The Limitations of Cupping in the Local Structure of the Enumeration Degrees

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The cupping property

Let $\langle \mathcal{A}, \leq, \vee, 1 \rangle$ be an upper semi-lattice with least and greatest element.

Definition

An element $\mathbf{a} \in \mathcal{A}$ is *cuppable* if there exists an element $\mathbf{b} \in \mathcal{A}$, $\mathbf{b} \neq \mathbf{1}$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{1}$.

The element **b** is called a *cupping partner* for **a**.

Results in the Turing degrees

- Posner, Robinson: Every nonzero degree in $\mathcal{D}_T(\leq 0')$ is cuppable.
- Cooper, Yates: There exists a nonzero c.e. Turing degree which cannot be cupped by any incomplete c.e. Turing degree.

Enumeration degrees

Definition

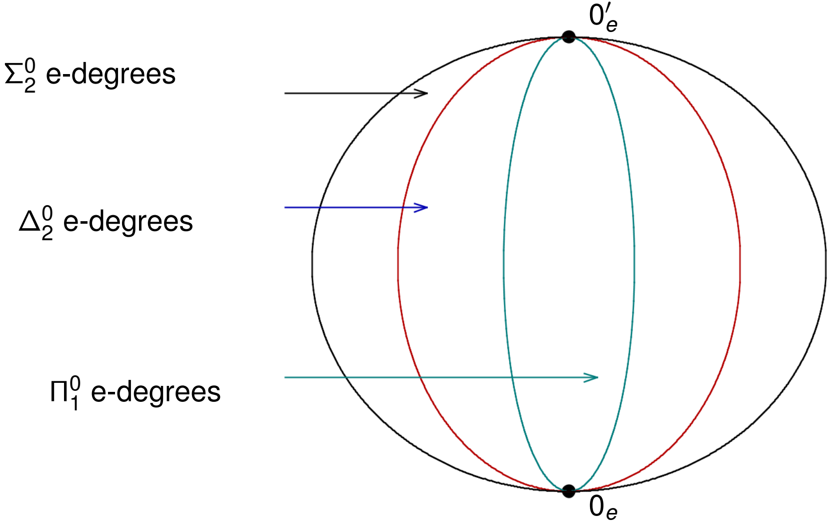
A set A is *enumeration reducible* ($\leq_e$) to a set B if there is a c.e. set Φ such that:

$$n \in A \Leftrightarrow \exists u(\langle n, u \rangle \in \Phi \wedge D_u \subseteq B),$$

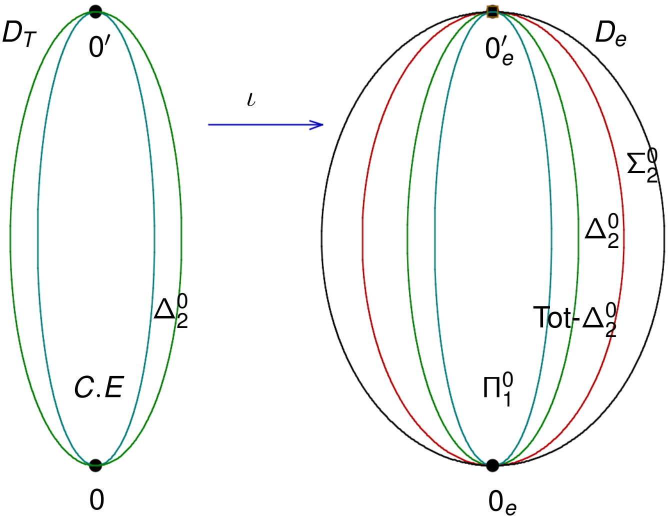
where D_u denotes the finite set with code u under the standard coding of finite sets.

- $A \equiv_e B \Leftrightarrow A \leq_e B \wedge B \leq_e A$
- $\langle \mathcal{D}_e, 0_e, \leq, \cup, ' \rangle$.
- $\iota : \mathcal{D}_T \rightarrow \mathcal{D}_e$.

