Moschovakis extension of effective topological spaces

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To the bright memory
of the colleague and friend,

the brilliant mathematician

Ivan Prodanov
(1935–1985)

Topology bases

Definition

A *base* for a topology $\mathcal{T}$ is a family of $\mathcal{T}$-open sets such that any $\mathcal{T}$-open set is a union of sets belonging to this family.

E. g. the open intervals with rational end-points in $\mathbb{R}$ form a base for the usual topology in $\mathbb{R}$.

A family of subsets of a set X is a base for some topology in X iff the set X and the intersection of any two sets of the family in question can be represented as unions of sets belonging to this family.

If a family of subsets of X has the above property then this family is a base for exactly one topology $\mathcal{T}$ in X – the one whose $\mathcal{T}$-open sets are all possible unions of sets belonging to the family in question.

Effective topological spaces

Definition

An *effective topological space (ETS)* is an ordered pair $(X, \mathcal{U})$, where X is a set, and $\mathcal{U} = \{\mathcal{U}_i\}_{i\in\mathbb{N}}$ is a base for a T_0 topology in X.

Example 1. Let $i \mapsto \mathcal{U}_i$ be a total enumeration of the set of all open intervals with rational end-points in $\mathbb{R}$. Then $(\mathbb{R}, \{\mathcal{U}_i\}_{i\in\mathbb{N}})$ is an ETS.

Example 2. Let X be the set of all partial functions from $\mathbb{N}$ to $\mathbb{N}$, and $f_0, f_1, f_2, \ldots$ be an effective listing of the ones with finite domains. For any $i \in \mathbb{N}$, let $\mathcal{U}_i$ be the set of all functions of X which are extensions of f_i. Then $(X, \{\mathcal{U}_i\}_{i\in\mathbb{N}})$ is an ETS.

Example 3. Let (X, d) be a separable metric space, A be a denumerable dense subset of it, $i \mapsto \mathcal{U}_i$ be a total enumeration of the set of the open balls in (X, d) with centers in A and radii of the form 2^{-k}, where $k \in \mathbb{N}$. Then $(X, \{\mathcal{U}_i\}_{i\in\mathbb{N}})$ is an ETS.

Computable ETS

Definition

An ETS $(X, \mathcal{U})$ is said to be *computable* if a recursively enumerable subset S of $\mathbb{N}^3$ exists such that $\mathcal{U}_i \cap \mathcal{U}_j = \bigcup\{\mathcal{U}_k \,|\, (i,j,k) \in S\}$ for all $i, j \in \mathbb{N}$.

Example 1′. If the enumeration $i \mapsto \mathcal{U}_i$ from Example 1 is computable then the ETS $(\mathbb{R}, \{\mathcal{U}_i\}_{i\in\mathbb{N}})$ is computable.

Example 2′. The ETS considered in Example 2 is computable.

Example 3′. Let X be $\mathbb{R}^K$, where K is a positive integer, d be the Euclidean metrics in $\mathbb{R}^K$, and A be $\mathbb{Q}^K$. If the enumeration $i \mapsto \mathcal{U}_i$ from Example 3 is computable then the ETS considered there is computable.

The set $S = \{(i,j,k) \in \mathbb{N}^3 \,|\, \mathcal{U}_k \subseteq \mathcal{U}_i \cap \mathcal{U}_j\})$ can be used in any of the above three examples. In the first two of them, one could also use $S = \{(i,j,k) \in \mathbb{N}^3 \,|\, \mathcal{U}_k = \mathcal{U}_i \cap \mathcal{U}_j\}$.

Computable elements of an ETS

Definition

Let $(X,\mathcal{U})$ be an ETS. We set $\mathcal{U}^{-1}(x) = \{i \in \mathbb{N} \mid x \in \mathcal{U}_i\}$ for any $x \in X$. The element x is said to be *$\mathcal{U}$-computable* if the set $\mathcal{U}^{-1}(x)$ is recursively enumerable.

Example 1″. If $\mathcal{U}$ is a computable enumeration of the kind considered in Example 1 then the $\mathcal{U}$-computability of a real number is equivalent to its computability in the usual sense.

Example 2″. If $\mathcal{U}$ is an enumeration of the kind considered in Example 2 then a partial function from $\mathbb{N}$ to $\mathbb{N}$ is $\mathcal{U}$-computable iff it is partial recursive.

Example 3″. Under the assumptions of Example 3′, an element of $\mathbb{R}^K$ is $\mathcal{U}$-computable iff its components are computable real numbers.

