Minimal Pairs and Quasi-Minimal degrees for Joint Spectra

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LC2005

Degree Spectra of Structures

Let $\mathfrak{A} = (\mathbb{N}; R_1, \ldots, R_k)$ be a countable abstract structure.

Definition

- **The Degree spectrum of** $\mathfrak{A}$ is the set

$$\mathrm{DS}(\mathfrak{A}) = \{d_e(f^{-1}(\mathfrak{A})) : f \text{ is an enumeration of } \mathfrak{A}\}.$$

- **The Co-spectrum of** $\mathfrak{A}$ is the set

$$\mathrm{CS}(\mathfrak{A}) = \{\mathrm{b} : (\forall \mathrm{a} \in \mathrm{DS}(\mathfrak{A}))(\mathrm{b} \leq \mathrm{a})\}.$$

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Joint Spectra of Structures

Let ζ be a recursive ordinal and let $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ be a sequence of abstract structures over the natural numbers.

Definition

- **The Joint Spectrum of the sequence** $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ is the set

$$\mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}) = \{\mathrm{a} : (\forall\xi \leq \zeta)(\mathrm{a}^{(\xi)} \in \mathrm{DS}(\mathfrak{A}_\xi))\}.$$

- **The αth Jump Spectrum of** $\{\mathfrak{A}_\xi\}_{\xi\leq\zeta}$ is the set

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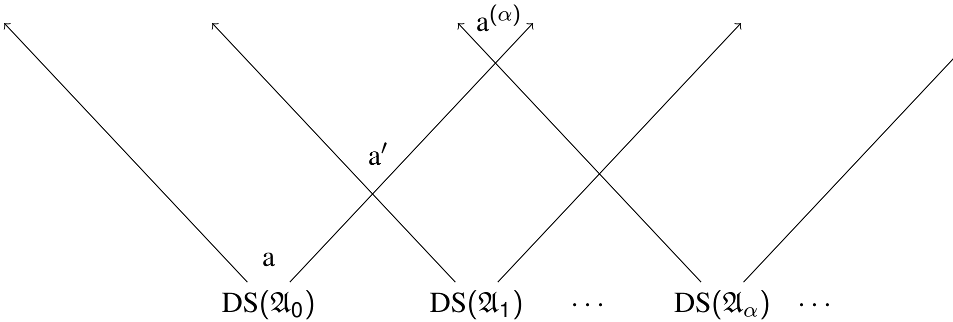

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The jump set

Let f_ξ be an enumeration of $\mathfrak{A}_\xi$ and $f = \{f_\xi\}_{\xi \leq \zeta}$.

Definition

The jump set $\mathcal{P}^f_\alpha$ of the sequence $\{\mathfrak{A}_\xi\}_{\xi \leq \zeta}$:

(i) $\mathcal{P}^f_0 = f_0^{-1}(\mathfrak{A}_0)$.

(ii) Let $\alpha = \beta + 1$. Then let $\mathcal{P}^f_\alpha = (\mathcal{P}^f_\beta)' \oplus f_\alpha^{-1}(\mathfrak{A}_\alpha)$.

(iii) Let $\alpha = \lim \alpha(p)$. Then set $\mathcal{P}^f_{<\alpha} = \{\langle p, x\rangle : x \in \mathcal{P}^f_{\alpha(p)}\}$ and let $\mathcal{P}^f_\alpha = \mathcal{P}^f_{<\alpha} \oplus f_\alpha^{-1}(\mathfrak{A}_\alpha)$.

