The high/low hierarchy in the local structure of the ω-enumeration degrees

Hristo Ganchev[a,1,2], Mariya Soskova[a,1,3]

[a]*Sofia University, Faculty of Mathematics and Informatics, 5 James Bourchier blvd. 1164 Sofia, Bulgaria*

Abstract

This paper gives two definability results in the local theory of the ω-enumeration degrees. First we prove that the local structure of the enumeration degrees is first order definable as a substructure of the ω-enumeration degrees. Our second result is the definability of the the classes H_n and L_n of the high$_n$ and low$_n$ ω-enumeration degrees. This allows us to deduce that the first order theory of true arithmetic is interpretable in the local theory of the ω-enumeration degrees.

1. Introduction

One of the oldest and most widely used mathematical approaches to understanding a structure is placing it in a wider context. This approach has often proved to be very effective, revealing properties of the structure that remain hidden in the smaller context. For example study of the enumeration degrees has been motivated largely (but not exclusively) by the fact that the structure of the Turing degrees is embedded in it, a result due to Myhill [8]. Support for this motivation has recently been given by Soskova and Cooper [14] who apply a structural result of the Σ^0_2 enumeration degrees to prove an extension of Harrington's non-splitting theorem for the Turing degrees.

The structure of the ω-enumeration degrees is a further attempt to widen the degree theoretic context. This structure is introduced by Soskov [12] and studied in the works of Soskov and Ganchev [4, 5, 13]. It is an upper semi-lattice with jump operation, where the building blocks of the degrees are of a higher type - sequences of sets of natural numbers. The main interest in this structure arises from the result that $\mathcal{D}_\omega$ is itself an extension of the structure of

the enumeration degrees $\mathcal{D}_e$. There is mapping κ of $\mathcal{D}_e$ into $\mathcal{D}_\omega$ which preserves the order, the least element, the least upper bound and the jump operation. In this case however the known relationship between the two structures are much stronger. Soskov and Ganchev [13] prove that the structure of the enumeration degrees is first order definable in the structure of the ω-enumeration degrees and furthermore the two structures have isomorphic automorphism groups.

In this article we will be mainly focused on the local structure of the ω-enumeration degrees, $\mathcal{G}_\omega$, and its connections to the local structure of the enumeration degrees, $\mathcal{G}_e$. The local structure of the ω-enumeration degrees consists of all ω-enumeration degrees that are below the first jump, $\mathbf{0}'_\omega$, of the least ω-enumeration degree. Previous work by Soskov and Ganchev [13] reveals the excessive information content of this structure. For instance for every natural number n there is an embedding of the interval $[\mathbf{0}_\omega^{(n)}, \mathbf{0}_\omega^{(n+1)}]$ of ω-enumeration degrees, where $\mathbf{0}_\omega^{(n)}$ denotes the n-th iteration of the jump on the least degree.

Our first result is a local analog of the above quoted relationship between the structures of the ω-enumeration degrees and the enumeration degrees.

Theorem 1. *The local structure of the enumeration degrees, viewed as a substructure of the ω-enumeration degrees, is first order definable in the local structure of the ω-enumeration degrees.*

Motivated by this connection we turn to the study of the high-low jump hierarchy of ω-enumeration degrees. For every n we denote with H_n the class of all ω-enumeration degrees in the local structure whose n-th jump is as high as possible, namely $\mathbf{0}_\omega^{(n+1)}$ and with L_n the class of all ω-enumeration degrees in the local structure whose n-th jump is as low as possible, namely $\mathbf{0}_\omega^{(n)}$. Our second result shows that these classes are also first order definable.

Theorem 2. *For every natural number n the classes H_n and L_n are first order definable in the local theory of the ω-enumeration degrees.*

Wether or not the classes of all high_n and all low_n enumeration degrees are definable in the local theory of the enumeration degrees is not known. As an immediate corollary of these two results we can obtain however that they are definable in the local theory of the ω-enumeration degrees.

A more significant application of Theorems 1 and 2 is the following result:

Theorem 3. *The first order theory of true arithmetic is interpretable in the local theory of the ω-enumeration degrees.*

This result gives further proof of the complexity of the local structure of the ω-enumeration degrees. It does not however characterize the strength of the theory completely, as unlike the theory of the c.e. degrees or the local theory of the enumeration degrees, it is not clear whether or not one can interpret the local theory of the ω-enumeration degrees in first order arithmetic.

The proof techniques used in this article use extensively the notation of a $\mathcal{K}$-pair. This notion is introduced and used by Kalimullin [6] to prove that the

enumeration jump is definable in the theory of the enumeration degrees. We leave formal definitions for Section 6, but note here that $\mathcal{K}$-pairs have very interesting properties. For example every $\mathcal{K}$-pair $\{a, b\}$ of Σ_2^0 enumeration degrees is a low quasi-minimal minimal pair. In Section 6 we study the properties and give a characterization of $\mathcal{K}$-pairs in the local ω-enumeration degrees.

Additionally we use a structural property of the enumeration degrees. Recall that the total degrees are the images of the Turing degrees under Rogers' embedding. Cooper, Sorbi and Yi [3] prove that every nonzero Δ_2^0 enumeration degree can be cupped to $\mathbf{0}_e'$ by a total incomplete Δ_2^0 e-degree. Later Soskova and Wu [15] show that every nonzero Δ_2^0 enumeration degree can be cupped by a non-total and low Δ_2^0 enumeration degree. We give an alternative proof of Soskova and Wu's result, which we see as structurally more informative.

Theorem 4. *For every non-zero Δ_2^0 enumeration degree $\mathbf{a}$ there exists a half of a nontrivial $\mathcal{K}$-pair $\mathbf{b}$, such that $\mathbf{a} \vee \mathbf{b} = 0'$.*

The proof of this theorem is rather technical and we will present it in the last section of this article. We also show how this proof can be relativized to prove the following.

Theorem 5. *For every total enumeration degree $\mathbf{g}$ and every degree $\mathbf{a}$, such that $\mathbf{g} \lneq \mathbf{a}$ and $\mathbf{a}$ contains a set Δ_2^0 relative to $\mathbf{g}$, there is a degree $\mathbf{b} > \mathbf{g}$ such that $\mathbf{b} \vee \mathbf{a} = \mathbf{g}' = \mathbf{b}'$.*

2. Preliminaries

We will use standard notation as can be found in [10] and [2]. We assume that the reader is familiar with basic degree theoretic notions and refer to Cooper [1] and Sorbi [11] for an extensive survey of result on both the global and local theory of the enumeration degrees. For completeness we outline basic notions used in this article.

Intuitively a set of natural numbers B is enumeration reducible ($\leq_e$) to a set of natural numbers A if one can obtain an enumeration of the set B given any enumeration of the set A. More formally:

Definition 1. *$B \leq_e A$ if there exists a c.e. set W such that*

$$B = \{n \mid \exists u(\langle x, u \rangle \in W \wedge D_u \subset A)\},$$

where D_u denotes the finite set with canonical index u.

The c.e. set W can be viewed as on operator on $\mathcal{P}(\mathbb{N})$ and will be referred to as an enumeration operator or e-operator. The elements of the set W will be called axioms. As each axiom consists of a natural number x and the code u of a finite set D_u, we will denote an axiom by $\langle x, D_u \rangle$.

The relation $\leq_e$ is a preorder on the powerset of the natural numbers and gives rise to a nontrivial equivalence relation $\equiv_e$. The equivalence classes under

this relation are called enumeration degrees and the their collection is denoted by $\mathbf{D}_e$. The enumeration degree of a set A is denoted by $\mathbf{d}_e(A)$. Enumeration reducibility between sets gives rise to a partial ordering $\leq_e$ on the enumeration degrees, namely

$$\mathbf{d}_e(A) \leq_e \mathbf{d}_e(B) \iff A \leq_e B.$$

We denote by $\mathcal{D}_e$ the partially ordered set $(\mathbf{D}_e, \leq_e)$. The enumeration degree of $\emptyset$, $\mathbf{0}_e$, is the least element in $\mathcal{D}_e$. Furthermore, the enumeration degree of $A \oplus B$ is the least upper bound of the degrees of A and B, so that $\mathcal{D}_e$ is an upper semi-lattice with least element.

The enumeration jump of a set A is defined as $A'_e = L_A^+$, where $L_A = \{\langle x, i \rangle \mid x \in W_i(A)\}$. This jump operation preserves enumeration reducibility and we can define $\mathbf{d}_e(A)' = \mathbf{d}_e(A'_e)$. Furthermore, $A \lneq_e A'_e$ and hence for an arbitrary enumeration degree $\mathbf{a}$, $\mathbf{a} \lneq_e \mathbf{a}'$. Finally we note that the jump operation is uniform in the sense that there exists a computable function g such that for arbitrary set A and B if $A = W_e(B)$ then $A' = W_{g(e)}(B')$.

The standard embedding ι of the partially ordered set of Turing degrees $\mathcal{D}_T$ in $\mathcal{D}_e$ is defined by $\iota(\mathbf{d}_T(A)) = \mathbf{d}_e(A^+)$. It preserves the order, the least element, the least upper bound and the jump operation. A set A is called total if $A \equiv_e A^+$ and a enumeration degree $\mathbf{a}$ is total if it contains a total set. Hence the range of ι consists exactly of the total enumeration degrees.

The jump operation gives rise to the local substructure, $\mathcal{G}_e$, consisting of all enumeration degrees below the jump, $\mathbf{0}'_e$, of the least degree. Cooper [1] proves that these are exactly the Σ_2^0 enumeration degree, i.e. the enumeration degrees of Σ_2^0 sets.

3. The ω-enumeration degrees

Soskov [12] introduces a reducibility, $\leq_\omega$, between sequences of sets of natural numbers. The original definition involves the so called *jump set* of a sequence and can be found in [12]. We use an equivalent definition in terms of uniform e-reducibility, which is more approachable. Before we define ω-reducibility we will need to introduce one more notation. Let $\mathcal{S}_\omega$ denote the class of all sequences of sets of natural numbers of length ω. With every member $\mathcal{A} \in \mathcal{S}_\omega$ we connect a jump sequence $P(\mathcal{A})$.

Definition 2. *Let $\mathcal{A} = \{A_n\}_{n<\omega} \in \mathcal{S}_\omega$. The* jump sequence *of the sequence $\mathcal{A}$, denoted by $P(\mathcal{A})$ is the sequence $\{P_n(\mathcal{A})\}_{n<\omega}$ defined inductively as follows:*

- $P_0(\mathcal{A}) = A_0$.

- $P_{n+1}(\mathcal{A}) = A_{n+1} \oplus P'_n(\mathcal{A})$, *where $P'_n(\mathcal{A})$ denotes the enumeration jump of the set $P_n(\mathcal{A})$.*

The jump sequence $P(\mathcal{A})$ transforms a sequence $\mathcal{A}$ into a monotone sequence of sets of natural numbers with respect to $\leq_e$. Every member of the jump sequence contains full information on previous members. The jump sequences

of sequence of natural numbers will be the objects that we are interested in, the building blocks of the ω-enumeration degrees. We define ω-reducibility appropriately so that every sequence turns out to be equivalent to its jump sequence.

Definition 3. *Let $\mathcal{A} = \{A_n\}_{n<\omega}, \mathcal{B} \in \mathcal{S}_\omega$. We shall say that $\mathcal{A}$ is ω-enumeration reducible to $\mathcal{B}$, denoted by $\mathcal{A} \leq_\omega \mathcal{B}$, if for every n we have $A_n \leq_e P_n(\mathcal{B})$ uniformly in n.*

Clearly "$\leq_\omega$" is a reflexive and transitive relation and defines a preorder on $\mathcal{S}_\omega$. The degree structure obtained from $\leq_\omega$ by the standard method is the structure of the ω-enumeration degrees, $\mathbf{D}_\omega$. We define the relation $\leq_\omega$ on ω-enumeration degrees by

$$\mathbf{d}_\omega(\mathcal{A}) \leq_\omega \mathbf{d}_\omega(\mathcal{B}) \iff \mathcal{A} \leq_\omega \mathcal{B},$$

The degree $\mathbf{0}_\omega$ of the sequence $\emptyset_\omega$, whose every member is the empty set, is the least element in $\mathbf{D}_\omega$ with respect to $\leq_\omega$.

For arbitrary sequences $\mathcal{A} = \{A_k\}_{k<\omega}$ and $\mathcal{B} = \{B_k\}_{k<\omega}$ we set

$$\mathcal{A} \oplus \mathcal{B} = \{A_k \oplus B_k\}_{k<\omega}.$$

It is not difficult to see that $\mathbf{d}_\omega(\mathcal{A} \oplus \mathcal{B})$ is the least upper bound of $\mathbf{d}_\omega(\mathcal{A})$ and $\mathbf{d}_\omega(\mathcal{B})$ and so the structure $\mathcal{D}_\omega = (\mathbf{D}_\omega, \leq_\omega)$ is an upper semilattice with least element.

Denote by $A \uparrow \omega$ the sequence $(A, \emptyset, \emptyset, \dots)$. It follows from the definition of ω-enumeration reducibility and the uniformity of the enumeration jump operation that for every pair of sets of natural numbers A and B:

$$A \uparrow \omega \leq_\omega B \uparrow \omega \iff A \leq_e B, \tag{3.1}$$

Using equivalence (3.1) we may define an embedding of the upper semilattice of the enumeration degrees in the upper semilattice of the ω-enumeration degrees. Indeed, consider the mapping $\kappa : \mathbf{D}_e \to \mathbf{D}_\omega$ defined by

$$\kappa(\mathbf{d}_e(A)) = \mathbf{d}_\omega(A \uparrow \omega).$$

It follows from (3.1) that κ is correctly defined and that

$$\forall \mathbf{a}, \mathbf{b} \in \mathbf{D}_e[\mathbf{a} \leq_e \mathbf{b} \iff \kappa(\mathbf{a}) \leq_\omega \kappa(\mathbf{b})],$$

which implies, that κ is order preserving and injective. Furthermore, note that for arbitrary sets A and B we have $(A \oplus B) \uparrow \omega = (A \uparrow \omega) \oplus (B \uparrow \omega)$ and hence

$$\kappa(\mathbf{a} \vee \mathbf{b}) = \kappa(\mathbf{a}) \vee \kappa(\mathbf{b}).$$

Finally we have that $\emptyset_\omega = \emptyset \uparrow \omega$, so that

$$\kappa(\mathbf{0}_e) = \mathbf{0}_\omega.$$

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We shall refer to κ as *the natural embedding* of the enumeration degrees in the ω-enumeration degrees. We denote the range of κ by $\mathbf{D}_1$ and call it *the natural copy* of the enumeration degrees.