Computability of partial functions from an ETS to an ETS

Definition

Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be ETS, and f be a partial function from X to Y. A mapping F of $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})$ is said to *realise* f if $F(\mathcal{U}^{-1}(x)) = \mathcal{V}^{-1}(f(x))$ for any $x \in \operatorname{dom}(f)$. The function f is $(\mathcal{U}, \mathcal{V})$*-computable* if there is some enumeration operator which realises f.

Example 1‴. If $\mathcal{U}$ is a computable enumeration of the kind considered in Example 1 then a partial function from $\mathbb{R}$ to $\mathbb{R}$ is $(\mathcal{U}, \mathcal{U})$-computable iff it is computable in the usual sense.

Example 3‴. Under the assumptions of Example 3′, if $\mathcal{V}$ is a computable enumeration of the kind considered in Example 1 then a partial function from $\mathbb{R}^K$ to $\mathbb{R}$ is $(\mathcal{U}, \mathcal{V})$-computable iff it is computable in the usual sense.

Some statements about preservation of computability

Proposition 1

If $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are ETS, f is a $(\mathcal{U},\mathcal{V})$-computable partial function from X to Y, and a is a $\mathcal{U}$-computable element of $\mathrm{dom}(f)$ then $f(a)$ is a $\mathcal{V}$-computable element of Y.

Proposition 2

If $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are ETS, and b is a $\mathcal{V}$-computable element of Y then the constant function from X to Y with value b is $(\mathcal{U},\mathcal{V})$-computable.

For any functions f and g, we will denote by gf the function $x \mapsto g(f(x))$ $(\mathrm{dom}(gf) = \{x \in \mathrm{dom}(f) \mid f(x) \in \mathrm{dom}(g)\})$.

Proposition 3

If $(X,\mathcal{U})$, $(Y,\mathcal{V})$ and $(Z,\mathcal{W})$ are ETS, f is a $(\mathcal{U},\mathcal{V})$-computable partial function from X to Y, and g is a $(\mathcal{V},\mathcal{W})$-computable partial function from Y to Z, then the partial function gf from X to Z is $(\mathcal{U},\mathcal{W})$-computable.

Computability of partial functions from an ETS to $\mathbb{N}$

Let $\mathcal{A} = \{\mathcal{A}_i\}_{i\in\mathbb{N}}$, where $\mathcal{A}_i = \{i\}$. Clearly $(\mathbb{N}, \mathcal{A})$ is a computable ETS and all elements of $\mathbb{N}$ are $\mathcal{A}$-computable.

Theorem

Let $(X,\mathcal{U})$ be an ETS, and f be a partial function from X to $\mathbb{N}$. For f to be $(\mathcal{U},\mathcal{A})$-computable, it is sufficient that an one-argument partial recursive function φ exists such that

$$f(x) = y \Leftrightarrow \mathcal{U}^{-1}(x) \cap \varphi^{-1}(y) \neq \varnothing$$

for all $x \in \mathrm{dom}(f)$ and all $y \in \mathbb{N}$. If $\mathrm{rng}(f) \subseteq \{0,1\}$ then for f to be $(\mathcal{U},\mathcal{A})$-computable, it is sufficient that some disjoint recursively enumerable subsets E_0 and E_1 of $\mathbb{N}$ exist such that

$$f(x) = y \Leftrightarrow \mathcal{U}^{-1}(x) \cap E_y \neq \varnothing, \quad y = 0, 1,$$

for all $x \in \mathrm{dom}(f)$. If the ETS $(X,\mathcal{U})$ is computable then the above conditions are also necessary.

Moschovakis extension of an ETS (I)

Let $(X,\mathcal{U})$ be an ETS. One constructs its Moschovakis extension X^* in the usual way: X^* is the closure of $X \cup \{o\}$ with respect to formation of ordered pairs, assuming that $o \notin X$ and no element of $X \cup \{o\}$ is an ordered pair. We define a family $\mathcal{U}^* = \{\mathcal{U}_j^*\}_{j\in\mathbb{N}}$ of subsets of X^* by means of the equalities

$$\mathcal{U}_0^* = \{o\},\ \mathcal{U}_{2i+2}^* = \mathcal{U}_i,\ \mathcal{U}_{2\langle m,n\rangle+1}^* = \mathcal{U}_m^* \times \mathcal{U}_n^*,$$

where $(m,n) \mapsto \langle m,n\rangle$ is some computable bijection from $\mathbb{N}^2$ to $\mathbb{N}$ such that $\langle m,n\rangle \geq \max(m,n)$ for all $m,n \in \mathbb{N}$.

