On a Relative Computability Notion for Real Functions

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Abstract. For any class of total functions in the set of natural numbers, we define what it means for a real function to be conditionally computable with respect to this class. This notion extends a notion of relative uniform computability of real functions introduced in a previous paper co-authored by Andreas Weiermann. If the given class consists of recursive functions then the conditionally computable real functions are computable in the usual sense. Under certain weak assumptions about the class in question, we show that conditional computability is preserved by substitution, that all conditionally computable real functions are locally uniformly computable, and that the ones with compact domains are uniformly computable. All elementary functions of calculus turn out to be conditionally computable with respect to one of the small subrecursive classes.

1 Introduction

In the paper [1], a notion of relative computability of real functions was introduced, namely, when a class $\mathcal{F}$ of total functions in $\mathbb{N}$ is given, certain real functions were called uniformly $\mathcal{F}$-computable. It was shown that the elementary functions of calculus are uniformly $\mathcal{M}^2$-computable if we consider them on compact subsets of their domains[1]. In the present paper, we introduce a wider notion of relative computability called conditional $\mathcal{F}$-computability[2]. This notion is close to uniform computability in a certain sense, but nevertheless all elementary functions of calculus, considered on their whole domains, turn out to be conditionally $\mathcal{M}^2$-computable. The supplementary feature of conditional $\mathcal{F}$-computability in comparison to the uniform one can be informally described

[1] The class $\mathcal{M}^2$ consists of the argumentless constants $0, 1, 2, \ldots$ and all functions in $\mathbb{N}$ which can be obtained from the successor function, the function $\lambda xy.x \dot{-} y$, the multiplication function and projection functions by finitely many applications of substitution and bounded least number operation.

[2] As in [1], we are interested mainly in the case when $\mathcal{F}$ is some of the small subrecursive classes (and especially in the case $\mathcal{F} = \mathcal{M}^2$).

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as follows. It is now allowed the approximation process to depend on an additional natural parameter. Its value at any point of the domain of the considered real function can be determined by means of a search until a certain term vanishes. The term in question must be constructed by using function symbols for functions in $\mathcal{F}$ and for the functions approximating the coordinates of the point[3].

We will use many definitions and results from [1] without explanation. Although not obligatory for the reading of the present paper, a more detailed acquaintance with the paper [1] could help for a better comprehension of the present one.

2 $\mathcal{F}$-Substitutional Mappings

For any $m \in \mathbb{N}$, we will denote by $\mathbb{T}_m$ the set of all m-argument total functions in $\mathbb{N}$. Let $\mathcal{F} \subseteq \bigcup_{m\in\mathbb{N}} \mathbb{T}_m$. For any $k, m \in \mathbb{N}$, certain mappings of $\mathbb{T}_1^k$ into $\mathbb{T}_m$ will be called *$\mathcal{F}$-substitutional*, as follows[4].

Definition 1. *We proceed by induction:*

1. *For any m-argument projection function h in $\mathbb{N}$ the mapping F defined by means of the equality $F(f_1,\dots,f_k) = h$ is $\mathcal{F}$-substitutional.*
2. *For any $i \in \{1,\dots,k\}$, if F_0 is a $\mathcal{F}$-substitutional mapping of $\mathbb{T}_1^k$ into $\mathbb{T}_m$ then so is the mapping F defined by means of the equality*

$$F(f_1,\dots,f_k)(n_1,\dots,n_m) = f_i(F_0(f_1,\dots,f_k)(n_1,\dots,n_m)).$$

3. *For any natural number r and any r-argument function f from $\mathcal{F}$, if $F_1,\dots,F_r$ are $\mathcal{F}$-substitutional mappings of $\mathbb{T}_1^k$ into $\mathbb{T}_m$ then so is the mapping F defined by means of the equality*

$$F(f_1,\dots,f_k)(n_1,\dots,n_m) = \\ f(F_1(f_1,\dots,f_k)(n_1,\dots,n_m),\dots,F_r(f_1,\dots,f_k)(n_1,\dots,n_m)).$$

Intuitively, a mapping F of $\mathbb{T}_1^k$ into $\mathbb{T}_m$ is $\mathcal{F}$-substitutional iff there is an expression for $F(f_1,\dots,f_k)(n_1,\dots,n_m)$ built from the variables $n_1,\dots,n_m$ by using function symbols $f_1,\dots,f_k$ and function symbols for functions from $\mathcal{F}$[5].

