Minimal Pairs and Quasi-Minimal Degrees for the Joint Spectra of Structures

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Degree Spectra of Structures

For any $A \subseteq \mathbb{N}^a$:

$f^{-1}(A) = \{\langle x_1 \dots x_a \rangle : (f(x_1), \dots, f(x_a)) \in A\}$.

Let $\mathfrak{A} = (\mathbb{N}; R_1, \dots, R_k)$ be a countable abstract structure. *An enumeration f of* $\mathfrak{A}$ is a total mapping from $\mathbb{N}$ onto $\mathbb{N}$.

Denote by $f^{-1}(\mathfrak{A}) = f^{-1}(R_1) \oplus \cdots \oplus f^{-1}(R_k)$.

The Degree spectrum of $\mathfrak{A}$ is the set

$\mathrm{DS}(\mathfrak{A}) = \{d_{\mathrm{e}}(f^{-1}(\mathfrak{A})) : f$ is an enumeration of $\mathfrak{A}\}$.

The Co-spectrum of $\mathfrak{A}$ is the set

$$\mathrm{CS}(\mathfrak{A}) = \{\mathrm{b} : (\forall \mathrm{a} \in \mathrm{DS}(\mathfrak{A}))(\mathrm{b} \leq \mathrm{a}\}.$$

The set A is **enumeration reducible** to the set B (notation: $A \leq_{\mathrm{e}} B$) if

$$(\exists z)(\forall x)[x \in A \iff (\exists u)[\langle x, u\rangle \in W_z \;\&\; D_u \subseteq B]].$$

The jump operation "$'$" denotes here the enumeration jump introduced by Cooper.

Let $B_0, \ldots, B_n$ be arbitrary subsets of $\mathbb{N}$. Define the set $\mathcal{P}(B_0, \ldots, B_i)$ as follows:

1. $\mathcal{P}(B_0) = B_0$;

2. If $i < n$, then

$$\mathcal{P}(B_0, \ldots, B_{i+1}) = (\mathcal{P}(B_0, \ldots, B_i))' \oplus B_{i+1}.$$

Joint Spectra of Structures

Let $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ be abstract structures on $\mathbb{N}$.

The Joint spectrum of $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ is the set

$$\mathrm{DS}(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) = \{\mathsf{a} : \mathsf{a} \in \mathrm{DS}(\mathfrak{A}_0), \mathsf{a}' \in \mathrm{DS}(\mathfrak{A}_1), \ldots, \mathsf{a}^{(n)} \in \mathrm{DS}(\mathfrak{A}_n)\}.$$

The k**th Jump spectrum of** $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$, $k \leq n$, is the set

$$\mathrm{DS}_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_n) = \{\mathsf{a}^{(k)} : \mathsf{a} \in \mathrm{DS}(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)\}.$$

The k**th Co-spectrum of** $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$, $k \leq n$, is the set

$$\mathrm{CS}_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_n) = \{\mathsf{b} : (\forall \mathsf{a} \in \mathrm{DS}_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_n))(\mathsf{b} \leq \mathbf{a})\}.$$

$$\mathrm{CS}_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_k, \ldots, \mathfrak{A}_n) = \mathrm{CS}_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_k).$$

Forcing k - definable sets

Let $f_0, \ldots, f_n$ be enumerations of $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$. Denote by $\bar{f} = (f_0, \ldots, f_n)$ and

$$\mathcal{P}_k^{\bar{f}} = \mathcal{P}(f_0^{-1}(\mathfrak{A}_0), \ldots, f_k^{-1}(\mathfrak{A}_k)),\ k = 0, \ldots, n.$$

1. $\bar{f} \models_0 F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&\ D_v \subseteq f_0^{-1}(\mathfrak{A}_0))$;

2. $\bar{f} \models_{i+1} F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \&$ $(\forall u \in D_v)(u = \langle 0, e_u, x_u\rangle \ \&\ \bar{f} \models_i F_{e_u}(x_u) \ \vee$ $u = \langle 1, e_u, x_u\rangle \ \&\ \bar{f} \models_i \neg F_{e_u}(x_u) \ \vee$ $u = \langle 2, x_u\rangle \ \&\ x_u \in f_{i+1}^{-1}(\mathfrak{A}_{i+1})))$;

3. $\bar{f} \models_i \neg F_e(x) \iff \bar{f} \not\models_i F_e(x)$.

$$A \leq_{\mathrm{e}} \mathcal{P}_k^{\bar{f}} \iff (\exists e)(A = \{x : \bar{f} \models_k F_e(x)\}).$$

The *finite parts* are $n+1$ tuples $\bar\tau = (\tau_0, \ldots, \tau_n)$ of finite mappings $\tau_0, \ldots, \tau_n$ of $\mathbb{N}$ in $\mathbb{N}$.

