A total degree splitting theorem and a jump inversion splitting theorem

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Abstract. We propose a meta-theorem from which some splitting theorem for total e-degrees can be derived.

1 Introduction

Let A and B be two sets of natural numbers. We say that $A \leq_e B$ iff there is $i \in \mathbb{N}$ such that

$$x \in A \iff \exists u(\langle x, u\rangle \in W_i \ \&\ D_u \subseteq B),$$

where W_i is the i-th r.e. set of natural numbers and D_u is the finite set with canonical code u. We also define the enumeration operators $\mathit{\Gamma}_i : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ with $\mathit{\Gamma}_i(B) = \{x \mid \langle x, u\rangle \in W_i \ \&\ D_u \subseteq B\}$ for arbitrary $B \subseteq \mathbb{N}$. Then

$$A \leq_e B \iff \exists i(A = \mathit{\Gamma}_i(B)).$$

The relation $\leq_e$ is reflexive and transitive, which allow us to define the equivalence relation $\equiv_e$ with $A \equiv_e B \iff A \leq_e B \ \&\ B \leq_e A$. The equivalence classes respect to $\equiv_e$ are called enumeration degrees. The equivalence class generated by A is denoted with $\mathbf{d}_e(A)$. The degrees bf a and $\mathbf{b}$satisfy the relation $\mathbf{a} \leq_e \mathbf{b}$ iff there is $A \in \mathbf{a}$ and $B \in \mathbf{b}$, such that $A \leq_e B$.

We will say that the set A is total iff $\overline{A} \leq_e A$. Thus we obtain that a set A is total iff $A \equiv_e f$ where f is a total function.

Let A and B be two sets of natural numbers. We set $A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$. It is clear that, if $A, B \leq_e C$, then $A \oplus B \leq_e C$. In this way $\mathbf{d}_e(A \oplus B)$ is the least upper bound for $\mathbf{d}_e(A)$ and $\mathbf{d}_e(B)$. We define $A^+ = A \oplus \overline{A}$. A^+ is a total set.

Finally we define the jump operator in the following way: Let A be an arbitrary set of natural numbers. Let $L_A = \{\langle x, n\rangle \mid x \in \mathit{\Gamma}_n(A)\}$. Then we define the jump of A to be $A' = L_A^+$.

2 Splitting theorems

We will say that τ is a (total) final part iff $\tau : \{0, \ldots, n-1\} \to \mathbb{N}$ for some $n \in \mathbb{N}$. We set lh(τ) = n. Every finite part is in fact a finite sequence of natural numbers and so we will suppose that an effective coding of all finite parts is fixed. Thus the set of codes of all finite parts is recursive. From now on we will not make a difference between the finite part and its code. If τ is a finite part and $x \in \mathbb{N}$ we will denote with $\tau * x$ the function from $\{0, \dots, \mathrm{lh}(\tau)\}$ in $\mathbb{N}$ for which: $(\tau * x)(k) = \tau(k)$ if $k < \mathrm{lh}(\tau)$ and $(\tau * x)(\mathrm{lh}(\tau)) = \mathrm{x}$.

Let B be a set of natural numbers. We say that the finite part τ is B-regular iff $\mathrm{lh}(\tau) = 2\mathrm{k}$ and $\tau(2i+1) \in B$ for all $i < k$. We say that the total function f is a regular numeration of B if $f(2\mathbb{N}+1) = B$.

Let now $\mathcal{T}$ be a set of finite parts such that $\emptyset \in \mathcal{T}$ and if x is arbitrary and $\tau \in \mathcal{T}$ then there is $\tau' \supseteq \tau * x$, such that $\tau' \in \mathcal{T}$. Let also $\mathcal{F}$ be the set of all partial functions such that infinitely many their finite parts are in $\mathcal{T}$. For example if B is a set of natural numbers then $\mathcal{T}$ can be the set of all B-regular finite parts and $\mathcal{F}$ is the set of all regular numerations of B.

Lemma 1. *Let R_1, R_2,..., R_n,... be a sequence of binary relations over $\mathcal{F}$. Let A be a total set such that $\mathcal{T} \leq_e A$ and there are recursive in A functions ϕ and γ, such that for all $n \in \mathbb{N}$ and all $\tau_1, \tau_2 \in \mathcal{T}$:*

(i) $\phi(n, \tau_1) \supseteq \tau_1$ and $\phi(n, \tau_1, \tau_2) \in \mathcal{T}$;
(ii) $\gamma(n, \tau_2) \supseteq \tau_2$ and $\gamma(n, \tau_1, \tau_2) \in \mathcal{T}$;
(iii) for all $f, g \in \mathcal{F}$ if $f \supseteq \phi(n, \tau_1)$ and $g \supseteq \gamma(n, \tau_2)$, then $R_n(f, g)$.

Then there are functions $f, g \leq_e A$ such that $f \oplus g \equiv_e A$ and for all $n \in \mathbb{N}$ $R_n(f, g)$.

Now from the Lemma above we will derive some splitting theorems.