In addition to embedding the enumeration degrees in $\mathcal{D}_\omega$, we can define a surjective order-preserving mapping from $\mathcal{D}_\omega$ onto $\mathcal{D}_e$. Consider the mapping $\lambda : \mathbf{D}_\omega \to \mathbf{D}_e$ acting by the rule

$$\lambda(\mathbf{d}_\omega(\mathcal{A})) = \mathbf{d}_e(P_0(\mathcal{A})).$$

By Definition 3 if $\mathcal{A} \leq_\omega \mathcal{B}$ then $P_0(\mathcal{A}) \leq_e P_0(\mathcal{B})$. From this it follows that λ is correctly defined and order preserving. To see that the mapping λ is onto, notice that for an arbitrary set A, we have that $P_0(A \uparrow \omega) = A$ and hence $A = \lambda(\mathbf{d}_\omega(A \uparrow \omega))$. On the other hand, as for any sequence $\mathcal{A}$ we have that $P_0(\mathcal{A}) \uparrow \omega \leq_\omega \mathcal{A}$ and therefore

$$\forall \mathbf{a} \in \mathbf{D}_\omega[\kappa(\lambda(\mathbf{a})) \leq_\omega \mathbf{a}].$$

Furthermore λ preserves least upper and greatest lower bounds (whenever they exist). The first one follows directly from the fact $P_0(\mathcal{A} \oplus \mathcal{B}) = P_0(\mathcal{A}) \oplus P_0(\mathcal{B})$. For the second one suppose that $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{D}_\omega$ and $\mathbf{a} \wedge \mathbf{b} = \mathbf{c}$. Fix $\mathbf{x} \in \mathbf{D}_e$, such that $\mathbf{x} \leq_e \lambda(\mathbf{a}), \lambda(\mathbf{b})$. Then $\kappa(\mathbf{x}) \leq_\omega \kappa(\lambda(\mathbf{a})) \leq_\omega \mathbf{a}$ and $\kappa(\mathbf{x}) \leq_\omega \kappa(\lambda(\mathbf{b})) \leq_\omega \mathbf{b}$. From here $\kappa(\mathbf{x}) \leq_\omega \mathbf{c}$ and therefore $\mathbf{x} = \lambda(\kappa(\mathbf{x})) \leq_\omega \lambda(\mathbf{c})$.

We summarize the properties of κ and λ described above in the following proposition.

Proposition 1. *The mappings κ and λ have the following properties:*

(K1) $\forall \mathbf{a}, \mathbf{b} \in \mathbf{D}_e[\mathbf{a} \leq_e \mathbf{b} \iff \kappa(\mathbf{a}) \leq_\omega \kappa(\mathbf{b})]$

(K2) $\forall \mathbf{a}, \mathbf{b} \in \mathbf{D}_e[\kappa(\mathbf{a} \vee \mathbf{b}) = \kappa(\mathbf{a}) \vee \kappa(\mathbf{b})]$

(K3) $\kappa(\mathbf{0}_e) = \mathbf{0}_\omega$

(L1) $\forall \mathbf{a}, \mathbf{b} \in \mathbf{D}_\omega[\mathbf{a} \leq_\omega \mathbf{b} \implies \lambda(\mathbf{a}) \leq_\omega \lambda(\mathbf{b})]$

(L2) $\forall \mathbf{a}, \mathbf{b} \in \mathbf{D}_\omega[\lambda(\mathbf{a} \vee \mathbf{b}) = \lambda(\mathbf{a}) \vee \lambda(\mathbf{b})]$

(L3) $\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{D}_\omega[\mathbf{a} \wedge \mathbf{b} = \mathbf{c} \implies \lambda(\mathbf{a}) \wedge \lambda(\mathbf{b}) = \lambda(\mathbf{c})]$

(KL1) $\forall \mathbf{a} \in \mathbf{D}_e[\lambda(\kappa(\mathbf{a})) = \mathbf{a}]$

(KL2) $\forall \mathbf{a} \in \mathbf{D}_\omega[\kappa(\lambda(\mathbf{a})) \leq_\omega \mathbf{a}]$

4. Jump operation and least jump invert

The jump operation in $\mathbf{D}_\omega$ is defined by Soskov and Ganchev [13].

Definition 4. *Let $\mathcal{A} \in \mathcal{S}_\omega$ be a sequence of sets of natural numbers. The ω-enumeration jump of $\mathcal{A}$ is the sequence $\mathcal{A}' = \{P_{1+k}(\mathcal{A})\}_{k<\omega}$.*

In other words, the jump of $\mathcal{A}$ is the jump sequence of $\mathcal{A}$ with deleted first element. For an arbitrary sequence $\mathcal{A}$ the jump operation has the properties $\mathcal{A} \lneq_\omega \mathcal{A}'$, and $\mathcal{A} \leq_\omega \mathcal{B} \implies \mathcal{A}' \leq_\omega \mathcal{B}'$, allowing to define a jump operation on ω-enumeration degrees by

$$(\mathbf{d}_\omega(\mathcal{A}))' = \mathbf{d}_\omega(\mathcal{A}').$$

As a direct consequence of the definition we obtain

$$\kappa(\mathbf{a}') = \kappa(\mathbf{a})'. \tag{4.1}$$

We define the iteration of the jump in the usual way, setting

$$\mathcal{A}^{(0)} = \mathcal{A} \text{ and } \mathcal{A}^{(n+1)} = (\mathcal{A}^{(n)})'.$$

Soskov [12] proves that for an arbitrary sequence $\mathcal{A}$, $P_k(\mathcal{A}) \equiv_e P_k(P(\mathcal{A}))$ uniformly in k and hence

$$\mathcal{A}^{(n)} \equiv_\omega \{P_{n+k}(\mathcal{A})\}_{k<\omega}. \tag{4.2}$$

From here we obtain

$$\lambda(\mathbf{d}_\omega(\mathcal{A}^{(n)})) = \mathbf{d}_e(P_n(\mathcal{A})). \tag{4.3}$$

In particular

$$\forall \mathbf{x} \in \mathbf{D}_\omega[\lambda(\mathbf{x})' \leq_e \lambda(\mathbf{x}')]. \tag{4.4}$$

Furthermore (4.3) together with Definition 3 give us the following characterization of the partial order $\leq_\omega$:

Proposition 2. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be arbitrary* ω*-enumeration degrees. For all* $n \in \mathbb{N}$

$$\mathbf{a} \leq_\omega \mathbf{b} \iff \forall 0 \leq k < n \left[\lambda(\mathbf{a}^{(k)}) \leq_e \lambda(\mathbf{b}^{(k)})\right] \ \& \ \mathbf{a}^{(n)} \leq_\omega \mathbf{b}^{(n)}.$$

Proof. The direction from left to right is clear. For the converse suppose that $\forall 0 \leq k < n \left[\lambda(\mathbf{a}^{(k)}) \leq_e \lambda(\mathbf{b}^{(k)})\right]$ and $\mathbf{a}^{(n)} \leq_\omega \mathbf{b}^{(n)}$. Take $\mathcal{A} \in \mathbf{a}$ and $\mathcal{B} \in \mathbf{b}$. The inequality $\mathbf{a}^{(n)} \leq_\omega \mathbf{b}^{(n)}$ yields $\mathcal{A}^{(n)} \leq_\omega \mathcal{B}^{(n)}$. From here, applying (4.2) and Definition 3, we obtain

$$\forall k[P_{n+k}(\mathcal{A}) \leq_e P_{n+k}(\mathcal{B}) \text{ uniformly in } k]. \tag{4.5}$$

The inequalities $\lambda(\mathbf{a}^{(k)}) \leq_e \lambda(\mathbf{b}^{(k)})$ for $0 \leq k < n$ together with (4.3) give us

$$\forall 0 \leq k < n \left[P_k(\mathcal{A}) \leq_e P_k(\mathcal{B})\right]. \tag{4.6}$$

Finally combining (4.5) and (4.6) we obtain $\mathcal{A} \leq_\omega \mathcal{B}$

$\square$

The jump operation on the ω-enumeration degrees exhibits a property (Soskov, Ganchev [13]), that neither the Turing nor the enumeration jump do. Namely, if $\mathbf{0}_\omega^{(n)} \leq_\omega \mathbf{a}$, then there exists a least solution to the equation

$$\mathbf{x}^{(n)} = \mathbf{a}.$$

We shall denote this solution by $I^n(\mathbf{a})$. We can give an explicit representative of $I^n(\mathbf{a})$ by setting for arbitrary $\mathcal{A} \in \mathcal{S}_\omega$

$$I^n(\mathcal{A}) = (\underbrace{\emptyset, \ldots, \emptyset}_{n}, A_0, A_1, \ldots).$$

Then for all $\mathbf{a} \in \mathbf{D}_\omega$ above $\mathbf{0}_\omega{}^{(n)}$

$$\mathcal{A} \in \mathbf{a} \iff I^n(\mathcal{A}) \in I^n(\mathbf{a}). \tag{4.7}$$

Lemma 1. *Let $n \in \mathbb{N}$ and let $\mathbf{a}, \mathbf{b} \in \mathbf{D}_\omega$ be such that $\mathbf{0}_\omega{}^{(n)} \leq_\omega \mathbf{a}, \mathbf{b}$. Then:*

(I0) $\forall k < n[\lambda(I^n(\mathbf{a})^{(k)}) = \mathbf{0}_e{}^{(k)}]$.

(I1) $\mathbf{a} \leq_\omega \mathbf{b} \iff I^n(\mathbf{a}) \leq_\omega I^n(\mathbf{b})$.

(I2) $\forall \mathbf{x} \in \mathbf{D}_\omega[\mathbf{x} \leq_\omega I^n(\mathbf{a}) \implies I^n(\mathbf{x}^{(n)}) = \mathbf{x}]$.

(I3) $I^n(\mathbf{a} \vee \mathbf{b}) = I^n(\mathbf{a}) \vee I^n(\mathbf{b})$.

(I4) $\mathbf{a} \wedge \mathbf{b} = \mathbf{c} \implies I^n(\mathbf{a}) \wedge I^n(\mathbf{b}) = I^n(\mathbf{c})$.

(I5) $\forall \mathbf{x} \in \mathbf{D}_\omega \forall k < n \left[\lambda \left((\mathbf{x} \vee I^n(\mathbf{a}))^{(k)} \right) = \lambda(\mathbf{x}^{(k)}) \right]$.

(I6) $\forall \mathbf{x} \in \mathbf{D}_\omega \left[(\mathbf{x} \vee I^n(\mathbf{a}))^{(n)} = \mathbf{x}^{(n)} \vee \mathbf{a} \right]$.

Proof. (I0) is a direct corollary of (4.6).

(I1) For the left to right direction suppose that $\mathbf{a} \leq_\omega \mathbf{b}$. Then $I^n(\mathbf{a})^{(n)} = \mathbf{a} \leq_\omega \mathbf{b} = I^n(\mathbf{b})^{(n)}$. On the other hand, for $k < n$, $\lambda(I^n(\mathbf{a})^{(k)}) = \lambda(I^n(\mathbf{a})^{(k)}) = \mathbf{0}_e{}^{(k)}$, so that applying Lemma 2 we obtain $I^n(\mathbf{a}) \leq_\omega I^n(\mathbf{b})$.
For the other direction suppose that $I^n(\mathbf{a}) \leq_\omega I^n(\mathbf{b})$. From the fact, that the jump operation is monotone, we obtain $\mathbf{a} = I^n(\mathbf{a})^{(n)} \leq_\omega I^n(\mathbf{b})^{(n)} = \mathbf{b}$.

(I2) Let $\mathbf{x} \leq_\omega I^n(\mathbf{a})$. According to Proposition 2 for $k < n$,

$$\lambda(\mathbf{x}^{(k)}) \leq_e \lambda(I^n(\mathbf{a})^{(k)}) = \mathbf{0}_e{}^{(k)} = \lambda(I^n(\mathbf{x}^{(n)})^{(k)}).$$

On the other hand $\mathbf{x}^{(n)} = I^n(\mathbf{x}^{(n)})$, so that applying again Proposition 2 we obtain $\mathbf{x} \leq_\omega I^n(\mathbf{x}^{(n)})$. Now the equality $\mathbf{x} = I^n(\mathbf{x}^{(n)})$ is obvious.

(I3) is again a direct application of (I0) and Proposition 2.

(I4) Let $\mathbf{a} \wedge \mathbf{b} = \mathbf{c}$ and let $\mathbf{x} \leq_\omega I^n(\mathbf{a}), I^n(\mathbf{b})$. According to (I2) $\mathbf{x} = I^n(\mathbf{x}^{(n)})$. Furthermore $\mathbf{x}^{(n)} \leq_\omega \mathbf{a}, \mathbf{b}$ and hence $\mathbf{x}^{(n)} \leq_\omega \mathbf{c}$. Thus from (I1) we obtain $\mathbf{x} = I^n(\mathbf{x}^{(n)}) \leq_\omega I^n(\mathbf{c})$.

(I5) and (I6) follow from that whenever $\mathcal{X} \in \mathbf{x}$ and $\mathcal{A} \in \mathbf{a}$, we have that $P_k(\mathcal{X} \oplus I^n(\mathcal{A})) \equiv_e P_k(\mathcal{X})$ for $k < n$ and $P_n(\mathcal{X} \oplus I^n(\mathcal{A})) \equiv_e P_n(\mathcal{X}) \oplus P_0(\mathcal{A})$. $\square$

Claims (I3) and (I4) of Lemma 1 show that the least jump invert operation preserves both least upper and greatest lower bounds in $\mathcal{D}_\omega$. The first author [4] proves that the jump operation on $\mathbf{D}_\omega$ preserves greatest lower bound, i.e.

$$\mathbf{a} \wedge \mathbf{b} = \mathbf{c} \implies \mathbf{a}' \wedge \mathbf{b}' = \mathbf{c}'. \tag{4.8}$$

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On the other hand one can easily see that the jump operation does not always preserve least upper bounds. Indeed, take a low splitting of $\mathbf{0}_\omega'$, i.e. ω-enumeration degrees $\mathbf{a}$ and $\mathbf{b}$, such that $\mathbf{a}' = \mathbf{b}' = \mathbf{a} \vee \mathbf{b} = \mathbf{0}_\omega'$ (such a splitting of $\mathbf{0}_\omega'$ exists, since it exists in the case of enumeration degrees). Then $\mathbf{a}' \vee \mathbf{b}' = \mathbf{0}_\omega' < \mathbf{0}_\omega'' = (\mathbf{a} \vee \mathbf{b})'$. We give a sufficient condition for the preservation of the least upper bound.

Lemma 2. *Let* $\mathbf{a}, \mathbf{b} \in \mathbf{D}_\omega$ *be such that* $(\lambda(\mathbf{a}) \vee \lambda(\mathbf{b}))' = \lambda(\mathbf{a})' \vee \lambda(\mathbf{b})'$. *Then* $(\mathbf{a} \vee \mathbf{b})' = \mathbf{a}' \vee \mathbf{b}'$.

Proof. Note that for arbitrary $\mathbf{x} \in \mathbf{D}_\omega$ we have

$$\mathbf{x} = \kappa(\lambda(\mathbf{x})) \vee I^1(\mathbf{x}').\tag{4.9}$$

Indeed, according to claims (I5) and (I6) of Lemma 1 we have

$$\lambda(\kappa(\lambda(\mathbf{x})) \vee I^1(\mathbf{x}')) = \lambda(\kappa(\lambda(\mathbf{x}))) = \lambda(\mathbf{x})$$

and

$$(\kappa(\lambda(\mathbf{x})) \vee I^1(\mathbf{x}'))' = \kappa(\lambda(\mathbf{x}))' \vee \mathbf{x}' = \mathbf{x}'.$$

which together with Proposition 2 yield (4.9).

