Ivan Soskov's Work on Computability on First-Order Structures

(in honour of his 60th birthday anniversary)

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Conference on Mathematical Logic,
dedicated to the memory of Ivan Soskov,
September 20–22, 2014, Gyolechitsa

Overview

Ivan Soskov was one of the active and successful researchers of computability on first-order structures. His research encompasses essential problems concerning the internal and the external approach to the notion and the interconnection between them. A number of valuable results in this field were presented in Soskov's Master, Ph. D. and Dr. Hab. Theses (defended at Sofia University in 1979, 1983 and 2001, respectively), as well as in a number of other works of him. An attempt to list the most interesting among these results will be done in the talk.

The Notions of Prime and Search Computability

A substantial part of Soskov's research concerns the notions of prime and search computability introduced by Y. N. Moschovakis in 1969.

Let B be a set, o be an object not in B, and B^* be the extension of B defined by Moschovakis as the closure of the set $B^0 = B \cup \{o\}$ under formation of ordered pairs (assuming a construction of them such that no element of B^0 is an ordered pair). Moschovakis defines prime and search computability for partial multiple-valued (p.m.v.) functions in B^*. These notions can be reduced to the notion of absolute prime computability introduced by him.

Without loss of generality, the consideration of arbitrary p.m.v. functions in B^* can be restricted to consideration of unary ones. In this case, a characterization of absolute prime computability is possible which is in the style of structured functional programming.

A Characterization of Absolute Prime Computability

As in Moschovakis' paper, let $\pi, \delta : B^* \to B^*$ be defined as follows: $\pi(o) = \delta(o) = o$; if $v \in B$ then $\pi(v) = \delta(v) = \langle o, o\rangle$; if $v = \langle t, u\rangle$ then $\pi(v) = t$ and $\delta(v) = u$.

A unary p.m.v. function ψ is *absolutely prime computable* in the unary p.m.v. functions $\varphi_1, \ldots, \varphi_l$ iff ψ can be generated from $\varphi_1, \ldots, \varphi_l, \pi, \delta$ by means of composition, combination and iteration, where, for any unary p.m.v. functions θ and χ in B^*:

- *the composition of θ and χ* is the p.m.v. function $\lambda s.\theta(\chi(s))$;
- *the combination of θ and χ* is the p.m.v. function
$$\lambda s.\theta(s) \times \chi(s);$$
- *the iteration of θ controlled by χ* is the p.m.v. function
$$\begin{aligned}\lambda s.\Big\{v \Big| \exists t_0, t_1, \ldots, t_m (t_0 = s \;\&\; t_m = v \\ \& \, \forall i (i < m \Rightarrow t_i \in \chi^{-1}(B^* \setminus B^0) \;\&\; t_{i+1} \in \theta(t_i)) \\ \& \; t_m \in \chi^{-1}(B^0))\Big\}.\end{aligned}$$

The Reduction to Absolute Prime Computability

Let $\psi, \varphi_1, \ldots, \varphi_l$ be p.m.v. functions in B^*, and let $A \subseteq B^*$. Then:

- ψ is *prime computable in* $\varphi_1, \ldots, \varphi_l$ *and constants from* A iff ψ is absolutely prime computable in $\varphi_1, \ldots, \varphi_l$ and finitely many constant functions from B^* to A;
- ψ is *absolutely search computable in* $\varphi_1, \ldots, \varphi_l$ iff ψ is absolutely prime computable in $\varphi_1, \ldots, \varphi_l$ and the p.m.v. function with graph $(B^*)^2$;
- ψ is *search computable in* $\varphi_1, \ldots, \varphi_l$ *and constants from* A iff ψ is absolutely search computable in $\varphi_1, \ldots, \varphi_l$ and finitely many constant functions from B^* to A.

Ivan Soskov's Master Thesis (I)

Ivan Soskov did his first research steps in his Master Thesis (Sofia University, 1979).

Suppose a partial structure $\mathfrak{A}$ with carrier B is given. Of course, all partial functions and predicates in B can be regarded as partial functions in B^* (after encoding the predicate values by the elements o and $\langle o, o\rangle$ of B^*). Let us call *prime computable in* $\mathfrak{A}$ the partial functions in B^* which are absolutely prime computable in the primitive functions and predicates of $\mathfrak{A}$.