The following statements are straightforward generalizations of statements from [1], and can be proved in the same way (i.e. by induction on F).

[3] In the situation studied in [3], also a dependance of the approximation process on an additional parameter is admitted, but its description uses the distance to the complement of the domain of the function and makes no use of $\mathcal{F}$.

[4] Only the cases $m = 1$ and $m = 2$ will be actually needed (the first of them is considered, by using a slightly different terminology, also in [1]).

[5] If $k = 0$ then such a mapping can be identified with an m-argument function in $\mathbb{N}$ which is explicitly definable through functions from $\mathcal{F}$.

Proposition 1. *Let $F : \mathbb{T}_1^k \to \mathbb{T}_m$ and $G_1, \ldots, G_m : \mathbb{T}_1^k \to \mathbb{T}_l$ be $\mathcal{F}$-substitutional. Then so is the mapping $H : \mathbb{T}_1^k \to \mathbb{T}_l$ defined by*

$$H(f_1, \ldots, f_k)(\overline{n}) = F(f_1, \ldots, f_k)(G_1(f_1, \ldots, f_k)(\overline{n}), \ldots, G_m(f_1, \ldots, f_k)(\overline{n})),$$

where $\overline{n}$ is an abbreviation for $n_1, \ldots, n_l$.

Proposition 2. *Let $F : \mathbb{T}_1^k \to \mathbb{T}_m$ and $G_1, \ldots, G_k : \mathbb{T}_1^l \to \mathbb{T}_1$ be $\mathcal{F}$-substitutional. Then so is the mapping $H : \mathbb{T}_1^l \to \mathbb{T}_m$ defined by*

$$H(g_1, \ldots, g_l) = F(G_1(g_1, \ldots, g_l), \ldots, G_k(g_1, \ldots, g_l)).$$

In the sequel, we will sometimes need to consider an expression with values in $\mathbb{N}$ as a function of some variable in it ranging over $\mathbb{N}$. Instead of a λ-notation for the function defined in this way, we will use another one obtained by replacing the variable in question by the symbol $\bullet$ (for instance, we could write $\bullet^2 + \bullet + 1$ instead of $\lambda x.x^2 + x + 1$). Despite having much more restricted usability than λ-notation, this notation will be sufficient for most of our needs here.

Proposition 2 can be generalized as follows.

Proposition 3. *Let $F : \mathbb{T}_1^k \to \mathbb{T}_m$ and $G_1, \ldots, G_k : \mathbb{T}_1^l \to \mathbb{T}_{p+1}$ be $\mathcal{F}$-substitutional. Then so is the mapping $H : \mathbb{T}_1^l \to \mathbb{T}_{p+m}$ defined by the equality*

$$H(g_1, \ldots, g_l)(\overline{u}, \overline{n}) = F(G_1(g_1, \ldots, g_l)(\overline{u}, \bullet), \ldots, G_k(g_1, \ldots, g_l)(\overline{u}, \bullet))(\overline{n}),$$

where $\overline{u}$ and $\overline{n}$ are abbreviations for $u_1, \ldots, u_p$ and $n_1, \ldots, n_m$, respectively.

Proof. If F is $\mathcal{F}$-substitutional by clause 1 of Definition 1 then so is H. Suppose now F has the form from clause 2 of Definition 1, and the mapping F_0 has the considered property. Then

$$H(g_1, \ldots, g_l)(\overline{u}, \overline{n}) = G_i(g_1, \ldots, g_l)(\overline{u}, \bullet)(H_0(g_1, \ldots, g_l)(\overline{u}, \overline{n})) = G_i(g_1, \ldots, g_l)(\overline{u}, H_0(g_1, \ldots, g_l)(\overline{u}, \overline{n})),$$

where

$$H_0(g_1, \ldots, g_l)(\overline{u}, \overline{n}) = F_0(G_1(g_1, \ldots, g_l)(\overline{u}, \bullet), \ldots, G_k(g_1, \ldots, g_l)(\overline{u}, \bullet))(\overline{n}).$$