Let $\mathbf{a}, \mathbf{b} \in \mathbf{D}_\omega$ be such that $(\lambda(\mathbf{a}) \vee \lambda(\mathbf{b}))' = \lambda(\mathbf{a})' \vee \lambda(\mathbf{b})'$. Applying (4.9) to $\mathbf{a}$ and $\mathbf{b}$ we obtain

$$\mathbf{a} \vee \mathbf{b} = \kappa(\lambda(\mathbf{a})) \vee I^1(\mathbf{a}') \vee \kappa(\lambda(\mathbf{b})) \vee I^1(\mathbf{b}') = \kappa(\lambda(\mathbf{a} \vee \mathbf{b})) \vee I^1(\mathbf{a}' \vee \mathbf{b}').$$

From here using (I6), the properties of κ and λ, and $(\lambda(\mathbf{a}) \vee \lambda(\mathbf{b}))' = \lambda(\mathbf{a})' \vee \lambda(\mathbf{b})'$ we obtain

$$(\mathbf{a} \vee \mathbf{b})' = (\kappa(\lambda(\mathbf{a} \vee \mathbf{b})) \vee I^1(\mathbf{a}' \vee \mathbf{b}'))' = \kappa(\lambda(\mathbf{a} \vee \mathbf{b}))' \vee \mathbf{a}' \vee \mathbf{b}' =$$

$$\kappa(\lambda(\mathbf{a} \vee \mathbf{b})') \vee \mathbf{a}' \vee \mathbf{b}' = \kappa((\lambda(\mathbf{a}) \vee \lambda(\mathbf{b}))') \vee \mathbf{a}' \vee \mathbf{b}' = \kappa(\lambda(\mathbf{a})' \vee \lambda(\mathbf{b})') \vee \mathbf{a}' \vee \mathbf{b}' =$$

$$\kappa(\lambda(\mathbf{a}))' \vee \kappa(\lambda(\mathbf{b}))' \vee \mathbf{a}' \vee \mathbf{b}' = \mathbf{a}' \vee \mathbf{b}'$$

$$\square$$

5. The local theory of the ω-enumeration degrees.

The local theory of the ω-enumeration degrees is the theory of the degrees that are in the interval with endpoints the least ω-enumeration degree and its first jump. It is considered for the first time by Soskov and Ganchev [13], who establish some basic properties of these degrees.

We shall denote by $\mathcal{G}_\omega$ the collection of the degrees below $\mathbf{0}_\omega'$, i.e.,

$$\mathcal{G}_\omega = \{\mathbf{x} \in \mathbf{D}_\omega \mid \mathbf{x} \leq_\omega \mathbf{0}_\omega'\}.$$

First of all note that in contrast to local structures in the Turing degrees and the enumeration degrees, there are degrees in $\mathcal{G}_\omega$ that can be explicitly defined. Indeed, consider the $n+1$-st jump of $\mathbf{0}_\omega$ for arbitrary natural number n. Set

$$\mathbf{o}_n = I^n(\mathbf{0}_\omega{}^{(n+1)}).$$

We have $\mathbf{o}_n^{(n)} = \mathbf{0}_\omega{}^{(n+1)} = (\mathbf{0}_\omega{}')^{(n)}$ and hence $\mathbf{o}_n \leq_\omega \mathbf{0}_\omega{}'$, i.e., $\mathbf{o}_n \in \mathcal{G}_\omega$. Since the operation I^n is first order definable in the structure $\mathcal{D}_\omega{}'$ (the partial order of the ω-enumeration degrees augmented with the jump operation), the degrees $\mathbf{o}_n$ are first order definable in $\mathcal{D}_\omega{}'$.

In addition we can show an explicit representative of $\mathbf{o}_n$. We have

$$(\emptyset_e^{(n+1)}, \emptyset_e^{(n+2)}, \emptyset_e^{(n+3)}, \ldots) \in \mathbf{0}_\omega{}^{(n+1)},$$

so applying (4.7) we obtain

$$(\underbrace{\emptyset, \ldots, \emptyset}_{n}, \emptyset_e^{(n+1)}, \emptyset_e^{(n+2)}, \emptyset_e^{(n+3)}, \ldots) \in \mathbf{o}_n. \tag{5.1}$$

Since $\mathbf{o}_n$ is the least n-th jump invert of $\mathbf{0}_\omega{}^{(n+1)}$ and every degree in $\mathcal{G}_\omega$ is bounded by $\mathbf{0}_\omega{}'$, we may conclude that

$$\forall \mathbf{x} \in \mathcal{G}_\omega[\mathbf{x}^{(n)} = \mathbf{0}_\omega{}^{(n+1)} \iff \mathbf{o}_n \leq_\omega \mathbf{x}]. \tag{5.2}$$

The degrees $\mathbf{x} \leq_\omega \mathbf{0}_\omega{}'$ having the property $\mathbf{x}^{(n)} = \mathbf{0}_\omega{}^{(n+1)}$ are called high$_n$, since their n-th jump is as high as possible. We shall denote the collection of the high$_n$ degrees by H_n. Thus

$$\forall \mathbf{x} \in \mathcal{G}_\omega \left[\mathbf{x} \in H_n \iff \mathbf{o}_n \leq_\omega \mathbf{x} \right]. \tag{5.3}$$

Conversely the low$_n$ degrees are the degrees from $\mathcal{G}_\omega$ with least possible n-th jump. In other words, $\mathbf{x} \in \mathcal{G}_\omega$ is low$_n$ if and only if $\mathbf{x}^{(n)} = \mathbf{0}_\omega{}^{(n)}$. We denote the collection of all low$_n$ degrees by L_n.

As in (5.3) we shall see that the degrees in L_n are exactly those satisfying an algebraic property involving $\mathbf{o}_n$. First we need to prove the following.

Proposition 3. *Let $\mathbf{x} \in \mathcal{G}_\omega$. Then for every natural number n*

$$\mathbf{x} \wedge \mathbf{o}_n = I^n(\mathbf{x}^{(n)}). \tag{5.4}$$

Proof. Let $\mathbf{x} \in \mathcal{G}_\omega$ and fix a natural number n. Since $\mathbf{x} \leq_\omega \mathbf{0}_\omega{}'$, we have $\mathbf{x}^{(n)} \leq_\omega \mathbf{0}_\omega{}^{(n+1)}$, so that using claim (I1) of Lemma 1 we obtain $I^n(\mathbf{x}^{(n)}) \leq_\omega \mathbf{o}_n$. On the other hand it is obvious, that $I^n(\mathbf{x}^{(n)}) \leq_\omega \mathbf{x}$.

Now take $\mathbf{y} \in \mathcal{G}_\omega$ such that $\mathbf{y} \leq_\omega \mathbf{x}, \mathbf{o}_n$. From $\mathbf{y} \leq_\omega \mathbf{o}_n$ and claim (I2) of Lemma 1 we obtain

$$\mathbf{y} = I^n(\mathbf{y}^{(n)}).$$

On the other hand $\mathbf{y} \leq_\omega \mathbf{x}$ implies $\mathbf{y}^{(n)} \leq_\omega \mathbf{x}^{(n)}$ from where we conclude

$$\mathbf{y} = I^n(\mathbf{y}^{(n)}) \leq_\omega I^n(\mathbf{x}^{(n)}).$$

$\square$

As a corollary of Proposition 3 we obtain

$$\forall \mathbf{x} \in \mathcal{G}_\omega [\mathbf{x} \in L_n \iff \mathbf{x} \wedge \mathbf{o}_n = \mathbf{0}_\omega]. \tag{5.5}$$

Indeed, for arbitrary $\mathbf{x} \leq_\omega \mathbf{0}_\omega{}'$

$$\mathbf{x}^{(n)} = \mathbf{0}_\omega{}^{(n)} \iff I^n(\mathbf{x}^{(n)}) = \mathbf{0}_\omega \iff \mathbf{x} \wedge \mathbf{o}_n = \mathbf{0}_\omega.$$

Another corollary of Proposition 3 is that the degrees $\mathbf{o}_n$ form a strictly descending sequence, i.e.,

$$\mathbf{0}_\omega{}' = \mathbf{o}_0 > \mathbf{o}_1 > \mathbf{o}_2 > \cdots > \mathbf{o}_n > \ldots$$

Indeed, $\mathbf{o}_n \wedge \mathbf{o}_{n+1} = I^{n+1}(\mathbf{o}_n^{(n)}) = I^{n+1}(\mathbf{0}_\omega{}^{(n+1)}) = \mathbf{o}_{n+1}$. Soskov and Ganchev [13] prove that this sequence does not converge to $\mathbf{0}_\omega$, meaning there is a nonzero degree $\mathbf{x}$, such that

$$\forall n [\mathbf{x} \leq_\omega \mathbf{o}_n]. \tag{5.6}$$

The degrees that have the property (5.6) are called *almost zero* (*a.z.*). Their representatives can be characterized by

$$\forall \ a.z. \ \mathbf{x} \left[\mathcal{X} \in \mathbf{x} \iff \mathcal{X} \leq_\omega \emptyset_\omega{}' \ \& \ \forall k [P_k(\mathcal{X}) \equiv_e \emptyset_e^{(k)}] \right]$$

In other words

$$\mathbf{x} \text{ is } a.z. \iff \mathbf{x} \leq_\omega \mathbf{0}_\omega{}' \ \& \ \forall k [\lambda(\mathbf{x}^{(k)}) = \mathbf{0}_e{}^{(k)}].$$

Let us denote by H and L all the degrees in $\mathcal{G}_\omega$ that are respectively high_n and low_n for some n. In [13] it is shown that the classes H and L can be characterized using the *a.z.* degrees. Namely, for arbitrary $\mathbf{a} \leq_\omega \mathbf{0}_\omega{}'$

$$\mathbf{a} \in H \iff \forall \ a.z. \ \mathbf{x}[\mathbf{x} \leq_\omega \mathbf{a}],$$

$$\mathbf{a} \in L \iff \forall \ a.z. \ \mathbf{x}[\mathbf{x} \leq_\omega \mathbf{a} \implies \mathbf{x} = \mathbf{0}_\omega].$$

From the second equivalence it follows that the only low_n *a.z.* degree is $\mathbf{0}_\omega$. On the other hand, according to (5.6) and (5.3) no *a.z.* degree is high_n for some n. Thus the *a.z.* degrees are intermediate, i.e., they belong to the class $I = \mathcal{G}_\omega - (H \cup L)$.

So far we have seen that $I \neq \emptyset$ and $H_{n+1} - H_n \neq \emptyset$ for arbitrary $n \geq 1$. In order to prove that the high/low jump hierarchy of $\mathcal{G}_\omega$ does not collapse, it remains to be shown that $L_{n+1} - L_n \neq \emptyset$. To prove this we shall use that for an arbitrary n there is an enumeration degree $\mathbf{x}$, such that

$$\mathbf{0}_e{}^{(n)} \lneq_e \mathbf{x} \ \& \ \mathbf{x}' = \mathbf{0}_e{}^{(n+1)}. \tag{5.7}$$

Fix a natural number n and an enumeration degree $\mathbf{x}$ satisfying (5.7). Note that $\mathbf{0}_e{}^{(n)} \lneq_e \mathbf{x} \leq_e \mathbf{0}_e{}^{(n+1)}$. Consider the ω-enumeration degree $I^n(\kappa(\mathbf{x}))$. From the properties of the mapping κ and the operation I^n we immediately conclude

that $I^n(\kappa(\mathbf{x})) \leq_\omega \mathbf{0}_\omega'$. Now, $(I^n(\kappa(\mathbf{x})))^{(n)} = \kappa(\mathbf{x}) > \mathbf{0}_\omega{}^{(n)}$, so that $I^n(\kappa(\mathbf{x})) \notin L_n$. On the other hand $(I^n(\kappa(\mathbf{x})))^{(n+1)} = \kappa(\mathbf{x})' = \kappa(\mathbf{x}') = \kappa(\mathbf{0}_e{}^{(n+1)}) = \mathbf{0}_\omega{}^{(n+1)}$ and hence $I^n(\kappa(\mathbf{x})) \in L_{n+1}$. Thus $L_{n+1} - L_n \neq \emptyset$.

The equivalences (5.3) and (5.5) give a first order definition of the classes H_n and L_n using as parameter the degree $\mathbf{o}_n$. Thus if the degree $\mathbf{o}_n$ is first order definable in $(\mathcal{G}_\omega, \leq_\omega)$, the classes H_n and L_n shall be also first order definable in $(\mathcal{G}_\omega, \leq_\omega)$. Besides, the next lemma shows that the first order definability of $\mathbf{o}_1$ in $\mathcal{G}_\omega$, leads to first order definability of the class $\mathbf{D}_1 \cap \mathcal{G}_\omega$.

Lemma 3. *Let* $\mathbf{x} \in \mathcal{G}_\omega$. *Then*

$$\mathbf{x} \in \mathbf{D}_1 \iff \forall \mathbf{y} \in \mathcal{G}_\omega \left[\mathbf{y} \vee \mathbf{o}_1 = \mathbf{x} \vee \mathbf{o}_1 \implies \mathbf{x} \leq_\omega \mathbf{y} \right].$$

Proof. Let $\mathbf{x} \in \mathcal{G}_\omega$ and suppose that $\mathbf{x} \in \mathbf{D}_1$. Let $\mathbf{y} \in \mathcal{G}_\omega$ be a degree for which $\mathbf{y} \vee \mathbf{o}_1 = \mathbf{x} \vee \mathbf{o}_1$. From claim (I5) of Lemma 1 we obtain

$$\lambda(\mathbf{y}) = \lambda(\mathbf{y} \vee \mathbf{o}_1) = \lambda(\mathbf{x} \vee \mathbf{o}_1) = \lambda(\mathbf{x}). \tag{5.8}$$

Since $\mathbf{x} \in \mathbf{D}_1$, we have $\mathbf{x}' = \kappa(\lambda(\mathbf{x})')$, so that (5.8) together with (4.1), (4.4) and (KL2) from Proposition 1 yield

$$\mathbf{x}' = \kappa(\lambda(\mathbf{x})') = \kappa(\lambda(\mathbf{y})') \leq_\omega \kappa(\lambda(\mathbf{y}')) \leq_\omega \mathbf{y}'. \tag{5.9}$$

Now the inequality $\mathbf{x} \leq_\omega \mathbf{y}$ follows from (5.8), (5.9) and Proposition 2.

For the converse suppose that for all $\mathbf{y} \in \mathcal{G}_\omega$ the implication

$$\mathbf{y} \vee \mathbf{o}_1 = \mathbf{x} \vee \mathbf{o}_1 \implies \mathbf{x} \leq_\omega \mathbf{y}$$

holds. Consider the degree $\kappa(\lambda(\mathbf{x}))$. According to claim (KL2) of Proposition 1 $\kappa(\lambda(\mathbf{x})) \leq_\omega \mathbf{x}$ and hence $\kappa(\lambda(\mathbf{x})) \in \mathcal{G}_\omega$. Applying claims (KL1) of Proposition 1 and (I5) from Lemma 1 we obtain

$$\lambda(\kappa(\lambda(\mathbf{x})) \vee \mathbf{o}_1) = \lambda(\kappa(\lambda(\mathbf{x}))) = \lambda(\mathbf{x}) = \lambda(\mathbf{x} \vee \mathbf{o}_1) \tag{5.10}$$

On the other hand $\kappa(\lambda(\mathbf{x}))' \leq_\omega \mathbf{x}' \leq_\omega \mathbf{0}_\omega''$, so that claim (I6) of Lemma 1 gives us

$$(\kappa(\lambda(\mathbf{x})) \vee \mathbf{o}_1)' = \kappa(\lambda(\mathbf{x}))' \vee \mathbf{0}_\omega'' = \mathbf{0}_\omega'' = \mathbf{x}' \vee \mathbf{0}_\omega'' = (\mathbf{x} \vee \mathbf{o}_1)'. \tag{5.11}$$

Applying Lemma 2 to (5.10) and (5.11) we obtain $\kappa(\lambda(\mathbf{x})) \vee \mathbf{o}_1 = \mathbf{x} \vee \mathbf{o}_1$ and therefore $\mathbf{x} \leq_\omega \kappa(\lambda(\mathbf{x}))$. Thus $\mathbf{x} = \kappa(\lambda(\mathbf{x}))$ and hence $\mathbf{x} \in \mathbf{D}_1$.

$\square$

6. $\mathcal{K}$-pairs in $\mathcal{G}_\omega$

The goal of this section is to prove that the degrees $\mathbf{o}_n$ are first order definable in $\mathcal{G}_\omega$ for arbitrary n and thus conclude the proof of Theorem 1 and Theorem 2. This shall be done using the notion of $\mathcal{K}$-pairs.

