}_k)$.

Theorem

For every $A \subseteq \mathbb{N}$ and $k \leq n$, the following are equivalent:

1. $d_e(A) \in \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$.
2. $A \leq_e \mathcal{P}^f_k$*, for every k-acceptable enumeration f of $\mathfrak{A}$ with respect to* $\mathfrak{A}_1, \dots, \mathfrak{A}_k$.

The Normal Form Theorem

The set A is *formally k-definable* on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ if there exists a recursive sequence $\{\Phi^{\gamma(x)}(W_1, \ldots, W_r)\}$ of Σ_k^+ formulae and elements $t_1, \ldots, t_r$ of $\mathbb{N}$ such that:

$x \in A \iff (\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) \models \Phi^{\gamma(x)}(W_1/t_1, \ldots, W_r/t_r)$.

- Σ_0^+ : $(\exists \overline{Y})(\beta_1 \ \& \ \ldots \ \& \ \beta_k)$;
- Σ_{k+1}^+: r.e. disjunction of $(\exists \overline{Y})\Phi(\overline{X}, \overline{Y})$,
  $\Phi = (\phi_1 \ \& \ \ldots \ \& \ \phi_l \ \& \ \beta)$.

Theorem

A set $A \in \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ if and only if A is formally k-definable on $\mathfrak{A}$ with respect to $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$.

The connection with the Joint Spectra

Definition

The Joint spectrum of $\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$ is the set

$$\mathrm{DS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \{\mathbf{a} : \mathbf{a} \in \mathrm{DS}(\mathfrak{A}), \mathbf{a}' \in \mathrm{DS}(\mathfrak{A}_1), \ldots, \mathbf{a}^{(\mathbf{n})} \in \mathrm{DS}(\mathfrak{A}_n)\}.$$

1. $\mathrm{CS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.
2. There are structures $\mathfrak{A}$ and $\mathfrak{A}_1$, for which $\mathrm{CS}_1(\mathfrak{A}, \mathfrak{A}_1) \neq \mathrm{CRS}_1(\mathfrak{A}, \mathfrak{A}_1)$.
3. The difference:
   - $A \leq_e \mathcal{P}(f^{-1}(\mathfrak{A}), f_1^{-1}(\mathfrak{A}_1), \ldots, f_n^{-1}(\mathfrak{A}_n))$ for every enumerations f of $\mathfrak{A}$, f_1 of $\mathfrak{A}_1, \ldots,$ f_n of $\mathfrak{A}_n$.
   - in the normal form $(\mathfrak{A}, \mathfrak{A}_1 \ldots, \mathfrak{A}_n)$ — as a many-sorted structure with separated sorts.

Minimal Pair Theorem

Theorem

For any structures $\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$, there exist enumeration degrees $\mathbf{f}$ *and* $\mathbf{g}$ *in* $\mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$*, such that for any enumeration degree* $\mathbf{a}$ *and each* $k \leq n$*:*

$$\mathbf{a} \leq \mathbf{f}^{(\mathbf{k})} \;\&\; \mathbf{a} \leq \mathbf{g}^{(\mathbf{k})} \Rightarrow \mathbf{a} \in \mathrm{CRS}_k(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n).$$

Quasi-Minimal Degree

Soskov An enumeration degree $\mathbf{q_0}$ is *quasi-minimal with respect to* $\mathrm{DS}(\mathfrak{A})$ if

- $\mathbf{q_0} \notin \mathrm{CS}(\mathfrak{A})$;
- for any total enumeration degree $\mathbf{a}$: $\mathbf{a} \geq \mathbf{q_0} \Rightarrow \mathbf{a} \in \mathrm{DS}(\mathfrak{A})$ and $\mathbf{a} \leq \mathbf{q_0} \Rightarrow \mathbf{a} \in \mathrm{CS}(\mathfrak{A})$.

Theorem

For any structures $\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n$ there exists an enumeration degree $\mathbf{q}$ such that:

1. $\mathbf{q} \notin \mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_\mathbf{1}, \dots, \mathfrak{A}_\mathbf{n})$;
2. *If $\mathbf{a}$ is a total degree and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \dots, \mathfrak{A}_n)$;*
3. *If $\mathbf{a}$ is a total degree and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in \mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1 \dots \mathfrak{A}_n)$.*

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Theorem

For any structures $\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$ there exists an enumeration degree $\mathbf{q}$ such that:

1. $\mathbf{q} \notin \mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_\mathbf{1}, \ldots, \mathfrak{A}_\mathbf{n})$;
2. *If $\mathbf{a}$ is a total degree and $\mathbf{a} \geq \mathbf{q}$, then $\mathbf{a} \in \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$;*
3. *If $\mathbf{a}$ is a total degree and $\mathbf{a} \leq \mathbf{q}$, then $\mathbf{a} \in \mathrm{CRS}(\mathfrak{A}, \mathfrak{A}_1 \ldots \mathfrak{A}_n)$.*

Relative degree spectra

- The Minimal pair theorem.
- The Quasi-minimal degree.

- Questions:
  - Find other specific properties of Relative spectra of structures?
  - For any structures $\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$, does there exist a structure $\mathfrak{B}$ such that $\mathrm{DS}(\mathfrak{B}) = \mathrm{RS}(\mathfrak{A}, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$?

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