Since G_i and H_0 are $\mathcal{F}$-substitutional, H is also $\mathcal{F}$-substitutional (by Proposition 1 and clause 1 of Definition 1). Finally suppose that F has the form from clause 3 of Definition 1, and the mappings $F_1, \ldots, F_r$ have the considered property. Then

$$H(g_1, \ldots, g_l)(\overline{u}, \overline{n}) = f(H_1(g_1, \ldots, g_l)(\overline{u}, \overline{n}), \ldots, H_r(g_1, \ldots, g_l)(\overline{u}, \overline{n})),$$

where

$$H_i(g_1, \ldots, g_l)(\overline{u}, \overline{n}) = F_i(G_1(g_1, \ldots, g_l)(\overline{u}, \bullet), \ldots, G_k(g_1, \ldots, g_l)(\overline{u}, \bullet))(\overline{n})$$

for $i = 1, \ldots, r$. Since $H_1, \ldots, H_r$ are $\mathcal{F}$-substitutional, H is also $\mathcal{F}$-substitutional (by clause 3 of Definition 1).

3 Conditional $\mathcal{F}$-Computability of Real Functions

As in [1], a triple $(f,g,h) \in \mathbb{T}_1^3$ is called to name a real number ξ if

$$\left|\frac{f(t)-g(t)}{h(t)+1} - \xi\right| < \frac{1}{t+1}$$

for all $t \in \mathbb{N}$.

Definition 2. *Let $N \in \mathbb{N}$ and $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^N$. The function θ will be called* conditionally computable with respect to $\mathcal{F}$ *or* conditionally $\mathcal{F}$-computable, *for short, if there exist $\mathcal{F}$-substitutional mappings $E:\mathbb{T}_1^{3N} \to \mathbb{T}_1$ and $F,G,H : \mathbb{T}_1^{3N} \to \mathbb{T}_2$ such that, whenever $(\xi_1,\dots,\xi_N) \in D$ and $(f_1,g_1,h_1),\dots,(f_N,g_N,h_N)$ are triples from $\mathbb{T}_1^3$ naming $\xi_1,\dots,\xi_N$, respectively, the following holds:*

1. *There exists a natural number s satisfying the equality*
$$E(f_1,g_1,h_1,\dots,f_N,g_N,h_N)(s) = 0. \tag{1}$$
2. *For any natural number s satisfying the equality (1), the number $\theta(\xi_1,\dots,\xi_N)$ is named by the triple $(\tilde{f},\tilde{g},\tilde{h})$, where*
$$\tilde{f} = F(f_1,g_1,h_1,\dots,f_N,g_N,h_N)(s,\bullet),$$
$$\tilde{g} = G(f_1,g_1,h_1,\dots,f_N,g_N,h_N)(s,\bullet),$$
$$\tilde{h} = H(f_1,g_1,h_1,\dots,f_N,g_N,h_N)(s,\bullet).$$

Example 1. If a real function is uniformly $\mathcal{F}$-computable in the sense of [1] then it is conditionally $\mathcal{F}$-computable. Indeed, let $(F^\circ,G^\circ,H^\circ)$ be an $\mathcal{F}$-substitutional (3,3)-computing system in the sense of [1] for the N-argument real function θ. Then we can satisfy the requirements of Definition 2 by setting

$$\begin{aligned}
E(f_1,g_1,h_1,\dots,f_N,g_N,h_N)(s) &= s,\\
F(f_1,g_1,h_1,\dots,f_N,g_N,h_N)(s,t) &= F^\circ(f_1,g_1,h_1,\dots,f_N,g_N,h_N)(t),\\
G(f_1,g_1,h_1,\dots,f_N,g_N,h_N)(s,t) &= G^\circ(f_1,g_1,h_1,\dots,f_N,g_N,h_N)(t),\\
H(f_1,g_1,h_1,\dots,f_N,g_N,h_N)(s,t) &= H^\circ(f_1,g_1,h_1,\dots,f_N,g_N,h_N)(t).
\end{aligned}$$

In the next three examples, we assume $\mathcal{F} = \mathcal{M}^2$.

Example 2. Despite not being uniformly $\mathcal{F}$-computable, the function $\theta(\xi) = \frac{1}{\xi}$ is conditionally $\mathcal{F}$-computable. To satisfy the requirements of Definition 2, we may set

$$\begin{aligned}
E(f,g,h)(s) &= (2h(s)+3) \dot{-} (s+1)|f(s)-g(s)|,\\
F(f,g,h)(s,t) &= (h(u(s,t))+1)\,\mathrm{sg}(f(u(s,t)) \dot{-} g(u(s,t))),\\
G(f,g,h)(s,t) &= (h(u(s,t))+1)\,\mathrm{sg}(g(u(s,t)) \dot{-} f(u(s,t))),\\
H(f,g,h)(s,t) &= |f(u(s,t))-g(u(s,t))| \dot{-} 1,
\end{aligned}$$

where $u(s,t) = s + (s+1)^2(t+1)$.

