Computability of Real Numbers by Using a Given Class of Functions in the Set of the Natural Numbers

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Abstract

Intuitively, a real number is computable if there is an effective method for constructing arbitrarily close rational approximations of that number. According to Church Thesis, this intuitive description has its mathematical counterpart – the well-known and extensively studied notion of a recursive real number. However, one could be possibly interested also in studying certain other notions corresponding to more restricted interpretations of the term "effective method" or to wider interpretations of "method" after dropping the adjective "effective". We present here a framework for such a kind of studies and prove some results concerning the arising computability notions for real numbers.

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1 Introduction

Intuitively, a real number is computable if there is an effective method for constructing arbitrarily close rational approximations of that number. According to Church Thesis, this intuitive description has its mathematical counterpart – the well-known and extensively studied notion of a recursive real number. However, one could be possibly interested also in studying certain other notions corresponding to more restricted interpretations of the term "effective method" or to wider interpretations of "method" after dropping the adjective "effective". We present here a framework for such a kind of studies and prove some results concerning the arising computability notions for real numbers.

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The approach we shall use can be described as follows. Given a class $\mathcal{F}$ of total functions in the set of the natural numbers, as long as $\mathcal{F}$ satisfies certain closedness conditions, a simple reasonable notion of expressibility through functions from $\mathcal{F}$ is defined for functions from the natural numbers into the rational ones. Then we consider the real numbers α satisfying the condition that for any natural number n a rational approximation $A(n)$ of α such that

$$(1) \qquad |A(n) - \alpha| \leq \frac{1}{n+1}$$

can be found by means of an expressible function A. Of course the notion of a recursive real number can be obtained in this way by taking $\mathcal{F}$ to be the class of all recursive functions. If the narrower class of the primitive recursive functions is taken as $\mathcal{F}$, then we get the notion of a primitive recursive real number studied for example in [11, 7, 8, 6, 4, 5, 2].[1] The set of all real numbers corresponds to the case when $\mathcal{F}$ contains all total functions in the set of the natural numbers. Some other choices of $\mathcal{F}$ will be indicated in the present paper, for example $\mathcal{F}$ can be any of the Grzegorczyk classes $\mathcal{E}^n$ (introduced in [1]) with $n \geq 2$.

2 The acceptable classes $\mathcal{F}$ and the corresponding fields of real numbers

Let $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ be the set of the non-negative integers, the set of the rational numbers and the set of the real numbers, respectively. A class $\mathcal{F}$ of total functions in $\mathbb{N}$ will be called *acceptable* if it is closed under substitution and contains the functions $\lambda n.n+1$, $\lambda mn.m \dot{-} n$, $\lambda mn.mn$ (where $m \dot{-} n = \max\{m-n, 0\}$), as well as the projection functions $\lambda n_1 \ldots n_k.n_i$, $k = 2, 3, 4, \ldots,$ $i = 1, 2, \ldots, k$. If $\mathcal{F}$ is an acceptable class of total functions in $\mathbb{N}$ then a function $A : \mathbb{N} \longrightarrow \mathbb{Q}$ will be called *expressible through* $\mathcal{F}$ or shortly $\mathcal{F}$*-expressible* if it can be represented in the form

$$(2) \qquad A(n) = \frac{u(n) - v(n)}{w(n) + 1} \ ,$$

where u, v, w are functions belonging to $\mathcal{F}$ (if all values of A are non-negative then clearly $u(n) - v(n) = u(n) \dot{-} v(n)$ and, taking $u(n) - v(n)$ as a new $u(n)$, one may use the representation (2) without the term $-v(n)$). Of course, one can treat quite similarly also those $\mathbb{Q}$-valued functions that depend on several natural arguments. A real number α will be called *computable through* a given acceptable class $\mathcal{F}$ or shortly $\mathcal{F}$*-computable* if there is an $\mathcal{F}$-expressible function $A : \mathbb{N} \longrightarrow \mathbb{Q}$ that satisfies the inequality (1) for all n in $\mathbb{N}$. The set of all real numbers computable through an acceptable class $\mathcal{F}$ will be denoted by $\mathbb{R}_{\mathcal{F}}$.

All classes $\mathcal{F}$ mentioned in Section 1 are acceptable in the sense of the above definition. Other examples are the class of all functions recursive in a given total function and the class of all functions primitive recursive in such a function.

[1]The preliminary version of this paper presented on the CCA 2001 Seminar under the title "Well computable real numbers" actually has been restricted to this particular instance of the present considerations (with some additions concerning topics specific for that case).

R e m a r k **1**. Obviously there is a smallest one among the acceptable classes (it can be defined by using the conditions from the definition of the notion of an acceptable class as an inductive definition). We note that each one-argument function belonging to that minimal class is almost a polynomial with integer coefficients, namely the function coincides with such a polynomial for all sufficiently large values of the argument (this can be shown by inductively proving the following statement: whenever we substitute in a function from the mentioned class some functions from $\mathbb{N}$ into $\mathbb{N}$ that are almost polynomials with integer coefficients, the function we get is also one of this sort). Therefore the minimal acceptable class is different from the ones listed above.

R e m a r k **2**. Let $\mathcal{F}$ be an acceptable class. Then clearly all constants from $\mathbb{N}$ belong to $\mathcal{F}$. Thanks to the equalities $n = (n+1)-1$, $m+n = (m+1)(n+1)-(mn+1)$, $|m-n| = (m \dot{-} n)+(n \dot{-} m)$, $\overline{\mathrm{sg}}(n) = 1 \dot{-} n$, $\mathrm{sg}(n) = \overline{\mathrm{sg}}(\overline{\mathrm{sg}}(n))$, $\min(m,n) = m \dot{-} (m \dot{-} n)$, $\max(m,n) = m+(n \dot{-} m)$, the functions $\lambda n.n$, $\lambda mn.m+n$, $\lambda mn.|m-n|$, $\overline{\mathrm{sg}}$, sg, min and max also belong to $\mathcal{F}$. Of course $\mathcal{F}$ contains as well all polynomials with coefficients from $\mathbb{N}$.

If $\mathcal{F}$ is an arbitrary acceptable class then all constant functions from $\mathbb{N}$ into $\mathbb{Q}$ are $\mathcal{F}$-expressible, therefore $\mathbb{Q} \subseteq \mathbb{R}_{\mathcal{F}}$. For some acceptable classes $\mathcal{F}$ there are no other $\mathcal{F}$-computable real numbers except for the rational ones. Such is the case for example when $\mathcal{F}$ is the minimal acceptable class.