Definition 5. *Let $\mathcal{D} = (\mathbf{D}, \leq)$ be a partial order. We say that $\{\mathbf{a}, \mathbf{b}\}$ is a $\mathcal{K}$-pair (strictly) over $\mathbf{u}$ for $\mathcal{D}$, if $\mathbf{a}, \mathbf{b}, \mathbf{u} \in \mathbf{D}$, $\mathbf{u} \leq \mathbf{a}, \mathbf{b}$ ($\mathbf{u} \lneq \mathbf{a}, \mathbf{b}$) and for all $\mathbf{x} \in \mathbf{D}$ such that $\mathbf{u} \leq \mathbf{x}$, the least upper bounds $\mathbf{x} \vee \mathbf{a}$, $\mathbf{x} \vee \mathbf{b}$ and the greatest lower bound $(\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b})$ exist, and the following equality holds:*

$$\mathbf{x} = (\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b}). \tag{6.1}$$

If $(\mathbf{D}, \leq)$ is a partially ordered set and $\mathbf{u}, \mathbf{v} \in \mathbf{D}$ we shall use the notation $[\mathbf{u}, \mathbf{v}]$ for the set $\{\mathbf{x} \in \mathbf{D} \mid \mathbf{u} \leq \mathbf{x} \leq \mathbf{v}\}$ together with the partial order inherited from $(\mathbf{D}, \leq)$. Note that if $\{\mathbf{a}, \mathbf{b}\}$ is a $\mathcal{K}$-pair (strictly) over $\mathbf{u}$ for $\mathcal{D}$, and $\mathbf{a}, \mathbf{b} \leq \mathbf{v} \in \mathbf{D}$, then $\{\mathbf{a}, \mathbf{b}\}$ is a $\mathcal{K}$-pair (strictly) over $\mathbf{u}$ for $[\mathbf{u}, \mathbf{v}]$.

The following two theorems of Kalimullin give important properties of $\mathcal{K}$-pairs in the enumeration degrees, which we shall use.

Theorem 6 (Kalimullin [6]). *Let A, B and U be sets of natural numbers.*

(i) *If for some $W \leq_e U$, we have $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$, then $\{\mathbf{d}_e(A \oplus U), \mathbf{d}_e(B \oplus U)\}$ is a $\mathcal{K}$-pair over $\mathbf{d}_e(U)$ for $\mathcal{D}_e$. If in addition*

$$\mathbf{d}_e(U) \lneq_e \mathbf{d}_e(A \oplus U), \mathbf{d}_e(B \oplus U) \leq_e \mathbf{d}_e(U)',$$

then $\mathbf{d}_e(A \oplus U)' = \mathbf{d}_e(B \oplus U)' = \mathbf{d}_e(U)'$

(ii) *If for no $W \leq_e U$, $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$, then there is a set $X \leq_e U' \oplus A^+ \oplus B^+$ such that*

$$\mathbf{d}_e(X) \neq (\mathbf{d}_e(X) \vee \mathbf{d}_e(A)) \wedge (\mathbf{d}_e(X) \vee \mathbf{d}_e(B)).$$

In particular if $A, B, \overline{A}, \overline{B} \leq_e U'$, then $X \leq_e U'$.

Theorem 7 (Kalimullin [6]). *For every $\mathbf{u} \in \mathbf{D}_e$ there is a $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ strictly over $\mathbf{u}$ for $\mathcal{D}_e$, such that*
$$\mathbf{a} \vee \mathbf{b} = \mathbf{u}' \text{ and } \mathbf{a}' = \mathbf{b}' = \mathbf{u}'.$$

A useful corollary of Theorem 6 is the following.

Corollary 1. *Let $\mathbf{u}, \mathbf{a}$ and $\mathbf{b}$ be enumeration degrees, such that $\mathbf{u} \leq_e \mathbf{a}, \mathbf{b} \leq_e \mathbf{u}'$ and $\mathbf{a}' = \mathbf{b}' = \mathbf{u}'$. If furthermore $\{\mathbf{a}, \mathbf{b}\}$ is a $\mathcal{K}$-pair over $\mathbf{u}$ for $[\mathbf{u}, \mathbf{u}']$, then for all $\mathbf{u} \leq_e \mathbf{x} \leq_e \mathbf{u}'$*

$$\mathbf{x} \lneq_e \mathbf{x} \vee \mathbf{a}, \mathbf{x} \vee \mathbf{b} \Longrightarrow (\mathbf{x} \vee \mathbf{a})' = (\mathbf{x} \vee \mathbf{b})' = \mathbf{x}'.$$

Proof. Let $\mathbf{u}$, $\mathbf{a}$ and $\mathbf{b}$ satisfy the conditions of the theorem, i.e., $\mathbf{u} \leq_e \mathbf{a}, \mathbf{b} \leq_e \mathbf{u}'$, $\mathbf{a}' = \mathbf{b}' = \mathbf{u}'$ and for all $\mathbf{u} \leq_e \mathbf{x} \leq_e \mathbf{u}'$

$$\mathbf{x} = (\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b}).$$

Fix $U \in \mathbf{u}$, $A \in \mathbf{a}$ and $\mathcal{B} \in \mathbf{b}$. From $A' \equiv_e B' \equiv_e U'$ we conclude $A, B, \overline{A}, \overline{B} \leq_e U'$ and hence from claim (ii) of Theorem 6 we obtain $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$ for some $W \leq_e U$ (from here it follows that $\{\mathbf{a}, \mathbf{b}\}$ is a $\mathcal{K}$-pair over $\mathbf{u}$ for $\mathcal{D}_e$).

Now fix $\mathbf{u} \leq_e \mathbf{x} \leq_e \mathbf{u}'$, such that $\mathbf{x} \lesssim_e \mathbf{x} \vee \mathbf{a}$, $\mathbf{x} \vee \mathbf{b}$ and let $X \in \mathbf{x}$. Since $W \leq_e U \leq_e X$ we conclude that $\{\mathbf{x} \vee \mathbf{a}, \mathbf{x} \vee \mathbf{b}\}$ is a $\mathcal{K}$-pair above $\mathbf{x}$ for $\mathcal{D}_e$. Furthermore we have

$$\mathbf{u} \leq_e \mathbf{x} \lesssim_e \mathbf{x} \vee \mathbf{a}, \ \mathbf{x} \vee \mathbf{b} \leq_e \mathbf{u}' \leq_e \mathbf{x}'.$$

From here and claim (i) of Theorem 6 we obtain

$$(\mathbf{x} \vee \mathbf{a})' = (\mathbf{x} \vee \mathbf{b})' = \mathbf{x}'.$$

$\square$

Our first goal is to characterize the K-pairs in $\mathcal{G}_\omega$. We start with two lemmas showing that the jump and least jump invert operations preserve the $\mathcal{K}$-pair property.

Lemma 4. *Let $n \in \mathbb{N}$ and let $\mathbf{a}, \mathbf{b} \in \mathbf{D}_\omega$ be such that $\{\mathbf{a}, \mathbf{b}\}$ is a $\mathcal{K}$-pair over $\mathbf{0}_\omega{}^{(n)}$ for $[\mathbf{0}_\omega{}^{(n)}, \mathbf{0}_\omega{}^{(n+1)}]$. Then $\{\mathbf{a}', \mathbf{b}'\}$ is a $\mathcal{K}$-pair over $\mathbf{0}_\omega{}^{(n+1)}$ for $[\mathbf{0}_\omega{}^{(n+1)}, \mathbf{0}_\omega{}^{(n+2)}]$. In particular if $\{\mathbf{a}, \mathbf{b}\}$ is a $\mathcal{K}$-pair over $\mathbf{0}_\omega$ for $\mathcal{G}_\omega$, then $\{\mathbf{a}^{(k)}, \mathbf{b}^{(k)}\}$ is a $\mathcal{K}$-pair over $\mathbf{0}_\omega{}^{(k)}$ for $[\mathbf{0}_\omega{}^{(k)}, \mathbf{0}_\omega{}^{(k+1)}]$.*

Proof. Suppose $\{\mathbf{a}, \mathbf{b}\}$ is a $\mathcal{K}$-pair above $\mathbf{0}_\omega{}^{(n)}$, i.e. $\mathbf{0}_\omega{}^{(n)} \leq_\omega \mathbf{a}$, $\mathbf{b}$ and

$$\forall \mathbf{0}_\omega{}^{(n)} \leq_\omega \mathbf{x} \leq_\omega \mathbf{0}_\omega{}^{(n+1)} \left[\mathbf{x} = (\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b})\right]. \tag{6.2}$$

Consider $\mathbf{0}_\omega{}^{(n+1)} \leq_\omega \mathbf{x} \leq_\omega \mathbf{0}_\omega{}^{(n+2)}$ and let $\mathbf{y} \leq_\omega \mathbf{x} \vee \mathbf{a}'$, $\mathbf{x} \vee \mathbf{b}'$. From claims (I1) and (I3) of Lemma 1 we obtain

$$I^1(\mathbf{y}) \leq_\omega I^1(\mathbf{x}) \vee I^1(\mathbf{a}), \ I^1(\mathbf{x}) \vee I^1(\mathbf{b}),$$

and hence

$$I^1(\mathbf{y}) \vee \mathbf{0}_\omega{}^{(n)} \leq_\omega (I^1(\mathbf{x}) \vee \mathbf{0}_\omega{}^{(n)}) \vee I^1(\mathbf{a}'), \ (I^1(\mathbf{x}) \vee \mathbf{0}_\omega{}^{(n)}) \vee I^1(\mathbf{b}'). \tag{6.3}$$

Since $\mathbf{0}_\omega{}^{(n+1)} \leq_\omega \mathbf{x} \leq_\omega \mathbf{0}_\omega{}^{(n+2)}$, we have $\mathbf{0}_\omega{}^{(n)} \leq_\omega I^1(\mathbf{x}) \vee \mathbf{0}_\omega{}^{(n)} \leq_\omega \mathbf{0}_\omega{}^{(n+1)}$. On the other hand $I^1(\mathbf{a}') \leq_\omega \mathbf{a}$ and $I^1(\mathbf{b}') \leq_\omega \mathbf{b}$, so that from (6.2) and (6.3) we obtain
$$I^1(\mathbf{y}) \vee \mathbf{0}_\omega{}^{(n)} \leq_\omega I^1(\mathbf{x}) \vee \mathbf{0}_\omega{}^{(n)}.$$
Now applying claim (I6) of Lemma 1 we get

$$\mathbf{y} = \mathbf{0}_\omega{}^{(n+1)} \vee \mathbf{y} = (\mathbf{0}_\omega{}^{(n)} \vee I^1(\mathbf{y}))' \leq_\omega (\mathbf{0}_\omega{}^{(n)} \vee I^1(\mathbf{x}))' = \mathbf{0}_\omega{}^{(n+1)} \vee \mathbf{x} = \mathbf{x}.$$

$\square$

Lemma 5. *Let $\{\mathbf{a}, \mathbf{b}\}$ be a $\mathcal{K}$-pair over $\mathbf{0}_\omega{}^{(n)}$ for $[\mathbf{0}_\omega{}^{(n)}, \mathbf{0}_\omega{}^{(n+1)}]$. Then the pair $\{I^n(\mathbf{a}), I^n(\mathbf{b})\}$ is a $\mathcal{K}$-pair over $\mathbf{0}_\omega$ for $\mathcal{G}_\omega$.*

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Proof. Suppose that $\mathbf{a}$ and $\mathbf{b}$ satisfy the condition of the lemma. It is clear that $I^n(\mathbf{a}), I^n(\mathbf{b}) \in \mathcal{G}_\omega$. Fix $\mathbf{x}, \mathbf{y} \in \mathcal{G}_\omega$, such that $\mathbf{y} \leq_\omega \mathbf{x} \vee I^n(\mathbf{a})$, $\mathbf{x} \vee I^n(\mathbf{b})$. Then applying (I5) and (I6) of Lemma 1 we obtain

$$\forall k < n[\lambda(\mathbf{y}^{(k)}) \leq_e \lambda(\mathbf{x}^{(k)})] \tag{6.4}$$

$$\mathbf{y}^{(n)} \leq_\omega \mathbf{x}^{(n)} \vee \mathbf{a}, \ \mathbf{x}^{(n)} \vee \mathbf{b}. \tag{6.5}$$

From (6.5) we conclude $\mathbf{y}^{(n)} \leq_\omega \mathbf{x}^{(n)}$, which together with (6.4) and Lemma 2 implies $\mathbf{y} \leq_\omega \mathbf{x}$.

$\square$

The next two lemmas show that the mapping λ preserves the $\mathcal{K}$-pair property for intervals with endpoints a degree $\mathbf{u}$ and its first jump, whereas the embedding κ preserves it in some special cases.

Lemma 6. *Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{u}$ be ω-enumeration degrees, such that $\{\mathbf{a}, \mathbf{b}\}$ is a $\mathcal{K}$-pair over $\mathbf{u}$ for the interval $[\mathbf{u}, \mathbf{u}']$. Then for all $\lambda(\mathbf{u}) \leq_e \mathbf{x} \leq_e \lambda(\mathbf{u})'$ we have*

$$\mathbf{x} = (\mathbf{x} \vee \lambda(\mathbf{a})) \wedge (\mathbf{x} \vee \lambda(\mathbf{b})).$$

In particular, if $\mathbf{u} \in \mathbf{D}_1$, then $\{\lambda(\mathbf{a}), \lambda(\mathbf{b})\}$ is a $\mathcal{K}$-pair over $\lambda(\mathbf{u})$ for $[\lambda(\mathbf{u}), \lambda(\mathbf{u})']$.

Proof. Let $\{\mathbf{a}, \mathbf{b}\}$ be $\mathcal{K}$-pair above $\mathbf{u}$ for the interval $[\mathbf{u}, \mathbf{u}']$ and consider the degrees $\lambda(\mathbf{a})$, $\lambda(\mathbf{b})$ and $\lambda(\mathbf{u})$. Note that since $\mathbf{a}, \mathbf{b} \in [\mathbf{u}, \mathbf{u}']$, we have $\lambda(\mathbf{a}), \lambda(\mathbf{b}) \in [\lambda(\mathbf{u}), \lambda(\mathbf{u}')]$. Fix $\lambda(\mathbf{u}) \leq_e \mathbf{x} \leq_e \lambda(\mathbf{u})'$ and suppose that $\mathbf{y} \leq_e \mathbf{x} \vee \lambda(\mathbf{a})$, $\mathbf{x} \vee \lambda(\mathbf{b})$. From claim (K1), (K2) and (KL2) of Proposition 1 we get

$$\kappa(\mathbf{y}) \leq_\omega \kappa(\mathbf{x}) \vee \kappa(\lambda(\mathbf{a})) \leq_\omega \kappa(\mathbf{x}) \vee \mathbf{a},$$

$$\kappa(\mathbf{y}) \leq_\omega \kappa(\mathbf{x}) \vee \kappa(\lambda(\mathbf{b})) \leq_\omega \kappa(\mathbf{x}) \vee \mathbf{b}.$$

From here

$$\mathbf{u} \vee \kappa(\mathbf{y}) \leq_\omega (\mathbf{u} \vee \kappa(\mathbf{x})) \vee \mathbf{a}, \ (\mathbf{u} \vee \kappa(\mathbf{x})) \vee \mathbf{b}. \tag{6.6}$$

Since $\lambda(\mathbf{u})' \leq_e \lambda(\mathbf{u})$, and $\mathbf{x}$ and $\mathbf{y}$ satisfy the inequalities

$$\mathbf{x}, \mathbf{y} \leq_e \lambda(\mathbf{u}'),$$

we have

$$\mathbf{u} \leq_\omega \mathbf{u} \vee \kappa(\mathbf{x}), \mathbf{u} \vee \kappa(\mathbf{y}) \leq_\omega \kappa(\lambda(\mathbf{u}')) \leq_\omega \mathbf{u}'.$$

From here, (6.6) and $\{\mathbf{a}, \mathbf{b}\}$ being a $\mathcal{K}$-pair over $\mathbf{u}$ for $[\mathbf{u}, \mathbf{u}']$, we conclude

$$\mathbf{u} \vee \kappa(\mathbf{y}) \leq_\omega \mathbf{u} \vee \kappa(\mathbf{x}).$$

Applying again Proposition 1 we finally obtain

$$\mathbf{y} \leq_e \lambda(\mathbf{u}) \vee \mathbf{y} = \lambda(\mathbf{u} \vee \kappa(\mathbf{y})) \leq_e \lambda(\mathbf{u} \vee \kappa(\mathbf{x})) = \lambda(\mathbf{u}) \vee \mathbf{x} = \mathbf{x}.$$

$\square$

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Lemma 7. *Let* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{u}$ *be enumeration degrees, such that* $\{\mathbf{a}, \mathbf{b}\}$ *is a* $\mathcal{K}$-*pair strictly over* $\mathbf{u}$ *for* $[\mathbf{u}, \mathbf{u}']$ *and* $\mathbf{a}' = \mathbf{b}' = \mathbf{u}'$. *Then* $\{\kappa(\mathbf{a}), \kappa(\mathbf{b})\}$ *is a* $\mathcal{K}$-*pair strictly over* $\kappa(\mathbf{u})$ *for* $[\kappa(\mathbf{u}), \kappa(\mathbf{u})']$.