Soskov gives a rigorous proof of the following intuitively plausible statement:

If $\mathfrak{A}$ has no primitive predicates then any partial function in B, which is prime computable in $\mathfrak{A}$ and has a non-empty domain, can be represented in $\mathfrak{A}$ by some term.

Ivan Soskov's Master Thesis (II)

Again in his Master Thesis, Soskov gives a characterization of the prime computable functions in $\mathfrak{A}$ for any $\mathfrak{A}$ with carrier ω and no primitive functions and predicates besides $S = \lambda x.x + 1$, $P = \lambda x.x - 1$ and the equality to 0 predicate Z.

A smart diagonal construction is used to prove the following result:

A recursive function $\varphi : \omega \to \{0, 1\}$ exists such that all partial recursive functions are prime computable in the partial structure $(\omega; P, S \restriction \varphi^{-1}(0), S \restriction \varphi^{-1}(1); Z)$, but the function S is not computable in it by means of a standard program.

This result was later published in the collection "Mathematical Theory and Practice of Program Systems" (A. P. Ershov, ed.), Computer Centre of the Siberian Branch of the Soviet Academy of Sciences, Novosibirsk, 1982 (in Russian).

Ivan Soskov's Ph. D. Thesis (I)

One of the results proved in Ivan Soskov's Ph. D. Thesis (Sofia University, 1983) reads as follows:

For functions in the carrier of a partial structure $\mathfrak{A}$, prime computablility in $\mathfrak{A}$ is equivalent to computability in $\mathfrak{A}$ by means of a standard program with counters and a stack, the stack being superfluous in the case when the primitive functions of $\mathfrak{A}$ are unary.

It follows from here that the result mentioned on the previous slide cannot be strengthened by replacing "a standard program" with "a standard program with counters" (in the structure in question, the function S will be computable by means of a standard program with counters).

Ivan Soskov's Ph. D. Thesis (II)

In Soskov's Ph. D. Thesis and several related papers of him, partial structures $\mathfrak{A}$ with carrier B are considered and additional function arguments are admitted which range over the set ω (the last one being identified with the subset $\{o, \langle o, o\rangle, \langle o, \langle o, o\rangle\rangle, \ldots\}$ of B^*). The set of all partial functions from $\omega^r \times B^j$ to B is denoted by $\mathcal{F}_{r,j}$ (of course they can be regarded as partial functions in B^*). A normal form theorem is proved for the functions in $\mathcal{F}_{r,j}$ which are prime computable in $\mathfrak{A}$. The following characterization of Shepherdson's REDS-computability is given.

For any $\varphi \in \mathcal{F}_{0,j}$, the REDS-computability of φ in $\mathfrak{A}$ is equivalent to the existence of some $\psi \in \mathcal{F}_{1,j}$ such that ψ is prime computable in $\mathfrak{A}$ and for every every $s_1, \ldots, s_j, t \in B$

$$\varphi(s_1, \ldots, s_j) = t \Leftrightarrow \exists z \in \omega(\psi(z, s_1, \ldots, s_j) = t).$$

Ivan Soskov's Ph. D. Thesis (III)

Next characterization of REDS-computability is also given:

A function from $\bigcup_{j\in\omega}\mathcal{F}_{0,j}$ is REDS-computable in $\mathfrak{A}$ if it is absolutely prime computable in the primitive functions and predicates of $\mathfrak{A}$ and the p.m.v. function whose graph is ω^2.

A function from $\mathcal{F}_{r,j}$ is called *search computable in* $\mathfrak{A}$ if it is absolutely search computable in the primitive functions and predicates of $\mathfrak{A}$. The following is shown:

Let $\mathfrak{A}_0 = (\omega; P; T_1, T_2)$, where $\mathrm{dom}(T_1) = \{0\}$, $\mathrm{dom}(T_2) = \{1\}$, $T_1(0) = T_2(1)$. Then S is search computable in $\mathfrak{A}_0$ without being REDS-computable in $\mathfrak{A}_0$, and $\mathrm{id}_\omega \restriction \{0, 2, 4, 6, \ldots\}$ is REDS-computable in $\mathfrak{A}_0$ without being prime computable in $\mathfrak{A}_0$.