For any two functions from $\mathbb{N}$ into $\mathbb{Q}$ that are expressible through an acceptable class $\mathcal{F}$ their sum and their product can be easily shown to be also $\mathcal{F}$-expressible. If all values of a function A from $\mathbb{N}$ into $\mathbb{Q}$ are distinct from 0 and A is expressible through an acceptable class $\mathcal{F}$, then the function $\lambda n.1/A(n)$ is also $\mathcal{F}$-expressible. In fact, if A has the representation (2), where u, v, w belong to $\mathcal{F}$, then we have the equality

$$1/A(n) = \frac{(w(n)+1)u(n) - (w(n)+1)v(n)}{||u(n)-v(n)|^2 - 1| + 1} \ .$$

Of course, all these statements can be immediately carried over to $\mathcal{F}$-expressible functions of several arguments.

If $\mathcal{F}$ is an acceptable class and A is an $\mathcal{F}$-expressible function from $\mathbb{N}$ into $\mathbb{Q}$ then (thanks to the closedness of $\mathcal{F}$ under substitution) the function $\lambda m.A(f(m))$ is also $\mathcal{F}$-expressible for any one-argument function f belonging to $\mathcal{F}$ (and quite similarly for functions from $\mathcal{F}$ and functions A depending on a greater number of arguments).

R e m a r k **3**. Let $\mathcal{F}$ be an acceptable class, α be a real number and A be an $\mathcal{F}$-expressible function from $\mathbb{N}$ into $\mathbb{Q}$ such that $n|A(n)-\alpha|$ remains bounded when n ranges over $\mathbb{N}$. Then $\alpha \in \mathbb{R}_{\mathcal{F}}$. To see this, it is sufficient to take a positive integer c which is an upper bound of $n|A(n)-\alpha|$ and to observe that for all n in $\mathbb{N}$

$$|A(cn+c)-\alpha| \leq \frac{1}{n+1} \ .$$

Since the number -1 belongs to $\mathbb{R}_{\mathcal{F}}$ for any acceptable class $\mathcal{F}$, the next statement implies that $\mathbb{R}_{\mathcal{F}}$ is a field for any such class.

P r o p o s i t i o n **1**. *Let $\mathcal{F}$ be an acceptable class. Then for any two numbers α and β from $\mathbb{R}_{\mathcal{F}}$ the numbers $\alpha+\beta$ and $\alpha\beta$ also belong to $\mathbb{R}_{\mathcal{F}}$. For any non-zero number α from $\mathbb{R}_{\mathcal{F}}$ the number $1/a$ also belongs to $\mathbb{R}_{\mathcal{F}}$.*

P r o o f . Let α and β belong to $\mathbb{R}_{\mathcal{F}}$, and let A and B be $\mathcal{F}$-expressible functions from $\mathbb{N}$ into $\mathbb{Q}$ such that

$$|A(n)-\alpha| \leq \frac{1}{n+1}, \quad |B(n)-\beta| \leq \frac{1}{n+1}$$

for all n in $\mathbb{N}$. Then

$$\begin{aligned} |(A(n)+B(n))-(\alpha+\beta)| &\leq |A(n)-\alpha|+|B(n)-\beta| \leq \frac{2}{n+1}\ , \\ |A(n)B(n)-\alpha\beta| &\leq |A(n)-\alpha||B(n)|+|\alpha||B(n)-\beta| \\ &\leq \frac{|B(n)|+|\alpha|}{n+1} \leq \frac{1+|\beta|+|\alpha|}{n+1}\ . \end{aligned}$$

Since $\lambda n.(A(n)+B(n))$ and $\lambda n.A(n)B(n)$ are $\mathcal{F}$-expressible, the above inequalities and Remark 3 lead to the conclusion that $\alpha+\beta$ and $\alpha\beta$ belong to $\mathbb{R}_{\mathcal{F}}$. Additionally suppose now that $\alpha \neq 0$. Let c be a number from $\mathbb{N}$ such that $(c+1)|\alpha| \geq 2$, and consider any n in $\mathbb{N}$ satisfying the inequality $n \geq c$. Then

$$|A(n)| \geq |\alpha|-|\alpha-A(n)| \geq \frac{2}{c+1}-\frac{1}{n+1} \geq \frac{1}{c+1}\ ,$$

hence $A(n) \neq 0$ and

$$\left|\frac{1}{A(n)}-\frac{1}{\alpha}\right| = \frac{|\alpha-A(n)|}{|A(n)||\alpha|} \leq \frac{h}{n+1}$$

with $h=(c+1)^2/2$. Now define a function C from $\mathbb{N}$ into $\mathbb{Q}$ as follows

$$C(m) = \frac{1}{A(m+k)}\ .$$

Then C is $\mathcal{F}$-expressible and

$$\left|C(m)-\frac{1}{\alpha}\right| \leq \frac{h}{m+k+1}$$

for all m in $\mathbb{N}$, hence $1/\alpha \in \mathbb{R}_{\mathcal{F}}$.

It is known [9, 5] that the fields of the recursive real numbers and of the primitive recursive ones have the property to contain the real roots of the one-argument polynomials with coefficients from these fields.[2] We shall present now a generalization of this.

An acceptable class $\mathcal{F}$ will be called *closed under bounded μ-operation* if, whenever f is a $k+1$-argument function from $\mathcal{F}$, the class $\mathcal{F}$ contains also the function

(3) $\qquad \lambda n_1 \dots n_k n_{k+1}.\min\{m \in \mathbb{N} \mid f(n_1,\dots,n_k,m)=0 \vee m=n_{k+1}\}\,.$

We note that all above-mentioned concrete acceptable classes except for the one considered in Remark 1 satisfy this condition.

[2]The proof in [5] (concerning the field of the primitive recursive real numbers) has a gap, since it supposes that not only the first derivative of the given polynomial is distinct from 0 for the considered root, but also the second one. This gap can be filled by multiplying the polynomial by a suitable linear function.

T h e o r e m **1**. *Let $\mathcal{F}$ be an acceptable class closed under bounded μ-operation, and let α_0, α_1, ..., α_{k-1}, α_k, where $\alpha_0 \neq 0$, belong to $\mathbb{R}_{\mathcal{F}}$. Then all real roots of the polynomial*

$$P(x) = \alpha_0 x^k + \alpha_1 x^{k-1} + \ldots + \alpha_{k-1}x + \alpha_k$$

also belong to $\mathbb{R}_{\mathcal{F}}$.