Proposition

$$\mathrm{d_e}(A) \in \mathrm{CS}^\alpha(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta}) \iff$$
$$(\textit{ for every enumeration } f = \{f_\xi\}_{\xi \leq \zeta})(A \leq_{\mathrm{e}} \mathcal{P}^f_\alpha) .$$

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Minimal Pair Theorem

Theorem (Soskov)

There exist elements f *and* g *of DS*$(\mathfrak{A})$ *such that for any enumeration degree* a *and any* $\alpha < \zeta$

$$\mathrm{a} \leq \mathrm{f}^{(\alpha)} \ \& \ \mathrm{a} \leq \mathrm{g}^{(\alpha)} \Rightarrow \mathrm{a} \in \mathrm{CS}^{\alpha}(\mathfrak{A}).$$

Theorem

There exist elements f *and* g *of* $\mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$, *such that for any enumeration degree* a *and* $\alpha < \zeta$*:*

$$\mathrm{a} \leq \mathrm{f}^{(\alpha)} \ \& \ \mathrm{a} \leq \mathrm{g}^{(\alpha)} \Rightarrow \mathrm{a} \in \mathrm{CS}^{\alpha}(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta}).$$

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Jump Inversion Theorem [JIT]

Theorem (Soskov, Baleva)

Let $\{B_\xi\}_{\xi \leq \zeta}$ be a sequence of sets, $A \not\leq_e \mathcal{P}_\alpha$. Then there exists a total set F with the following properties:

1. *For all $\xi \leq \zeta$, $B_\xi \leq_e F^{(\xi)}$;*
2. *$A \not\leq_e F^{(\alpha)}$.*

Minimal Pair Theorem

Minimal Pair Theorem.

- $g = \{g_\xi\}_{\xi \leq \zeta}$-arbitrary enumeration.
- By [JIT] there is a total set F, $g_\xi^{-1}(\mathfrak{A}_\xi) \leq_{\mathrm{e}} F^{(\xi)}$.
- $h = \{h_\xi\}_{\xi \leq \zeta}$, $h_\xi^{-1}(\mathfrak{A}_\xi) \equiv_{\mathrm{e}} F^{(\xi)}$ for $\xi \leq \zeta$.
- By [JIT] here is a total set G, $G^{(\alpha)}$ omits any $A \leq_{\mathrm{e}} \mathcal{P}^h_\alpha$ not forcing α-definable.
- If $X \leq_{\mathrm{e}} F^{(\alpha)}$ and $X \leq_{\mathrm{e}} G^{(\alpha)}$ and X is a total set then $d_{\mathrm{e}}(X) \in \mathrm{CS}^\alpha(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$.
- Set $d_{\mathrm{e}}(F) = \mathrm{f}$ and $d_{\mathrm{e}}(G) = \mathrm{g}$.

□

Quasi-Minimal Degree

Definition (Soskov)

An enumeration degree q_0 is quasi-minimal with respect to $\mathrm{DS}(\mathfrak{A}_0)$ if

- $\mathrm{q}_0 \notin \mathrm{CS}(\mathfrak{A}_0)$
- for every total degree a: if $\mathrm{a} \geq \mathrm{q}_0$, then $\mathrm{a} \in \mathrm{DS}(\mathfrak{A}_0)$
- if $\mathrm{a} \leq \mathrm{q}_0$, then $\mathrm{a} \in \mathrm{CS}(\mathfrak{A}_0)$.

Theorem

There exists an enumeration degree q such that:

1. $\mathrm{q}^{(\alpha)} \in \mathrm{DS}(\mathfrak{A}_\alpha), \alpha < \zeta$, $\mathrm{q} \notin \mathrm{CS}(\{\mathfrak{A}_\xi\}_{\xi<\zeta})$;
2. *If* a *is a total degree and* $\mathrm{a} \geq \mathrm{q}$, *then* $\mathrm{a} \in \mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi<\zeta})$;
3. *If* a *is a total degree and* $\mathrm{a} \leq \mathrm{q}$, *then* $\mathrm{a} \in \mathrm{CS}(\{\mathfrak{A}_\xi\}_{\xi<\zeta})$.

Quasi-Minimal Degree

Definition (Soskov)

An enumeration degree q_0 is $\mathrm{q}$uasi-minimal with respect to $\mathrm{DS}(\mathfrak{A}_0)$ if

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- if $\mathrm{a} \leq \mathrm{q}_0$, then $\mathrm{a} \in \mathrm{CS}(\mathfrak{A}_0)$.