The local structure of the enumeration degrees $\mathcal{D}_e(\leq 0'_e)$



Transferring results from the Turing degrees



Cupping results in the enumeration degrees

- Negative Results:
  (Cooper, Sorbi, Yi): There exists a nonzero Σ^0_2 enumeration degree that is not cuppable.
- Positive Results:
  (Cooper, Sorbi and Yi): Every nonzero Δ^0_2 e-degree is cuppable by a total incomplete Δ^0_2 e-degree.

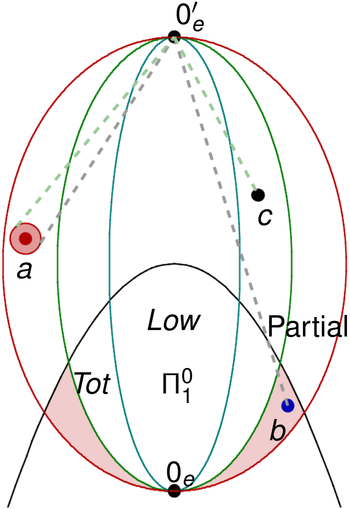
  (S, Wu): Every nonzero Δ^0_2 e-degree is cuppable by a partial and low Δ^0_2 e-degree.

Cupping partners

Question

How much further can we limit the search for cupping partners.

Δ_2^0 e-degrees



Reaching the first limit

Theorem

Let $\{\mathbf{a}_i\}_{i<\omega}$ be a Δ^0_2-computably enumerable sequence of enumeration degrees. There exists a nonzero Δ^0_2 enumeration degree $\mathbf{b}$ such that for every $i < \omega$ if $\mathbf{a}_i$ is incomplete then $\mathbf{a}_i \vee \mathbf{b} \neq 0'_e$.

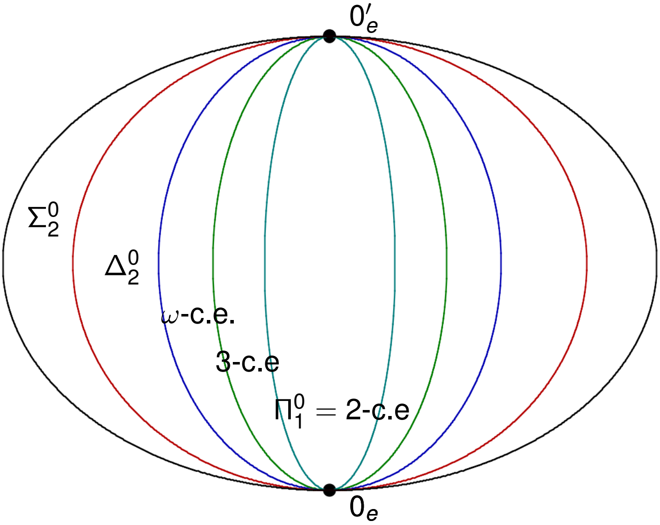
Here a class $\{\mathbf{a}_i\}_{i<\omega}$ of Δ^0_2 enumeration degrees is Δ^0_2-computably enumerable if there is a computable sequence of Δ^0_2 approximations $\{A_i[s]\}_{i,s<\omega}$ to representatives A_i of every degree $\mathbf{a}_i$ in the class.

The Difference Hierracy

Definition (Ershov)

1. A set A is n-c.e. if there is a computable function f such that for each x, $f(x, 0) = 0$, $|\{s+1 \mid f(x,s) \neq f(x,s+1)\}| \leq n$ and $A(x) = \lim_s f(x,s)$.
2. A is ω-c.e. if there are two computable functions $f(x,s), g(x)$ such that for all x, $f(x,0) = 0$, $|\{s+1 \mid f(x,s) \neq f(x,s+1)\}| \leq g(x)$ and $\lim_s f(x,s) = A(x)$.
3. A degree **a** is n-c.e.(ω-c.e.) if it contains a n-c.e.(ω-c.e.) set.

