THE GROUPS $\mathrm{AUT}(\mathcal{D}_\omega{}')$ AND $\mathrm{AUT}(\mathcal{D}_e)$ ARE ISOMORPHIC

HRISTO GANCHEV AND IVAN SOSKOV

In the present paper we continue the study of the partial ordering $\mathcal{D}_\omega$ of the ω-enumeration degrees initiated in [2]. We show that the enumeration degrees are first order definable in the structure $\mathcal{D}_\omega{}'$ of the ω-enumeration degrees augmented by the jump operator and that the groups of the automorphisms of enumeration degrees and of the automorphisms of $\mathcal{D}_\omega{}'$ are isomorphic.

1. The ω-enumeration degrees

Denote by $\mathcal{S}$ the set of all sequences $\mathcal{B} = \{B_k\}_{k<\omega}$ of sets of natural numbers. Consider an element $\mathcal{B}$ of $\mathcal{S}$ and let the *jump class* $J_\mathcal{B}$ defined by $\mathcal{B}$ be the set of the Turing degrees of all $X \subseteq \mathbb{N}$ such that $(\forall k)(B_k$ is r.e. in $X^{(k)}$ uniformly in $k)$.

Given two sequences $\mathcal{A}$ and $\mathcal{B}$ let $\mathcal{A} \leq_u \mathcal{B}$ ($\mathcal{A}$ is *uniformly reducible to* $\mathcal{B}$) if $J_\mathcal{B} \subseteq J_\mathcal{A}$ and $\mathcal{A} \equiv_u \mathcal{B}$ if $J_\mathcal{B} = J_\mathcal{A}$. Clearly "$\leq_u$" is a reflexive and transitive relation on $\mathcal{S}$ and "$\equiv_u$" is an equivalence relation on $\mathcal{S}$.

For every sequence $\mathcal{B}$ let $d_\omega(\mathcal{B}) = \{\mathcal{A} : \mathcal{A} \equiv_u \mathcal{B}\}$ and let $\mathcal{D}_\omega = \{d_\omega(\mathcal{B}) : \mathcal{B} \in \mathcal{S}\}$. The elements of $\mathcal{D}_\omega$ are called the *ω-enumeration degrees.*

The ω-enumeration degrees can be ordered in the usual way. Given two elements $\mathbf{a} = d_\omega(\mathcal{A})$ and $\mathbf{b} = d_\omega(\mathcal{B})$ of $\mathcal{D}_\omega$, let $\mathbf{a} \leq_\omega \mathbf{b}$ if $\mathcal{A} \leq_u \mathcal{B}$. Clearly $\mathcal{D}_\omega = (\mathcal{D}_\omega, \leq_\omega)$ is a partial ordering with least element $\mathbf{0}_\omega = d_\omega(\emptyset_\omega)$, where all members of $\emptyset_\omega$ are equal to $\emptyset$.

Given two sequences $\mathcal{A} = \{A_k\}$ and $\mathcal{B} = \{B_k\}$ of sets of natural numbers let $\mathcal{A} \oplus \mathcal{B} = \{A_k \oplus B_k\}$. Is it easy to see that $J_{\mathcal{A}\oplus\mathcal{B}} = J_\mathcal{A} \cap J_\mathcal{B}$ and hence every two elements $\mathbf{a} = d_\omega(\mathcal{A})$ and $\mathbf{b} = d_\omega(\mathcal{B})$ of $\mathcal{D}_\omega$ have a least upper bound $\mathbf{a} \cup \mathbf{b} = d_\omega(\mathcal{A} \oplus \mathcal{B})$.

There is a natural embedding of the enumeration degrees into the ω-enumeration degrees. Given a set A of natural numbers denote by $A \uparrow \omega$ the sequence $\{A_k\}_{k<\omega}$, where $A_0 = A$ and for all $k \geq 1$, $A_k = \emptyset$.