Proof. Let $\{\mathbf{a}, \mathbf{b}\}$ be a $\mathcal{K}$-pair strictly over $\mathbf{u}$ for $[\mathbf{u}, \mathbf{u}']$, i.e., $\mathbf{u} \lneqq_e \mathbf{a}$, $\mathbf{b} \leq_\omega \mathbf{u}'$ and

$$\forall \mathbf{u} \leq_e \mathbf{x} \leq_e \mathbf{u}' \left[\mathbf{x} = (\mathbf{x} \vee \mathbf{a}) \wedge (\mathbf{x} \vee \mathbf{b}) \right].$$

Consider $\kappa(\mathbf{a})$, $\kappa(\mathbf{b})$ and $\kappa(\mathbf{u})$. It is clear that $\kappa(\mathbf{u}) \lneqq_\omega \kappa(\mathbf{a})$, $\kappa(\mathbf{b}) \leq_\omega \kappa(\mathbf{u})'$. Let $\kappa(\mathbf{u}) \leq_\omega \mathbf{x} \leq_\omega \kappa(\mathbf{u})$ and fix $\widetilde{\mathbf{y}} \leq_\omega \mathbf{x} \vee \kappa(\mathbf{a})$, $\mathbf{x} \vee \kappa(\mathbf{b})$. Consider the degree $\mathbf{y} = \widetilde{\mathbf{y}} \vee \kappa(\mathbf{u})$. Since $\kappa(\mathbf{u}) \leq_\omega \mathbf{x}$, $\kappa(\mathbf{a})$, $\kappa(\mathbf{b})$, we have

$$\widetilde{\mathbf{y}} \leq_\omega \mathbf{y} \leq_\omega \mathbf{x} \vee \kappa(\mathbf{a}), \ \mathbf{x} \vee \kappa(\mathbf{b}).$$

We shall consider two cases. First suppose that $\mathbf{a} \leq_e \lambda(\mathbf{x})$ (or respectively $\mathbf{b} \leq_e \lambda(\mathbf{x})$). From claim (LK2) of Proposition 1 we obtain

$$\kappa(\mathbf{a}) \leq_\omega \kappa(\lambda(\mathbf{x})) \leq_\omega \mathbf{x},$$

so that $\mathbf{a} \vee \mathbf{x} = \mathbf{x}$ and hence $\widetilde{\mathbf{y}} \leq_\omega \mathbf{y} \leq_\omega \mathbf{x} \vee \kappa(\mathbf{a}) = \mathbf{x}$.

Now suppose that $\mathbf{a}, \mathbf{b} \not\leq_e \lambda(\mathbf{x})$. From claims (L1), (L2) and (KL1) of Proposition 1 we obtain

$$\mathbf{u} \leq_e \lambda(\mathbf{x}), \lambda(\mathbf{y}) \leq_e \mathbf{u}',$$

$$\lambda(\mathbf{y}) \leq_e \lambda(\mathbf{x} \vee \kappa(\mathbf{a})) = \lambda(\mathbf{x}) \vee \mathbf{a},$$

$$\lambda(\mathbf{y}) \leq_e \lambda(\mathbf{x} \vee \kappa(\mathbf{a})) = \lambda(\mathbf{x}) \vee \mathbf{a}.$$

Now, since $\{\mathbf{a}, \mathbf{b}\}$ is a $\mathcal{K}$-pair strictly above $\mathbf{u}$ for the interval $([\mathbf{u}, \mathbf{u}'], \leq_e)$, we get

$$\lambda(\mathbf{y}) \leq_e \lambda(\mathbf{x}). \tag{6.7}$$

Now to prove that $\mathbf{y} \leq_\omega \mathbf{x}$ it suffices to prove that $\mathbf{y}' \leq_\omega \mathbf{x}'$. From $\mathbf{a}, \mathbf{b} \not\leq_e \lambda(\mathbf{x})$, the equalities $\mathbf{a}' = \mathbf{b}' = \mathbf{u}'$ and Corollary 1 we conclude

$$(\lambda(\mathbf{x}) \vee \mathbf{a})' = (\lambda(\mathbf{x}) \vee \mathbf{b})' = \lambda(\mathbf{x})'.$$

In particular $(\lambda(\mathbf{x}) \vee \mathbf{a})' = \lambda(\mathbf{x})' \vee \mathbf{a}'$ and $(\lambda(\mathbf{x}) \vee \mathbf{b})' = \lambda(\mathbf{x})' \vee \mathbf{b}'$. Thus Lemma 2 yields

$$(\mathbf{x} \vee \kappa(\mathbf{a}))' = \mathbf{x}' \vee \kappa(\mathbf{a})' = \mathbf{x}' \vee \mathbf{u}' = \mathbf{x}'.$$

But $\mathbf{y} \leq_\omega \mathbf{x} \vee \kappa(\mathbf{a})$ and hence

$$\mathbf{y}' \leq_\omega \mathbf{x}'. \tag{6.8}$$

Now Lemma 2, (6.7) and (6.8) imply

$$\mathbf{y} \leq_\omega \mathbf{x}.$$

$\square$

At this point we will need to use the structural result of the enumeration degrees announced in the introduction. Recall that Theorem 5 states that for

every total degree $\mathbf{u}$ and every $\Delta_2^0(\mathbf{u})$ degree $\mathbf{a} \geq_e \mathbf{u}$ there is a low over $\mathbf{u}$ degree $\mathbf{b}$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{u}'$. Here a degree $\mathbf{a}$ is $\Delta_2^0(\mathbf{u})$ if $\mathbf{a}$ contains a set which is Δ_2^0 relative to a representative of $\mathbf{u}$. In particular every total degree $\mathbf{x}$ in the interval $[\mathbf{u}, \mathbf{u}']$ is $\Delta_2^0(\mathbf{u})$. As a corollary of this result we prove that a $\mathcal{K}$-pair for an interval $[\mathbf{u}, \mathbf{u}']$ is also quasi-minimal over $\mathbf{u}$.

Corollary 2. *Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{u}$ be enumeration degrees, such that $\{\mathbf{a}, \mathbf{b}\}$ is a $\mathcal{K}$-pair over $\mathbf{u}$ for $[\mathbf{u}, \mathbf{u}']$. If furthermore $\mathbf{u}$ is total and there is a total degree $\mathbf{x}$, such that $\mathbf{u} \leq_e \mathbf{x} \leq_e \mathbf{a}$, then $\mathbf{b} = \mathbf{u}$.*

Proof. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{u}$ and $\mathbf{x}$ satisfy the conditions of the corollary. Since $\mathbf{x}$ is total and $\mathbf{u} \leq_e \mathbf{x}$, Theorem 5 implies that $\mathbf{x} \vee \mathbf{y} = \mathbf{u}'$ for some enumeration degree $\mathbf{u} \leq_e \mathbf{y}$, having the property $\mathbf{y}' = \mathbf{u}'$. Note that, since $\mathbf{x} \leq_e \mathbf{a} \leq_e \mathbf{u}'$, we have $\mathbf{a} \vee \mathbf{y} = \mathbf{u}'$. Thus we obtain

$$\mathbf{y} = (\mathbf{y} \vee \mathbf{a}) \wedge (\mathbf{y} \vee \mathbf{b}) = \mathbf{u}' \wedge (\mathbf{y} \vee \mathbf{b}) = \mathbf{y} \vee \mathbf{b}.$$

Therefore $\mathbf{b} \leq_e \mathbf{y}$. From here we obtain $\mathbf{u}' \leq_e \mathbf{b}' \leq_e \mathbf{y}' \leq_e \mathbf{u}'$, i.e., $\mathbf{b}' = \mathbf{u}'$. Now we can apply Theorem 5 to $\mathbf{b}$ and reasoning as above we obtain $\mathbf{a}' = \mathbf{u}'$ from where we may conclude that $\mathbf{x}' = \mathbf{u}'$.

Note that since $\mathbf{u} \leq_e \mathbf{x} \leq_e \mathbf{a}$, the pair $\{\mathbf{x}, \mathbf{b}\}$ is also a $\mathcal{K}$-pair above $\mathbf{u}$ for the interval $[\mathbf{u}, \mathbf{u}']$. Applying Theorem 6 to $\mathbf{x}$, $\mathbf{b}$ and $\mathbf{u}$ we obtain $X \in \mathbf{x}$, $B \in \mathbf{b}$ and $W \leq_e U \in \mathbf{u}$, such that X is total and

$$X \times B \subseteq W, \quad \overline{X} \times \overline{B} \subseteq \overline{W}. \tag{6.9}$$

Kalimullin [6] proves that (6.9) implies $B \leq_e W \oplus \overline{X}$. But $\overline{X} \equiv_e X$, so that $B \leq_e X$. Thus $\mathbf{b} \leq_e \mathbf{x}$ and hence $\mathbf{b} = \mathbf{u}$.

$\square$

Corollary 3. *Let $\{\mathbf{a}, \mathbf{b}\}$ be a $\mathcal{K}$-pair over $\mathbf{0}_\omega$ for $\mathcal{G}_\omega$ and let for some $n \geq 0$, $\mathbf{0}_e^{(n)} \leq_e \lambda(\mathbf{a}^{(n)})$ and $\lambda(\mathbf{a}^{(n)})' = \mathbf{0}_e^{(n+1)}$. Then $\mathbf{b} \in L_{n+1}$.*

Proof. Suppose that the degrees $\mathbf{a}, \mathbf{b} \in \mathcal{G}_\omega$ satisfy the conditions of the corollary. Since $\mathbf{0}_e^{(n)} \leq_e \lambda(\mathbf{a}^{(n)}) \leq_e \mathbf{0}_e^{(n+1)}$, the equality $\lambda(\mathbf{a}^{(n)})' = \mathbf{0}_e^{(n+1)}$ together with Theorem 5 yield

$$\lambda(\mathbf{a}^{(n)}) \vee \mathbf{x} = \mathbf{x}' = \mathbf{0}_e^{(n+1)}$$

for some $\mathbf{0}_e^{(n)} \leq_e \mathbf{x} \leq_e \mathbf{0}_e^{(n+1)}$. Consider the ω-enumeration degree $\kappa(\mathbf{x})$. We have $\kappa(\mathbf{x}) \in [\mathbf{0}_\omega^{(n)}, \mathbf{0}_\omega^{(n+1)}]$, so that $I^n(\kappa(\mathbf{x})) \in \mathcal{G}_\omega$. Therefore

$$I^n(\kappa(\mathbf{x})) = (I^n(\kappa(\mathbf{x})) \vee \mathbf{a}) \wedge (I^n(\kappa(\mathbf{x})) \vee \mathbf{b}),$$

from where, Lemma 1, and (4.8) we obtain

$$\kappa(\mathbf{x}) = (\kappa(\mathbf{x}) \vee \mathbf{a}^{(n)}) \wedge (\kappa(\mathbf{x}) \vee \mathbf{b}^{(n)}).$$

By the choice of $\mathbf{x}$ we have $\mathbf{a}^{(n)} \vee \kappa(\mathbf{x}) = \mathbf{0}_\omega^{(n+1)}$, so that we may conclude that $\mathbf{b}^{(n)} \leq_\omega \kappa(\mathbf{x})$. But $\kappa(\mathbf{x})' = \mathbf{0}_\omega^{(n+1)}$ and hence $\mathbf{b}^{(n+1)} = \mathbf{0}_\omega^{(n+1)}$.

$\square$

Now we are ready to characterize the $\mathcal{K}$-pairs in $\mathcal{G}_\omega$.

Theorem 8. *Let $\{\mathbf{a}, \mathbf{b}\}$ be $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ for $\mathcal{G}_\omega$. Then exactly one of the following assertions holds:*

(i) *Both $\mathbf{a}$ and $\mathbf{b}$ are a.z..*

(ii) *There is a natural number n and a $\mathcal{K}$-pair of enumeration degrees $\{\widetilde{\mathbf{a}}, \widetilde{\mathbf{b}}\}$ for the interval $[\mathbf{0}_e{}^{(n)}, \mathbf{0}_e{}^{(n+1)}]$ with the property $\widetilde{\mathbf{a}}' = \widetilde{\mathbf{b}}' = \mathbf{0}_e{}^{(n+1)}$, such that*

$$\mathbf{a} = I^n(\kappa(\widetilde{\mathbf{a}})) \ and \ \mathbf{b} = I^n(\kappa(\widetilde{\mathbf{b}})).$$

Proof. Let $\{\mathbf{a}, \mathbf{b}\}$ be a $\mathcal{K}$-pair strictly over $\mathbf{0}_\omega$ for $(\mathcal{G}_\omega, \leq_\omega)$ and suppose that at least one of the degrees $\mathbf{a}$, $\mathbf{b}$ is not *a.z.*. Then, without loss of generality, we may fix a natural number n such that

$$\mathbf{0}_e{}^{(n)} \lneqq_e \lambda(\mathbf{a}^{(n)}) \ \& \ \forall k < n[\lambda(\mathbf{a}^{(k)}) = \lambda(\mathbf{b}^{(k)}) = \mathbf{0}_e{}^{(k)}]. \tag{6.10}$$

First we shall prove that $\lambda(\mathbf{a}^{(n)})' = \mathbf{0}_e{}^{(n+1)}$. Towards a contradiction assume that $\mathbf{0}_e{}^{(n+1)} \lneqq \lambda(\mathbf{a}^{(n)})'$. According to Lemma 4 and Lemma 5 the pair $\{\lambda(\mathbf{a}^{(n+1)}), \lambda(\mathbf{b}^{(n+1)})\}$ is a $\mathcal{K}$-pair over $\mathbf{0}_e{}^{(n+1)}$ for $[\mathbf{0}_e{}^{(n+1)}, \mathbf{0}_e{}^{(n+2)}]$. But $\mathbf{0}_\omega{}^{(n+1)} \lneqq_e \lambda(\mathbf{a}^{(n)})' \leq_e \lambda(\mathbf{a}^{(n+1)})$, so that Corollary 2 and Corollary 3 imply that

$$\lambda(\mathbf{b}^{(n+1)}) = \mathbf{0}_e{}^{(n+1)} \ \& \ \mathbf{b}^{(n+2)} = \mathbf{0}_\omega{}^{(n+2)}. \tag{6.11}$$

From here we conclude that $\lambda(\mathbf{b}^{(n)})' = \mathbf{0}_e{}^{(n+1)}$. Now, if $\mathbf{0}_e{}^{(n)} \lneqq_e \lambda(\mathbf{b}^{(n)})$ applying Corollary 3 we would obtain $\mathbf{a}^{(n+1)} = \mathbf{0}_\omega{}^{(n+1)}$, which is not the case and hence $\lambda(\mathbf{b}^{(n)}) = \mathbf{0}_e{}^{(n)}$. From here, (6.10), (6.11) and Lemma 2 we obtain that $\mathbf{b} = \mathbf{0}_\omega$. A contradiction.

