

A gentle introduction to Harrington non-splitting and beyond

Mariya I. Soskova

University of Leeds
Department of Pure Mathematics

14.02.2007

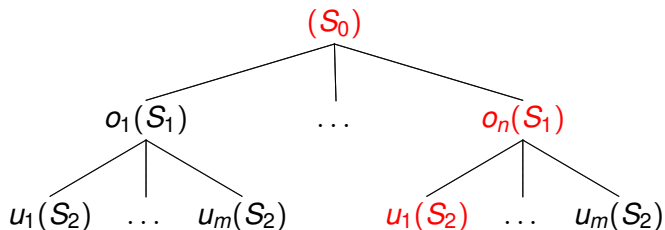
The Priority Method

- 1958 Post's Problem and Friedberg and Muchnik's solution
- 1963 Sacks Jump Theorem and 1964 Density theorem
- 1975 Lachlan's Monster Theorem and the priority tree method

The priority tree method

- Construct a set A satisfying R_0, R_1, \dots
- Priority ordering $R_0 < R_1 \dots$
- Strategies and outcomes: S_0 with outcomes o_1, \dots, o_n , S_1 with outcomes u_1, u_2, \dots, u_m .
- The tree of strategies is a computable tree of all possible ways that the construction might go.

The construction

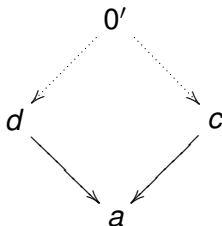


- At stage s we construct a finite path through the tree, approximating the true path.
- We *injure* strategies to the right.
- If a strategy is not injured infinitely many times and is visited on infinitely many stages it satisfies its requirement.
- The most left infinite path each initial segment of which is visited infinitely many times is the true path.

Harrington's Nonsplitting theorem

Theorem

There exists a c.e. degree $a < 0'$ such that $0'$ can not be split over a .



The Requirements

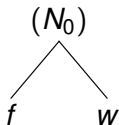
We will construct the c.e. sets A and E

- $N_\Psi : E \neq \Psi^A$ - hence A is not complete
- $P_{\Theta, U, V} : E = \Theta^{U, V} \Rightarrow (\exists \Gamma, \Lambda)[K = \Gamma^{U, A} \vee K = \Lambda^{V, A}]$

Priority: $N_0 < P_0 < N_1 \dots$

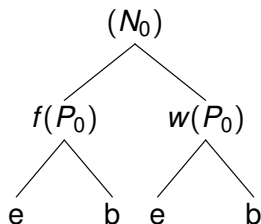
We start off with $A = E = \emptyset$.

The first N -strategy



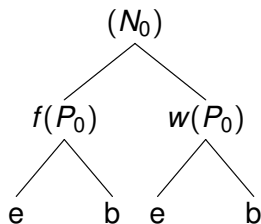
- $N_0 : E \neq \Psi_0^A$
- Select a witness x for N_0
- Wait for $\Psi_0^{A_s}(x) = 0$
- Enumerate x in E and restrain $A \upharpoonright \psi(x)$.

The first P strategy



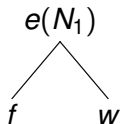
- $P_{\Theta, U, V} : E = \Theta^{U, V} \Rightarrow (\exists \Gamma, \Lambda)[K = \Gamma^{U, A} \vee K = \Lambda^{V, A}]$
- Monitor the length of agreement $l(s) = l(E_s, \Theta^{U, V}[s])$.
- If the length of agreement is bounded, then $E \neq \Theta^{U, V}$.
- If $E = \Theta^{U, V}$ then we have infinitely many expansionary stages.

The first P strategy



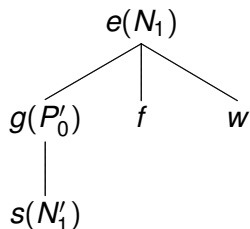
- $P_{\Theta, U, V} : (\exists \Gamma, \Lambda)[K = \Gamma^{U, A} \vee K = \Lambda^{V, A}]$
- At expansionary stages construct a Turing operator Γ , so that $\Gamma^{U, A} = K$.
- Γ is a c.e. set of axioms of the form $\Gamma^{\tau_1, \tau_2}(z) = v$.
- For each $z < l$: axiom $\Gamma^{U_s \upharpoonright (u(z)+1), A_s \upharpoonright (\gamma(z)+1)}(z) = K_s(z)$.
- We are allowed to enumerate new axioms only if the previous ones are not valid anymore.
- If $K(z)$ changes, enumerate $\gamma(z)$ in A and rectify Γ .

The second N - strategy



- A -restraint by N_1 conflicts with the need to rectify Γ by P_0 .
- Expansionary stages: $\Theta^{U,V}(x) = E(x) = 0$
- Enumerating the witness x in E ensures a change in the set $U \oplus V \upharpoonright \theta(x)$.
- A U -change enables us to move the markers $\gamma(n)$ above $\psi(x)$ without changing A .