P r o o f . Let ξ be a real root of the polynomial in question. Without loss of generality (at least non-constructively), we may assume that $P'(\xi) \neq 0$. Then there are rational numbers a, b, c and d such that $a < \xi < b$, $0 < c < d$ and

$$c|x - \xi| \leq |P(x)| \leq d|x - \xi|$$

whenever $a \leq x \leq b$. Making use of the computability through $\mathcal{F}$ of the numbers α_0, α_1, ..., α_{k-1}, α_k, let us consider $\mathcal{F}$-expressible functions A_0, A_1, ..., A_{k-1}, A_k from $\mathbb{N}$ into $\mathbb{Q}$ such that

$$|A_i(n) - \alpha_i| \leq \frac{1}{n+1}, \quad i = 0, 1, \ldots, k-1, k, \quad n = 0, 1, 2, 3, \ldots$$

Consider now the polynomials

$$P_n(x) = A_0(n)x^k + A_1(n)x^{k-1} + \ldots + A_{k-1}(n)x + A_k(n), \quad n = 0, 1, 2, 3, \ldots$$

Clearly there is a rational constant h such that

$$|P_n(x) - P(x)| \leq \frac{h}{n+1} \ , \quad n = 0, 1, 2, 3, \ldots,$$

whenever $a \leq x \leq b$. We shall define now a function X from $\mathbb{N}$ into $\mathbb{Q}$. Given any n in $\mathbb{N}$, let us divide the interval between a and b into $n+1$ equally long subintervals. Let M_n be the set of the middle points of these subintervals. There is at least one number x in M_n satisfying the inequality

$$(4) \qquad |P_n(x)| \leq \frac{d(b-a) + 2h}{2(n+1)} \ .$$

In fact, there is some x in M_n such that

$$|x - \xi| \leq \frac{b-a}{2(n+1)} \ ,$$

and for any such x we have the inequalities

$$|P_n(x)| \leq |P(x)| + \frac{h}{n+1} \leq d\frac{b-a}{2(n+1)} + \frac{h}{n+1} = \frac{d(b-a) + 2h}{2(n+1)} \ .$$

We set $X(n)$ to be the leftmost x in M_n satisfying the inequality (4). Then

$$c|X(n) - \xi| \leq |P(X(n))| \leq |P_n(X(n))| + \frac{h}{n+1} \leq \frac{d(b-a) + 4h}{2(n+1)} \ .$$

Therefore the product $n|X(n) - \xi|$ remains bounded when n ranges over $\mathbb{N}$, and, having in mind Remark 3, it is sufficient to show the expressibilty of the function X through $\mathcal{F}$. For that purpose we shall use the fact that

$$(5) \qquad X(n) = a + (b-a)\frac{2g(n) + 1}{2n + 2} \ ,$$

where $g(n)$ is the least $m \in \{0, 1, 2, \ldots, n\}$ such that (4) is satisfied by

$$x = a + (b-a)\frac{2m+1}{2n+2} \ .$$

Thus we have

$$g(n) = \min\left\{m \in \{0,1,2,\ldots,n\} \;\middle|\; P_n\left(a+(b-a)\frac{2m+1}{2n+2}\right) - \frac{h'}{n+1} \le 0\right\} \ ,$$

where $h' = d(b-a)/2 + h$. Since

$$P_n\left(a+(b-a)\frac{2m+1}{2n+2}\right) = \sum_{i=0}^{k} A_i(n)\left(a+(b-a)\frac{2m+1}{2n+2}\right)^{k-i} \ ,$$

it is not difficult to see that the function

$$\lambda nm.P_n\left(a+(b-a)\frac{2m+1}{2n+2}\right) - \frac{h'}{n+1}$$

is expressible through $\mathcal{F}$. Hence this function can be represented in the form

$$\lambda nm.\frac{u(n,m)-v(n,m)}{w(n,m)+1} \ ,$$

where u, v, w are some functions from $\mathcal{F}$. If we set $f(n,m) = u(n,m) \dot{-} v(n,m)$, then $f \in \mathcal{F}$ and we shall have $g(n) = g_0(n,n)$, where the function g_0 is defined by means of the equality $g_0(n_1,n_2) = \min\{m \in \mathbb{N} | f(n_1,m) = 0 \vee m = n_2\}$. The closedness of $\mathcal{F}$ under bounded μ-operation allows us to conclude that $g_0 \in \mathcal{F}$ and hence $g \in \mathcal{F}$. This, together with the equality (5), shows the expressibility of the function X through $\mathcal{F}$.

R e m a r k **4**. It is natural to consider also complex numbers that are computable through a given acceptable class $\mathcal{F}$, i.e. such ones both components of which are $\mathcal{F}$-computable. Theorem 1 remains valid after replacing "real" by "complex" in its formulation. This can be shown by means of certain natural changes in the above proof (one replaces intervals by squares, divides the initial square into $(n+1)^2$ ones and so on). Another way to see the validity of the mentioned generalization of Theorem 1 has been indicated to the author by Alex Simpson. Namely he noted that, once such a theorem is proved, its generalization to complex numbers can be immediately obtained by applying results from [12].

The following remark and the proposition after it will be useful later.

R e m a r k **5**. If an acceptable class $\mathcal{F}$ is closed under bounded μ-operation then the functions

$$\lambda ik.\left[\frac{i}{k+1}\right] \ , \quad \lambda ik.i \bmod (k+1)$$

belong to $\mathcal{F}$ thanks to the equalities

$$\left[\frac{i}{k+1}\right] = \min\{m \in \mathbb{N} |\, f(i,k,m) = 0 \vee m = i\} \ , \quad i \bmod (k+1) = i - \left[\frac{i}{k+1}\right](k+1) \ ,$$

where $f(i,k,m) = \overline{\mathrm{sg}}((m+1)(k+1) \dot{-} i)$.