Ivan Soskov's Ph. D. Thesis (IV)

A partial structure with carrier ω is called *complete* if any partial recursive function is REDS-computable in this structure. A necessary and sufficient condition for completeness is given in the spirit of V. A. Nepomniaschy's results. A simplification of this condition is given for the case of partial recursiveness of the primitive functions and predicates.

A partial structure with carrier ω is called *complete WRT prime computability* if any partial recursive function is prime computable in this structure. Completeness WRT other kinds of computability is defined similarly. Examples are given which show the mutual non-equivalence of the following four notions: completeness, completeness WRT prime computability, completeness WRT computability by means of standard programs with counters, completeness WRT computability by means of standard programs.

Ivan Soskov's Ph. D. Thesis (V)

Here are the examples mentioned on the previous slide:

$$\mathfrak{A}_1 = (\omega; S, P; T_1, T_2), \quad \mathfrak{A}_2 = (\omega; \varphi; E), \quad \mathfrak{A}_3 = (\omega; \theta_1, \theta_2; Z),$$

where T_1 and T_2 are as in the partial structure $\mathfrak{A}_0$, E is the equality predicate,

$$\varphi(s,t) = \begin{cases} s+1 & \text{if } t = (s+1) - 2^{\lfloor \log_2(s+1) \rfloor}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\theta_1(s) = \begin{cases} s \dot{-} 1 & \text{if } s \in A, \\ s+1 & \text{otherwise,} \end{cases} \qquad \theta_2(s) = \begin{cases} s+1 & \text{if } s \in A, \\ s \dot{-} 1 & \text{otherwise} \end{cases}$$

with $A = \left\{ s \,\middle|\, \exists t \left(2^{2t} \leq s < 2^{2t+1} \right) \right\}$.

For $i = 1, 2, 3$, the structure $\mathfrak{A}_i$ is complete in the sense of the i-th notion without being complete in the sense of the $(i+1)$-th one.

Ivan Soskov's Study of Functional Combinatory Spaces (I)

The study in question is presented in a paper published in the collection "Mathematical Theory of Programming" (A. P. Ershov and D. Skordev, eds.), Novosibirsk, 1985 (in Russian).

Let M be an infinite set and $\mathcal{F}$ be the set of all partial unary functions in M. Let $a, b \in M$, $a \neq b$, $J : M^2 \to M$, $D, L, R \in \mathcal{F}$, $\mathrm{rng}(D) = \{a, b\}$, $D(a) = a$, $D(b) = b$, $L(J(s,t)) = s$ and $R(J(s,t)) = t$ for all $s, t \in M$. Let $\Pi : \mathcal{F}^2 \to \mathcal{F}$ be defined as follows:

$$\Pi(\theta, \chi)(s) = J(\theta(s), \chi(s)).$$

If $\theta, \chi \in \mathcal{F}$ then *the iteration of θ controlled by χ (through D, a, b)* is the function τ defined by

$$\tau(s) = v \Leftrightarrow \exists t_0, t_1, \ldots, t_m(t_0 = s \ \&\ t_m = v \\ \&\ \forall i(i < m \Rightarrow D(\chi(t_i)) = b \ \&\ t_{i+1} = \theta(t_i)) \ \&\ D(\chi(t_m)) = a).$$

Ivan Soskov's Study of Functional Combinatory Spaces (II)

If $\varphi_1, \ldots, \varphi_l \in \mathcal{F}$ then the elements of $\mathcal{F}$ *recursive in* $\{\varphi_1, \ldots, \varphi_l\}$ are those ones which can be generated from $\varphi_1, \ldots, \varphi_l, L, R, \lambda s.a, \lambda s.b$ by means of composition, Π and iteration.

