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On Some Computability Notions for Real Functions

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Abstract. A computability notion for real functions which is in Grzegorczyk's spirit is shown to be equivalent to one in the spirit of Tent and Ziegler under some general assumptions.

Keywords: Computability, Function, Operator, Uniformity, Subrecursive

1. Introduction

A widely used approach to computability of real functions is the one in Grzegorczyk's style originating from [1]. This approach uses computable transformations of infinitistic names of real numbers, as well as general quantifiers over these names. Other approaches allow avoiding the use of such names at least in some cases. An approach of this other kind is, for instance, the one of Tent and Ziegler from [6]. In the present paper, the equivalence of a certain approach in Grzegorczyk's spirit and one in the spirit of Tent and Ziegler is shown under some general assumptions.

2. Uniform Computability of a Real Function by Means of a Class of Total Operators (a Notion in Grzegorczyk's Spirit)

We will denote by $\mathcal{T}_m$ the class of all m-ary total functions in $\mathbb{N}$. The mappings of $\mathcal{T}_1^k$ into $\mathcal{T}_1$ will be called *k-ary total operators*, and the k-ary total operators for all $k \in \mathbb{N}$ will be generally called *total operators* (the adjective "total" will be omitted sometimes).

Definition 2.1 (Of naming of a real number). A triple $(f, g, h) \in \mathcal{T}_1^3$ will be said to *name* a real number ξ if

$$\left|\frac{f(n) - g(n)}{h(n) + 1} - \xi\right| < \frac{1}{n + 1}$$

for any $n \in \mathbb{N}$.

Definition 2.2 (Of the notion of a computing system for a real function). Let $N \in \mathbb{N}$ and $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^N$. We will call a *computing system* for θ any triple (F, G, H) of $3N$-ary total operators such that, whenever $(\xi_1, \ldots, \xi_N) \in D$ and the triples $(f_1, g_1, h_1), \ldots, (f_N, g_N, h_N) \in \mathcal{T}_1^3$ name $\xi_1, \ldots, \xi_N$, respectively, the $3N$-tuple $(f_1, g_1, h_1, \ldots, f_N, g_N, h_N)$ is transformed by the operators F, G, H into the components of a triple, which names $\theta(\xi_1, \ldots, \xi_N)$.

Definition 2.3 (Of a computability notion for real functions). Let $\mathbf{O}$ be a class of total operators, $N \in \mathbb{N}$ and $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^N$. The function θ will be said to be *uniformly* $\mathbf{O}$*-computable* if there exists a computing system (F, G, H) for θ such that $F, G, H \in \mathbf{O}$.

Remark. If $\mathbf{O}$ is the class of all computable total operators then the uniform $\mathbf{O}$-computability of a real function coincides with its computability in the sense of [1] (naturally extended to functions of any number of variables).

Let us note that this is not the most general effective computability of real functions. There are quite simple (not everywhere defined) real functions effectively computable in a natural more general sense but not computable in the sense of [1], since any function computable in the sense of [1] is uniformly continuous in the bounded subsets of its domain. As is well-known (cf. [2]), one should also allow using partial operators (i.e. mappings of proper subclasses of $\mathcal{T}_1^k$ into $\mathcal{T}_1$) in order to encompass such functions.[1]

3. Acceptability of a Pair of a Class of Functions in $\mathbb{N}$ and a Class of Total Operators

Definition 3.1 (Of acceptability). Let $\mathcal{F}$ be a subclass of $\bigcup_{m=1}^{\infty}\mathcal{T}_m$, and $\mathbf{O}$ be a class of total operators. The pair $(\mathcal{F},\mathbf{O})$ will be called *acceptable* if the following conditions are satisfied:

(i) The projection functions in $\mathbb{N}$, the successor function, the multiplication function, as well as the functions $\lambda xy.\,x\dot{-}y$ and $\lambda xy.\left\lfloor\frac{x}{y+1}\right\rfloor$ belong to $\mathcal{F}$.

(ii) The set $\mathcal{F}$ is closed under substitution.

(iii) For any $k\in\mathbb{N}$, the k-ary operator F defined by $F(f_1,\ldots,f_k)=\mathrm{id}_{\mathbb{N}}$ belongs to $\mathbf{O}$.