The second N - strategy



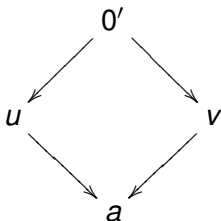
- A V - change is not useful at all - we have to try again with a new witness.
- It would be useful if we were constructing $\Lambda^{V,A} = K$.
- A backup strategy P'_0 will work only when the attack ends with a V -change.
- A copy of N_1 - will now be able to satisfy its requirement.

Further generalizations

- The requirement that both degrees above a are c.e. is strong.
- Even mildly weakening it is not possible: Arslanov's Splitting Theorem

Theorem

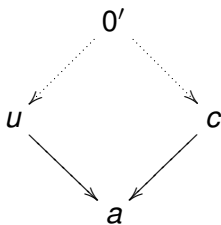
There is a d.c.e. splitting of $0'$ above each c.e. degree $a < 0'$.



The strongest non-splitting theorem

Theorem

There exists a c.e. degree $a < 0'$ such that there exists no nontrivial splitting of $0'$ into a c.e. and a Δ_2 degree above a .



Embedding the Turing degrees into the enumeration degrees

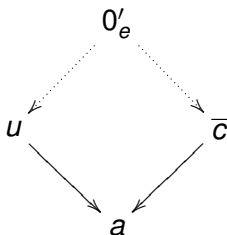
There exists an order theoretic embedding $\iota : D_T \rightarrow D_e$ with following properties.

- 1 ι preserves least element, joins and jump operators
- 2 The c.e. Turing degrees embed exactly onto the Π_1 enumeration degrees
- 3 There are partial Δ_2 degrees.

A theorem in the e-degrees

Theorem

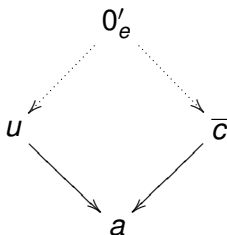
There exists a Π_1 e-degree $a < 0'_e$ such that there exist no nontrivial splittings of $0'_e$ into a Π_1 e-degree and Δ_2 e-degree above a .



A more general theorem in the e-degrees

Theorem

There exists a Π_1 e-degree $a < 0'_e$ such that there exist no nontrivial splittings of $0'_e$ into a Π_1 e-degree and a Σ_2 e-degree above a .



The Requirements

We will construct the Π_1 sets A and E

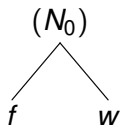
- For all enumeration operators Ψ :

$$N_\Psi : E \neq \Psi^A$$

- For each pair of a Σ_2 set U and a Π_1 set \overline{W} and each enumeration operator Θ :

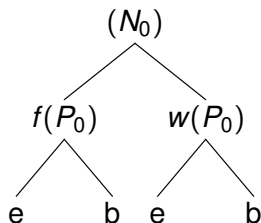
$$P_{\Theta, U, W} : E = \Theta^{U, \overline{W}} \Rightarrow (\exists \Gamma, \Lambda)[\overline{K} = \Gamma^{U, A} \vee \overline{K} = \Lambda^{\overline{W}, A}]$$

The first N strategy



- Select a witness $x \in E$ for N_Ψ
- Wait for $x \in \Psi^A[s]$, i.e. for an axiom $\langle x, A_x \rangle \in \Psi_s$ with $A_x \subset A_s$.
- Extract x from E and restrain each $y \in A \upharpoonright \psi(x)$.

The first P -strategy



- Monitor the length of agreement $l(s) = l(E_s, \Theta^{U, \bar{W}}[s])$.
- At expansionary stages construct an enumeration operator Γ , so that $\Gamma^{U, A}[s] = \bar{K}[s]$
- For each $n < l$ such that $n \in \bar{K}_s$: axiom $\langle n, U_s \upharpoonright (u(n) + 1), A_s \upharpoonright (\gamma(n) + 1) \rangle \in \Gamma$.
- For each $n < l$ such that $n \notin \bar{K}_s$: make all previously defined axioms invalid by extracting $\gamma(n)$ from A .

Complications

- The set U is now Σ_2 .
- A Σ_2 -approximation U_s gives us:
 - 1 If $a \in U$ then $a \in U_s$ for almost all s .
 - 2 If $a \notin U$ then $a \notin U_s$ for infinitely many s
- It could happen that $I(E, \Theta^{U, \overline{W}})$ is bounded but $E = \Theta^{U, \overline{W}}$.
- We may never see the right approximation to $U \upharpoonright u(n)$.