The main result in the paper mentioned on the previous slide reads as follows:

An element of $\mathcal{F}$ is recursive in $\{\varphi_1, \ldots, \varphi_l\}$ iff it is prime computable in the partial structure

$$(M; \varphi_1, \ldots, \varphi_l, J, L, R, \lambda s.a, \lambda s.b; \overline{D}),$$

where

$$\overline{D}(s) = \begin{cases} \text{true} & \text{if } D(s) = a, \\ \text{false} & \text{if } D(s) = b. \end{cases}$$

Ivan Soskov's Study of Horn Programs with Parameters

An extension of the notion of Horn program is given in a paper of Soskov published in 1990 in the Annual of Sofia University. Namely predicate symbols can be used as parameters which correspond to arbitrary subsets of the corresponding Cartesian degrees of the domain of a structure. For the operators defined by means of such programs, a version of the first recursion theorem is proved.

Soskov's Results Making Use of the External Approach (I)

- External characterizations of various computabilities on denumerable first-order structures.

  These results are presented in several papers published in 1989 and 1990 (one of them written jointly with Alexandra Soskova), as well as in Soskov's Dr. Hab. Thesis (Sofia University, completed 2000, defended 2001). The papers in question are published in the J. of Symbolic Logic, the Annual of Sofia Univ., Arch. Math. Logic and the proceedings of the Heyting'88 Conference.

- Studies on maximal concepts of computabilities on abstract structures.

  The results of these studies are presented in a technical report written in 1989, in an abstract in the J. of Symbolic Logic published in 1992, and, in detail, in Soskov's Dr. Hab. Thesis.

Soskov's Results Making Use of the External Approach (II)

- External characterization of inductive definability on denumerable first-order structures.

  The results are presented in a paper in Math. Logic Quarterly published in 1996, and in Soskov's Dr. Hab. Thesis.

- Results on effective admissibility.

  They are presented in another paper in Math. Logic Quarterly published in 1996, and in Soskov's Dr. Hab. Thesis.

Associates of Functions and Predicates

Definition

Let α be a partial mapping of ω onto a set B. An m-argument p.m.v. function φ on ω is called an *α-associate* of an m-argument p.m.v. function θ on B if

$$\varphi(x_1,\ldots,x_m) \subseteq \mathrm{dom}(\alpha),\ \ \theta(\alpha(x_1),\ldots,\alpha(x_m)) = \alpha(\varphi(x_1,\ldots,x_m))$$

for all $x_1,\ldots,x_m \in \mathrm{dom}(\alpha)$. An m-argument partial predicate χ on ω is called an *α-associate* of an m-argument partial predicate π on B if

$$\pi(\alpha(x_1),\ldots,\alpha(x_m)) = \chi(x_1,\ldots,x_m)$$

for all $x_1,\ldots,x_m \in \mathrm{dom}(\alpha)$.

Enumerations and Admissibility

Let $\mathfrak{A} = (B; \theta_1, \ldots, \theta_n; \pi_1, \ldots, \pi_k)$ be a partial structure with $|B| \leq \aleph_0$.

Definition

An *enumeration* of $\mathfrak{A}$ is any ordered pair $(\alpha, \mathfrak{B})$, where α is a partial mapping of ω onto B, and $\mathfrak{B} = (\omega; \varphi_1, \ldots, \varphi_n; \chi_1, \ldots, \chi_k)$ is a partial structure such that:

- φ_i is an α-associate of θ_i for $i = 1, \ldots, n$;
- χ_j is an α-associate of π_j for $j = 1, \ldots, k$.

The enumeration $(\alpha, \mathfrak{B})$ is called *total* (*injective*) if $\mathrm{dom}(\alpha) = \omega$ (α is injective).

A p.m.v. function θ in B is called *p.r. admissible* (μ*-admissible*) in an enumeration $(\alpha, \mathfrak{B})$ of $\mathfrak{A}$ if θ has an α-associate which is partial recursive (μ-recursive) in the primitive functions and predicates of $\mathfrak{B}$.

Characterization Theorems

External definition		Internal definition
Enumerations class	**Admissibility**	**Computability**
all enumerations	μ	$\mathbf{PC}(\mathfrak{A})$
all enumerations	p.r.	$\mathbf{REDS}(\mathfrak{A})$
the injective ones	μ	$\mathbf{PC}(\mathfrak{A}^{=})$
the injective ones	p.r.	$\mathbf{REDS}(\mathfrak{A}^{=})$
the total ones	μ	?
the total ones	p.r.	$\mathbf{SC}(\mathfrak{A})$
the injective total ones	μ	?
the injective total ones	p.r.	$\mathbf{SC}(\mathfrak{A}^{=})$

$\mathbf{PC}(\mathfrak{A})$, $\mathbf{REDS}(\mathfrak{A})$ and $\mathbf{SC}(\mathfrak{A})$ mean, respectively, prime, REDS- and search computability in the primitive functions and predicates of $\mathfrak{A}$ and constants from B; $\mathfrak{A}^{=}$ is $\mathfrak{A}$ with equality added.