P r o p o s i t i o n **2**. *Let $\mathcal{F}$ be an acceptable class closed under bounded μ-operation, f be a $k+1$-argument function belonging to $\mathcal{F}$, and the functions $f_\vee$ and $f_\wedge$ from $\mathbb{N}^{k+1}$ into $\mathbb{N}$ be defined by means of the equalities*

$$f_\vee(n_1,\dots,n_k,n_{k+1}) = \max\{f(n_1,\dots,n_k,m) \mid m\in\mathbb{N}, m\le n_{k+1}\},$$
$$f_\wedge(n_1,\dots,n_k,n_{k+1}) = \min\{f(n_1,\dots,n_k,m) \mid m\in\mathbb{N}, m\le n_{k+1}\}.$$

Then $f_\vee$ and $f_\wedge$ also belong to $\mathcal{F}$.

P r o o f . We shall carry out the proof only for the function $f_\vee$ (for the other one it would be very similar). Let the functions h_0 and h_1 from $\mathbb{N}^{k+2}$ into $\mathbb{N}$ be defined as follows:

$$h_0(l,n_1,\dots,n_k,i) = \min\{m\in\mathbb{N} \mid (l+1) \dot{-} f(n_1,\dots,n_k,m) = 0 \vee m = i\},$$
$$h_1(n_1,\dots,n_k,n_{k+1},j) = (n_{k+1}+1) \dot{-} h_0(f(n_1,\dots,n_k,j),n_1,\dots,n_k,n_{k+1}+1).$$

Then $h_0\in\mathcal{F}$ and for all $l,n_1,\dots,n_k,i\in\mathbb{N}$ the equality $i \dot{-} h_0(l,n_1,\dots,n_k,i)=0$ is equivalent to the condition l to be an upper bound of $\{f(n_1,\dots,n_k,m) \mid m\in\mathbb{N}, m<i\}$. Therefore h_1 also belongs to $\mathcal{F}$ and, whenever $n_1,\dots,n_k,n_{k+1},j\in\mathbb{N}$ and $j\le n_{k+1}$, we have the equivalence

$$h_1(n_1,\dots,n_k,n_{k+1},j)=0 \ \Leftrightarrow\ f(n_1,\dots,n_k,j) = f_\vee(n_1,\dots,n_k,n_{k+1})\,.$$

Thus if we set

$$g(n_1,\dots,n_k,n_{k+1}) = \min\{j\in\mathbb{N} \mid h_1(n_1,\dots,n_k,n_{k+1},j)) = 0 \vee j = n_{k+1}\}$$

then $g\in\mathcal{F}$ and

$$f_\vee(n_1,\dots,n_k,n_{k+1}) = f(n_1,\dots,n_k,g(n_1,\dots,n_k,n_{k+1}))\,,$$

hence $f_\vee\in\mathcal{F}$.

3 $\mathcal{F}$-convergence

Let $\mathcal{F}$ be an acceptable class. An infinite sequence $\alpha_0,\alpha_1,\alpha_2,\dots$ of real numbers will be called *$\mathcal{F}$-convergent* if there is an one-argument function $f\in\mathcal{F}$ such that for any n in $\mathbb{N}$

$$(6)\qquad |\alpha_i-\alpha_j| \le \frac{1}{n+1}$$

whenever $j>i\ge f(n)$. The sequence $\alpha_0,\alpha_1,\alpha_2,\dots$ will be called *computable through* $\mathcal{F}$ or shortly *$\mathcal{F}$-computable* if there is an $\mathcal{F}$-expressible function $A:\mathbb{N}^2\longrightarrow\mathbb{Q}$ such that

$$(7)\qquad |A(m,n)-\alpha_m| \le \frac{1}{n+1},\quad m,n=0,1,2,3,\dots$$

Clearly any $\mathcal{F}$-convergent infinite sequence of real numbers has a limit, and the members of an $\mathcal{F}$-computable infinite sequence of real numbers always belong to $\mathbb{R}_\mathcal{F}$. We note also the $\mathcal{F}$-computability of the sequence of the values of any $\mathcal{F}$-expressible function from $\mathbb{N}$ into $\mathbb{Q}$.

P r o p o s i t i o n **3**. *Let $\alpha_0,\alpha_1,\alpha_2,\dots$ be an $\mathcal{F}$-convergent infinite sequence of real numbers that is $\mathcal{F}$-computable. Then* $\lim_{m\to\infty}\alpha_m\in\mathbb{R}_\mathcal{F}$.

P r o o f . Let $\alpha = \lim_{m\to\infty} \alpha_m$. If f is a function with the properties from the definition of $\mathcal{F}$-convergence, and A is a function with the properties from the other definition above, then

$$|A(f(n),n) - \alpha| \leq |A(f(n),n) - \alpha_{f(n)}| + |\alpha_{f(n)} - \alpha| \leq \frac{2}{n+1}$$

for any n in $\mathbb{N}$, and the function $\lambda n.A(f(n),n)$ is $\mathcal{F}$-expressible, hence Remark 3 is applicable.

The notion of $\mathcal{F}$-convergence is transferred in a natural way to infinite series, namely an infinite series of real numbers will be called *$\mathcal{F}$-convergent* if the sequence of its partial sums is $\mathcal{F}$-convergent.

Next two propositions show that certain often used tools of calculus for showing the convergence of infinite series always or usually establish in fact their $\mathcal{F}$-convergence.

P r o p o s i t i o n **4**. *Let $\tau_0, \tau_1, \tau_2, \ldots$ be an infinite sequence of real numbers such that the series*

$$\sum_{k=0}^{\infty} |\tau_k| \tag{8}$$

is convergent by D'Alembert's, Cauchy's or Raabe's test. Then the series

$$\sum_{k=0}^{\infty} \tau_k \tag{9}$$

is $\mathcal{F}$-convergent.