1. $\bar\tau \Vdash_0 F_e(x) \iff (\exists v)(\langle v, x\rangle \in W_e \ \& \ D_v \subseteq \tau_0^{-1}(\mathfrak{A}_0))$;

2. $\bar\tau \Vdash_{i+1} F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \&$
   $(\forall u \in D_v)(u = \langle 0, e_u, x_u\rangle \ \& \ \bar\tau \Vdash_i F_{e_u}(x_u) \ \vee$
   $u = \langle 1, e_u, x_u\rangle \ \& \ \bar\tau \Vdash_i \neg F_{e_u}(x_u) \ \vee$
   $u = \langle 2, x_u\rangle \ \& \ x_u \in \tau_{i+1}^{-1}(\mathfrak{A}_{i+1})))$;

3. $\bar\tau \Vdash_i \neg F_e(x) \iff (\forall \bar\rho \supseteq \bar\tau)(\bar\rho \nVdash_i F_e(x))$.

Generic enumerations

An enumeration $\bar{f}$ of $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ is k-**generic** if for every $j < k$, $e, x \in \mathbb{N}$

$$(\forall \bar{\tau} \subseteq \bar{f})(\exists \bar{\rho} \supseteq \bar{\tau})(\bar{\rho} \Vdash_j F_e(x)) \Longrightarrow (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \Vdash_j F_e(x)).$$

1. If $\bar{f}$ is a k-generic enumeration, then

$$\bar{f} \models_k F_e(x) \iff (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \Vdash_k F_e(x)).$$

2. If $\bar{f}$ is a $(k+1)$-generic enumeration, then

$$\bar{f} \models_k \neg F_e(x) \iff (\exists \bar{\tau} \subseteq \bar{f})(\bar{\tau} \Vdash_k \neg F_e(x)).$$

The set $A \subseteq \mathbb{N}$ is **forcing k-definable** on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$ if there exist a finite part $\bar{\delta}$ and $e \in \mathbb{N}$ such that

$$x \in A \iff (\exists \bar{\tau} \supseteq \bar{\delta})(\bar{\tau} \Vdash_k F_e(x)).$$

Theorem. *For every $A \subseteq \mathbb{N}$, the following are equivalent:*

1. *$d_e(A) \in \mathrm{CS}_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$.*
2. *$A \leq_e \mathcal{P}_k^{\bar{f}}$, for all $\bar{f} = (f_0, \ldots, f_k)$ enumerations of $\mathfrak{A}_0, \ldots, \mathfrak{A}_k$.*
3. *A is forcing k-definable on $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$.*

Minimal Pair Theorem

Theorem. (Soskov) *For each constructive ordinal α there exist elements* f *and* g *of $DS(\mathfrak{A})$ such that for any enumeration degree* a *and any $\beta < \alpha$*

$$\mathsf{a} \le \mathsf{f}^{(\beta)} \ \& \ \mathsf{a} \le \mathsf{g}^{(\beta)} \Rightarrow \mathsf{a} \in CS_\beta(\mathfrak{A}).$$

For all structures $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$:

Theorem. *There exist enumeration degrees* f *and* g *in* $\mathrm{DS}(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$, *such that for any enumeration degree* a *and $k \le n$:*

$$\mathsf{a} \le \mathsf{f}^{(k)} \ \& \ \mathsf{a} \le \mathsf{g}^{(k)} \Rightarrow \mathsf{a} \in \mathrm{CS}_k(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n) \ .$$

Proof *We begin with arbitrary enumerations* $g_0, \ldots, g_n$ *of* $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$. *By* Jump Inversion Theorem [Soskov] *there exists a total set* F, *such that:* $g_k^{-1}(\mathfrak{A}_k) \leq_{\mathrm{e}} F^{(k)}$, *for each* $k \leq n$, *and* $d_{\mathrm{e}}(F) \in \mathrm{DS}(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

Hence, there exist enumerations $h_0, \ldots, h_n$ *of* $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$, *respectively, such that* $h_k^{-1}(\mathfrak{A}_k) \equiv_{\mathrm{e}} F^{(k)}$ *and then* $F^{(k)} \equiv_{\mathrm{e}} \mathcal{P}_k^{\bar{h}}$.

Then, we construct $(n+1)$*-generic enumerations* $\bar{f}$ *such that if a set* $A \leq_{\mathrm{e}} \mathcal{P}_k^{\bar{f}}$ *and* $A \leq_{\mathrm{e}} \mathcal{P}_k^{\bar{h}}$, *then* A *is forcing* k*-definable on* $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$.

By Jump Inversion Theorem [Soskov] *(omitting variant) there is a total set* $G \in \mathrm{DS}(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$, *such that* $f_k^{-1}(\mathfrak{A}_k) \leq_{\mathrm{e}} G^{(k)}$ *and* $G^{(k)}$ *omits any* $A \leq_{\mathrm{e}} \mathcal{P}_k^{\bar{h}}$ *which is not forcing* k*-definable on* $\mathfrak{A}_0, \ldots, \mathfrak{A}_n$.