Example 3. The function $\theta(\xi) = \exp(\xi)$ is not uniformly $\mathcal{F}$-computable, but it is conditionally $\mathcal{F}$-computable. To show the conditional $\mathcal{F}$-computability of θ, we may use Theorem 7 of [1]. According to it, $\min(\exp(\xi), \eta)$ is a uniformly $\mathcal{M}^2$-computable function of ξ and η. Let $(F^\circ, G^\circ, H^\circ)$ be an $\mathcal{M}^2$-substitutional (3,3)-computing system for this function. To satisfy the requirements of Definition 2, we may set

$$\begin{aligned}
E(f,g,h)(s) &= (f(0)+h(0)+1) \dot{-} ((s+1)_1(h(0)+1)+g(0)),\\
F(f,g,h)(s,t) &= F^\circ(f,g,h,\lambda x.s+1,\lambda x.0,\lambda x.0)(t),\\
G(f,g,h)(s,t) &= G^\circ(f,g,h,\lambda x.s+1,\lambda x.0,\lambda x.0)(t),\\
H(f,g,h)(s,t) &= H^\circ(f,g,h,\lambda x.s+1,\lambda x.0,\lambda x.0)(t),
\end{aligned}$$

where $(s+1)_1$ is the exponent of the prime number 3 in $s+1$.

Example 4. Any partial recursive function in $\mathbb{N}$ regarded as a function in $\mathbb{R}$ is conditionally $\mathcal{F}$-computable. Indeed, let θ be an N-argument partial recursive function. Then θ has a representation of the form

$$\theta(x_1,\dots,x_N) = U(\mu y[T(x_1,\dots,x_N,y)=0]),$$

where $T, U \in \mathcal{F}$. To satisfy the requirements of Definition 2, we may set

$$\begin{aligned}
E(f_1,g_1,h_1,\dots,f_N,g_N,h_N)(s) &= T(x_1,\dots,x_N,s) + \max_{y<s} \overline{\mathrm{sg}}\, T(x_1,\dots,x_N,y),\\
F(f_1,g_1,h_1,\dots,f_N,g_N,h_N)(s,t) &= U(s),\\
G(f_1,g_1,h_1,\dots,f_N,g_N,h_N)(s,t) &= 0,\\
H(f_1,g_1,h_1,\dots,f_N,g_N,h_N)(s,t) &= 0,
\end{aligned}$$

where

$$x_i = \left\lfloor \frac{f_i(1) \dot{-} g_i(1)}{h_i(1)+1} + \frac{1}{2} \right\rfloor, \quad i=1,\dots,N.$$

It is easy to see that all conditionally $\mathcal{F}$-computable real functions are computable in the usual sense, whenever all functions in $\mathcal{F}$ are recursive ones.

4 Substitution in Conditionally $\mathcal{F}$-Computable Real Functions

Theorem 1. *Let the class $\mathcal{F}$ contain the addition function and one-argument functions L and R such that $\{(L(s),R(s)) \mid s \in \mathbb{N}\} = \mathbb{N}^2$. Then the substitution operation on real functions preserves conditional $\mathcal{F}$-computability.*

Proof. To avoid writing excessively long expressions, we will restrict ourselves to the case of one-argument functions. Let θ_0 and θ_1 be conditionally $\mathcal{F}$-computable one-argument real functions. We will show the conditional $\mathcal{F}$-computability of the function θ defined by $\theta(\xi) = \theta_0(\theta_1(\xi))$. For $i = 0, 1$, let E_i, F_i, G_i, H_i be $\mathcal{F}$-substitutional mappings such that $\exists s(E_i(f,g,h)(s) = 0)$ and

$$\forall s(E_i(f,g,h)(s) = 0 \Rightarrow (F_i(f,g,h)(s,\bullet), G_i(f,g,h)(s,\bullet), H_i(f,g,h)(s,\bullet)) \text{ names } \theta_i(\xi))$$

for any $\xi \in \text{dom}(\theta_i)$ and any triple (f,g,h) naming ξ. We will show that the requirements of Definition 2 for the function θ are satisfied through the mappings E, F, G, H defined as follows:

$$\begin{aligned}
&E(f,g,h)(s) = E_1(f,g,h)(R(s)) + \\
&\quad E_0(F_1(f,g,h)(R(s),\bullet), G_1(f,g,h)(R(s),\bullet), H_1(f,g,h)(R(s),\bullet))(L(s)),\\
&F(f,g,h)(s,t) = \\
&\quad F_0(F_1(f,g,h)(R(s),\bullet), G_1(f,g,h)(R(s),\bullet), H_1(f,g,h)(R(s),\bullet))(L(s),t),\\
&G(f,g,h)(s,t) = \\
&\quad G_0(F_1(f,g,h)(R(s),\bullet), G_1(f,g,h)(R(s),\bullet), H_1(f,g,h)(R(s),\bullet))(L(s),t),\\
&H(f,g,h)(s,t) = \\
&\quad H_0(F_1(f,g,h)(R(s),\bullet), G_1(f,g,h)(R(s),\bullet), H_1(f,g,h)(R(s),\bullet))(L(s),t)
\end{aligned}$$

for all $s, t \in \mathbb{N}$. By Propositions 1 and 3, these mappings are also $\mathcal{F}$-substitutional. Suppose now $\xi \in \text{dom}(\theta)$ and (f,g,h) is a triple naming ξ. By the conditional $\mathcal{F}$-computability of θ_1, there exists $s_1 \in \mathbb{N}$ such that

$$E_1(f,g,h)(s_1) = 0, \tag{2}$$

and if we choose such an s_1 then the number $\theta_1(\xi)$ is named by the triple (f_1, g_1, h_1), where

$$f_1 = F_1(f,g,h)(s_1,\bullet),\ g_1 = G_1(f,g,h)(s_1,\bullet),\ h_1 = H_1(f,g,h)(s_1,\bullet). \tag{3}$$

By the conditional $\mathcal{F}$-computability of θ_0, there exists $s_0 \in \mathbb{N}$ such that

$$E_0(f_1,g_1,h_1)(s_0) = 0. \tag{4}$$

If s is a natural number such that $L(s) = s_0$, $R(s) = s_1$, then $E(f,g,h)(s) = 0$. Consider now any natural number s such that $E(f,g,h)(s) = 0$. Let $s_0 = L(s)$, $s_1 = R(s)$. The equality $E(f,g,h)(s) = 0$ implies the equality (2), as well as the equality (4) for the functions f_1, g_1, h_1 defined by means of the equalities (3). It follows from the equality (2) that (f_1, g_1, h_1) names $\theta_1(\xi)$, and, together with the equality (4), this fact implies that $\theta(\xi)$ is named by the triple

$$(F(f,g,h)(s,\bullet), G(f,g,h)(s,\bullet), H(f,g,h)(s,\bullet)).$$

Example 5. The function $\theta(\xi) = \ln \xi$ is conditionally $\mathcal{M}^2$-computable, although it is not uniformly $\mathcal{M}^2$-computable . To prove the conditional $\mathcal{M}^2$-computability of θ, let us consider the function θ° having domain $\{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid \xi_1 > 0, \xi_1\xi_2 \geq 1\}$

and defined by $\theta^\circ(\xi_1,\xi_2) = \ln\xi_1$. This function is uniformly $\mathcal{M}^2$-computable by Theorem 6 of [1], hence (by Example 1) it is conditionally $\mathcal{M}^2$-computable. Since $\theta(\xi) = \theta^\circ(\xi, 1/\xi)$ for all $\xi \in \mathrm{dom}(\theta)$, the conditional $\mathcal{M}^2$-computability of θ follows from here by Theorem 1 and Example 2.

Corollary 1. *All elementary functions of calculus are conditionally computable with respect to the class* $\mathcal{M}^2$.

Proof. Any elementary function of calculus can be obtained by means of substitution from some functions shown in [1] to be uniformly $\mathcal{M}^2$-computable and the functions shown in Examples 2, 3 and 5 to be conditionally $\mathcal{M}^2$-computable. By Theorem 1, this implies the conditional $\mathcal{M}^2$-computability of all elementary functions of calculus.