(iv) Whenever $i\in\{1,\ldots,k\}$ and F_0 is a k-ary operator belonging to $\mathbf{O}$, the operator F defined by

$$F(f_1,\ldots,f_k)(n)=f_i(F_0(f_1,\ldots,f_k)(n))$$

also belongs to $\mathbf{O}$.

(v) Whenever $m\in\mathbb{N}, f\in\mathcal{T}_m\cap\mathcal{F}$, and $F_1,\ldots,F_m$ are k-ary operators belonging to $\mathbf{O}$, the operator F defined by

$$F(f_1,\ldots,f_k)(n)=f(F_1(f_1,\ldots,f_k)(n),\ldots,F_m(f_1,\ldots,f_k)(n))$$

also belongs to $\mathbf{O}$.

(vi) Whenever $f_1,\ldots,f_k\in\mathcal{T}_{l+1}\cap\mathcal{F}$ and F is a k-ary operator belonging to $\mathbf{O}$, the function

$$\lambda s_1\ldots s_l n.F(\lambda t.f_1(t,s_1,\ldots,s_l),\ldots,\lambda t.f_k(t,s_1,\ldots,s_l))(n) \tag{1}$$

belongs to $\mathcal{F}$.

(vii) For any positive integer k and any k-ary operator $F\in\mathbf{O}$, there exists a unary operator $F^{\nabla}\in\mathbf{O}$ such that $F(f_1,\ldots,f_k)(e)=F(f_1',\ldots,f_k')(e)$, whenever $f_1,\ldots,f_k,f_1',\ldots,f_k'\in\mathcal{T}_1$, $e\in\mathbb{N}$, g is a monotonically increasing function from $\mathcal{T}_1$ dominating $f_1,\ldots,f_k,f_1',\ldots,f_k'$, and $f_1(t)=f_1'(t),\ \ldots,\ f_k(t)=f_k'(t)$ for any natural number $t\le F^{\nabla}(g)(e)$.

Grzegorczyk's paper [1] suggests the following example of an acceptable pair $(\mathcal{F},\mathbf{O})$: the class $\mathcal{F}$ consists of all computable functions from $\bigcup_{m=1}^{\infty}\mathcal{T}_m$, and the class $\mathbf{O}$ consists of all computable total operators. In particular, the conditions (vi) and (vii) are satisfied in this case due to Property 8 in § 2 of [1] and to the Uniformity Theorem proved in that paper.[2]

Two other examples of acceptable pairs $(\mathcal{F},\mathbf{O})$ can be obtained by using the subrecursive versions of Grzegorczyk's Uniformity Theorem proved in [3]. In the first of them, $\mathcal{F}$ is the class of all primitive recursive functions from $\bigcup_{m=1}^{\infty}\mathcal{T}_m$, and $\mathbf{O}$ is the class of all primitive recursive operators (i.e. the class of all total operators F such that $\lambda f_1\ldots f_k n.F(f_1,\ldots,f_k)(n)$, where k is the arity of F, is a primitive recursive functional). In the second one, $\mathcal{F}$ is the

[1] In the case of N-argument real functions, the wider computability notion obtained this way is equivalent to (ρ^N,ρ)-computability in the sense of [7], hence this wider notion can be regarded as the most general natural computability notion for such functions.

[2] The property and the theorem in question concern computable functionals, but there is a straightforward reduction of the notion of computable total operator to the notion of computable functional. Namely, a k-ary total operator F is computable if and only if $\lambda f_1\ldots f_k n.F(f_1,\ldots,f_k)(n)$ is a computable functional in the sense of [1].

class of all functions from $\bigcup_{m=1}^{\infty}\mathcal{T}_m$ which are elementary in Kalmár's sense, and $\mathbf{O}$ is the class of all elementary recursive operators (i.e. the class of all total operators F such that $\lambda f_1 \ldots f_k n.F(f_1,\ldots,f_k)(n)$, where k is the arity of F, is an elementary recursive functional). In both cases, condition (vi) can be shown to be satisfied by inductively proving that

$$\lambda s_1 \ldots s_l n_1 \ldots n_m.\Phi(\lambda t.f_1(t,s_1,\ldots,s_l),\ldots,\lambda t.f_k(t,s_1,\ldots,s_l),n_1,\ldots,n_m) \in \mathcal{F},$$

whenever Φ is a functional of the corresponding kind with k function arguments and m number arguments, and $f_1,\ldots,f_k \in \mathcal{T}_{l+1} \cap \mathcal{F}$.

