Moschovakis extension of multi-represented spaces

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The Moschovakis extension of a set B is the set B^* defined in [Mo69]. Assuming without loss of generality that no element of the set B is an ordered pair, one builds B^* as the closure of $B \cup \{o\}$ under formation of ordered pairs, where o (denoted by 0 in [Mo69]) is some object which does not belong to B and also is not an ordered pair. Certain relative computability notions for functions in B^* are introduced and studied in the mentioned paper. The functions considered there are, in general, multi-valued. In the case of single-valued functions, one of these notions, namely absolute prime computability, seems to be able to cover any reasonable kind of deterministic computability by means of programs using some given functions.

In [Sk18], a certain link between computability of the above-mentioned kind and TTE computability is indicated in the case when some representation of the set B is given. An appropriate related representation of B^* is used for that purpose. The present paper generalizes to the case of multi-representations some of the results from [Sk18] and extends them. Two kinds of TTE computability are considered. The first of them is the usual computability via realizations, with the restriction that only single-valued realizations may be used. The other computability we consider is a Brattka style one – it is in the sense of [Br03, Definition 7.1], appropriately generalized for the case of multi-representations.[1] We establish some statements which are in the same vein as [Br96, Theorem 31], [We00, Theorems 3.1.6 and 3.1.7], [Br03, Theorem 8.3] and the results in [We08]. For single-valued functions, we prove the TTE computability of any function which is absolutely prime computable in some TTE computable functions. A similar result holds for multi-valued functions, but with an analog of absolute prime computability.

Let γ be a multi-representation of the set B. We construct a multi-representation γ^* of the set B^* with the following properties, where the mentioned computability is the usual one:

- the identity function of B is both (γ, γ^*)- and (γ^*, γ)-computable;
- the ordered pair operation in B^* is $(\gamma^*, \gamma^*, \gamma^*)$-computable;
- the two unary partial functions in B^* which transform ordered pairs into their first and their second components are (γ^*, γ^*)-computable;
- the element o is γ^*-computable, and so is the mapping of B^* into $\mathbb{N}$ which maps o, the elements of B and all other elements of B^* into 0, 1 and 2, respectively.

We prove that any two multi-representations of the set B^* with these properties are equivalent.

When considering functions in B^*, it is not an essential restriction to confine oneself to unary ones. Let $\mathcal{F}$ be the set of all unary partial multi-valued functions in B^* (the set of the

[1] The corresponding definition reads as follows: if γ and γ' are multi-representations of X and X', respectively, and φ is a partial multi-valued function from X to X' then φ is said to be *Brattka style* (γ, γ')*-computable* if a computable partial mapping β of $\mathrm{dom}(\gamma) \times \mathbb{N}^{\mathbb{N}}$ into $\mathrm{dom}(\gamma')$ exists such that the following holds whenever $p \in \mathrm{dom}(\gamma)$ and $x \in \gamma(p) \cap \mathrm{dom}(\varphi)$:

- $(p, q) \in \mathrm{dom}(\beta)$ and $\gamma'(\beta(p, q)) \cap \varphi(x) \neq \varnothing$ for all $q \in \mathbb{N}^{\mathbb{N}}$;
- $\varphi(x) \subseteq \bigcup \{\gamma'(\beta(p, q)) \mid q \in \mathbb{N}^{\mathbb{N}}\}$.

single-valued ones can be regarded as a subset of $\mathcal{F}$). It is shown in [Sk92] that a function $\varphi \in \mathcal{F}$ is absolutely prime computable in some given functions $\psi_1, \ldots, \psi_l \in \mathcal{F}$ iff φ can be obtained from $\psi_1, \ldots, \psi_l$ and the functions π and δ from [Mo69] by finitely many applications of three natural operations, namely the usual composition in $\mathcal{F}$ and the following two ones: $\theta_1, \theta_2 \mapsto \lambda x.\theta_1(x) \times \theta_2(x)$ (this operation is called *combination* in [Sk92] and *juxtaposition* in [Br96, Br03, We08]) and $\theta_1, \theta_2 \mapsto \iota$, where $y \in \iota(x)$ iff a finite sequence $z_0, z_1, \ldots, z_n$ of elements of B^* exists such that $z_0 = x$, $z_n = y$,

$$z_i \in \mathrm{dom}(\theta_1) \cap \mathrm{dom}(\theta_2) \ \& \ \theta_2(z_i) \setminus (B \cup \{o\}) \neq \varnothing \ \& \ z_{i+1} \in \theta_1(z_i) \tag{1}$$

for all $i < n$, and $z_n \in \mathrm{dom}(\theta_2)$, $\theta_2(z_n) \cap (B \cup \{o\}) \neq \varnothing$ (the function ι is called *the iteration of* θ_1 *controlled by* θ_2 in [Sk92]). Making use of this characterization of absolute prime computability, we prove the (γ^*, γ^*)-computability of any function from $\mathcal{F}$ which is absolutely prime computable in some single-valued (γ^*, γ^*)-computable functions from $\mathcal{F}$.

To get a similar result for arbitrary functions in $\mathcal{F}$, we consider a new triple of operations, namely we replace the first and the last of the above three operations with certain modifications of them, the results of applying the modified operations being appropriate restrictions of the results of applying the original ones. The modified composition is one used in [Br96, We00, Br03, We08], and the modified iteration operation is $\theta_1, \theta_2 \mapsto \iota \restriction E$, where ι is the same as above, and E is the set of the elements x of B^* such that

- no infinite sequence $z_0, z_1, z_2, \ldots$ of elements of B^* exists with $z_0 = x$ and (1) holding for all i;
- $z_n \in \mathrm{dom}(\theta_2)$ & $\big(\theta_2(z_n) \setminus (B \cup \{o\}) \neq \varnothing \Rightarrow z_n \in \mathrm{dom}(\theta_1)\big)$ whenever $z_0, z_1, \ldots, z_n$ is a finite sequence of elements of B^* with $z_0 = x$ and (1) holding for all $i < n$.

It turns out that (γ^*, γ^*)-computability (including the Brattka style one) is preserved by each of the operations from the new triple.

Remark. A natural ternary operation of branching is omitted in both lists because it is expressible through the three listed operations.

The First Recursion Theorem from [Sk92] can be used to show the (γ^*, γ^*)-computability of certain least fixed points.

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