Comments on the Table on the Previous Slide

The two question marks in the table should be replaced with $\mathbf{SSC}(\mathfrak{A})$ and $\mathbf{SSC}(\mathfrak{A}^{=})$, where **SSC** means "sequential search computability". This is a new computability notion. Under appropriate definitions, the functions computable in this sense are the single-valued sequentially existentially definable ones.

The last line in the table on the previous slide generalizes Moschovakis' characterization of Lacombe's $\forall$-recursiveness through search computability.

The implication from left to right in all lines of the table is proved by using an ingeniuos general Cohen type forcing-construction.

The results are actually shown for a wider class of p.m.v. functions, namely additional arguments ranging over the set ω are allowed and functions with values in it are additionally considered.

Applications to Semi-Decidability

Definition

A subset A of B^m is called *semi-decidable* (REDS-*semidecidable*) in $\mathfrak{A}$ if the function mapping all elements of A into 0 belongs to **SC**$(\mathfrak{A})$ (**REDS**$(\mathfrak{A})$).

Corollary

A subset A of B^m is semi-decidable (REDS-semidecidable) in $\mathfrak{A}$ iff, for any total enumeration (for any enumeration) $(\alpha, \mathfrak{B})$ of $\mathfrak{A}$, the set $\alpha^{-1}(A)$ is (is the intersection of $\mathrm{dom}(\alpha)$ with some set) partial recursive in $\mathfrak{B}$.

Corollary

A subset A of B^m is semi-decidable (REDS-semidecidable) in $\mathfrak{A}^=$ iff, for any injective total enumeration (for any injective enumeration) $(\alpha, \mathfrak{B})$ of $\mathfrak{A}$, the set $\alpha^{-1}(A)$ is (is the intersection of $\mathrm{dom}(\alpha)$ with some set) partial recursive in $\mathfrak{B}$.

Applications to Effective Enumerability

Definition

A subset of B is called *effectively enumerable* in $\mathfrak{A}$ if it is the range of some function from $\mathbf{SC}(\mathfrak{A})$. If $(\alpha, \mathfrak{B})$ is a total enumeration of $\mathfrak{A}$ then the images under α of the subsets of ω partial recursive in $\mathfrak{B}$ are called *weakly admissible* in $(\alpha, \mathfrak{B})$ subsets of B. The last notion is easily generalized to subsets of B^m.

Theorem

A subset of B is effectively enumerable in $\mathfrak{A}$ iff it is weakly admissible in any total enumeration of $\mathfrak{A}$.

Corollary

The subsets of B semi-decidable in $\mathfrak{A}$ coincide with the ones which are effectively enumerable in $\mathfrak{A}$ iff the equality relation in B is semi-decidable in $\mathfrak{A}$.

Applications to Completeness

Suppose $B = \omega$.

Theorem

The following two conditions are equivalent:

- The system $\mathfrak{A}$ is complete with respect to search computability.
- For any total enumeration $(\alpha, \mathfrak{B})$ of $\mathfrak{A}$, the function α is partial recursive in $\mathfrak{B}$.

Theorem

The following two conditions are equivalent:

- The system $\mathfrak{A}^=$ is complete with respect to search computability.
- For any injective total enumeration $(\alpha, \mathfrak{B})$ of $\mathfrak{A}$, the function α is partial recursive in $\mathfrak{B}$.

Characterization of the Computability by Logic Programs

Suppose 'true' is the only value of any of the primitive predicates of the partial structure $\mathfrak{A}$. Let $\mathcal{L} = \{F_1, \ldots, F_n, T_1, \ldots, T_k\}$ be a first-order language with $F_1, \ldots, F_n, T_1, \ldots, T_k$ interpreted as the primitive functions and predicates of $\mathfrak{A}$. Let $\mathcal{L}_C = \mathcal{L} \cup C$, where C consists of constant symbols for all elements of B, and let $\partial(\mathfrak{A})$ be the set of all closed atomic formulas of $\mathcal{L}_C$ which are true in $\mathfrak{A}$.