P r o o f . Let us set

$$\alpha_m = \sum_{k<m} \tau_k \ , \quad m = 0, 1, 2, 3, \ldots$$

To show the $\mathcal{F}$-convergence of (9) in the case when the series (8) is convergent by D'Alembert's or Cauchy's test, we consider a number γ such that $0 < \gamma < 1$ and for all sufficiently large k in $\mathbb{N}$ the inequality $|\tau_k| \leq \gamma^k$ holds. If $i, j \in \mathbb{N}$, $j > i$ and i is suffiently large then

$$|\alpha_i - \alpha_j| \leq \sum_{i\leq k<j} |\tau_k| \leq \sum_{i\leq k<j} \gamma^k = \frac{\gamma^i - \gamma^j}{1-\gamma} < \frac{\gamma^i}{1-\gamma}$$

Since $(i+1)\gamma^i$ tends to 0 when i tends to infinity,

$$\frac{\gamma^i}{1-\gamma} < \frac{1}{i+1}$$

for all sufficiently large i in $\mathbb{N}$. Hence there is a number $c \in \mathbb{N}$ such that

$$|\alpha_i - \alpha_j| < \frac{1}{i+1}$$

whenever $i, j \in \mathbb{N}$ and $j > i \geq c$. By setting $f(n) = \max(n, c)$ we get a function $f \in \mathcal{F}$ such that (6) holds whenever $j > i \geq f(n)$. Suppose now that the series (8) is convergent by Raabe's test. This implies the existence of a positive integer a such that for all sufficiently large k in $\mathbb{N}$ the inequality $|\tau_k| \leq (k+1)^{-1/a} - (k+2)^{-1/a}$ holds. Let c be such a number from $\mathbb{N}$ that the above inequality holds whenever $k \geq c$. If $i, j \in \mathbb{N}$ and $j > i \geq c$ then $|\alpha_i - \alpha_j| \leq (i+1)^{-1/a} - (j+1)^{-1/a} < (i+1)^{-1/a}$. Thus by setting $f(n) = \max((n+1)^a - 1, c)$ we shall again get a function $f \in \mathcal{F}$ such that (6) holds whenever $j > i \geq f(n)$.

C o r o l l a r y **1**. *Each power series with a non-zero radius of convergence is $\mathcal{F}$-convergent at any point inside its convergence interval.*

P r o p o s i t i o n **5**. *Let $\tau_0, \tau_1, \tau_2, \ldots$ be a monotonically decreasing infinite sequence of real numbers that is $\mathcal{F}$-convergent with limit 0. Then the infinite series*

$$\sum_{k=0}^{\infty} (-1)^k \tau_k$$

is $\mathcal{F}$-convergent.

P r o o f . If we set

$$\alpha_m = \sum_{k<m} (-1)^k \tau_k \ , \quad m = 0, 1, 2, 3, \ldots$$

then, as it is well-known, $|\alpha_i - \alpha_j| \leq \tau_i$ whenever $i, j \in \mathbb{N}$, $j > i$.

4 Strongly acceptable classes

Let $\mathcal{F}$ be an acceptable class. The class $\mathcal{F}$ will be called *strongly acceptable* if $\mathcal{F}$ contains the function $\lambda n.2^n$ and $\mathcal{F}$ is closed under bounded primitive recursion, i.e. under such primitive recursion that produces a function bounded by some function from $\mathcal{F}$. All above-mentioned acceptable classes except for $\mathcal{E}^2$ and the one from Remark 1 are in fact strongly acceptable.

R e m a r k **6**. Making use of the inequalities $m^n \leq 2^{mn}$, $n! \leq 2^{n^2}$, as well as of the primitive recursive equations for the functions $\lambda mn.m^n$ and $\lambda n.n!$, we see that these functions belong to any strongly acceptable class.

P r o p o s i t i o n **6**. *Any strongly acceptable class is closed under bounded μ-operation.*

P r o o f . Let $\mathcal{F}$ be a strongly acceptable class, f be a $k+1$-argument function belonging to $\mathcal{F}$, and g be the corresponding function (3). Then $g \in \mathcal{F}$ thanks to the inequality $g(n_1, \ldots, n_k, n_{k+1}) \leq n_{k+1}$ and the equalities $g(n_1, \ldots, n_k, 0) = 0$, $g(n_1, \ldots, n_k, n_{k+1}+1) = g(n_1, \ldots, n_k, n_{k+1}) + \mathrm{sg}(f(n_1, \ldots, n_k, g(n_1, \ldots, n_k, n_{k+1})))$.

R e m a r k **7**. The converse statement to Proposition 6 is not true. For example the class of the functions in $\mathbb{N}$ that are bounded by polynomials is an acceptable class closed under bounded μ-operation, but it does not contain the function $\lambda n.2^n$.

P r o p o s i t i o n **7**. *Let $\mathcal{F}$ be a strongly acceptable class, f be a $k+1$-argument function belonging to $\mathcal{F}$, and the functions f_Σ and f_Π from $\mathbb{N}^{k+1}$ into $\mathbb{N}$ be defined by means of the equalities*

$$\begin{aligned} f_\Sigma(n_1,\ldots,n_k,n_{k+1}) &= \sum_{m<n_{k+1}} f(n_1,\ldots,n_k,m) \ , \\ f_\Pi(n_1,\ldots,n_k,n_{k+1}) &= \prod_{m<n_{k+1}} f(n_1,\ldots,n_k,m) \ . \end{aligned}$$

Then f_Σ and f_Π also belong to $\mathcal{F}$.

P r o o f . Let the function $f_\vee$ be defined as in Proposition 2. By that proposition and Proposition 6 $f_\vee \in \mathcal{F}$. Therefore we may use the primitive recursive equations for the functions f_Σ and f_Π, as well as the inequalities

$$\begin{aligned} &f_\Sigma(n_1,\ldots,n_k,n_{k+1}) \leq f_\vee(n_1,\ldots,n_k,n_{k+1} \dot{-} 1)\, n_{k+1} \ , \\ &f_\Pi(n_1,\ldots,n_k,n_{k+1}) \leq f_\vee(n_1,\ldots,n_k,n_{k+1} \dot{-} 1)^{n_{k+1}} \ . \end{aligned}$$

Proposition 7 has its analogue for $\mathcal{F}$-expressible $\mathbb{Q}$-valued functions. We shall formulate and prove here only the part concerning summation.

P r o p o s i t i o n **8**. *Let $\mathcal{F}$ be a strongly acceptable class, $A : \mathbb{N}^{k+1} \longrightarrow \mathbb{Q}$ be $\mathcal{F}$-expressible, and the function A_Σ from $\mathbb{N}^{k+1}$ into $\mathbb{Q}$ be defined by means of the equality*

$$A_\Sigma(n_1,\ldots,n_k,n_{k+1}) = \sum_{m<n_{k+1}} A(n_1,\ldots,n_k,m) \ .$$

Then A_Σ is also $\mathcal{F}$-expressible.

