

A Locally Definable Set of Low Enumeration Degrees

Work in progress

Mariya I. Soskova¹

Faculty of Mathematics and Informatics
Sofia University

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Outline

- 1 \mathcal{K} -pairs: definition and properties
- 2 The main theorem and the definition of a set of low degrees
- 3 Basic ideas in the construction

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\mathcal{K} -pairs: Definition

Iskander Kalimullin: Definability of the jump operator in the enumeration degrees

Journal of Mathematical Logic (2003)

Definition

Let A , B and U be sets of natural numbers. The pair (A, B) is a \mathcal{K} -pair over U (U -e-ideal) if there exists a set $W \leq_e U$ such that $A \times B \subseteq W$ and $\bar{A} \times \bar{B} \subseteq \bar{W}$.

If $U = \emptyset$ then (A, B) is called a K -pair.

\mathcal{K} -pairs: A trivial example

Example

Let V be a c.e set. Then (V, A) is a \mathcal{K} -pair for any set of natural numbers A .

Let $W = V \times \mathbb{N}$. Then $V \times A \subseteq W$ and $\overline{V} \times \overline{A} \subseteq \overline{W}$.

\mathcal{K} -pairs: A more interesting example

Definition (Jockusch 1968)

A set of natural numbers A is semi-recursive if there is a computable function s_A such that for every pair of natural numbers (x, y) :

- 1 $s_A(x, y) \in \{x, y\}$.
- 2 If $x \in A$ or $y \in A$ then $s_A(x, y) \in A$.

Example

Let A be a semi-recursive set. Then (A, \overline{A}) is a \mathcal{K} -pair.

Let s_A be the selector function for A .

Set $W = \{(s_A(x, y), \overline{s_A(x, y)}) \mid x, y \in \mathbb{N}\}$,

where $\overline{s_A(x, y)} = x$ if $s_A(x, y) = y$ and $\overline{s_A(x, y)} = y$ if $s_A(x, y) = x$.

An order theoretic characterization of \mathcal{K} -pairs

Theorem (Kalimullin)

(A, B) is a \mathcal{K} -pair if and only if the degrees $\mathbf{a} = d_e(A)$ and $\mathbf{b} = d_e(B)$ have the following property:

$$(\forall \mathbf{c} \in \mathcal{D}_e)((\mathbf{a} \vee \mathbf{c}) \wedge (\mathbf{b} \vee \mathbf{c}) = \mathbf{c})$$

- 1 A pair of degrees (\mathbf{a}, \mathbf{b}) is a \mathcal{K} -pair
 \Leftrightarrow There is a \mathcal{K} -pair of sets (A, B) , such that $A \in \mathbf{a}$ and $B \in \mathbf{b}$.
 \Leftrightarrow For all $A \in \mathbf{a}$ and $B \in \mathbf{b}$ the pair of sets (A, B) is a \mathcal{K} -pair.
- 2 If (\mathbf{a}, \mathbf{b}) is a \mathcal{K} -pair then (\mathbf{a}, \mathbf{b}) is a minimal pair:

$$\underbrace{(\mathbf{a} \vee \mathbf{0}_e)}_{\mathbf{a}} \wedge \underbrace{(\mathbf{b} \vee \mathbf{0}_e)}_{\mathbf{b}} = \mathbf{0}_e$$

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\mathcal{K} -pairs in the local structure \mathcal{G}_e

Lemma (Kalimullin)

Let A and B be Σ_2^0 sets with Σ_2^0 approximation $\{A^{\{s\}}\}_{s < \omega}$ and $\{B^{\{s\}}\}_{s < \omega}$ such that:

$$(\forall s)(A^{\{s\}} \subseteq A \vee B^{\{s\}} \subseteq B).$$

Then (A, B) is a \mathcal{K} -pair.

Proof:

Let $W = \bigcup_{s < \omega} A^{\{s\}} \times B^{\{s\}}$.