Theorem

The ordered pair $(X^*,\mathcal{U}^*)$ is an ETS, and the identity mapping id_X is a $(\mathcal{U},\mathcal{U}^*)$-computable function from X to X^*, as well as a $(\mathcal{U}^*,\mathcal{U})$-computable partial function from X^* to X.

By definition, $0^* = o$, $(n+1)^* = (n^*,o)$ for any $n \in \mathbb{N}$.

Theorem

The function $n \mapsto n^*$ from $\mathbb{N}$ to X^* is $(\mathcal{A},\mathcal{U}^*)$-computable, and its inverse function is $(\mathcal{U}^*,\mathcal{A})$-computable.

Moschovakis extension of an ETS (II)

Corollary

If $(X^*, \mathcal{U}^*)$ is the Moschovakis extension of an ETS $(X, \mathcal{U})$ then:

- *For any partial function from X to X, its $(\mathcal{U}, \mathcal{U})$-computability is equivalent to anyone of the following three properties: $(\mathcal{U}, \mathcal{U}^*)$-computability as a partial function from X to X^*, $(\mathcal{U}^*, \mathcal{U})$-computability as a partial function from X^* to X, $(\mathcal{U}^*, \mathcal{U}^*)$-computability as a partial function from X^* to X^*.*
- *A partial function from X to $\mathbb{N}$ is $\mathcal{U}$-computable iff it is $\mathcal{U}^*$-computable as a partial function from X^* to $\mathbb{N}$.*
- *A partial function f from X^* to $\mathbb{N}$ is $\mathcal{U}^*$-computable iff the partial function $z \mapsto f(z)^*$ from X^* to X^* is $(\mathcal{U}^*, \mathcal{U}^*)$-computable.*

From now on, we will suppose that an ETS $(X, \mathcal{U})$ is given and some Moschovakis extension $(X^*, \mathcal{U}^*)$ of it is specified.

The Moschovakis extension of any computable ETS is computable

Theorem

If the ETS $(X,\mathcal{U})$ is computable, then the ETS $(X^*,\mathcal{U}^*)$ is computable too.

Proof. Let $S \subseteq \mathbb{N}^3$, S be recursively enumerable and

$$\mathcal{U}_i \cap \mathcal{U}_j = \bigcup\{\mathcal{U}_k \mid (i,j,k) \in S\}$$

for all $i, j \in \mathbb{N}$. We define $S^* \subseteq \mathbb{N}^3$ in the following inductive way:

- $(0,0,0) \in S^*$.
- $(2i+2, 2j+2, 2k+2) \in S^*$, whenever $(i,j,k) \in S$.
- If $(m, \bar{m}, r) \in S^*$ and $(n, \bar{n}, s) \in S^*$ then $(2\langle m, n\rangle + 1, 2\langle \bar{m}, \bar{n}\rangle + 1, 2\langle r, s\rangle + 1) \in S^*$.

The set S^* is recursively enumerable and

$$\mathcal{U}_i^* \cap \mathcal{U}_j^* = \bigcup\{\mathcal{U}_k^* \mid (i,j,k) \in S^*\}$$

for all $i, j \in \mathbb{N}$. □

Computable elements of the Moschovakis extension

Proposition 1

An element of X is $\mathcal{U}^*$-computable iff it is $\mathcal{U}$-computable. The element o is $\mathcal{U}^*$-computable.

Proposition 2

For any x and y in X^*, the element (x, y) is $\mathcal{U}^*$-computable iff x and y are $\mathcal{U}^*$-computable.

Corollary

An element of X^* is $\mathcal{U}^*$-computable iff it can be obtained from $\mathcal{U}$-computable elements of X and o by finitely many applications of the ordered pair operation.

$(\mathcal{U}^*, \mathcal{U}^*)$-computability of Moschovakis's functions π and δ

The functions π and δ from X^* to X^* are defined as follows:

$$\pi(x,y) = x,\ \delta(x,y) = y,\ \pi(o) = \delta(o) = o,$$
$$\pi(z) = \delta(z) = (o,o) \text{ for } z \in X.$$

Theorem

The functions π and δ are $(\mathcal{U}^*, \mathcal{U}^*)$-computable.