P r o o f . Let A has the representation

$$A(n_1,\ldots,n_k,m) = \frac{u(n_1,\ldots,n_k,m) - v(n_1,\ldots,n_k,m)}{w(n_1,\ldots,n_k,m)+1} \ ,$$

where the functions u, v and w belong to $\mathcal{F}$. After setting

$$h(n_1,\ldots,n_k,n_{k+1}) = \prod_{m<n_{k+1}} (w(n_1,\ldots,n_k,m)+1)$$

we have

$$A_\Sigma(n_1,\ldots,n_k,n_{k+1}) = \frac{\bar{u}(n_1,\ldots,n_k,n_{k+1}) - \bar{v}(n_1,\ldots,n_k,n_{k+1})}{\bar{w}(n_1,\ldots,n_k,n_{k+1})+1} \ ,$$

where $\bar{w}(n_1,\ldots,n_k,n_{k+1}) = h(n_1,\ldots,n_k,n_{k+1}) \dot{-} 1$,

$$\bar{u}(n_1,\ldots,n_k,n_{k+1}) = \sum_{m<n_{k+1}} u(n_1,\ldots,n_k,m) \left[\frac{h(n_1,\ldots,n_k,n_{k+1})}{w(n_1,\ldots,n_k,m)+1}\right]$$

and similarly for $\bar{v}(n_1,\ldots,n_k,n_{k+1})$.

The presented results enable us to show for many concrete real numbers playing a role in analysis that they belong to $\mathbb{R}_{\mathcal{F}}$ for any strongly acceptable class $\mathcal{F}$. This can be done by using appropriate representations as sums of infinite series with rational terms and applying the case $k=0$ of Proposition 8 together with some of the sufficient conditions from Section 3 for $\mathcal{F}$-convergence of infinite series. For example the representation $\ln 2 = 1-1/2+1/3-1/4+\dots$ can be used in this way in combination with Proposition 5 for showing that $\ln 2 \in \mathbb{R}_{\mathcal{F}}$. For doing similar things in the case of infinite series with not necessarily rational terms, the following statement can be often helpful (in fact, this statement and Taylor series for the basic elementary functions of analysis can be used to show that all values of such functions for $\mathcal{F}$-computable values of the argument are also $\mathcal{F}$-computable).

P r o p o s i t i o n **9**. *Let $\mathcal{F}$ be a strongly acceptable class, and $\tau_0, \tau_1, \tau_2, \dots$ be an $\mathcal{F}$-computable infinite sequence of real numbers. Let*

$$\alpha_m = \sum_{k<m} \tau_k \ , \quad m = 0, 1, 2, 3, \dots$$

Then the sequence $\alpha_0, \alpha_1, \alpha_2, \dots$ is also $\mathcal{F}$-computable.

P r o o f . Let A be an $\mathcal{F}$-expressible function from $\mathbb{N}^2$ into $\mathbb{Q}$ satisfying the condition (7). Then

$$\left| \sum_{k<m} A(k,n) - \sum_{k<m} \tau_i \right| \le \sum_{k<m} |A(k,n) - \tau_k| \le \frac{m}{n+1}$$

for all m and n in $\mathbb{N}$. Hence by setting

$$B(m,n) = \sum_{k<m} A(k, m(n+1))$$

we get an $\mathcal{F}$-computable function B from $\mathbb{N}$ into $\mathbb{Q}$ such that

$$\left| B(m,n) - \sum_{k<m} \tau_k \right| \le \frac{m}{m(n+1)+1} < \frac{1}{n+1} \ .$$

For any strongly acceptable class $\mathcal{F}$ we shall prove a characterization of the real numbers in $\mathbb{R}_{\mathcal{F}}$ by means of infinite signed-digit binary fractions. The characterization will be obtained as a corollary from the next statement.

L e m m a **1**. *Let the class $\mathcal{F}$ be strongly acceptable, and let $\alpha \in \mathbb{R}$, $0 \le \alpha \le 4$. Then the following two conditions are equivalent:*

1. The number α belongs to $\mathbb{R}_{\mathcal{F}}$.

2. There is an one-argument function f in $\mathcal{F}$ with values in $\{0,1,2\}$ such that

$$\alpha = \sum_{i=0}^{\infty} \frac{f(i)}{2^i} \ . \tag{10}$$

P r o o f . First suppose condition 2 is satisfied. We have to show that condition 1 is also satisfied. Let

$$A(n) = \sum_{i<n} \frac{f(i)}{2^i}\ , \quad n = 0, 1, 2, 3, \dots$$

The function A is $\mathcal{F}$-expressible by Proposition 8. Since

$$|A(n) - \alpha| = \sum_{i=n}^{\infty} \frac{f(i)}{2^i} \le \frac{4}{2^n} \le \frac{4}{n+1}\ ,$$

the product $n|A(n) - \alpha|$ is bounded, and the implication from condition 2 to condition 1 is thus proved. To prove the converse implication, suppose condition 1 is satisfied. Let A be an $\mathcal{F}$-expressible function from $\mathbb{N}$ into $\mathbb{Q}$ satisfying for any n in $\mathbb{N}$ the inequality (1). By setting $\tilde{A} = \lambda n.A(2^{n+1} - 1)$ we get an $\mathcal{F}$-expressible function $\tilde{A} : \mathbb{N} \longrightarrow \mathbb{Q}$ satisfying the inequality $|\tilde{A}(n) - \alpha| \le 2^{-n-1}$ for all n in $\mathbb{N}$. For any rational number r let $\varphi(r) = 2$ if $r > 5/2$, $\varphi(r) = 1$ if $5/2 \ge r > 3/2$, and $\varphi(r) = 0$ otherwise. We define a function $u : \mathbb{N} \longrightarrow \mathbb{N}$ in the following recursive way:

$$u(0) = 0,\ u(n+1) = 2u(n) + \varphi(2^n\tilde{A}(n) - 2u(n))\,.$$

By setting $f = \lambda n.u(n+1) - 2u(n)$ we get a function f from $\mathbb{N}$ into $\{0, 1, 2\}$. We shall prove that equality (10) holds and f belongs to $\mathcal{F}$. For proving (10) it is sufficient to prove that for any n in $\mathbb{N}$ the equality

$$u(n) = \sum_{i<n} f(i)2^{n-1-i} \tag{11}$$

holds, as well as the inequalities $0 \le 2^n\alpha - 2u(n) \le 4$, since these statements imply the inequalities

$$0 \le \alpha - \sum_{i<n} \frac{f(i)}{2^i} \le \frac{4}{2^n}\ .$$

The proof of (11) is by an easy induction using the equality $u(0) = 0$ and the definition of f. The proof of the inequalities $0 \le 2^n\alpha - 2u(n) \le 4$ is also by induction. These inequalities hold for $n = 0$ since $0 \le \alpha \le 4$. Now suppose that they hold for a certain n in $\mathbb{N}$. Then we set $r = 2^n\tilde{A}(n) - 2u(n)$ and note that $|2^n\alpha - 2u(n) - r| \le 1/2$, hence $\max\{r - 1/2, 0\} \le 2^n\alpha - 2u(n) \le \min\{r + 1/2, 4\}$. Consequently

$$\max\{r - 1/2, 0\} - \varphi(r) \le 2^n\alpha - u(n+1) \le \min\{r + 1/2, 4\} - \varphi(r).$$