1.1. Proposition. *For every $A, B \subseteq \mathbb{N}$, $A \uparrow \omega \leq_u B \uparrow \omega \iff A \leq_e B$.*

Let $\mathcal{D}_1 = \{d_\omega(A \uparrow \omega) : A \subseteq \mathbb{N}\}$ and $\mathcal{D}_1 = (\mathcal{D}_1, \mathbf{0}_\omega, \cup, \leq_\omega)$.

Define the mapping $\kappa : \mathcal{D}_e \to \mathcal{D}_1$ by $\kappa(d_e(A)) = d_\omega(A \uparrow \omega)$. Clearly κ is an isomorphism from $\mathcal{D}_e$ to $\mathcal{D}_1$ and hence κ is an embedding of $\mathcal{D}_e$ into $\mathcal{D}_\omega$.

The elements of $\mathcal{D}_1$ form a base of the automorphisms of $\mathcal{D}_\omega$. Indeed given a sequence $\mathcal{A}$ let $J^e_\mathcal{A} = \{\mathbf{a} : \mathbf{a} \in \mathcal{D}_e \ \&\ d_\omega(\mathcal{A}) \leq_\omega \kappa(\mathbf{a})\}$. Let ι be the Roger's embedding of the Turing degrees $\mathcal{D}_T$ into the enumeration degrees. It is easy to see that $\iota(J_\mathcal{A}) = J^e_\mathcal{A} \cap \iota(\mathcal{D}_T)$ and hence

$$\mathcal{A} \leq_u \mathcal{B} \iff J^e_\mathcal{B} \subseteq J^e_\mathcal{A}.$$

This work was partially supported by Sofia University Science Fund.

Suppose that φ is an automorphism of $\mathcal{D}_\omega$ and $\varphi(\mathbf{x}) = \mathbf{x}$ for $\mathbf{x} \in \mathcal{D}_1$. Let $\mathbf{a} \in \mathcal{D}_\omega$ and $\mathcal{A} \in \mathbf{a}$, $\mathcal{B} \in \varphi(\mathbf{a})$. Clearly

$$J^e_{\mathcal{B}} = \{\kappa^{-1}(\varphi(\mathbf{x})) : \mathbf{a} \leq_\omega \mathbf{x} \ \& \ \mathbf{x} \in \mathcal{D}_1\} = \{\kappa^{-1}(\mathbf{x}) : \mathbf{a} \leq_\omega \mathbf{x} \ \& \ \mathbf{x} \in \mathcal{D}_1\} = J^e_{\mathcal{A}}.$$

Hence $\mathcal{A} \equiv_u \mathcal{B}$ and $\mathbf{a} = \varphi(\mathbf{a})$.

2. The jump operator on the ω-enumeration degrees

Given a sequence $\mathcal{B} = \{B_k\}_{k<\omega}$ of sets of natural numbers we define the respective *jump sequence* $\mathcal{P}(\mathcal{B}) = \{\mathcal{P}_k(\mathcal{B})\}_{k<\omega}$ by induction on k:

(i) $\mathcal{P}_0(\mathcal{B}) = B_0$;
(ii) $\mathcal{P}_{k+1}(\mathcal{B}) = \mathcal{P}_k(\mathcal{B})' \oplus B_{k+1}$.

We shall assume fixed an effective coding of all finite sets of natural numbers. By D_v we shall denote the finite set with code v.

Given two sets of W and A of natural numbers, let

$$W(A) = \{x : (\exists v)(\langle x, v\rangle \in W \ \& \ D_v \subseteq A)\}.$$

Let $W_0, W_1, \dots$ be a Gödel numbering of the recursively enumerable sets.