Theorem

There exists an enumeration degree q *such that:*

1. $\mathrm{q}^{(\alpha)} \in \mathrm{DS}(\mathfrak{A}_\alpha)$,$\alpha < \zeta$, $\mathrm{q} \notin \mathrm{CS}(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$;
2. *If* a *is a total degree and* $\mathrm{a} \geq \mathrm{q}$, *then* $\mathrm{a} \in \mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$;
3. *If* a *is a total degree and* $\mathrm{a} \leq \mathrm{q}$, *then* $\mathrm{a} \in \mathrm{CS}(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$.

Quasi-Minimal Degree

Theorem

For any sequence of sets $\{B_\xi\}_{\xi\leq\zeta}$ there exists a set of natural numbers F having the following properties:

1. $B_0 <_e F$;
2. *For all* $1 \leq \alpha \leq \zeta$, $B_\alpha \leq_e F^{(\alpha)}$;
3. *For any total set A, if $A \leq_e F$, then $A \leq_e B_0$.*

The set F is called `quasi-minimal over` B_0. The proof is by the technique of partial regular enumerations.

Quasi-Minimal Degree

Proof.

- Let q_0 be a quasi-minimal degree q_0 with respect to $\mathrm{DS}(\mathfrak{A}_0)$ [Soskov].
- Let $B_0 \subseteq \mathbb{N}$, $d_\mathrm{e}(B_0) = \mathrm{q}_0$, and $\{f_\xi\}_{\xi \leq \zeta}$ be fixed total enumerations of $\{\mathfrak{A}_\xi\}_{\xi \leq \zeta}$.
- There is quasi-minimal over B_0 set F, such that
  - $B_0 <_\mathrm{e} F$,
  - $f_\alpha^{-1}(\mathfrak{A}_\alpha) \leq_\mathrm{e} F^{(\alpha)}$, $\alpha < \zeta$
  - if $A \leq_\mathrm{e} F$, then $A \leq_\mathrm{e} B_0$, for any total set A.
- Then $\mathrm{q} = d_\mathrm{e}(F)$ is quasi-minimal with respect to $\mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$.

□

Quasi-Minimal Degree

Continued.

- Since q_0 is quasi-minimal with respect to $\mathrm{DS}(\mathfrak{A}_0)$, $\mathrm{q}_0 \notin \mathrm{CS}(\mathfrak{A}_0)$.
- But $\mathrm{q}_0 < \mathrm{q}$ and thus $\mathrm{q} \notin \mathrm{CS}(\mathfrak{A}_0)$. Hence $\mathrm{q} \notin \mathrm{CS}(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$.
- $\mathrm{q}^{(\alpha)} \in \mathrm{DS}(\mathfrak{A}_\alpha)$.
- X — total, $X \geq_{\mathrm{e}} F$. Then $d_{\mathrm{e}}(X) \geq \mathrm{q}_0$. But q_0 is quasi-minimal, thus $d_{\mathrm{e}}(X) \in \mathrm{DS}(\mathfrak{A}_0)$. Since $X^{(\alpha)} \geq_{\mathrm{e}} F^{(\alpha)} \geq_{\mathrm{e}} f_\alpha^{-1}(\mathfrak{A}_\alpha)$, and $X^{(\alpha)}$ is a total, $d_{\mathrm{e}}(X^{(\alpha)}) \in \mathrm{DS}(\mathfrak{A}_\alpha)$, and hence $d_{\mathrm{e}}(X) \in \mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$.
- X — total, $X \leq_{\mathrm{e}} F$. Then, $X \leq_{\mathrm{e}} B_0$. From the quasi-minimality of q_0, $d_{\mathrm{e}}(X) \in \mathrm{CS}(\mathfrak{A}_0) = \mathrm{CS}(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$.

□

Properties of joint spectra of sequence of structures

- The Minimal pair theorem.
- The Quasi-minimal degree.

- Questions:
  - Another specific properties of Joint spectra of structures?
  - Do there exist a structure $\mathfrak{A}$ such that $\mathrm{DS}(\mathfrak{A}) = \mathrm{DS}(\{\mathfrak{A}_\xi\}_{\xi \leq \zeta})$?

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