Good approximations

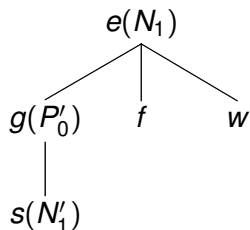
Definition

An approximation U_s to a set U is good if it has the following properties:

- 1 For all n there exists a good stage s such that $U \upharpoonright n \subset U_s \subset U$.
- 2 For all n there exists a stage s such that if $t > s$ is a good stage then $U \upharpoonright n \subset U_t$

We define a good Σ_2 approximation to U and $U \oplus \overline{W}$ with infinitely many common good stages.

The second N -strategy



- Choose a witness x and try to achieve $\gamma(n) > \psi(x)$ before the imposition of the restraint.
- Extract x from E . Return of $I(E, \Theta^{U, \bar{W}})$ forces U or \bar{W} to change below $\theta(x)$.
- Only trust \bar{W} -changes - start over with new witness, implement the backup strategy which insures $\Lambda^{\bar{W}, A} = \bar{K}$
- Otherwise: lift the gamma markers and preserve the restraint, but keep an eye on \bar{W} , for further changes.

The overall picture for the enumeration degrees

- The strongest non-splitting theorem
- Theorem(Sorbi, Arslanov) There is a Δ_2 splitting of $0'_e$ above each Δ_2 degree.
- What about splitting/non-splitting above a Σ_2 degree?

Motivation

- Non-cuppable degrees:
 - 1 There is non-cuppable c.e. degree in the Turing degrees. Cooper, Yates
 - 2 Every Δ_2 e-degree is Δ_2 cuppable. Cooper, Sorbi, Xiaoding Yi.
 - 3 There is a Σ_2 non-cuppable e-degree. Cooper, Sorbi, Xiaoding Yi.
- Non-bounding degree
 - 1 There is non-bounding c.e. degree in the Turing degrees. Lachlan
 - 2 Every Δ_2 e-degree bounds a minimal pair. Cooper, Li, Sorbi, Yang
 - 3 There is a Σ_2 non-bounding e-degree. Cooper, Li, Sorbi, Yang
- Non-splitting
 - 1 Harrington's Theorem
 - 2 Arslanov and Sorbi's Theorem

Non-splitting in the enumeration degrees

We will construct a Σ_2 set A and a Π_1 set E

- For all enumeration operators Ψ :

$$N_\Psi : E \neq \Psi^A$$

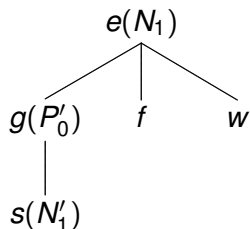
- For each pair of a Σ_2 set U and V and each enumeration operator Θ :

$$P_{\Theta,U,V} : E = \Theta^{U,V} \Rightarrow (\exists \Gamma, \Lambda)[\bar{K} = \Gamma^{U,A} \vee \bar{K} = \Lambda^{V,A}]$$

Differences

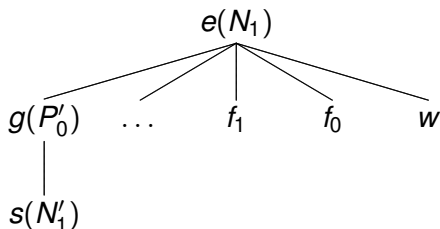
- Now we need to deal with two Σ_2 sets.
- We use good approximation again.
- We lose the stability that using the Π_1 set \overline{W} gave us.

The Problem in Detail










- Lets Look at the second N -strategy
- When we attack with a witness x , we get a change in $U \oplus V$.
- The pair of Σ_2 sets can now trick us - giving us false information: a V -change that is later on corrected.
- The N -requirement that counted on this V -change will be injured.

Longer memory



- We keep a record of all previous attempts - detailed information about each witness $x_0 < x_1 < \dots$.
- Every time we attack - first take a look at what happened to previous witnesses.
- Only when we have all changes in V_0 , in V_1, \dots do we let the backup strategy work.
- Delayed successfulness of previous attacks.

Bibliography

-  S. B. Cooper, *Computability Theory*, Chapman & Hall/CRC Mathematics, Boca Raton, FL, 2004.
-  P. G. Odifreddi, *Classical Recursion Theory, Volume II*, North-Holland/Elsevier, Amsterdam, Lausanne, New York, Oxford, Shannon, Singapore, Tokyo 1999.
-  R. I. Soare, *Recursively enumerable sets and degrees*, Springer-Verlag, Heidelberg, 1987.
-  L. Harrington, *Understanding Lachlan's Monster Paper*, Notes
-  A.H. Lachlan, R.A. Shore, *The n -re Enumeration Degrees are Dense*, Arch. Math. Logic (1992)31 : 277-285.
-  S.D. Leonhardi, *Generalized Nonsplitting in the Recursively Enumerable Degrees*
-  R. Soare, *Notes on Lachlan's Monster Theorem*