If $(x, y) \in A \times B$ then there is a stage s such that $(x, y) \in A^{\{s\}} \times B^{\{s\}}$.

If $(x, y) \in \bar{A} \times \bar{B}$ then for all stages s , $A^{\{s\}} \subseteq A$ or $B^{\{s\}} \subseteq B$, hence $x \notin A^{\{s\}}$ or $y \notin B^{\{s\}}$, so $(x, y) \notin W$.

Constructing a nontrivial \mathcal{K} -pair

We want to construct Σ_2^0 approximations $\{A^{[s]}\}$ and $\{B^{[s]}\}$ to sets A and B such that for all e :

$$\mathcal{N}_e^A : W_e \neq A$$

$$\mathcal{N}_e^B : W_e \neq B$$

and for all s

$$\mathcal{K}_s : A^{[s]} \subseteq A \vee B^{[s]} \subseteq B.$$

Strategy for satisfying \mathcal{N} -requirements

To satisfy \mathcal{N}_e^A at every stage s do the following:

- 1 If a witness x_e is not yet selected then select a witness x_e as a fresh number and enumerate it in $A^{\{s\}}$.
- 2 If $x_e \in W_e^{\{s\}}$ then extract x_e from $A^{\{s\}}$.

To satisfy \mathcal{N}_e^B at every stage s do the following:

- 1 If a witness y_e is not yet selected then select a witness y_e as a fresh number and enumerate it in $B^{\{s\}}$.
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To satisfy \mathcal{N}_e^B at every stage s do the following:

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Conflict and solution

Conflict: If x_e is enumerated in $A^{\{s_{start}\}}$ and extracted from $A^{\{s_{end}\}}$ then for all t such that $s_{start} \leq t < s_{end}$, $A^{\{t\}} \not\subseteq B$. We must ensure that $B^{\{t\}} \subseteq B$.

Solution: We order the \mathcal{N} requirements linearly by priority:

$$\mathcal{N}_0^A < \mathcal{N}_0^B < \mathcal{N}_1^A < \mathcal{N}_1^B < \dots$$

and add injury:

If an \mathcal{N} -strategy is injured then the value of its witness is cancelled and the witness is enumerated back in the corresponding set A or B .

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Final construction

At every stage s activate the first s \mathcal{N} -strategies in order of their priority:

If the strategy \mathcal{N}_e^A is activated at stage s the:

- 1 If a witness x_e is not yet selected then select a witness x_e as a fresh number and enumerate it in $A^{\{s\}}$.
- 2 If $x_e \in W_e^{\{s\}}$ then extract x_e from $A^{\{s\}}$ and injure all lower priority strategies.

If the strategy \mathcal{N}_e^B is activated at stage s the:

- 1 If a witness y_e is not yet selected then select a witness y_e as a fresh number and enumerate it in $B^{\{s\}}$.
- 2 If $y_e \in W_e^{\{s\}}$ then extract y_e from $B^{\{s\}}$ and injure all lower priority strategies.

Properties of Σ_2^0 \mathcal{K} -pairs

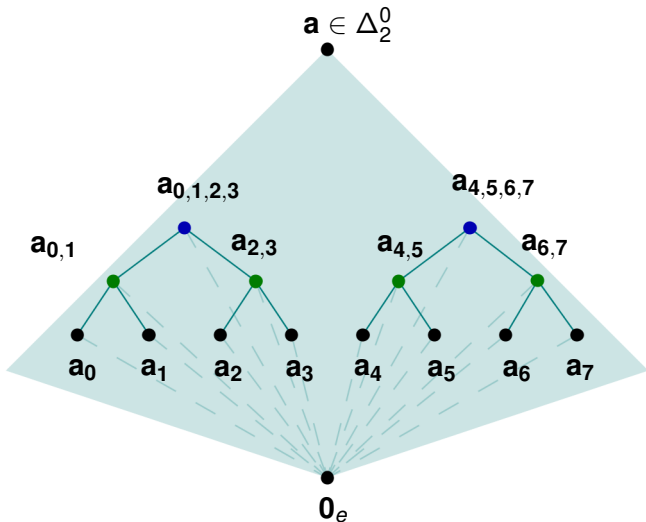
Theorem (Kallimulin)