Theorem 1. *Let A and B be sets of natural numbers such that $A' \leq_e B$ and B is total. Then there, are total functions f and g, such that $A \leq_e f, g$, $f \not\leq_e g$, $g \not\leq_e f$ and $f \oplus g \equiv_e B$*

Proof. Let $\mathcal{T}$ be the set of all A-regular finite parts. Then $\mathcal{F}$ is the set of all regular numerations of A. Set the relations R_n over $\mathcal{F}$ be:

$$(f, g) \in R_{2n} \iff g \neq \Gamma_n(f)$$

$$(f, g) \in R_{2n+1} \iff f \neq \Gamma_n(g).$$

Then consider the function $\mathcal{U}(\tau, n, k)$ that returns the less $\rho \in \mathcal{T}$ for which $\tau \subset \rho$ and $\rho \vdash F_n(\langle x, y \rangle)$ for some $x, y \in \mathbb{N}$, where $x > k$, if such a ρ exists and returns τ otherwise. It is clear that $\mathcal{U}$ is recursive in A' and thus recursive in B. It is also clear that, if $\mathcal{U}(\tau, n, k) = \tau$, then for all $f \in \mathcal{F}$ that satisfy $\tau \subseteq f$, if $\Gamma_n(f)$ is a function, then it is a finite part. Now using $\mathcal{U}$ we can construct the functions ϕ and γ of the lemma by setting:

$$\phi(2n, \tau_1, \tau_2) = \mathcal{U}(\tau_1, n, \mathrm{lh}(\tau_2))$$

and $\gamma(2n, \tau_1, \tau_2) = \tau_2$ if $U(\tau_1, n, \mathrm{lh}(\tau_2)) = \tau_1$ and $\mu\rho \in \mathcal{T}[\rho \supseteq \tau \& \rho(x) \neq y]$ otherwise. For $2n+1$, ϕ and γ are defined analogously. As $\mathcal{T}$ and $\mathcal{U}$ are recursive in B, ϕ and γ are also recursive in B and so we can apply the lemma. Thus we obtain the desired total functions f and g.

□

Thus we obtain the following corollary:

Corollary 1. *Let A be a total set with $0' \leq_e A$. Then*

$$A \leq_e B \iff (\forall X\text{−total})(X \leq_e A \Rightarrow X \leq_e B).$$

Let us now consider the sets B_0, B_1,..., B_n. We define the "polynomials" $\mathcal{P}(B_0, \ldots, B_k)$ in the following way:

1)$\mathcal{P}(B_0) = B_0$;
2)$\mathcal{P}(B_0, \ldots, B_k) = \mathcal{P}(B_0, \ldots, B_{k-1})' \oplus B_k$ for $1 \leq k \leq n$

In [1] Soskov proved the following theorem:

Theorem 2. *Let $n > k \geq 0$ and $B_0, \ldots, B_n$ be arbitrary sets of natural numbers. Let $A \subseteq \mathbb{N}$ and Q be a total set such that $\mathcal{P}(B_0, \ldots, B_n) \leq_e Q$ and $A^+ \leq_e Q$. Suppose also that $A \not\leq_e \mathcal{P}(B_0, \ldots, B_k)$. Then there exists total set F having the following properties:*
(1) *for all $i \leq n$ $B_i \leq_e F^{(i)}$;*
(2) *for all $1 \leq i \leq n$ $F^{(i)} \equiv_e F \oplus \mathcal{P}(B_0, \ldots, B_{i-1})$;*
(3) *$F^{(n)} \equiv_e Q$;*
(4) *$A \not\leq_e F^{(k)}$.*

In order to prove the theorem Soskov introduced the notion of n-regular finite parts and the notion of n-rank of n-regular finite part. He proved that the set of all n-regular finite parts is e-reducible to Q. He formulated conditions ψ_i for the n-regular finite parts, such that if f is a total function such that for all $i \in \mathbb{N}$ there is a n-regular finite part $\tau \subseteq f$ for which $\psi_i(\tau)$, then f satisfies the properties of F in the theorem. He showed a recursive in Q procedure to obtain n-regular τ with $\psi_{i+1}(\tau)$ from $\rho * x$, where $x \in \mathbb{N}$ and ρ is n-regular satisfying ψ_i.

Let $\mathcal{T}$ be the set of all n-regular finite parts and define R_i as follows:

$$(f, g) \in R_i \iff (\exists \tau \in \mathcal{T})(\tau \subseteq f \;\&\; \psi_i(\tau)) \;\&\; (\exists \tau \in \mathcal{T})(\tau \subseteq g \;\&\; \psi_i(\tau)).$$

The conditions of the Lemma are satisfied and so we obtain:

Theorem 3. *Let $n > k \geq 0$ and $B_0, \ldots, B_n$ be arbitrary sets of natural numbers. Let $A \subseteq \mathbb{N}$ and Q be a total set such that $\mathcal{P}(B_0, \ldots, B_n) \leq_e Q$ and $A^+ \leq_e Q$. Suppose also that $A \not\leq_e \mathcal{P}(B_0, \ldots, B_k)$. Then there exists total sets F and G having the following properties:*
(1) *for all $i \leq n$ $B_i \leq_e F^{(i)}$ and $B_i \leq_e G^{(i)}$;*
(2) *for all $1 \leq i \leq n$ $F^{(i)} \equiv_e F \oplus \mathcal{P}(B_0, \ldots, B_{i-1})$ and $G^{(i)} \equiv_e G \oplus \mathcal{P}(B_0, \ldots, B_{i-1})$;*
(3) *$F^{(n)} \equiv_e Q$ and $G^{(n)} \equiv_e Q$;*
(4) *$A \not\leq_e F^{(k)}$ and $A \not\leq_e G^{(k)}$;*
(5) *$F \oplus G \equiv_e Q$.*

We can obtain the following

Corollary 2. *Let Q be a total set such that $0^{(n)} \leq_e Q$. Then there exists total sets F and G such that $F^{(n)} \equiv_e G^{(n)} \equiv_e Q$ and $F \oplus G \equiv_e Q$.*

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