By considering separately the three cases in the definition of $\varphi(r)$ it is easy to check that always $0 \le \max\{r - 1/2, 0\} - \varphi(r)$ and $\min\{r + 1/2, 4\} - \varphi(r) \le 2$. Therefore $0 \le 2^n\alpha - u(n+1) \le 2$, hence $0 \le 2^{n+1}\alpha - 2u(n+1) \le 4$. Thus the proof of the equality (10) is completed and it remains only to prove that $f \in \mathcal{F}$. For doing this it is sufficient to show that $u \in \mathcal{F}$. Since $u(n) \le 2^{n+1} - 2$ for all n in $\mathbb{N}$, it would be enough to prove that $\lambda nm.2m + \varphi(2^n\tilde{A}(n) - 2m)$ belongs to $\mathcal{F}$. For that purpose we first note that $\lambda nm.2^n\tilde{A}(n) - 2m$ is an $\mathcal{F}$-expressible function, i.e. there are two-argument functions $\bar{u}$, $\bar{v}$ and $\bar{w}$ in $\mathcal{F}$ such that

$$2^n\tilde{A}(n) - 2m = \frac{\bar{u}(n,m) - \bar{v}(n,m)}{\bar{w}(n,m) + 1}$$

for all n and m in $\mathbb{N}$. But it is not difficult to see that the above equality implies

$$\varphi(2^n \tilde{A}(n) - 2m) = h(\bar{u}(n,m), \bar{v}(n,m), \bar{w}(n,m)),$$

where $h(i,j,k) = \mathrm{sg}(2i \dot{-} (2j+5k+5)) + \mathrm{sg}(2i \dot{-} (2j+3k+3))$, and this completes the proof of the lemma.

The mentioned characterization by means of infinite signed-digit binary fractions reads as follows.

T h e o r e m **2**. *Let $\mathcal{F}$ be a strongly acceptable class, α be a real number, and k be an integer such that $2^{k+1} \geq |\alpha|$. Then the following two conditions are equivalent:*

1. The number α belongs to $\mathbb{R}_{\mathcal{F}}$.

2. There is an one-argument function f in $\mathcal{F}$ with values in $\{0,1,2\}$ such that

$$\alpha = 2^k \sum_{i=0}^{\infty} \frac{f(i)-1}{2^i} \quad . \tag{12}$$

P r o o f . Let $\alpha' = 2^{-k}\alpha + 2$. Then $0 \leq \alpha' \leq 4$, and $\alpha \in \mathbb{R}_{\mathcal{F}}$ iff $\alpha' \in \mathbb{R}_{\mathcal{F}}$. On the other hand, for any one-argument function f in $\mathcal{F}$ with values in $\{0,1,2\}$ the equality (12) is equivalent to the equality

$$\alpha' = \sum_{i=0}^{\infty} \frac{f(i)}{2^i} \quad .$$

Thus it is sufficient to apply Lemma 2 to the number α'.

R e m a r k **8**. In the case of non-negative α one could be interested in replacing the signed-digit binary fractions in the above theorem by ordinary binary fractions. In general, such a replacement is not possible (although it is possible for example if $\mathcal{F}$ is the class of all recursive functions). The impossibility in question can be seen by taking as $\mathcal{F}$ the class of the primitive recursive functions and using the existence of a primitive recursive number not representable as a primitive recursive infinite binary fraction. The existence of such a number is clear from the generalization in [2] of the example in [11] of a primitive recursive real number not representable as a primitive recursive infinite decimal fraction.[3]

R e m a r k **9**. If $\mathcal{F}$ is a strongly acceptable class, and $\alpha_0, \alpha_1, \alpha_2, \ldots$ is an $\mathcal{F}$-computable sequence of real numbers then there is some $\mathcal{F}$-computable real number that is distinct from all members of the given sequence. To prove this, one can take an interval with rational end points and use a refined version of the classical diagonal procedure to construct an $\mathcal{F}$-computable number belonging to the chosen interval and

[3] It is also possible to proceed by means of a slight modification of the mentioned example. Namely one may consider infinite hexadecimal fractions instead of decimal ones and set

$$\alpha = \sum_{k=0}^{\infty} \frac{\phi(k)}{16^k} \ ,$$

where the primitive recursive function ϕ is defined as in [11], but with values 1, 5, 9 instead of 1, 3, 5, respectively. Then α is a primitive recursive real number, hence 3α is also primitive recursive. Nevertheless, a reasoning similar to the one in [11] shows that 3α cannot be represented in the form of a primitive recursive infinite hexadecimal fraction, hence 3α is not representable also in the form of a primitive recursive infinite binary fraction.

distinct from all members of the sequence in question (cf. the proof of Theorem 4.2.6 in [13] for a similar construction in the case when the considered computability is recursiveness). Intuitively, the construction can be described as follows. We divide the given interval into three equally long subintervals and find a rational approximation of the number α_0 sufficiently close to it for enabling either the conclusion that α_0 does not belong to the leftmost of the three subintervals or the conclusion that α_0 does not belong to the rightmost of them (it would be enough if the distance between this approximation and α_0 is less than 1/6 of the length of the given interval). Then we take such one among the leftmost and the rightmost subintervals that does not contain α_0, and we proceed with it and α_1 in the same way to find a three times shorter subinterval with rational end points that does not contain α_1. By continuing this ad infinitum we get a sequence of nested intervals with a real number α belonging to all of them and therefore distinct from all members of the sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$ It is a routine task to transform this intuitive description into a precise mathematical definition of the number α and to prove that α is $\mathcal{F}$-computable (of course by making use of the $\mathcal{F}$-computability of the sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$).[4]