As mentioned in [3], similar versions of the Uniformity Theorem can be proved for many other subrecursive classes of functionals. Examples of acceptable pairs can also be obtained in a natural way from such versions – for instance, from the version for lower elementary functionals.[3] The next theorem yields a family of somewhat different examples which have the additional feature that the second term of the pair is the least possible for the first one. The theorem makes use of the notion of $\mathcal{F}$-substitutional operator which was defined in [5]. By its definition, whenever $\mathcal{F} \subseteq \bigcup_{m=1}^{\infty}\mathcal{T}_m$, the $\mathcal{F}$-substitutional operators form the least class of total operators satisfying conditions (iii)–(v) of Definition 3.1. We will denote the class of the $\mathcal{F}$-substitutional operators by $\mathbf{O}_{\mathcal{F}}$.

Theorem 3.2. *Let $\mathcal{F}$ be a subclass of $\bigcup_{m=1}^{\infty}\mathcal{T}_m$ satisfying conditions (i) and (ii) of Definition 3.1, as well as the following condition:*

(a) Whenever $f \in \mathcal{T}_m \cap \mathcal{F}$, there exists a function from $\mathcal{T}_m \cap \mathcal{F}$, which dominates f and is monotonically increasing with respect to any of its arguments.

Then $(\mathcal{F},\mathbf{O}_{\mathcal{F}})$ is an acceptable pair.

Proof. Let $\mathbf{O} = \mathbf{O}_{\mathcal{F}}$. Then, of course, the conditions (i)–(v) of Definition 3.1 are satisfied. The remaining two conditions (vi), (vii) will be shown to hold by induction on the construction of the $\mathcal{F}$-substitutional operator F (i.e. by using the minimality of $\mathbf{O}_{\mathcal{F}}$ among the classes satisfying (iii)–(v)).

For the case of (vi), suppose $f_1,\ldots,f_k$ are arbitrary functions from $\mathcal{T}_{l+1} \cap \mathcal{F}$. If F has the form from condition (iii) then

$$F(\lambda t.f_1(t,s_1,\ldots,s_l),\ldots,\lambda t.f_k(t,s_1,\ldots,s_l))(n) = n$$

for all $s_1,\ldots,s_l,n \in \mathbb{N}$, hence (1) is a projection function in $\mathbb{N}$, and therefore belongs to $\mathcal{F}$. If F has the form from condition (iv), and the function

$$\lambda s_1 \ldots s_l n.F_0(\lambda t.f_1(t,s_1,\ldots,s_l),\ldots,\lambda t.f_k(t,s_1,\ldots,s_l))(n)$$

belongs to $\mathcal{F}$ then the function (1) also belongs to $\mathcal{F}$, since the value of this function at an arbitrary $(l+1)$-tuple $(s_1,\ldots,s_l,n)$ of natural numbers equals

$$f_i(F_0(\lambda t.f_1(t,s_1,\ldots,s_l),\ldots,\lambda t.f_k(t,s_1,\ldots,s_l))(n),s_1,\ldots,s_l).$$

Suppose now F has the form from condition (v), and the functions

$$\lambda s_1 \ldots s_l n.F_i(\lambda t.f_1(t,s_1,\ldots,s_l),\ldots,\lambda t.f_k(t,s_1,\ldots,s_l))(n),\ i = 1,\ldots,m,$$

belong to $\mathcal{F}$. Then (1) is the result of the substitution of these functions into f, hence (1) again belongs to $\mathcal{F}$.

Before going to the case of (vii), we will first prove the following auxiliary statement: for any k-ary $\mathcal{F}$-substitutional operator F, there exists a unary $\mathcal{F}$-substitutional operator $F^{\triangle}$ such that $F(f_1,\ldots,f_k)$ is dominated

[3] The notion of lower elementary functional can be defined by means of the definition used in [3] for the notion of elementary recursive functional, but with omitted bounded multiplication.