5 Local Uniform $\mathcal{F}$-Computability of the Conditionally $\mathcal{F}$-Computable Functions

Definition 3. *Let* $N \in \mathbb{N}$ *and* $\theta : D \to \mathbb{R}$, *where* $D \subseteq \mathbb{R}^N$. *The function* θ *will be called* locally uniformly $\mathcal{F}$-computable *if any point of* D *has some neighbourhood* U *such that the restriction of* θ *to* $D \cap U$ *is uniformly* $\mathcal{F}$*-computable.*

Theorem 2. *Let for any* $a, b \in \mathbb{N}$ *the class* $\mathcal{F}$ *contain the two-argument function whose value at* (x, y) *is* b *or* y *depending on whether or not* $x = a$. *Let also all one-argument constant functions in* $\mathbb{N}$ *belong to* $\mathcal{F}$. *Then all conditionally* $\mathcal{F}$*-computable real functions are locally uniformly* $\mathcal{F}$*-computable.*

Proof. Let θ be a conditionally $\mathcal{F}$-computable real function, and ξ_0 be a point of its domain (for the sake of simplicity, we assume additionally that θ is a one-argument function). Let E, F, G, H be such $\mathcal{F}$-substitutional mappings as in Definition 2 (with $N = 1$). Let (f_0, g_0, h_0) be a triple naming ξ_0 , and let s_0 be a natural number satisfying the equality $E(f_0, g_0, h_0)(s_0) = 0$. There exists a finite set A of natural numbers such that $E(f, g, h)(s_0) = 0$, whenever f, g, h are functions from $\mathbb{T}_1$ coinciding, respectively, with f_0, g_0, h_0 on A. The assumptions imposed on $\mathcal{F}$ imply the existence of $\mathcal{F}$-substitutional mappings P, Q, R of $\mathbb{T}_1$ into itself, such that, for any $f, g, h \in \mathbb{T}_1$, the functions $P(f), Q(g), R(h)$ coincide, respectively, with the functions f_0, g_0, h_0 on A and with the functions f, g, h on $\mathbb{N} \setminus A$. Let

$$U = \left(\max_{i \in A} \left(\frac{f_0(i) - g_0(i)}{h_0(i) + 1} - \frac{1}{i+1} \right), \min_{i \in A} \left(\frac{f_0(i) - g_0(i)}{h_0(i) + 1} + \frac{1}{i+1} \right) \right) .$$

Then $\xi_0 \in U$, and, whenever a triple (f, g, h) names a number belonging to U, the triple $(P(f), Q(g), R(h))$ also names this number. Let us set

$$F'(f, g, h) = F(P(f), Q(g), R(h))(s_0, \bullet),$$
$$G'(f, g, h) = G(P(f), Q(g), R(h))(s_0, \bullet),$$
$$H'(f, g, h) = H(P(f), Q(g), R(h))(s_0, \bullet).$$

The mappings F', G', H' are also $\mathcal{F}$-substitutional. Since

$$E(P(f), Q(g), R(h))(s_0) = 0$$

for any $f, g, h \in \mathbb{T}_1$, it is clear that, whenever (f, g, h) names a number $\xi \in U$ that belongs to the domain of θ, the triple $(F'(f,g,h), G'(f,g,h), H'(f,g,h))$ names $\theta(\xi)$. Thus (F', G', H') is an $\mathcal{F}$-substitutional 3,3-computing system for the restriction of θ to $D \cap U$.

Theorem 2 and the Characterization Theorem from [2] imply that, under the assumptions about $\mathcal{F}$ in them, if θ is a conditionally $\mathcal{F}$-computable function then each point of $\mathrm{dom}(\theta)$ has some neighbourhood U such that θ is uniformly continuous in $\mathrm{dom}(\theta) \cap U$.[6] Since these assumptions are satisfied when $\mathcal{F} = \bigcup_{m \in \mathbb{N}} \mathbb{T}_m$, it follows that there exist computable real functions, which are not conditionally $\mathcal{F}$-computable, whatever be the class $\mathcal{F}$, e.g. the function $\theta(\xi) = \sum_{k=1}^{\infty} \frac{1}{2^k} \sigma\left(\xi - \frac{1}{k}\right)$, where σ is the restriction of the sign function to $\mathbb{R} \setminus \{0\}$.

6 Uniform $\mathcal{F}$-Computability of the Locally Uniformly $\mathcal{F}$-Computable Functions with Compact Domains

The conclusion of the next theorem is natural to be expected under some assumptions about the class $\mathcal{F}$. The premise of the theorem contains a choice of such ones which allows a proof along more or less usual lines.