Thus indeed $\lambda(\mathbf{a}^{(n)})' = \mathbf{0}_e{}^{(n+1)}$ and hence (6.10) and Corollary 3 imply $\mathbf{b}^{(n+1)} = \mathbf{0}_\omega{}^{(n+1)}$. But $\mathbf{b} \neq \mathbf{0}_\omega$ and hence it should be the case

$$\mathbf{0}_e{}^{(n)} \lneqq_e \lambda(\mathbf{b}^{(n)}) \ \& \ \lambda(\mathbf{b}^{(n)})' = \mathbf{0}_e{}^{(n+1)}.$$

Thus Corollary 3 yields $\mathbf{a}^{(n+1)} = \mathbf{0}_\omega{}^{(n+1)}$ and hence

$$\mathbf{a} = I^n(\kappa(\lambda(\mathbf{a}^{(n)}))) \ and \ \mathbf{b} = I^n(\kappa(\lambda(\mathbf{b}^{(n)}))).$$

$\square$

Corollary 4. *Let $\{\mathbf{a}, \mathbf{b}\}$ be a $\mathcal{K}$-pair strictly above $\mathbf{0}_\omega$ for $\mathcal{G}_\omega$. Then for every natural number n*

$$\forall \mathbf{x} \lneqq_\omega \mathbf{o}_n[\mathbf{a} \vee \mathbf{x} \lneqq \mathbf{o}_n] \iff \mathbf{a}, \mathbf{b} \leq_\omega \mathbf{o}_{n+1}.$$

Proof. An obvious application of Theorem 8 and Theorem 5.

$\square$

Finally we have the necessary tools sufficient to give us a first order definition of $\mathbf{o}_n$ in $(\mathcal{G}_\omega, \leq_\omega)$ for every n and thus by Lemma 3 and the equivalences (5.3) and (5.5) we conclude the proofs of Theorem 1 and Theorem 2.

Theorem 9. *For arbitrary $n \geq 0$, $\mathbf{o}_{n+1}$ is the greatest degree which is the least upper bound of a $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ strictly above $\mathbf{0}_\omega$ for $\mathcal{G}_\omega$, such that*

$$\forall \mathbf{x} \lneqq \mathbf{o}_n [\mathbf{a} \vee \mathbf{x} \leqq_\omega \mathbf{o}_n].$$

Proof. Theorem 7, Lemma 7 and Lemma 5 give us a $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ for $\mathcal{G}_\omega$, such that $\mathbf{0}_\omega \lneqq \mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \vee \mathbf{b} = \mathbf{o}_{n+1}$. Now the theorem follows from Corollary 4.
$\square$

7. The first order theory of $\mathcal{G}_\omega$

In this section we apply the results from the previous section to prove Theorem 3. We give an interpretation of true arithmetic in $(\mathcal{G}_\omega, \leq_\omega)$.

Consider the class

$$\mathcal{R}_1 = \{\mathbf{a} \wedge \mathbf{o}_1 \mid \mathbf{a} \in \mathbf{D}_1 \cap \mathcal{G}_\omega\}.$$

According to Theorem 9 and Theorem 1 $\mathcal{R}_1$, is first order definable in $\mathcal{G}_\omega$. Furthermore, using Proposition 3 we obtain

$$\mathbf{x} \in \mathcal{R}_1 \iff \exists \mathbf{a} \in \mathbf{D}_1 \cap \mathcal{G}_\omega [\mathbf{x} = I^1(\mathbf{a}')]. \tag{7.1}$$

From here we obtain the following.

Proposition 4. *The partially ordered set of the Π_2^0 enumeration degrees greater or equal $\mathbf{0}_e{}'$ is isomorphic to $(\mathcal{R}_1, \leq_\omega)$.*

Proof. Denote by $\mathbf{\Pi}_2$ the set of all enumeration degrees above or equal to $\mathbf{0}_e{}'$, which contain a Π_2^0 set. According to McEvoy [7]

$$\mathbf{x} \in \mathbf{\Pi}_2 \iff \exists \mathbf{a} \leq_e \mathbf{0}_e{}' [\mathbf{x} = \mathbf{a}']. \tag{7.2}$$

Let the mapping $\varphi : \mathbf{\Pi}_2 \to \mathcal{G}_\omega$ act by the rule

$$\varphi(\mathbf{x}) = I^1(\kappa(\mathbf{x})).$$

The equivalences (7.1) and (7.2) give us

$$\mathrm{Range}(\varphi) = \mathcal{R}_1.$$

On the other hand claim (K1) of Proposition 1 and claim (I1) of Lemma 1 yield

$$\forall \mathbf{x}, \mathbf{y} \in \mathbf{\Pi}_2 [\mathbf{x} \leq_e \mathbf{y} \iff \varphi(\mathbf{x}) \leq_\omega \varphi(\mathbf{y})],$$

and hence φ is the isomorphism we are looking for.
$\square$

The class $\mathbf{\Pi}_2$ consists exactly of the enumeration degrees, which are images, under the standard embedding ι of $\mathcal{D}_T$ into $\mathcal{D}_e$, of the Turing degrees c.e. in and above $\mathbf{0}_T{}'$. In other words

$$\mathbf{\Pi}_2 = \iota[\mathcal{R}_{\mathbf{0}_{T'}}],$$

where $\mathcal{R}_{\mathbf{0}_{T'}} = \{\mathbf{d}_T(X) \mid \emptyset'_T \leq_T X \ \& \ X \text{ is c.e. in } \emptyset'_T\}$. Thus Proposition 4 implies

$$(\mathcal{R}_{\mathbf{0}_{T'}}, \leq_T) \cong (\mathcal{R}_1, \leq_\omega).$$

Nies, Shore and Slaman [9] prove that first order true arithmetic is interpretable in $(\mathcal{R}_{\mathbf{0}_{T'}}, \leq_T)$, so that we obtain an interpretation of the first order theory of true arithmetic in $(\mathcal{G}_\omega, \leq_\omega)$ and conclude the proof of Theorem 3.

8. Proof of Theorem 4

In this section we present the construction for the simple case of Theorem 5, as stated in Theorem 4. For every nonzero Δ^0_2 degree $\mathbf{a}$ we construct a $\mathcal{K}$-pair strictly over $\mathbf{0}_e$, $\{\mathbf{b}, \mathbf{c}\}$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{0}'_e$. In the next section we describe a method for relativizing the presented construction above any total degree $\mathbf{u}$.

As we will be dealing with sets and not degrees, to simplify notation we shall say that $\{A, B\}$ is a $\mathcal{K}$-pair over U if $\{d_e(A), d_e(B)\}$ form a $\mathcal{K}$-pair in $\mathcal{D}_e$ over $d_e(U)$. When U is a c.e. set we shall say that $\{A, B\}$ is a $\mathcal{K}$-pair.

We need a dynamic characterization of the Σ^0_2 $\mathcal{K}$-pairs of sets given by the following lemma:

Lemma 8 (Kalimullin[6]). *Let B and C be two Σ^0_2 sets. B and C form a $\mathcal{K}$-pair if and only if there are Σ^0_2 approximations $\{B^{\{s\}}\}_{s<\omega}$ and $\{C^{\{s\}}\}_{s<\omega}$ to B and C respectively such that for every stage s we have that $B^{\{s\}} \subseteq B$ or $C^{\{s\}} \subseteq C$.*

Of course as every nontrivial $\mathcal{K}$-pair of Σ^0_2 sets consists of low sets it follows that if B and C are not c.e. then they are Δ^0_2. With this characterization in mind we proceed to describe the proof of Theorem 4.

Fix a Δ^0_2 representative A of the given nonzero degree $\mathbf{a}$. We will construct Δ^0_2 approximations to sets B and C so that the following three types of requirements are satisfied:

1. First we want to ensure the cupping property. We will construct an enumeration operator Γ so that

$$\mathcal{S} : \Gamma(A, B) = \overline{K}.$$

   Here $\Gamma(A, B)$ is considered as being enumerated relative to two sources. We write $\Gamma(A, B)$, instead of $\Gamma(A \oplus B)$. Naturally we will assume that an axiom of the operator Γ has the structure $\langle n, D_A, D_B \rangle$ and that it is valid if an only if $D_A \subseteq A$ and $D_B \subseteq B$. $\overline{K}$ is any Π^0_1 representative of the degree $\mathbf{0}'_e$.

2. To ensure that the set B is not complete it will be enough to prove that C is not c.e. and that $\{B, C\}$ form a $\mathcal{K}$-pair. Fix a computable enumeration of all c.e. sets $\{W_e\}_{e<\omega}$. For every e we have a requirements:

$$\mathcal{N}_e : W_e \neq C.$$

3. Finally we ensure the $\mathcal{K}$-pair property. For every s we have the following requirement:

$$\mathcal{K}_s : B^{\{s\}} \subseteq B \vee C^{\{s\}} \subseteq C.$$

Intuitive description of the strategies. Fix a Δ_2^0 approximation $\{A^{\{s\}}\}_{s<\omega}$ to the given set A and a Π_1^0 approximation $\{\overline{K}^{\{s\}}\}_{s<\omega}$ to the set $\overline{K}$. Every c.e. set W_e will be approximated by its standard c.e. approximation. The construction will run in stages. At every stage s we construct $C^{\{s\}}$ and $B^{\{s\}}$ from their previous values at stage $s-1$, by activating certain strategies. An activated strategy will perform actions (e.g. modify the approximation to the constructed sets or the value of parameters) needed in order to satisfy its corresponding requirement. We start by describing the intuition behind each such strategy.

The $\mathcal{S}$-strategy. The global $\mathcal{S}$-strategy constructs the enumeration operator Γ so that ultimately we have $\Gamma(A, B)$. At every stage s it ensures that for every element $n \leq s$ we have $\Gamma(A, B)^{\{s\}}(n) = \overline{K}^{\{s\}}(n)$. Every element n will be assigned a current A-marker $a(n)$ and current B-marker $b(n)$ by the $\mathcal{N}$-strategies. If the element $n \in \overline{K}$ then the strategy enumerates in Γ an axiom of the form $\langle n, A^{\{s\}} \upharpoonright a(n) + 1, B^{\{s\}} \upharpoonright b(n) + 1 \rangle$. If later on the element exits the approximation to $\overline{K}$ the strategy invalidates the previously enumerated axioms by extracting the current B-marker from the set B.

The $\mathcal{N}$-strategies. To ensure that $C \neq W_e$ we use the standard strategy. When first activated at stage s the strategy chooses a witness x_e and enumerates it in the set $C^{\{s\}}$. If this witness never enters the approximation to W_e then the requirement is satisfied and no further actions are needed. If at stage s^+ the witness x_e does enter the approximation to the set W_e the strategy extracts the witness x_e from the set $C^{\{s^+\}}$, and in this case as well succeeds to satisfy the requirement.

Incorporating the $\mathcal{K}$-requirements. There will be no explicit strategy activated during the construction to ensure that $\mathcal{K}$-requirements are satisfied. Instead it will be incorporated in the the $\mathcal{N}_e$-strategies. If a witness is enumerated at stage s and then extracted at stage s^+ then for all t in the interval $[s, s^+)$ the approximation to the set C is wrong, namely $C^{\{t\}} \nsubseteq C$. To ensure that the $\mathcal{K}$-requirements for such stages are respected, we must therefore ensure that $B^{\{t\}} \subseteq B$. So any B-marker that appears in the set $B^{\{t\}}$ must remain in B. This might obstruct the global $\mathcal{S}$-strategy as it might require the extraction of such a B-marker in order to keep Γ rectified.

To resolve this conflict the $\mathcal{N}_e$-strategies must be modified. Before a witness x_e is extracted from the set C, the construction must ensure that restraining certain elements in the set B will not affect the operator Γ. As every axiom in Γ is composed of two parts: a finite set D_A and a finite set D_B, to invalidate an axiom while at the same time restraining $D_B \subseteq B$ would be possible if there is a useful extraction from the set A. Every $\mathcal{N}_e$ strategy shall therefore try to force such an extraction.

To every $\mathcal{N}_e$ strategy we assign a threshold d_e - the e-th element of $\overline{K}$. The threshold d_e marks the point where $\mathcal{N}_e$ takes control over the axioms enumerated in Γ. Of course we do not know initially which is the e-th element of $\overline{K}$ but after finitely many wrong guesses we will eventually find the right one. The strategy then assigns current A and B-markers to the threshold d_e. Through a priority ordering and initialization of the $\mathcal{N}$-strategies we ensure that all axioms for $n \geq d_e$ are extensions of an axiom for d_e in Γ. Thus invalidating the axioms for d_e will have the effect of invalidating all axioms for elements $n \geq d$ and moving the activity of the $\mathcal{S}$ strategy above any pre-fixed restraint. $\mathcal{N}_e$ will select a witness x_e and try to diagonalize with it against W_e. If the witness x_e enters W_e then the strategy shall not immediately extract it from C, instead it shall try to force a change in A by initiating the construction of a c.e. approximation to A and starting a new cycle with a new witness. If this process is repeated infinitely often with no useful extraction from A then we can argue that the set A is c.e. Thus after finitely many unsuccessful attempts at diagonalization the strategy will eventually be successful.

Construction. We order the $\mathcal{N}$-requirements linearly:
$$\mathcal{N}_0 < \mathcal{N}_1 < \mathcal{N}_2 < \ldots$$
and assign a strategy to every requirement. Next we will define when a strategy requires attention and the actions that it makes if it is activated.

The $\mathcal{S}$-strategy. The global $\mathcal{S}$-strategy requires attention at every stage. If activated as stage s, the $\mathcal{S}$-strategy operates as follows:

For every element $n \leq s$ perform the following actions:

$\mathcal{S}.1$ If $n \in \Gamma(A,B)^{\{s\}} \setminus \overline{K}^{\{s\}}$ then find all valid axioms for n in $\Gamma^{\{s\}}$. For each such axiom, $\langle n, A_n, B_n \rangle$, the finite set B_n ends in an old B-marker $b(n)$ defined at a previous stage. Extract $b(n)$ from $B^{\{s\}}$.

$\mathcal{S}.2$ If $n \in \overline{K}^{\{s\}}$ and the current A and B markers for n are defined then enumerate in $\Gamma^{\{s\}}$ the axiom $\langle n, A^{\{s\}} \restriction a(n)+1, B^{\{s\}} \restriction b(n)+1 \rangle$.

The $\mathcal{N}$-strategies. Fix an $\mathcal{N}$-strategy $\mathcal{N}_e$. The strategy $\mathcal{N}_e$ is equipped with the parameters listed below. Whenever a parameter is cancelled, it gets its initial value.