- 1 If (\mathbf{a}, \mathbf{b}) are a Σ_2^0 \mathcal{K} -pair then \mathbf{a} and \mathbf{b} are low and quasi-minimal.
- 2 Every Δ_2^0 enumeration degree bounds a \mathcal{K} -pair.
- 3 There is a Δ_2^0 \mathcal{K} -pair (\mathbf{a}, \mathbf{b}) which splits $\mathbf{0}'_e$.
- 4 The set of degrees \mathbf{c} which form a \mathcal{K} -pair with a fixed degree \mathbf{a} is an ideal.

Theorem (Ganchev, S)

Every Δ_2^0 enumeration degree bounds a countable \mathcal{K} -system, i.e. a sequence $(\mathbf{a}_0, \mathbf{a}_1, \dots)$ of Δ_2^0 enumeration degrees such that for every $i \neq j$ the pair $(\mathbf{a}_i, \mathbf{a}_j)$ is a \mathcal{K} -pair.

Properties of Σ_2^0 \mathcal{K} -pairs



Definability of \mathcal{K} -pairs

$\mathcal{K}(\mathbf{a}, \mathbf{b}) =$ " (\mathbf{a}, \mathbf{b}) is a \mathcal{K} -pair" is definable in \mathcal{D}_e :

$$\mathcal{K}(\mathbf{a}, \mathbf{b}) \Leftrightarrow (\forall \mathbf{c})((\mathbf{a} \vee \mathbf{c}) \wedge (\mathbf{b} \vee \mathbf{c}) = \mathbf{c}).$$

Theorem (Kalimullin)

If $(d_e(A), d_e(B))$ is not a \mathcal{K} -pair then there is a witness C computable from $A \oplus B \oplus K$ such that:

$$(d_e(A) \vee d_e(C)) \wedge (d_e(B) \vee d_e(C)) \neq d_e(C)$$

- If A and B are Δ_2^0 then C is also Δ_2^0 and hence in the structure of the Δ_2^0 enumeration degrees the property $\mathcal{K}(\mathbf{a}, \mathbf{b})$ is definable and $\{\mathbf{a} \mid (\exists \mathbf{b})\mathcal{K}(\mathbf{a}, \mathbf{b})\}$ is a definable set of low degrees.
- If A and B are Σ_2^0 then C is Δ_3^0 .

Question

Is the property $\mathcal{K}(\mathbf{a}, \mathbf{b})$ first order definable in \mathcal{G}_e ?

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- If A and B are Σ_2^0 then C is Δ_3^0 .

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The main theorem

Theorem (S,Wu)

Every nonzero Δ_2^0 enumeration degree \mathbf{a} is low cuppable, i.e. there is a low \mathbf{b} such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$.

Theorem (Cooper, Sorbi, Yi)

There are non-cuppable Σ_2^0 enumeration degrees.

Theorem (The main theorem)

If \mathbf{u} and \mathbf{v} are Σ_2^0 enumeration degrees such that $\mathbf{u} \vee \mathbf{v} = \mathbf{0}'_e$ then \mathbf{u} is low cuppable or \mathbf{v} is low cuppable.

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A definition of a low set

Corollary

The formula $\mathcal{L}(\mathbf{a}) \Leftrightarrow (\exists \mathbf{b})(\mathcal{K}(\mathbf{a}, \mathbf{b}) \wedge \mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e)$ defines in \mathcal{G}_e a nonempty set of low enumeration degrees.

Proof:

There is a Δ_2^0 \mathcal{K} -pair which splits $\mathbf{0}'_e$ so the set $\{\mathbf{a} \mid \mathcal{G}_e \vdash \mathcal{L}(\mathbf{a})\} \neq \emptyset$.