Theorem 3. *Let the class $\mathcal{F}$ be closed under substitution, and let $\mathcal{F}$ contain the projection functions, the successor function, the addition function, the function $\lambda xy.x \dot{-} y$ and the function $\lambda xy.x(1 \dot{-} y)$. Then all locally uniformly $\mathcal{F}$-computable real functions with compact domains are uniformly $\mathcal{F}$-computable.*

Proof. The assumptions of the theorem imply that any constant function in $\mathbb{N}$ with a non-zero number of arguments belongs to $\mathcal{F}$. Since $1 \dot{-} y = \overline{\mathrm{sg}}\, y$, and $\mathrm{sg}\, y = \overline{\mathrm{sg}}\, \overline{\mathrm{sg}}\, y$, the class $\mathcal{F}$ contains also the functions $\lambda xy.x\, \overline{\mathrm{sg}}\, y$ and $\lambda xy.x\, \mathrm{sg}\, y$. Suppose now $N \in \mathbb{N}$, $\theta : D \to \mathbb{R}$, where D is a compact subset of $\mathbb{R}^N$, and θ is locally uniformly $\mathcal{F}$-computable. Then there exist a natural number n, rational numbers a_{ij} $(i = 1, \ldots, n,\ j = 1, \ldots, N)$ and positive rational numbers $d_1, \ldots, d_n$ such that $D \subseteq U_1 \cup \ldots \cup U_n$, where, for $i = 1, \ldots, n$,

$$U_i = \left\{ (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N \,\middle|\, |\xi_1 - a_{i1}| < d_i, \ldots, |\xi_N - a_{iN}| < d_i \right\}$$

and the restriction of θ to $D \cap U_i$ is uniformly $\mathcal{F}$-computable. We will prove that θ is also uniformly $\mathcal{F}$-computable. In order to do this, we consider the continuous function $\delta(\xi_1, \ldots, \xi_N) = \max_{i=1,\ldots,n} \min_{j=1,\ldots,N} (d_i - |\xi_j - a_{ij}|)$. Since $\delta(\overline{\xi}) > 0$ for all $\overline{\xi} \in D$, there exists a natural number k, such that $\delta(\overline{\xi}) \geq \frac{2}{k+1}$

[6] The above-mentioned theorem from [2] characterizes the uniformly $\mathcal{F}$-computable real functions by essentially the same property which is required on the last line of Definition 3.1 in [3].

for any $\overline{\xi} \in D$. For such a k, as it is easy to see, whenever $(\xi_1, \dots, \xi_N) \in D$, and $x_1, \dots, x_N$ are rational numbers satisfying the inequalities $|x_j - \xi_j| < \frac{1}{k+1}$ $(j = 1, \dots, N)$, at least one of the numbers $r_1 = \min_{j=1,\dots,N}(d_1 - |x_j - a_{1j}|)$, $\dots$, $r_n = \min_{j=1,\dots,N}(d_n - |x_j - a_{nj}|)$ will be greater than $\frac{1}{k+1}$, and $(\xi_1, \dots, \xi_N)$ will belong to U_i for any $i \in \{1, \dots, n\}$ such that $r_i > \frac{1}{k+1}$. In particular, that will be the case, whenever $(\xi_1, \dots, \xi_N) \in D$, (f_1, g_1, h_1), $\dots$, (f_N, g_N, h_N) are triples naming $\xi_1, \dots, \xi_N$, respectively, and

$$x_j = \frac{f_j(k) - g_j(k)}{h_j(k) + 1}$$

for $j = 1, \dots, N$. Let, for $i = 1, \dots, n$, the triple (F_i, G_i, H_i) be a $\mathcal{F}$-substitutional 3,3-computing system for the restriction of θ to $D \cap U_i$. We define mappings $F, G, H : \mathbb{T}_1^{3N} \to \mathbb{T}_1$ by setting

$$\begin{aligned} F(f_1, g_1, h_1, \dots, f_N, g_N, h_N)(t) &= F_l(f_1, g_1, h_1, \dots, f_N, g_N, h_N)(t), \\ G(f_1, g_1, h_1, \dots, f_N, g_N, h_N)(t) &= G_l(f_1, g_1, h_1, \dots, f_N, g_N, h_N)(t), \\ H(f_1, g_1, h_1, \dots, f_N, g_N, h_N)(t) &= H_l(f_1, g_1, h_1, \dots, f_N, g_N, h_N)(t), \end{aligned}$$