- A threshold d_e, defined as the e-th element of $\overline{K}^{\{s\}}$. Important attributes of the threshold will be its first B-marker which will be denoted by $b_0(d_e)$ and its current B-marker denoted by $b(d_e)$. The threshold and its attributes are initially undefined.

- A current witness x_e - the witness from the current cycle, and an old witness y_e - the witness from the previous cycle. These witnesses are initially undefined. Whenever a witness is cancelled it is enumerated back in the current approximation to C.

- Finally the strategy keeps a parameter G_e, which is meant to approximate the set A and is referred to as the *current guess*. Initially $G_e = \emptyset$.

We list the cases in which $\mathcal{N}_e$ requires attention and the actions it makes.

$\mathcal{N}.0$ The strategy is in initial state or the threshold $d_e \notin \overline{K}$.

 Action: Define the threshold $d_e \in \overline{K}^{\{s\}}$ as the e-th member of $\overline{K}^{\{s\}}$. Define its first current B-marker $b(d_e) = b_0(d_e)$ and its current A-marker $a(d_e)$- as fresh numbers, numbers that have not appeared in the construction so far. Enumerate $b(d_e)$ in the set $B^{\{s\}}$. All other parameters: x_e, y_e and G_e, are cancelled.

$\mathcal{N}.1$ A B-marker for an element $n < d_e$ has been extracted from B at stage s.

 Action: Extract from the set $B^{\{s\}}$ all B-markers for the threshold d_e that are greater than or equal to $b_0(d_e)$ and have been defined until stage s. Define the current marker $b(d_e)$ as a fresh number and enumerate it in the set $B^{\{s\}}$. All other parameters: x_e, y_e and G_e, are cancelled.

$\mathcal{N}.2$ $G_e \subseteq A^{\{s\}}$ and $y_e \downarrow \notin C^{\{s\}}$.

 Action: Extract from the set $B^{\{s\}}$ all B-markers for the threshold d_e that are greater than or equal to $b_0(d_e)$ and have been defined until stage s. Enumerate y_e in $C^{\{s\}}$. Cancel the current witness x_e. Define the current marker $b(d_e)$ as a fresh number and enumerate it in the set $B^{\{s\}}$.

$\mathcal{N}.3$ $G_e \nsubseteq A^{\{s\}}$ and $y_e \downarrow \in C^{\{s\}}$.

 Action: Let $s_{start}(y_e)$ be the first stage at which y_e is enumerated in the approximation to the set C. For every stage t, such that $s_{start}(y_e) \le t \le s$ and $y_e \in C^{\{t\}}$ enumerate all $n \in B^{\{t\}}$ such that $n \ge b_0(d_e)$ in the set $B^{\{s\}}$. Extract y_e from $C^{\{s\}}$. Cancel the current witness x_e. Define the current marker $b(d_e)$ as a fresh number and enumerate it in the set $B^{\{s\}}$.

$\mathcal{N}.4$ $y_e \uparrow$ or $y_e \downarrow \in C^{\{s\}}$, and $G_e \subseteq A^{\{s\}}$ and the witness x_e is not defined.

 Define the witness x_e as a fresh number and enumerate it in $C^{\{s\}}$. Extract from the set $B^{\{s\}}$ all B-markers for the threshold d_e that are greater than or equal to $b_0(d_e)$ and have been defined until stage s. Define the current marker $b(d)$ as a fresh number and enumerate it in the set $B^{\{s\}}$.

$\mathcal{N}.5$ $y_e \uparrow$ or $y_e \downarrow \in C^{\{s\}}$, and $G_e \subseteq A^{\{s\}}$ and $x_e \in W_e^{\{s\}}$.

 Action: Let s_{start} be the first stage at which x_e is enumerated in the approximation to the set C. Set the guess G_e to be the set $\bigcap_{s_{start} \le t \le s} A^{\{t\}} \upharpoonright a(d_e)$. Set $y_e = x_e$ and cancel x_e, i.e. the current witness becomes the old witness and the value of the current witness is cancelled. Extract all B-markers for the threshold d_e that are greater than or equal to $b_0(d_e)$ and have been defined until stage s from the set $B^{\{s\}}$. Redefine the the current A-marker $a(d_e)$ of the threshold as a new fresh number. Define the current marker $b(d_e)$ as a fresh number and enumerate it in $B^{\{s\}}$.

 The complete construction. At stage 0 we set $\Gamma^{\{0\}} = C^{\{0\}} = B^{\{0\}} = \emptyset$, all markers and parameters are undefined. At stage $s > 0$ let $\mathcal{N}_e$ be the highest priority $\mathcal{N}$-strategy which requires attention. The strategy requires attention at stage s under the least step which applies.

 Case 1: $\mathcal{N}_e$ requires attention under step $\mathcal{N}.3$. Then $\mathcal{N}_e$ receives attention.

 Case 2: Otherwise the global strategy $\mathcal{S}$ receives attention and executes its actions. This may cause a higher priority $\mathcal{N}$-strategy to require attention under

step $\mathcal{N}.1$. Let $\mathcal{N}_j$ be the highest priority strategy which requires attention after the action of the global $\mathcal{S}$-strategy. $\mathcal{N}_j$ receives attention.

In both cases all $\mathcal{N}$-strategies with lower priority than the $\mathcal{N}$-strategy which receives attention are initialized. All of their parameters are cancelled. $B^{\{s\}}$ and $C^{\{s\}}$ are the final values of the constructed sets at the end of stage s.

This completes the construction.

Below we prove that the construction produces the desired sets B and C. The proof is divided into a series of simple propositions. We start with a very simple property of the construction.

Proposition 5. *The global strategy $\mathcal{S}$ receives attention at infinitely many stages.*

Proof. Towards a contradiction assume that there is a stage s such that at all $t \geq s$ the strategy $\mathcal{S}$ does not receive attention. Then at each stage $t \geq s$ an $\mathcal{N}$-strategy requires and receives attention under step $\mathcal{N}.3$. Let $\mathcal{N}_e$ be the highest priority strategy which requires attention at a stage $t > s$. By construction $\mathcal{N}_e$ receives attention, and performs action that ensure that it does not require attention under step $\mathcal{N}.3$ at the next stage $t+1$, e.g. enumerates its old witness y_e in the set $C^{\{t\}}$. All lower priority strategies are initialized at stage t and require attention under $\mathcal{N}.1$ at the next stage. It follows that at stage $t+1$ no strategy will require attention under step $\mathcal{N}.3$ and $\mathcal{S}$ will receive attention contradicting our assumption. $\square$

Fix an $\mathcal{N}$-strategy $\mathcal{N}_e$ and assume that no higher priority $\mathcal{N}$-strategy requires attention after stage $s_i(e)$. We will prove that there is a stage $s_f(e)$ such that $\mathcal{N}_e$ does not require attention at stages $t > s_f(e)$ and that the requirement $\mathcal{N}_e$ satisfies its requirement. We will do this in three steps, incorporating the inductive proof that the $\mathcal{S}$ strategy succeeds as well.

Proposition 6. *There is a least stage $s_1(e)$ such that $\mathcal{N}_e$ does not require attention under steps $\mathcal{N}.0$ and $\mathcal{N}.1$ at stages $t > s_1(e)$.*

Proof. As $\mathcal{N}_e$ is not initialized after stage $s_i(e)$, whenever $\mathcal{N}_e$ requires attention, it receives attention and no other strategy can cancel the values of its parameters.

At stage $s_i(e) + 1$ the strategy $\mathcal{N}_e$ requires attention under step $\mathcal{N}.0$ and assigns to d_e the e-th element of the approximation to $\overline{K}$. The set $\overline{K}$ is infinite and approximated by a Π_1^0 approximation. There will be a least stage $s_0(e) \geq s_i(e)$ such that $\overline{K}^{\{t\}}$ correctly approximates $\overline{K}$ on all numbers less than or equal to its e-th member at all stages $t \geq s_0(e)$. At stage $s_0(e)$ the strategy $\mathcal{N}_e$ requires attention for the last time under step $\mathcal{N}.0$. At all further stages the value of its threshold d_e together with its first B-marker $b_0(d_e)$ do not change. All lower priority strategies are initialized.

There are finitely many numbers $n < d_e$ which do not belong to the set $\overline{K}$ and for each such element finitely many axioms are enumerated in Γ. All of these axioms are enumerated before stage $s_0(e)$, as after stage $s_0(e)$ the approximation to $\overline{K} \upharpoonright d_e + 1$ does not change. Hence each such axiom ends

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in a B-marker $b(n) < b_0(d_e)$. Thus if an axiom for an element $n < d_e$ is invalidated by the global $\mathcal{S}$ strategy at a stage $t > s_0(e)$, a marker $b(n) < b_0(d_e)$ is extracted from the set $B^{\{t\}}$. This marker will not be re-enumerated in the set B by any $\mathcal{N}$-strategy at any further stage, as higher priority $\mathcal{N}$-strategies are not activate, $\mathcal{N}_e$ and lower priority strategies enumerate elements only under step $\mathcal{N}.3$, hence elements that are larger than the current value of the first marker of their threshold, defined after stage $s_0(e)$, hence larger than $b(n)$.

It follows that there is a last stage $s_1(e)$ at which the global strategy $\mathcal{S}$ extracts a B-marker for an element $n < d$. After this stage $\mathcal{N}_e$ does not require attention under step $\mathcal{N}.1$. $\qquad\square$

From this point on we will not indicate a stage when talking about the value of the threshold d_e and its first B-marker $b_0(d_e)$ as by Proposition 6 the value of these parameters do not change after stage $s_0(e)$.

Corollary 5. *At stages $t > s_1(e)$, $B^{\{t\}} \upharpoonright b_0(d_0) = B \upharpoonright b_0(d)$. The operator Γ is correct on all numbers $n < d_0$.*

Proof. The first part of this lemma is straightforward. After stage $s_1(e)$ the global strategy does not modify the approximation to the set $B \upharpoonright b_0(d)$ and it follows from the proof of Proposition 6 than no $\mathcal{N}$-strategy does either.

If $n \notin \overline{K}$, as the global strategy receives attention at infinitely many stages by Proposition 5, it will ensure that all axioms in Γ for n are invalid at infinitely many stages, hence $n \notin \Gamma(A, B)$. If $n \in \overline{K}$ then n is a threshold of a higher priority strategy $\mathcal{N}_k$, $k < e$. By construction $\mathcal{N}_k$ always ensures that the current A- and B-markers of its threshold are defined and is the only strategy which can modify their values. After stage $s_i(e)$ the strategy $\mathcal{N}_k$ does not require attention, hence the values of $a(n)^{\{t\}}$ and $b(n)^{\{t\}}$ are defined at all $t > s_i(e)$ and do not change. Denote their final values by $a(n)$ and $b(n)$. The approximation to $A \upharpoonright a(n) + 1$ will eventually settle down and there will be a stage $s_n \geq s_i(e)$ such that $(\forall t \geq s_n)(A^{\{t\}} \upharpoonright a(n) + 1 = A \upharpoonright a(n) + 1)$. At stage s_n the global strategy enumerates the axiom $\langle n, A^{\{s_n\}} \upharpoonright a(n) + 1, B^{\{s_n\}} \upharpoonright b(n) + 1 \rangle$ which is valid at all further stages. $\qquad\square$

Lemma 9. *The set $G_e = \bigcup_{t \geq s_1(e)} G_e^{\{t\}}$ is computably enumerable.*

Proof. We will prove that $G_e^{\{t\}} \subseteq G_e^{\{t+1\}}$ for all $t \geq s_1(e)$. Let s be a stage such that $G_e^{\{s\}} \neq G^{\{s+1\}}$. Then at stage $s+1$ the value of the guess is changed, hence $\mathcal{N}_e$ executes step $\mathcal{N}.5$.

Denote the previous value of the guess, $G_e^{\{s\}}$, by G_e^- and the previous value of the old witness, $y_e^{\{s\}}$ by y_e^-. Let $s^- \geq s_1(e)$ be the stage at which these values are assigned to the guess. At all stages t in the interval $[s^-, s]$, $G_e^{\{t\}} = G_e^-$ and $y_e^{\{t\}} = y_e^-$.

At stage $s+1$ the strategy $\mathcal{N}_e$ has a current witness $x_e^{\{s\}}$ defined at stage $s_{start}(x_e) > s^-$ and sets $G_e^{\{s+1\}} = \bigcap_{s_{start}(x_e) \leq t \leq s+1} A^{\{t\}} \upharpoonright a(d_e)$. We will prove that $G_e^- = G_e^{\{t\}} \subseteq A^{\{t\}} \upharpoonright a(d_e)$ for all t in the interval $[s_{start}(x_e), s+1]$. First

we note that maximal element of G_e^- is less than $a(d_e)^{\{s+1\}}$ as every time a new value of the guess is defined the value of the marker $a(d_e)$ is shifted to a greater number.

Secondly note that every time step $\mathcal{N}.3$ or $\mathcal{N}.2$ are executed the value of the current witness is cancelled. It follows that these steps are not executed at any stage t in the interval $[s_{start}(x_e), s+1]$. At stage $s_{start}(x_e)$ step $\mathcal{N}.4$ is executed, hence the old witness y_e^- is in the approximation to the set C. The old witness can only be extracted under step $N.3$, hence at all stages t in the interval $[s_{start}(x_e), s+1]$, $y_e^- \in C^{\{t\}}$. Again by the fact that $\mathcal{N}_e$ does not require attention under $\mathcal{N}.0$, $\mathcal{N}.1$, $\mathcal{N}.2$ and $\mathcal{N}.3$ and $y_e^- \in C^{\{t\}}$ at all stages t in the interval $[s_{start}(x_e), \le s+1]$, it follows that $G_e^- = G_e^{\{t\}} \subseteq A^{\{t\}}$ and hence $G_e^{\{s\}} \subseteq G_e^{\{s+1\}}$.

$\square$

Corollary 6. *There is a least stage $s_5(e) \ge s_1(e)$ such that $\mathcal{N}_e$ does not require attention under step $\mathcal{N}.5$ at any stage $t > s_5(e)$.*

Proof. Assume towards a contradiction that $\mathcal{N}_e$ requires attention under step $\mathcal{N}.5$ at infinitely many stages. Then at infinitely many stages t we have that $G_e^{\{t\}} \subseteq A^{\{t\}}$ and the value of the guess is redefined. We will prove that $A = G_e = \bigcup_{t \ge s_1(e)} G_e^{\{t\}}$. Fix n and let t_n be a stage such that $(\forall t \ge t_n)(A^{\{t\}}(n) = A(n))$. If $n \in A$ then let $t_2 > t_1 > t_n$ be two stages at which $\mathcal{N}_e$ executes $\mathcal{N}.5$. At stage t_1 the current witness is cancelled and the marker $a(d_e)$ is redefined to a fresh number larger than any that has appeared in the construction so far, hence larger than n. The current witness at stage t_2 is defined at stage $s_{start}(x) > t_1 > t_n$. Since $n \in A^{\{t\}}$ at all $t \ge s_{start}(x)$ and $a(d_e)^{\{t_2\}} > n$, the element n enters the approximation to the guess G_e at stage t_2. If $n \notin A$ and we assume that $n \in G_e^{\{t_0\}}$ at some stage $t_0 > t_n$ then at all $t > t_0$ $G_e^{\{t\}} \nsubseteq A^{\{t\}}$, contradicting the assumption that $\mathcal{N}_e$ executes $\mathcal{N}.5$ at infinitely many stages. However by Lemma 9 $G_e = A$ is a c.e. set, and we have reached the desired contradiction. $\square$

We are ready for the final third step in the proof of the satisfaction of the $\mathcal{N}$-requirements.