R e m a r k **10**. For some strongly acceptable class $\mathcal{F}$ it may happen that a larger strongly acceptable class $\mathcal{F}'$ contains a two-argument function ω that is universal for the one-argument functions in $\mathcal{F}$, i.e. the functions obtainable by substitution of constants from $\mathbb{N}$ for the first argument of ω are exactly the one-argument functions from $\mathcal{F}$ (a well-known example of such a situation is the case when $\mathcal{F}$ is the class of the primitive recursive functions, and $\mathcal{F}'$ is the class of the recursive functions). If such $\mathcal{F}$, $\mathcal{F}'$ and ω are given then let us define an infinite sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$ of real numbers by setting

$$\alpha_m = 2^{\omega(m,0)} \sum_{i=0}^{\infty} \frac{\min(\omega(m,i+1),2)-1}{2^i}, \quad m = 0,1,2,\ldots$$

By Theorem 2, the set of the members of this sequence is exactly $\mathbb{R}_{\mathcal{F}}$, and, on the other hand, it can be shown that the sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$ is $\mathcal{F}'$-computable.[5] As a particular instance of this we get the conclusion that the primitive recursive real numbers can be effectively enumerated – a thing that is not obvious from the definition of the notion of such a number. By Remark 9, the sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$ constructed above can be used to show that $\mathbb{R}_{\mathcal{F}}$ is a proper subset of $\mathbb{R}_{\mathcal{F}'}$ in the considered situation.[6]

There is another way of proving the existence of a number in $\mathbb{R}_{\mathcal{F}'} \setminus \mathbb{R}_{\mathcal{F}}$ in the situation considered in Remark 10. Namely, if ω is a two-argument function from

[4]Some of the details will be similar to ones in the proof of Lemma 1, but powers of 3 will play a role now instead of powers of 2.

[5]To satisfy the requirement of the corresponding definition, one could set

$$A(m,n) = 2^{\omega(m,0)} \sum_{i=0}^{n+\omega(m,0)} \frac{\min(\omega(m,i+1),2)-1}{2^i}, \quad m,n = 0,1,2,\ldots$$

[6]The existence of recursive numbers that are not primitive recursive (proved in [11]) will be a particular instance of this result.

$\mathcal{F}'$ universal for the one-argument functions in $\mathcal{F}$, then the one-argument function $\lambda i.\overline{\mathrm{sg}}(\omega(i,i))$ belongs to $\mathcal{F}' \setminus \mathcal{F}$ and all its values belong to the set $\{0,1\}$. We shall show now that the existence of such a function is sufficient for the existence of a number in $\mathbb{R}_{\mathcal{F}'} \setminus \mathbb{R}_{\mathcal{F}}$, even without an assumption that $\mathcal{F} \subseteq \mathcal{F}'$.[7]

T h e o r e m **3**. *Let $\mathcal{F}$ and $\mathcal{F}'$ be strongly acceptable classes, $f : \mathbb{N} \longrightarrow \{0,1\}$ be a function belonging to $\mathcal{F}' \setminus \mathcal{F}$, and let*

$$\alpha = \sum_{i=0}^{\infty} \frac{f(i)}{3^i} \ .$$

Then $\alpha \in \mathbb{R}_{\mathcal{F}'} \setminus \mathbb{R}_{\mathcal{F}}$.

P r o o f . To show that $\alpha \in \mathbb{R}_{\mathcal{F}'}$, let us set

$$A(n) = \sum_{i<n} \frac{f(i)}{3^i} \ , \quad n = 0,1,2,3,\ldots$$

The function A is $\mathcal{F}'$-expressible (by Proposition 8), and the product $n|A(n) - \alpha|$ is bounded, hence $\alpha \in \mathbb{R}_{\mathcal{F}'}$. We shall indicate now a specific way of computing the values of f on the base of arbitrary sufficiently close rational approximations of α. Namely, whenever $i \in \mathbb{N}$, $r \in \mathbb{Q}$ and

(13) $\quad |r - \alpha| \leq 3^{-i-2}$,

we shall see that $3^i r + 1/9$ is non-negative and the following equality holds:

(14) $\quad f(i) = [3^i r + 1/9] \bmod 3$.

In fact, the definition of α implies that $[3^i\alpha] \bmod 3 = f(i)$, $3^i\alpha \leq [3^i\alpha] + 1/2$. On the other hand, the inequality (13) implies that $3^i\alpha \leq 3^i r + 1/9 \leq 3^i\alpha + 2/9$, hence $[3^i\alpha] \leq 3^i r + 1/9 \leq [3^i\alpha] + 1/2 + 2/9 < [3^i\alpha] + 1$. Therefore $[3^i\alpha] = [3^i r + 1/9]$, and from here the equality (14) follows. Suppose now $\alpha \in \mathbb{R}_{\mathcal{F}}$, i.e. some function $A : \mathbb{N} \longrightarrow \mathbb{Q}$ expressible through $\mathcal{F}$ satisfies the inequality (1) for any n in $\mathbb{N}$. Then we can satisfy (13) by taking $r = A(3^{i+2} - 1)$, but the equality (14) with this choice of r easily leads to the contradictory conclusion that $f \in \mathcal{F}$ (one may use Remark 5 and the fact that $\lambda i.3^i A(3^{i+2}-1)+1/9$ is a non-negative function expressible through $\mathcal{F}$). Hence $\alpha \notin \mathbb{R}_{\mathcal{F}}$.

R e m a r k **11**. Theorem 3 generalizes a variation of an example given in [10]. Up to denotations, the mentioned example concerns a real number α defined as in the theorem, but with 4^i in the denominator instead of 3^i. It is shown in the example that such a number is recursive, but not primitive recursive, if $f : \mathbb{N} \longrightarrow \{0,1\}$ is recursive, but not primitive recursive.[8] The replacement of 4^i by 3^i became possible thanks to a simplification in [3] of the reasoning in the case considered in [10] (if $\mathcal{F}$ and $\mathcal{F}'$

[7] We did not explicitly used this assumption in the situation considered above, but anyway the existence of the universal function ω implies the mentioned inclusion.

[8] Unfortunately, at the time of writing [10] the author did not know that the same statement (presented in a slightly more complicated form) has been proved in [5] (cf. the proof of Theorem 1.27 there).

are the class of the primitive recursive functions and the class of the recursive ones, respectively, the above proof can be regarded as a straightforward adaptation of the simplified reasoning given in [3]). Let us note that a replacement of the denominator by 2^i is not possible in general (cf. Remark 8).

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