The following result from [3] gives an explicit definition of the uniform reducibility:

2.1. Theorem. *Let $\mathcal{A} = \{A_k\}$ and $\mathcal{B} = \{B_k\}$ belong to $\mathcal{S}$. Then $\mathcal{A} \leq_u \mathcal{B}$ if and only if there exists a recursive function h such that $(\forall k)(A_k = W_{h(k)}(\mathcal{P}_k(\mathcal{B})))$.*

2.2. Corollary. *For every $\mathcal{A} \in \mathcal{S}$, $\mathcal{A} \equiv_u \mathcal{P}(\mathcal{A})$.*

Given a sequence $\mathcal{A} \in \mathcal{S}$ set $\mathcal{A}' = \{\mathcal{P}_{1+n}(\mathcal{A})\}$. We have that $J_{\mathcal{A}'} = \{\mathbf{a}' : \mathbf{a} \in J_{\mathcal{A}}\}$ and hence $\mathcal{A} \leq_u \mathcal{B}$ implies $\mathcal{A}' \leq_u \mathcal{B}'$. So we may define a jump operator "$'$" on the ω-enumeration degrees by letting $d_\omega(\mathcal{A})' = d_\omega(\mathcal{A}')$. On can easily see that the jump operator is monotone and for every ω-enumeration degree $\mathbf{a}$, $\mathbf{a} <_\omega \mathbf{a}'$. Moreover the jump operator agrees with the enumeration jump under the embedding κ defined in the previous section. Namely for every enumeration degree $\mathbf{a}$, $\kappa(\mathbf{a}') = \kappa(\mathbf{a})'$.

Some of the properties of the jump operator on the ω-enumeration degrees are surprising. For example we have the following jump inversion theorem:

2.3. Theorem. *Let $n > 0$. Let $\mathbf{a}, \mathbf{b} \in \mathcal{D}_\omega$ be such that $\mathbf{a}^{(n)} \leq_\omega \mathbf{b}$. Then the equation $\mathbf{x}^{(n)} = \mathbf{b}$ has a least solution above $\mathbf{a}$.*

Proof. Let $\mathbf{a} = d_\omega(\mathcal{A})$ and $\mathbf{b} = d_\omega(\mathcal{B})$. Since $\mathbf{a}^{(n)} \leq_\omega \mathbf{b}$ we have that $\{P_{k+n}(\mathcal{A})\} \leq_u \mathcal{B}$, and therefore $P_n(A) \leq_e B_0$. Consider the sequence $\mathcal{X} = \{X_k\}_{k<\omega}$, where for $0 \leq k < n$, $X_k = A_k$, and for $k \geq n$, $X_k = B_{k-n}$. We have that for $0 \leq k < n$, $P_k(\mathcal{A}) = P_k(\mathcal{X})$ and for $k \geq 0$, $P_{k+n}(\mathcal{X}) \equiv_e P_k(\mathcal{B})$ uniformly in k. Thus we obtain that $\mathcal{A} \leq_u \mathcal{X}$ and $\mathcal{X}^{(n)} \equiv_u \mathcal{B}$.

Now suppose that $\mathcal{Y} \in \mathcal{S}$ is such that $\mathcal{A} \leq_u \mathcal{Y}$ and $\mathcal{Y}^{(n)} \equiv_u \mathcal{B}$. Then we have that for $0 \leq k < n$, $P_k(\mathcal{A}) \leq_e P_k(\mathcal{Y})$ and for all k, $P_{n+k}(\mathcal{Y}) \equiv_e P_k(\mathcal{B})$ uniformly in k. Therefore $\mathcal{X} \leq_u \mathcal{Y}$. □

The last theorem shows that the structures ${\mathcal{D}_e}'$ and ${\mathcal{D}_\omega}'$ are not elementary equivalent.

Now using this property of the ω-enumeration degrees we will show that the set of all elements of $\mathcal{D}_1$ is first order definable in ${\mathcal{D}_\omega}'$.

For $\mathbf{a} \in \mathcal{D}_\omega$, by $I(\mathbf{a})$ we shall denote the set of all least jump inverts over $\mathbf{a}$, i.e.,

$$I(\mathbf{a}) = \{\mathbf{x} \mid \mathbf{a} \leq_\omega \mathbf{x} \ \&\ \forall \mathbf{y}(\mathbf{a} \leq_\omega \mathbf{y} <_\omega \mathbf{x} \Longrightarrow \mathbf{y}' <_\omega \mathbf{x}')\}.$$

$I(\mathbf{a})$ has the following properties:

(i) $\mathbf{a} \leq_\omega \mathbf{z} \leq_\omega \mathbf{x} \ \&\ \mathbf{x} \in I(\mathbf{a}) \Longrightarrow \mathbf{z} \in I(\mathbf{a})$
(ii) $\mathbf{x}_1, \mathbf{x}_2 \in I(\mathbf{a}) \Longrightarrow \mathbf{x}_1 \cup \mathbf{x}_2 \in I(\mathbf{a})$
(iii) $I(\mathbf{a}_1) \subseteq I(\mathbf{a}_2) \Longrightarrow \mathbf{a}_2 \leq_\omega \mathbf{a}_1$
(iv) $\mathbf{d}_\omega(\mathcal{B}) \in I(\mathbf{d}_\omega(\mathcal{A})) \Longrightarrow B_0 \equiv_e A_0$
(v) $I(\mathbf{d}_\omega(\mathcal{A})) \subseteq I(\mathbf{d}_\omega(\mathcal{B})) \iff \mathcal{B} \leq_u \mathcal{A} \ \&\ A_0 \equiv_e B_0$
(vi) $\mathbf{d}_\omega(\mathcal{B}) \in I(\kappa(A)) \iff B_0 \equiv_e A$

Proof. (iv) Suppose that $\mathbf{d}_\omega(\mathcal{B}) \in I(\mathbf{d}_\omega(\mathcal{A}))$. From the definition of $I(\mathbf{d}_\omega(\mathcal{A}))$ we obtain that $\mathcal{A} \leq_u \mathcal{B}$ and hence $A_0 \leq_e B_0$. Now assume that $A_0 <_e B_0$ and consider the sequence $\mathcal{Y} = (A_0, P_1(\mathcal{B}), P_2(\mathcal{B}), \ldots)$. Then it is clear, that $\mathcal{Y} <_u \mathcal{B}$ and $\mathcal{Y}' \equiv_u \mathcal{B}'$. Besides, from $\mathcal{A} \leq_u \mathcal{B}$ we obtain $\mathcal{A} \leq_u \mathcal{Y}$. Therefore $d_\omega(\mathcal{A}) \leq_\omega d_\omega(\mathcal{Y}) <_\omega d_\omega(\mathcal{B})$ and $d_\omega(\mathcal{Y})' = d_\omega(\mathcal{B})'$. But this contradicts $\mathbf{d}_\omega(\mathcal{B}) \in I(\mathbf{d}_\omega(\mathcal{A}))$ and thus the statement $A_0 \equiv_e B_0$ is proven.

The other five properties are corollaries of property (iv) and the definition of $I(\mathbf{a})$.

□

Using (v) and (vi) we obtain a characterization of the degrees in $\mathcal{D}_1$ in the form: An ω-enumeration degree $\mathbf{a}$ is in $\mathcal{D}_1$ iff $I(\mathbf{a})$ is a maximal element of $\{I(\mathbf{x}) \mid \mathbf{x} \in \mathcal{D}_\omega\}$ with respect to the set theoretical inclusion.

From here we obtain that the set of all degrees in $\mathcal{D}_1$ is first order definable in $\mathcal{D}_\omega{}'$. Indeed, consider the binary predicate I defined by:

$$I(\mathbf{a}, \mathbf{x}) \iff \mathbf{a} \leq_\omega \mathbf{x} \ \&\ \forall \mathbf{y}(\mathbf{a} \leq_\omega \mathbf{y} <_\omega \mathbf{x} \Longrightarrow \mathbf{y}' <_\omega \mathbf{x}').$$

Clearly $\mathbf{x} \in I(\mathbf{a}) \iff I(\mathbf{a}, \mathbf{x})$ and therefore:

$$\mathbf{a} \in \mathcal{D}_1 \iff I(\mathbf{a}) \text{ is maximal } \iff$$

$$\forall \mathbf{b}(\forall \mathbf{x}(I(\mathbf{a}, \mathbf{x}) \Rightarrow I(\mathbf{b}, \mathbf{x})) \Longrightarrow \forall \mathbf{x}(I(\mathbf{b}, \mathbf{x}) \Rightarrow I(\mathbf{a}, \mathbf{x}))),$$

which is a first order formula.