Lemma 10. *There is a least stage $s_f(e) \ge s_5$ after which the strategy $\mathcal{N}_e$ does not require attention. The requirement $\mathcal{N}_e$ is satisfied.*

Proof. Let $G_e = G_e^{\{s_5(e)\}}$ be the final value of the guess and $y_e = y_e^{\{s_5(e)\}}$ be the final value of the old witness (if defined). We have two cases depending on whether G_e is a subset of A or not.

Suppose that $G_e \subseteq A$ and let $s_3(e) \ge s_5(e)$ be the least stage such that $G_e \subseteq A^{\{t\}}$ at all $t \ge s_3(e)$. Then at stages $t \ge s_3(e)$ the strategy does not require attention under step $\mathcal{N}.3$. If at stage $s_3(e)$ the old witness $y_e \downarrow \notin C^{\{s_3(e)\}}$ the strategy will require attention once under step $\mathcal{N}.2$ and enumerate $y_e \in C^{\{s\}}$ permanently. Hence after a least stage $s_2(e) \ge s_3(e)$ the strategy will not require attention under step $N.2$. In all cases at stage $s_2(e) \ge s_3(e) \ge s_5(e)$ the current

witness is not defined as it is cancelled at steps $\mathcal{N}.1$, $\mathcal{N}.2$, $\mathcal{N}.3$ and $\mathcal{N}.5$. At stage $s_2(e)$ the strategy $\mathcal{N}_e$ requires attention once under step $\mathcal{N}.4$, defines the final value of the witness current x_e and enumerates it in $C^{\{s_2(e)\}}$ and after this the strategy does not require attention under $\mathcal{N}.4$. Hence $\mathcal{N}_e$ does not require attention at any stage $t > s_f(e) = s_2(e)$. Furthermore the final witness x_e never enters the approximation to W_e, or else $\mathcal{N}_e$ would require attention under step $\mathcal{N}.5$. Hence $x_e \in C \setminus W_e$.

The second case is $G_e \nsubseteq A$. In this case $G_e \neq \emptyset$ hence at stage $s_5(e)$ the strategy executes step $\mathcal{N}.5$, assigns as the final value of the old witness y_e an element which belongs to the set W_e. Let $s_f(e) > s_5(e)$ be the least stage such that $G_e \nsubseteq A^{\{t\}}$ at all $t \geq s_f(e)$. At $s_f(e)$ the strategy $\mathcal{N}_e$ requires attention under step $\mathcal{N}.3$ and extracts the old witness y_e from the set $C^{\{s_f(e)\}}$. The strategy does not require attention at stages $t > s_f(e)$ and hence $y_e \in W_e \setminus C$.
□

So far we have proved that the $\mathcal{S}$- and $\mathcal{N}$-requirements are satisfied. To conclude the proof we need to show that the $\mathcal{K}$-requirements are as well respected. For this we shall need to following.

Lemma 11. *Suppose $\mathcal{N}_e$ does not require attention after stage $s_f(e)$. Let $b(d_e)$ be the current marker of the threshold at stage $s_f(e)$. The approximation to $B \restriction b(d_e) + 1$ does not change at stages $t > s_f(e)$.*

Proof.　At stages $t > s_f(e)$ higher priority strategies than $\mathcal{N}_e$ do not require attention. Lower priority strategies are initialized at stage $s_f(e)$ and their parameters are cancelled. At further stages they modify B only on elements larger than the first marker of their threshold, defined after stage $s_f(e)$ and larger than $b(d_e)$.

This leaves the global $\mathcal{S}$-strategy which may modify the approximation to B at stage $t > s_f(e)$ in order to invalidate an axiom for an element $n \notin \overline{K}^{\{t\}}$.

Suppose that this is the case. It follows that $n > d_e$ (otherwise $\mathcal{N}_e$ would require attention under step $\mathcal{N}.0$ or $\mathcal{N}.1$ at sage $t > s_f(e)$) and that this axiom is enumerated in $\Gamma^{\{s\}}$ at stage $s \leq s_f(e)$ as it ends in a B-marker for n which is less than $b(d_e)$. By Corollary 5 the approximation to $B \restriction b_0(d_e)$ does not change at stages $t > s_1(e)$ hence this marker is larger than $b_0(d)$ and $s_0(e) \leq s \leq s_f(e)$, where $s_0(e)$ is the last stage at which $\mathcal{N}_e$ executes $\mathcal{N}.0$.

After stage $s_0(e)$ the current A and B-markers of the threshold d_e are always defined. Hence at stage s an axiom is enumerated in Γ for the threshold d_e as well. Let that be $\langle d_e, A_{d_e}, B_{d_e} \rangle$, ending in the current B-marker for d_e at stage s, say $b_s(d_e) \geq b_0(d_e)$. We will show that the axiom $\langle d_e, A_{d_e}, B_{d_e} \rangle$ is invalid at all stages $t \geq s_f(e)$. As the axiom for n enumerated at stage s is an extension of this axiom we will reach the desired contradiction. There are two cases depending on the outcome of $\mathcal{N}_e$.

If $\mathcal{N}_e$ is satisfied by $C \nsubseteq W_e$ by Lemma 10 at stage $s_f(e)$ the strategy executes $\mathcal{N}.4$ and extracts permanently all B-markers for the threshold defined after the first one, including $b_s(d_e)$. It follows that the axiom $\langle d_e, A_{d_e}, B_{d_e} \rangle$ is invalid at all stages $t \geq s_f(e)$.

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If $\mathcal{N}_e$ is satisfied by $W_e \nsubseteq C$ then the strategy has a final old witness y_e, defined at stage $s_{sart}(y)$ and a defined final value of the guess G_e, such that $G_e \nsubseteq A^{\{t\}}$ at all stages $t \geq s_f(e)$.

The marker $b_s(d_e)$ is the current marker for the threshold d_e at stages in the interval $[s1, s2]$ where $s1$ and $s2$ are two consecutive visits of the strategy $\mathcal{N}_e$ and $s1 < s \leq s2$. If $s1 < s_{start}(y)$ then $b_s(d_e)$ is permanently extracted at stage $s_{start}(y) \geq s2$ under the actions of $\mathcal{N}.4$. Suppose $s_{start}(y) \leq s1$. If $y_e \notin C^{\{s_1\}}$ then at stage s_2 the strategy $\mathcal{N}_e$ executes a step different from $\mathcal{N}.3$ and the marker $b_s(d_e)$ is as well permanently extracted from the set B. Finally if $y_e \in C^{\{s_1\}}$ then at stage s the guess $G_e \subseteq A^{\{s\}}$, otherwise $\mathcal{N}_e$ would require and receive attention under $\mathcal{N}.3$ and $\mathcal{S}$ would not be activated. In this case however $G_e \subseteq A_{d_e}$ and again the axiom is invalid at all stages $t > s_f(e)$.
$\qquad\square$

Finally we can conclude the proof of the theorem

Corollary 7. *All $\mathcal{N}$-requirements are satisfied. $\Gamma(A, B) = \overline{K}$. The sets B and C form a $\mathcal{K}$-pair.*

Proof. That all $\mathcal{N}$-requirements are satisfied follows by an induction on their priority. Suppose that all requirements of higher priority than $\mathcal{N}_e$ are satisfied and their corresponding strategies do not require attention after stage s_i. . Then by Lemma 10 $\mathcal{N}_e$ satisfies its requirement and does not require attention after stage $s_f(e)$.

To prove that $\Gamma(A, B) = \overline{K}$ fix a number n and select an $\mathcal{N}$-strategy $\mathcal{N}_k$ with permanent threshold $d_k > n$. By Corollary 5 the operator Γ is correct on the element n.

Finally to prove that B and C form a $\mathcal{K}$-pair we will show that for every stage s if $C^{\{s\}} \nsubseteq C$ then $B^{\{s\}} \subseteq B$.

Fix s such that $C^{\{s\}} \nsubseteq C$ and assume towards a contradiction that $B^{\{s\}} \nsubseteq B$. Let $y \in C^{\{s\}} \setminus C$. Then y is the old witness of an $\mathcal{N}$-strategy $\mathcal{N}_e$, which is defined at stage $s_{start}(y) \leq s$, never cancelled and eventually permanently extracted from the set C at stage $s_f(e) > s$.

As $s > s_1(e)$ by Corollary 5 $B^{\{s\}} \restriction b_0(d_e) \subseteq B$. On the other hand by Lemma 10 at stage $s_f(e)$ the strategy $\mathcal{N}_e$ executes step $\mathcal{N}.3$ and enumerates in the set $B^{\{s_f(e)\}}$ all elements $n \in B^{\{s\}}$ such that $n \geq b_0(d_e)$ and then sets the final value of the B-marker for the threshold $b(d_e)$ to a fresh number larger than $\max B^{\{s\}}$. Finally by Lemma 11 the approximation to $B \restriction b(d) + 1$ does not change at stages $t > s_f(e)$, hence $B^{\{s\}} \subseteq B$. $\qquad\square$

9. Proof of Theorem 5

Theorem 5 is a relativized version of Theorem 4. In order to prove it we will first show some basic concepts that will allow us to carry out the construction described in the previous section above any total enumeration degree.

Let G be a total representative of a total enumeration degree $\mathbf{g}$. We can define a $\Sigma_2^0(G)$ approximation to a set A to be a uniformly computable from

G sequence of finite sets $\{A^{\{s\}}\}_{s<\omega}$ such that $n \in A$ if and only if $(\exists s)(\forall t > s)(n \in A^{\{s\}})$. A $\Sigma_1^0(G)$ approximation to a set A is a $\Sigma_2^0(G)$ approximation $\{A^{\{s\}}\}_{s<\omega}$ with the additional property that for every s the $A^{\{s\}} \subseteq A^{\{s+1\}}$. A $\Pi_1^0(G)$ approximation to a set A is a $\Sigma_2^0(G)$ approximation $\{A^{\{s\}}\}_{s<\omega}$ with the additional property that for every s and n if $n \in A^{\{s\}} \setminus A^{\{s+1\}}$ then $n \notin A^{\{t\}}$ at all $t \geq s+1$. A $\Delta_2^0(G)$ approximation to a set A is a $\Sigma_2^0(G)$ approximation $\{A^{\{s\}}\}_{s<\omega}$ with the additional property that for every n the limit $\lim_s A^{\{s\}}(n)$ exists. These are natural definitions motivated by the fact that a set A is $\Sigma_2^0(G)$ ($\Sigma_1^0(G)$, $\Pi_1^0(G)$ or $\Delta_2^0(G)$) if and only if it has a $\Sigma_2^0(G)(\Sigma_1^0(G)$, $\Pi_1^0(G)$ or $\Delta_2^0(G))$ approximation.

The following lemma is true of any set G, not necessarily a total set G.

Lemma 12. *A set Y is enumeration reducible to $G \oplus X$ if and only if there is a set $W \leq_e G$ such that $Y = W(X) = \{n \mid (\exists \langle n, D\rangle \in W)(D \subseteq X)\}$.*

Proof. Suppose $Y = \Gamma(G, X)$ where Γ is an e-operator, i.e. a c.e. set. Then consider the set $W = \{\langle n, D_x\rangle | \langle n, D_g, D_x\rangle \in \Gamma \wedge D_g \subseteq G\}$. Then $W \leq_e G$ and $Y = W(X)$.

On the other hand if $Y = W(X)$ and $W = \Lambda(G)$, where Λ is an e-operator, then let $\Gamma = \{\langle n, D_g, D_x\rangle | \langle\langle n, D_x\rangle, D_g\rangle \in \Lambda\}$. Then $Y = \Gamma(G, X)$.
$\square$

Thus, as our set G is total, it follows that the set Y is enumeration reducible to the set $G \oplus X$ if and only if there is a set W which is c.e. in G, such that $Y = W(X)$. Of course a set W is c.e. in G if an only if it is $\Sigma_1^0(G)$ if and only if it has a $\Sigma_1^0(G)$ approximation.

Finally we turn to $\mathcal{K}$-pairs with respect to G.

Lemma 13 (Relativized $\mathcal{K}$-pair approximation property). *Let B and C be $\Sigma_2^0(G)$ sets with $\Sigma_2^0(G)$ approximations $\{B^{\{s\}}\}_{s<\omega}$ and $\{C^{\{s\}}\}_{s<\omega}$ such that for every s either $B^{\{s\}} \subseteq B$ or $C^{\{s\}} \subseteq C$. Then B and C form a $\mathcal{K}$-pair over G.*

Proof. Let $W = \bigcup_{s<\omega} A^{\{s\}} \times B^{\{s\}}$. Then W is c.e. in G, hence enumeration reducible to G. Furthermore for every pair $(b, c) \in B \times C$ then there are stages s_b and s_c such that $(\forall t \geq s_b)(b \in B^{\{t\}})$ and $(\forall t \geq s_c)(c \in C^{\{t\}})$, hence at stage $s = \max(s_b, s_c)$ we have $(b, c) \in B^{\{s\}} \times C^{\{s\}} \subseteq W$, hence $B \times C \subseteq W$. Now fix $(b, c) \in \overline{B} \times \overline{C}$. If $b \in B^{\{s\}}$ then $\mathcal{B}^{\{s\}} \not\subseteq B$ and hence by the property of the approximations $C^{\{s\}} \subseteq C$. But in this case $c \notin \mathcal{C}^{\{s\}}$. As this is true for every stage s it follows that $(b, c) \notin W$ and hence $\overline{B} \times \overline{C} \subseteq \overline{W}$. Now applying Theorem 6 we get that B and C form a $\mathcal{K}$-pair over G. $\square$

These properties are sufficient to prove the desired relativization. Fix A such that A is $\Delta_2^0(G)$ and $A \not\leq_e G$. Let $\overline{K}_G$ be a $\Pi_1^0(G)$ representative of the degree g'. We construct $\Delta_2^0(G)$ sets B and C and a c.e. in G set Γ so that the following requirements are satisfied:

$$\mathcal{S} : \Gamma(A, B) = \overline{K}_G.$$

This will ensure that $A \oplus B \oplus G = \overline{K}_G$ hence the degree of A is cupped to g' by the degree of $B \oplus G$.

Fix a computable enumeration of all c.e. in G sets $\{W_e^G\}_{e<\omega}$. For every e we have a requirements:

$$\mathcal{N}_e : W_e^G \neq C.$$

Finally we ensure the $\mathcal{K}$-pair property. For every s we have the following requirement:

$$\mathcal{K}_s : B^{\{s\}} \subseteq B \vee C^{\{s\}} \subseteq C.$$

To satisfy these requirements we carry out precisely the same construction as in the previous section but relative to the oracle G. Finally we set $\mathbf{b} = d_e(B \oplus G)$ and $\mathbf{c} = d_e(C \oplus G)$. Then from the $\mathcal{S}$-requirement we get that $\mathbf{b} \vee \mathbf{a} = \mathbf{g}'$. Hence $\mathbf{b} > \mathbf{g}$. From the $\mathcal{N}$-requirements we get that $\mathbf{c} > \mathbf{g}$, as $C \not\leq_e G$. From the $\mathcal{K}$-requirement we get that $\mathbf{b}$ and $\mathbf{c}$ is a $\mathcal{K}$-pair. Indeed $\{B^{\{s\}} \oplus G \upharpoonright s\}$ and $\{C^{\{s\}} \oplus G \upharpoonright s\}$ are $\Delta_2^0(G)$ approximations to the sets $B \oplus G$ and $C \oplus G$ with the relativized $\mathcal{K}$-approximation property. Finally as $\mathbf{b}$ and $\mathbf{c}$ are $\mathcal{K}$-pairs over $\mathbf{g}$ it follows that both $\mathbf{b}$ and $\mathbf{c}$ are low over $\mathbf{g}$ and hence $\mathbf{b} < \mathbf{g}'$.

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