Let \mathbf{a} be a Σ_2^0 degree such that $\mathcal{G}_e \vdash \mathcal{L}(\mathbf{a})$. Let \mathbf{b} a witness such that $\mathcal{K}(\mathbf{a}, \mathbf{b}) \wedge \mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$.

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Proof:

Case 1: \mathbf{b} is low cuppable. Let \mathbf{c} be a low Δ_2^0 e-degree which cups \mathbf{b} .

$$(\mathbf{a} \vee \mathbf{c}) \wedge \underbrace{(\mathbf{b} \vee \mathbf{c})}_{\mathbf{c}} = \mathbf{c}$$

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Hence $\mathbf{a} \leq \mathbf{c}$ and $\mathbf{a}' \leq \mathbf{c}' = \mathbf{0}'_e$ and the degree \mathbf{a} is low.

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So \mathbf{b} is low cuppable and we are back to Case 1.

A definition of a low set

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Main theorem

Theorem (The main theorem)

If \mathbf{u} and \mathbf{v} are Σ_2^0 enumeration degrees such that $\mathbf{u} \vee \mathbf{v} = \mathbf{0}'_e$ then \mathbf{u} is low cuppable or \mathbf{v} is low cuppable.

Proof:(Sketch)

Uses a construction very similar to the construction of a non-splitting enumeration degree.

Non-splitting as a strong form of cupping

Theorem (S)

There is an incomplete Σ_2^0 e -degree \mathbf{a} such that there is no splitting of $\mathbf{0}'_e$ in the e -degrees above \mathbf{a} .

Proof idea: Construct a Π_1^0 set E and a Σ_2^0 set A such that the following requirements are satisfied:

$$\mathcal{N}_\Psi : \Psi_e(A) \neq E.$$

$$\mathcal{R}_{\Theta, U, V} : \Theta(U, V) = E \Rightarrow \exists \Gamma, \Lambda (\Gamma(U, A) = \bar{K} \vee \Lambda(V, A) = \bar{K}).$$

So the constructed degree $\mathbf{a} = d_e(A)$ has the following property:

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Level 1: gaining control over the give sets

- Let U and V be Σ_2^0 degrees such that $U \oplus V \equiv_e \overline{K}$. Fix good Σ_2^0 approximations to U , V and $U \oplus V$.
- We will construct a Π_1^0 set E , which we use to force changes in the approximations to U and V .
- As $U \oplus V$ has complete e-degree there is an enumeration operator Θ such that $\Theta(U, V) = E$.
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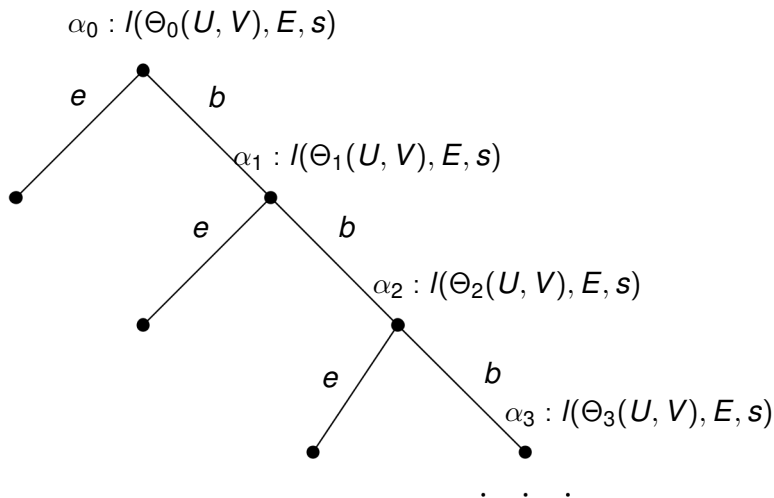
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Level 2: Working below outcome e

Suppose α is monitoring $I(\Theta(U, V), E, s)$ and see infinitely many expansionary stages.