Proof. The $(\mathcal{U}^*, \mathcal{U}^*)$-computability of the function π will be shown, and the reasoning about δ is similar. Let $z \in X^*$. The condition $\pi(z) \in \mathcal{U}_j^*$ holds for a number $j \in \mathbb{N}$ iff some of the following cases is present:

- $z \in \mathcal{U}^*_{2\langle j,n\rangle+1}$ for some $n \in \mathbb{N}$;
- $j = 0$ and $z \in \mathcal{U}_0^*$;
- $j = 1$ and $z \in \mathcal{U}_2^* \cup \mathcal{U}_4^* \cup \mathcal{U}_6^* \cup \cdots$.

It is clear now that π is realized by the mapping $F : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ which is defined as follows:

$$F(M) = \{j \in \mathbb{N} \mid \exists n \in \mathbb{N}\,(2\langle j,n\rangle + 1 \in M) \vee (j = 0 \ \&\ 0 \in M)$$
$$\vee\,(j = 1 \ \&\ M \cap \{2,4,6,\ldots\} \neq \varnothing).\} \qquad \square$$

Preservation of $(\mathcal{U}^*,\mathcal{U}^*)$-computability under combination

Theorem

Let f, g, e be partial functions from X^* to X^* with $\mathrm{dom}(e) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$ and $e(z) = (f(z), g(z))$ for all $z \in \mathrm{dom}(e)$. If the functions f and g are $(\mathcal{U}^*,\mathcal{U}^*)$-computable, then the function e is $(\mathcal{U}^*,\mathcal{U}^*)$-computable too.

Proof. Let F and G be enumeration operators realising the function f and the function g, respectively. For any $z \in \mathrm{dom}(e)$ and any $j \in \mathbb{N}$,

$$\begin{aligned} & j \in \mathcal{U}^{*-1}(e(z)) \\ & \quad \Leftrightarrow \exists m \in \mathcal{U}^{*-1}(f(z))\, \exists n \in \mathcal{U}^{*-1}(g(z))\, (j = 2\langle m, n\rangle + 1) \\ & \quad\quad \Leftrightarrow \exists m \in F(\mathcal{U}^{*-1}(z))\, \exists n \in G(\mathcal{U}^{*-1}(z))\, (j = 2\langle m, n\rangle + 1). \end{aligned}$$

Thus $U^{*-1}(e(z)) = E(\mathcal{U}^{*-1}(z))$ for all $z \in \mathrm{dom}(e)$, where $E : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is defined by means of the equality $E(M) = \{j \mid \exists m \in F(M)\, \exists n \in G(M)\, (j = 2\langle m, n\rangle + 1)\}$. □

The function e considered in the theorem will be denoted by (f, g).

Preservation of $(\mathcal{U}^*,\mathcal{U}^*)$-computability under branching

If f, g and h are partial functions from X^* to X^* then $(h \supset f, g)$ is the function e (*branching to f or g controlled by h*) defined as follows: $e(z) = z'$ iff

$$(z \in h^{-1}(X \cup \{o\}) \,\&\, f(z) = z') \vee (z \in h^{-1}(X^* \setminus (X \cup \{o\})) \,\&\, g(z) = z').$$

Theorem

If f, g h are $(\mathcal{U}^*,\mathcal{U}^*)$-computable partial functions from X^* to X^* then the function $(h \supset f, g)$ is $(\mathcal{U}^*,\mathcal{U}^*)$-computable too.

Proof. Let F, G and H be enumeration operators which realise the functions f, g and h, respectively. If $e = (h \supset f, g)$, $z \in \mathrm{dom}(e)$ and $j \in \mathbb{N}$ then

$$\begin{aligned} e(z) \in \mathcal{U}_j^* \Leftrightarrow (&h(z) \in \mathcal{U}_0^* \cup \mathcal{U}_2^* \cup \mathcal{U}_4^* \cup \cdots \;\&\; f(z) \in \mathcal{U}_j^*) \\ &\vee (h(z) \in \mathcal{U}_1^* \cup \mathcal{U}_3^* \cup \mathcal{U}_5^* \cup \cdots \;\&\; g(z) \in \mathcal{U}_j^*), \end{aligned}$$

i. e.

$$j \in \mathcal{U}^{*-1}(e(z)) \Leftrightarrow (H(M) \cap \{0, 2, 4, \ldots\} \neq \varnothing \;\&\; j \in F(M))$$

$$\vee (H(M) \cap \{1, 3, 5, \ldots\} \neq \varnothing \;\&\; j \in G(M)), \text{ where } M = \mathcal{U}^{*-1}(z). \quad \square$$