The definability of $\mathcal{D}_1$ shows that every automorphism of $\mathcal{D}_\omega{}'$ induces an automorphism of the structure $\mathcal{D}_1$. On the other hand, since $\mathcal{D}_1$ is a base of the automorphisms of $\mathcal{D}_\omega$ we have that if two automorphisms of $\mathcal{D}_\omega$ induce the same automorphism of $\mathcal{D}_1$ then they coincide. In particular every nontrivial automorphism of $\mathcal{D}_\omega$ induces a nontrivial automorphism of $\mathcal{D}_1$.

Now we are going to prove that every automorphism of $\mathcal{D}_e{}'$ can be extended to an automorphism of $\mathcal{D}_\omega{}'$. We will use the following Theorem:

2.4. Theorem. *Let φ be an automorphism of $\mathcal{D}_e{}'$. Then for all $\mathbf{a} \geq \mathbf{0}^{(4)}$, $\varphi(\mathbf{a}) = \mathbf{a}$.*

The proof of the last theorem follows along the lines of the proof of the theorem that every automorphism of $\mathcal{D}_T{}'$ is identity on the cone above $\mathbf{0}'''$ presented in [1].

Suppose that φ is an automorphism of $\mathcal{D}_e{}'$. We shall show, that given a sequence $\mathcal{A} \in \mathcal{S}$ one can construct a sequence $\mathcal{B}$ such that $J^e_{\mathcal{B}} = \{\varphi(\mathbf{a}) : \mathbf{a} \in J^e_{\mathcal{A}}\}$. Indeed let $\mathbf{p}_k = d_e(\mathcal{P}_k(\mathcal{A}))$. Notice that if $k \geq 4$ then $\mathbf{p_k} \geq \mathbf{0}^{(4)}$ and hence $\varphi(\mathbf{p_k}) = \mathbf{p_k}$.

Fix some elements B_0, B_1, B_2, B_3 of $\varphi(\mathbf{p_0}), \varphi(\mathbf{p_1}), \varphi(\mathbf{p_2})$ and $\varphi(\mathbf{p_3})$ respectively and let for $k \geq 4$, $B_k = \mathcal{P}_k(\mathcal{A})$.

Now let $\mathbf{x} \in J^e_{\mathcal{A}}$ and $X \in \mathbf{x}$. Then $d_\omega(\mathcal{A}) \leq_\omega \kappa(\mathbf{x})$ and hence $\mathcal{P}(\mathcal{A}) \leq_u X \uparrow \omega$. Since for all k, $\mathcal{P}_k(X \uparrow \omega)$ is enumeration equivalent to $X^{(k)}$ uniformly in k we have that for all k, $\mathcal{P}_k(\mathcal{A}) \leq_e X^{(k)}$ uniformly in k. Now let $Y \in \varphi(\mathbf{x})$. We have to show that $\mathcal{B} \leq_u Y \uparrow \omega$. For it is sufficient to show that for all k, $B_k \leq_e Y^{(k)}$ uniformly in k. Notice that for all k, $X^{(k)} \in \mathbf{x}^{(k)}$ and $Y^{(k)} \in \varphi(\mathbf{x}^{(k)})$. Hence $X^{(4)} \equiv_e Y^{(4)}$. From here it follows immediately that for all $k \geq 4$, $B_k \leq_e Y^{(k)}$ uniformly in k and hence $\mathcal{B} \leq_u Y \uparrow \omega$.

So we have proved the inclusion $\varphi(J^e_{\mathcal{A}}) \subseteq J^e_{\mathcal{B}}$. The proof of the reverse inclusion is similar.

Let $\Phi : \mathcal{D}_\omega \to \mathcal{D}_\omega$ be defined as follows. Given $\mathbf{a} \in \mathcal{D}_\omega$, let $\mathcal{A} \in \mathbf{a}$ and $\mathcal{B}$ be such that $J^e_{\mathcal{B}} = \{\varphi(\mathbf{x}) : \mathbf{x} \in J^e_{\mathcal{A}}\}$. Let $\Phi(\mathbf{a}) = d_\omega(\mathcal{B})$.