If $X \leq_{\mathrm{e}} F^{(k)}$ *and* $X \leq_{\mathrm{e}} G^{(k)}$ *and* X *is a total set then* $d_{\mathrm{e}}(X) \in \mathrm{CS}_k(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$.

Quasi-Minimal Degree

We call an enumeration degree q_0 **quasi-minimal with respect to** $\mathrm{DS}(\mathfrak{A}_0)$ if $\mathsf{q}_0 \notin \mathrm{CS}(\mathfrak{A}_0)$ and for every total degree a: if $\mathsf{a} \geq \mathsf{q}_0$, then $\mathsf{a} \in \mathrm{DS}(\mathfrak{A}_0)$ and if $\mathsf{a} \leq \mathsf{q}_0$, then $\mathsf{a} \in \mathrm{CS}(\mathfrak{A}_0)$. [Soskov]

For all structures $\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n$:

Theorem. *There exists an enumeration degree* q *such that:*

1. $\mathsf{q}' \in \mathrm{DS}(\mathfrak{A}_1), \ldots, \mathsf{q}^{(n)} \in \mathrm{DS}(\mathfrak{A}_n)$,
   $\mathsf{q} \notin \mathrm{CS}(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$;

2. *If* a *is a total degree and* $\mathsf{a} \geq \mathsf{q}$, *then* $\mathsf{a} \in \mathrm{DS}(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$;

3. *If* a *is a total degree and* $\mathsf{a} \leq \mathsf{q}$, *then* $\mathsf{a} \in \mathrm{CS}(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

Proof (Sketch) *Let* q_0 *be a quasi-minimal degree* q_0 *with respect to* $\mathrm{DS}(\mathfrak{A}_0)$ *[Soskov].*

Let $B_0 \subseteq \mathbb{N}$, $d_{\mathrm{e}}(B_0) = \mathrm{q}_0$, *and* $f_1, \ldots, f_n$ *be fixed total enumerations of* $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. *There is quasi-minimal over* B_0 *set* F, *such that* $B_0 <_{\mathrm{e}} F$, $f_k^{-1}(\mathfrak{A}_k) \leq_{\mathrm{e}} F^{(k)}$, $k \leq n$ *and if* $A \leq_{\mathrm{e}} F$, *then* $A \leq_{\mathrm{e}} B_0$, *for any total set* A. *Then* $\mathrm{q} = d_{\mathrm{e}}(F)$ *is quasi-minimal with respect to* $\mathrm{DS}(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$.

Since q_0 is quasi-minimal with respect to $\mathsf{DS}(\mathfrak{A}_0)$, $\mathsf{q}_0 \notin \mathsf{CS}(\mathfrak{A}_0)$. But $\mathsf{q}_0 < \mathsf{q}$ and thus $\mathsf{q} \notin \mathsf{CS}(\mathfrak{A}_0)$. Hence $\mathsf{q} \notin \mathsf{CS}(\mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_n)$.

For each $1 \leq k \leq n$, $\mathsf{q}^{(k)} \in \mathsf{DS}(\mathfrak{A}_k)$.

Let X be a total set, such that $X \geq_{\mathsf{e}} F$. Then $d_{\mathsf{e}}(X) \geq \mathsf{q}_0$. From the fact that q_0 is quasi-minimal with respect to $\mathsf{DS}(\mathfrak{A}_0)$ it follows that $d_{\mathsf{e}}(X) \in \mathsf{DS}(\mathfrak{A}_0)$. Moreover for each $1 \leq k \leq n$, $X^{(k)} \geq_{\mathsf{e}} F^{(k)} \geq_{\mathsf{e}} f_k^{-1}(\mathfrak{A}_k)$, and $X^{(k)}$ is a total set. Then for each $k \leq n$, $d_{\mathsf{e}}(X^{(k)}) \in \mathsf{DS}(\mathfrak{A}_k)$, and hence $d_{\mathsf{e}}(X) \in \mathsf{DS}(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$.

Suppose that X is a total set and $X \leq_{\mathsf{e}} F$. Then, from the choice of F, since X is total, $X \leq_{\mathsf{e}} B_0$. Apply again the quasi-minimality of q_0 and then $d_{\mathsf{e}}(X) \in \mathsf{CS}(\mathfrak{A}_0)$.
But $\mathsf{CS}(\mathfrak{A}_0, \ldots, \mathfrak{A}_n) = \mathsf{CS}(\mathfrak{A}_0)$ and therefore $d_{\mathsf{e}}(X) \in \mathsf{CS}(\mathfrak{A}_0, \ldots, \mathfrak{A}_n)$.

The existence of the set F — quasi-minimal over B_0, uses the technique of partial regular enumerations.