The strategy α starts the construction of a set A and an operator Γ so that $\Gamma(U, A) = \overline{K}$.

On every visit it rectifies Γ on all $n \leq s$.

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Making A low

Definition

A set A is 1-generic if for every c.e. set W viewed as a set of finite binary strings

$$(\exists \tau \subseteq A)(\tau \in W \vee (\forall \mu \supseteq \tau)\mu \notin W).$$

Theorem (Copestake)

Every Δ_2^0 1-generic set is low.

For each c.e. set W_e add a strategy β_e ensuring:

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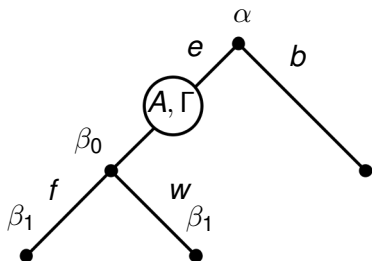
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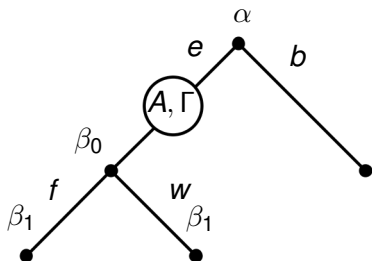
Naive strategy for β

- 1 Select a threshold $d \in \overline{K}$.
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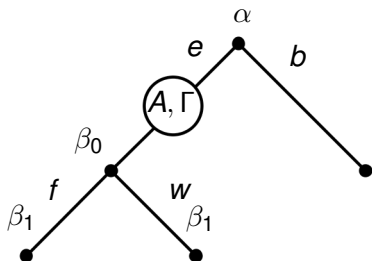
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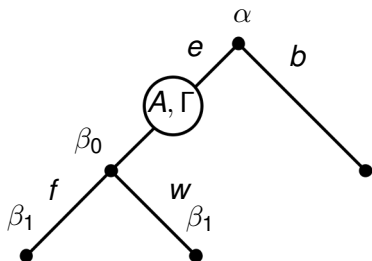
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Including the set E to force changes

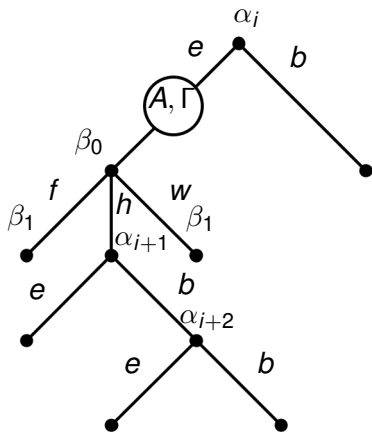
Solution: β will use an agitator.

- 1 Select an agitator $e \in E$.
- 2 As β is working under the assumption that $\Theta(U, V) = E$ so $e \in \Theta(U, V)$ with axiom $\langle e, U_e, V_e \rangle$.
- 3 Extracting e from the set E forces e out of $\Theta(U, V)$ and $\langle e, U_e, V_e \rangle$ is not valid at further stages. This is called an attack.
- 4 Arrange things so that $U_e \not\subseteq U$ invalidates all axioms in Γ for elements $n \geq d$.
- 5 This moves α 's influence on A on elements larger than $|\mu|$ and we can preserve $\mu \subseteq A$.

Honestification before an attack

- Make sure that $u(d) > \max U_e$ so that $U_e \not\subseteq U$ ensures $\langle d, U \upharpoonright u(n) + 1, A \upharpoonright a(d) + 1 \rangle$ invalid.
- This must be repeated every time the valid axiom for e in Θ changes before the attack.
- If this happens infinitely many times then there are no valid axioms for e in Θ and $\Theta(U, V) \neq E$.
- A new outcome h is added to signify this.

Honestification



Transferring results from the Turing degrees

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