Since for every two sequences $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \leq_u \mathcal{B} \iff J^e_{\mathcal{B}} \subseteq J^e_{\mathcal{A}}$ the mapping Φ is well defined and is an automorphism of $\mathcal{D}_\omega$. It is easy to see also that for every element $\mathbf{a}$ of $\mathcal{D}_1$, $\Phi(\mathbf{a}) = \kappa(\varphi(\kappa^{-1}(\mathbf{a})))$. Hence for every element $\mathbf{x}$ of $\mathcal{D}_1$ we have that $\Phi(\mathbf{x}') = \Phi(\mathbf{x})'$. From here it follows that Φ is an automorphism of $\mathcal{D}_\omega{}'$ by means of the following:

2.5. Theorem. *Let Φ be an automorphism of $\mathcal{D}_\omega$ such that for all $\mathbf{x} \in \mathcal{D}_1$, $\Phi(\mathbf{x}') = \Phi(\mathbf{x})'$. Then Φ is an automorphism of $\mathcal{D}_\omega{}'$.*

Thus we have shown that there is a mapping $\pi : \mathrm{Aut}(\mathcal{D}_e{}') \to \mathrm{Aut}(\mathcal{D}_\omega{}')$, acting by the rule $\pi(\varphi) = \Phi$, where Φ is defined from φ as above. It is clear that π is a homomorphism of groups and since $\mathcal{D}_1$ is an automorphism base for $\mathcal{D}_\omega{}'$, we have that π is one to one. In order to show that π is an isomorphism of groups it remains to show that π is onto. Indeed, suppose that $\phi \in \mathrm{Aut}(\mathcal{D}_\omega{}')$. Then $\varphi = \kappa^{-1} \circ \phi_{|\mathcal{D}_1} \circ \kappa$ is an automorphism of $\mathcal{D}_e{}'$. Now, we have that

$$\pi(\varphi)_{|\mathcal{D}_1} = \kappa \circ \varphi \circ \kappa^{-1} = \kappa \circ \kappa^{-1} \circ \phi_{|\mathcal{D}_1} \circ \kappa \circ \kappa^{-1} = \phi_{|\mathcal{D}_1}$$

and hence $\phi = \pi(\varphi)$.

So we have proven the following theorem.

2.6. Theorem. *The groups* $\mathrm{Aut}(\mathcal{D}_e{}')$ *and* $\mathrm{Aut}(\mathcal{D}_\omega{}')$ *are isomorphic.*

Finally, according to [4] the jump operation in $\mathcal{D}_e$ is first order definable and hence every automorphism of $\mathcal{D}_e$ is an automorphism of $\mathcal{D}_e{}'$ and vice versa, i.e., $\mathrm{Aut}(\mathcal{D}_e) = \mathrm{Aut}(\mathcal{D}_e{}')$. So we can reformulate Theorem 2.6 in the form:

2.7. Theorem. *The groups* $\mathrm{Aut}(\mathcal{D}_e)$ *and* $\mathrm{Aut}(\mathcal{D}_\omega{}')$ *are isomorphic.*

References

[1] M. Lerman, *Degrees of unsolvability*, Springer-Verlag, Berlin Heidelberg New York Tokyo, 1983.

[2] I. N. Soskov, *The ω-enumeration degrees*, Journal of Logic and Computation, to appear.

[3] Ivan Soskov and Bogomil Kovachev, *Uniform regular enumerations*, Mathematical Structures in Comp. Sci. **16** (2006), no. 5, 901–924.

[4] I. Sh. Kalimullin, *Definability of the Jump Operator in the Enumeration Degrees*, Journal of Mathematical Logic, Vol. 3, No. 2 (2003), 257-267

Faculty of Mathematics and Computer Science, Sofia University, 5 James Bourchier Blvd, 1164 Sofia, Bulgaria

E-mail address: h.ganchev@gmail.com, soskov@fmi.uni-sofia.bg