Preservation of $(\mathcal{U}^*,\mathcal{U}^*)$-computability under iteration (I)

If f and h are partial functions from X^* to X^* then *the iteration of f controlled by h* is the partial function e from X^* to X^* which is defined as follows: $e(z) = z'$ iff a finite sequence $z_0, z_1, z_2, \ldots, z_n$ of element of X^* exists such that

- $z_0 = z$, $z_n = z'$;
- $z_k \in h^{-1}(X^* \setminus (X \cup \{o\}))$ and $z_{k+1} = f(z_k)$ for all $k < n$;
- $z_n \in h^{-1}(X \cup \{o\})$.

We will denote this function by $[f, h]$.

Theorem

Let f and h be partial functions from X^* to X^*. If f and h are $(\mathcal{U}^*,\mathcal{U}^*)$-computable then $[f, h]$ is $(\mathcal{U}^*,\mathcal{U}^*)$-computable too.

Preservation of $(\mathcal{U}^*, \mathcal{U}^*)$-computability under iteration (II)

Proof. Let F and H be enumeration operators which realise, respectively, the functions f and h, and let $e = [f, h]$. For any $m \in \mathbb{N}$, let D_m be the finite subset of $\mathbb{N}$ with canonical index m. Suppose $z \in \mathrm{dom}(e)$. One verifies that a natural number j belongs to $\mathcal{U}^{*-1}(e(z))$ iff some finite sequence $m_0, m_1, m_2, \ldots, m_n$ of natural numbers satisfies the following conditions:

1. $D_{m_0} \subseteq \mathcal{U}^{*-1}(z)$, $j \in D_{m_n}$;
2. $H(D_{m_k}) \cap \{1, 3, 5, \ldots\} \neq \varnothing$ and $D_{m_{k+1}} \subseteq F(D_{m_k})$ for all $k < n$;
3. $H(D_{m_n}) \cap \{0, 2, 4, \ldots\} \neq \varnothing$.

Therefore $\mathcal{U}^{*-1}(e(z)) = E(\mathcal{U}^{*-1}(z))$, where $E : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is defined as follows: a natural number j belongs to $E(M)$ iff some finite sequence $m_0, m_1, m_2, \ldots, m_n$ of natural numbers satisfies the conditions $D_{m_0} \subseteq M$, $j \in D_{m_n}$ and the conditions 2 and 3 above. $\square$

A computability notion for operators in the set of the partial functions from X^* to X^*

Let $\mathcal{F}$ be the set of all partial functions from X^* to X^*. Some mappings of $\mathcal{F}$ into itself will be called $(\mathcal{U}^*,\mathcal{U}^*)$-*computable by construction (CBC)*. This will be arranged by means of the following inductive definition:

- The identity mapping $\mathrm{id}_{\mathcal{F}}$ is CBC.
- If a mapping assigns to any function from $\mathcal{F}$ one and the same $(\mathcal{U}^*,\mathcal{U}^*)$-computable function from $\mathcal{F}$ then this mapping is CBC.
- If Γ_1 and Γ_2 are CBC mappings of $\mathcal{F}$ into itself then the mappings $f \mapsto \Gamma_1(f)\Gamma_2(f)$, $f \mapsto (\Gamma_1(f), \Gamma_2(f))$ and $f \mapsto [\Gamma_1(f), \Gamma_2(f)]$ are CBC too.
- If Γ_1, Γ_2 and Γ_3 are CBC mappings of $\mathcal{F}$ into itself then the mapping $f \mapsto (\Gamma_3(f) \supset \Gamma_1(f), \Gamma_2(f))$ is CBC too.

Example. The mapping $f \mapsto (\mathrm{id}_{X^*} \supset \mathrm{id}_{X^*}, (f\delta, f\pi))$ is CBC.

The mapping from the above example has exactly one fixed point (its domain is X^*).

First Recursion Theorem for the computability in $(X^*, \mathcal{U}^*)$

Theorem

If Γ is a CBC mapping of $\mathcal{F}$ into itself then Γ has a least fixed point with respect to the usual partial ordering of $\mathcal{F}$, and this fixed point is $(\mathcal{U}^*, \mathcal{U}^*)$-computable.

Proof. One makes use of the recursion theorem from [2] and the preservation of $(\mathcal{U}^*, \mathcal{U}^*)$-computability under composition, combination, branching and iteration. □

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