A finer partition of the Δ^0_2 enumeration degrees



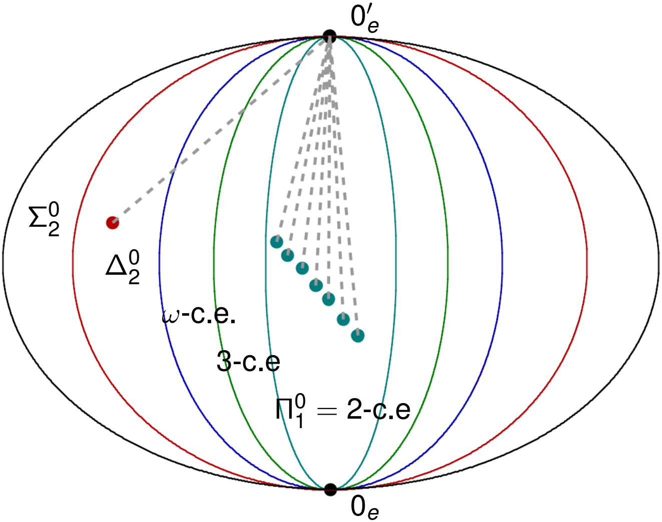
Consequences

Corollary

There exists a nonzero Δ^0_2 enumeration degree that cannot be cupped by any incomplete ω-c.e. degree.

Wu, S: For every nonzero ω-c.e. enumeration degree **a** there exists an incomplete 3-c.e. enumeration degree **b** that cups **a**.

Cupping classes of enumeration degrees



(Cooper, Seetapun and Li): There exists a single incomplete Δ^0_2 Turing degree that cups every nonzero c.e. Turing degree.

The second limitation

For any larger subclass, which contains the nonzero 3-c.e enumeration degrees this cannot be done as:

Theorem

Let $\mathbf{a}$ *be an incomplete* Σ^0_2 *enumeration degree. There exists a nonzero* 3*-c.e. enumeration degree* $\mathbf{b}$ *such that* $\mathbf{a} \vee \mathbf{b} \neq \mathbf{0}'_e$*.*

Proof(ideas)

Let A be a representative of the given Σ_2^0 e-degree. We construct two 3-c.e. sets X and Y so that:

- For every natural number e:

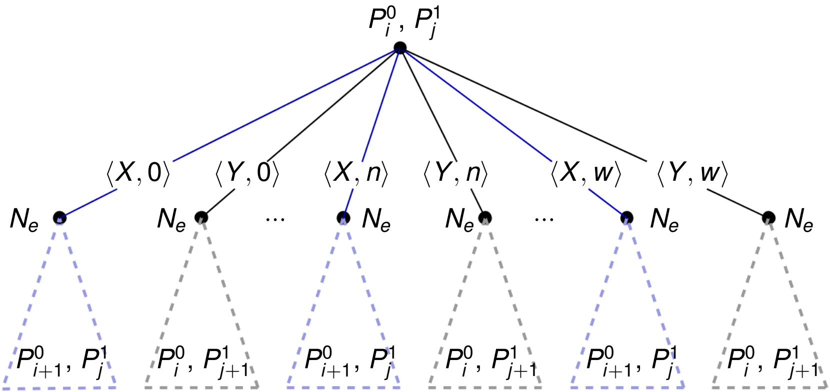
$$\mathcal{N}_e : W_e \neq X \wedge W_e \neq Y.$$

- If for some i the requirement $\mathcal{P}_i^0$ is not satisfied then every i the requirement $\mathcal{P}_i^1$ is satisfied , where:

$$\mathcal{P}_i^0 : \Theta_i^{A,X} \neq \overline{K};$$

$$\mathcal{P}_i^1 : \Psi_i^{A,Y} \neq \overline{K}.$$

The tree


P_i^0, P_j^1
$\langle X, 0\rangle$
$\langle Y, 0\rangle$
$\langle X, n\rangle$
$\langle Y, n\rangle$
$\langle X, w\rangle$
$\langle Y, w\rangle$
N_e
N_e
...
N_e
N_e
...
N_e
N_e
P_{i+1}^0, P_j^1
P_i^0, P_{j+1}^1
P_{i+1}^0, P_j^1
P_i^0, P_{j+1}^1
P_{i+1}^0, P_j^1
P_i^0, P_{j+1}^1

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