where l is the least of the numbers $i \in \{1, \dots, n\}$ satisfying the inequality

$$\min_{j=1,\dots,N} \left(d_i - \left| \frac{f_j(k) - g_j(k)}{h_j(k) + 1} - a_{ij} \right| \right) > \frac{1}{k+1}, \tag{5}$$

if there exists such an i, and

$$F(f_1, g_1, h_1, \dots, f_N, g_N, h_N)(t) = G(f_1, g_1, h_1, \dots, f_N, g_N, h_N)(t) = \\ H(f_1, g_1, h_1, \dots, f_N, g_N, h_N)(t) = 0$$

otherwise. The triple (F, G, H) is a 3,3-computing system for θ. This triple is $\mathcal{F}$-substitutional. To show this, we note that the inequality (5) is equivalent to the conjunction of the inequalities

$$d_i - \left| \frac{f_j(k) - g_j(k)}{h_j(k) + 1} - a_{ij} \right| > \frac{1}{k+1}, \quad j = 1, \dots, N. \tag{6}$$

For $i = 1, \dots, n$, $j = 1, \dots, N$, let

$$a_{ij} = \frac{p_{ij} - q_{ij}}{r_{ij} + 1},$$

where p_{ij}, q_{ij}, r_{ij} are natural numbers. Let m be a positive integer divisible by $k+1$, by all numbers $r_{ij}+1$ and by the denominators of all d_i. Then the numbers

$$e = \frac{m}{k+1}, \; e_i = m d_i, \; e_{ij} = \frac{m}{r_{ij} + 1}$$

are positive integers, and the inequality (6) is equivalent to the inequality

$$e_i(h_j(k) + 1) > \\ e(h_j(k) + 1) + |(m f_j(k) + q_{ij} e_{ij}(h_j(k) + 1)) - (m g_j(k) + p_{ij} e_{ij}(h_j(k) + 1))|.$$

In its turn, this inequality is equivalent to the equality $K_{ij}(f_j, g_j, h_j) = 0$, where

$$K_{ij}(f_j, g_j, h_j) = \\ (e(h_j(k)+1) + |(mf_j(k) + q_{ij}e_{ij}(h_j(k)+1)) - (mg_j(k) + p_{ij}e_{ij}(h_j(k)+1))| + 1) \\ \dot{-}\, e_i(h_j(k)+1).$$

Therefore the inequality (5) is equivalent to the equality $\sum_{j=1}^{N} K_{ij}(f_j, g_j, h_j) = 0$. Let us define mappings $F'_1, \ldots, F'_n : \mathbb{T}_1^{3N} \to \mathbb{T}_1$ as follows:

$$F'_i(f_1, g_1, h_1, \ldots, f_N, g_N, h_N)(t) = \\ F_i(f_1, g_1, h_1, \ldots, f_N, g_N, h_N)(t)\ \overline{\mathrm{sg}} \sum_{j=1}^{N} K_{ij}(f_j, g_j, h_j) \prod_{i'=1}^{i-1} \mathrm{sg} \sum_{j=1}^{N} K_{i'j}(f_j, g_j, h_j).$$

Then we have the equality

$$F(f_1, g_1, h_1, \ldots, f_N, g_N, h_N)(t) = \sum_{i=1}^{n} F'_i(f_1, g_1, h_1, \ldots, f_N, g_N, h_N)(t),$$

and it implies that F is a $\mathcal{F}$-substitutional mapping. It is seen in a similar way that G and H are also $\mathcal{F}$-substitutional mappings.

Corollary 2. *If the class $\mathcal{F}$ satisfies the assumptions of Theorem 3 then all conditionally $\mathcal{F}$-computable real functions with compact domains are uniformly $\mathcal{F}$-computable.*

Proof. The assumptions of Theorem 3 imply the assumptions of Theorem 2.

Notes and Comments. The conditional $\mathcal{F}$-computability of real functions has some similarity in its spirit with the notion of a real function in $\mathcal{F}$ introduced in [3] (under some restrictions on the class $\mathcal{F}$) for functions whose domains are open sets. However, there are many essential differences between the two notions. For instance, if $\mathcal{F}$ is the class of the lower elementary functions then the class of the real functions in $\mathcal{F}$ is not closed under substitution, it is not true that it contains all elementary functions of calculus, and there are real functions in $\mathcal{F}$ which are not computable in the usual sense.

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