Definition

A subset A of B^m is called *definable in $\mathfrak{A}$ by means of a logic program* if there exist an extension $\mathcal{L}'$ of $\mathcal{L}_C$, an m-ary predicate symbol H of $\mathcal{L}'$ and a formula P in $\mathcal{L}'$ such that $H \notin \mathcal{L}$, P is the universal closure of a Horn formula, and, for any $s_1, \ldots, s_m \in B$, $(s_1, \ldots, s_m) \in A$ iff $P \cup \partial(\mathfrak{A}) \vdash H(\lambda_1, \ldots, \lambda_m)$ for some ground terms $\lambda_1, \ldots, \lambda_m$ of $\mathcal{L}_C$ whose values are $s_1, \ldots, s_m$, respectively.

Theorem

A subset of B^m is definable in $\mathfrak{A}$ by means of a logic program iff it is weakly admissible in any total enumeration of $\mathfrak{A}$.

Results on Universal Computabilities

Applications are given to certain classification of the concepts of computability on abstract first-order structures and to programming languages with maximal expressive power within certain classes of such ones. For instance, let $\mathcal{A}$ is a class of structures which is closed under homomorphic preimages. Then, after the necessary definitions are formulated, a universality is shown of search computability for the class of the effective computabilities over $\mathcal{A}$ which are invariant under homomorphisms. A programming language, whose programs are codes of recursively enumerable sets of existential conditional expressions, is shown to be universal for the class of the effective programming languages over $\mathcal{A}$ which are invariant under homomorphisms.

External Characterization of the Inductive sets

Let $\mathfrak{A}$ be a structure with a countable carrier B and with primitive predicates only.

Definition

A subset A of B^n is called Π^1_1-*admissible* in a bijection $\alpha : \omega \to B$ if the set

$$\alpha^{-1}(A) = \{(x_1, \dots, x_n) \in \omega^n \mid (\alpha(x_1), \dots, \alpha(x_n)) \in A\}$$

is Π^1_1 relative to the diagram of the structure $\alpha^{-1}(\mathfrak{A})$.

Theorem

A subset A of B^n is inductively definable on the least acceptable extension of $\mathfrak{A}$ iff A is Π^1_1-admissible in any bijection of ω onto B.

The result is used for transfering some results of the classical recursion theory to the abstract case. In particular, a hierarchy for the hyperelementary sets is obtained which is similar to the classical Suslin-Kleene hierarchy.

Results on Effective Admissibility

Let again A be a structure with a countable carrier B and with primitive predicates only.

Definition

A subset A of B^n is called *relatively intrinsically hyperarithmetical* on $\mathfrak{A}$ if, for any bijection $\alpha : \omega \to B$, the set $\alpha^{-1}(A)$ is hyperarithmetical relative to the diagram of the structure $\alpha^{-1}(\mathfrak{A})$. In the case when a bijection $\alpha : \omega \to B$ exists such that the diagram of $\alpha^{-1}(\mathfrak{A})$ is a recursive set, a subset A of B^n is called *intrinsically hyperarithmetical* on A if the set $\alpha^{-1}(A)$ is a hyperarithmetical set for any such α.

Theorem

If a bijection $\alpha : \omega \to B$ exists such that the diagram of $\alpha^{-1}(\mathfrak{A})$ is a recursive set then the intrinsically hyperarithmetical sets coincide with the relatively intrinsically hyperarithmetical sets.

Some applications of Soskov's results, ideas and technique

Soskov's results and ideas are substantially used in a paper of Goncharov, Harizanov, Knight and Shore, where some additions are made to the results on effective admissibility. For instance, an analog for intrinsically Π_1^1 sets is proved there of the coincidence of intrinsical and relative intrinsical hyperarithmeticity on a recursive structure.

Soskov himself applies the technique developped by him for the external characterization of the inductive sets also to a problem of degree theory.

Thank you for the attention!