by $F^{\triangle}(g)$, whenever $f_1,\dots,f_k \in \mathcal{T}_1$ and g is a monotonically increasing function from $\mathcal{T}_1$ dominating $f_1,\dots,f_k$. We will proceed by induction on the construction of F. If F has the form from condition (iii) then we may set

$$F^{\triangle}(g) = \mathrm{id}_{\mathbb{N}}.$$

If F has the form from condition (iv), and $F_0^{\triangle}$ is such that $F_0^{\triangle}(g)$ dominates $F_0(f_1,\dots,f_k)$, whenever g is a monotonically increasing function from $\mathcal{T}_1$ dominating $f_1,\dots,f_k$, we may define $F^{\triangle}$ by setting

$$F^{\triangle}(g)(n) = g(F_0^{\triangle}(g)(n)).$$

Suppose now F has the form from condition (v), and the $\mathcal{F}$-substitutional unary operators $F_1^{\triangle},\dots,F_m^{\triangle}$ are such that, for $i=1,\dots,m$, $F_i^{\triangle}(g)$ dominates $F_i(f_1,\dots,f_k)$, whenever g is a monotonically increasing function from $\mathcal{T}_1$ dominating $f_1,\dots,f_k$. Then we may define $F^{\triangle}$ by setting

$$F^{\triangle}(g)(n) = \tilde{f}(F_1^{\triangle}(g)(n),\dots,F_m^{\triangle}(g)(n)),$$

where $\tilde{f}$ is a function from $\mathcal{T}_m \cap \mathcal{F}$ dominating f and monotonically increasing with respect to any of its arguments.

We mention also the equalities

$$0 = n \dot{-} n,\ n_1+n_2 = (n_1+1)(n_2+1) \dot{-} (n_1n_2+1),\ \max(n_1,n_2) = n_1 + (n_2 \dot{-} n_1),$$
$$\max(n_1,\dots,n_l,n_{l+1}) = \max(\max(n_1,\dots,n_l),n_{l+1}).$$

By these equalities and properties (i) and (ii) of $\mathcal{F}$, the constant 0, the addition function and the maximum function of any number of arguments belong to $\mathcal{F}$.

Now the statement of condition (vii) will be proved as follows. If F has the form from condition (iii) then we may set

$$F^{\nabla}(g)(e) = 0.$$

Let F have the form from condition (iv), and the $\mathcal{F}$-substitutional unary operator F_0^{∇} be such that

$$F_0(f_1,\dots,f_k)(e) = F_0(f_1',\dots,f_k')(e),$$

whenever $f_1,\dots,f_k,\ f_1',\dots,f_k' \in \mathcal{T}_1$, $e \in \mathbb{N}$, g is a monotonically increasing function from $\mathcal{T}_1$ dominating $f_1,\dots,f_k,f_1',\dots,f_k'$, and $f_1(t)=f_1'(t),\ \dots,\ f_k(t)=f_k'(t)$ for any natural number $t \le F_0^{\nabla}(g)(e)$. Then we may define F^{∇} by setting

$$F^{\nabla}(g)(e) = \max(F_0^{\nabla}(g)(e), F_0^{\triangle}(g)(e)),$$

where $F_0^{\triangle}$ is the $\mathcal{F}$-substitutional unary operator corresponding to F_0 according to the auxiliary statement. Finally, let F have the form from condition (v), and the $\mathcal{F}$-substitutional unary operators $F_1^{\nabla},\dots,F_m^{\nabla}$ be such that, for $i=1,\dots,m$, $F_i(f_1,\dots,f_k)(e) = F_i(f_1',\dots,f_k')(e)$, whenever $f_1,\dots,f_k,f_1',\dots,f_k' \in \mathcal{T}_1$, $e \in \mathbb{N}$, g is a monotonically increasing function from $\mathcal{T}_1$ dominating $f_1,\dots,f_k,f_1',\dots,f_k'$, and $f_1(t)=f_1'(t),\ \dots,\ f_k(t)=f_k'(t)$ for any natural number $t \le F_i^{\nabla}(g)(e)$. Then we may define F^{∇} by setting

$$F^{\nabla}(g)(e) = \max(F_1^{\nabla}(g)(e),\dots,F_m^{\nabla}(g)(e)). \qquad \square$$

In the examples of acceptable pairs $(\mathcal{F},\mathbf{O})$ indicated above except for the ones whose construction makes use of Theorem 3.2, the class $\mathbf{O}$ is larger than the class $\mathbf{O}_{\mathcal{F}}$. This can be seen, for instance, by inductively proving the following statement: for any k-ary $\mathcal{F}$-substitutional operator F, there exists a natural number j (the number of the occurrences of the symbols $f_1,\dots,f_k$ in the expression for $F(f_1,\dots,f_k)(n)$) with the property that, for any $f_1,\dots,f_k \in \mathcal{T}_1$

and any $n \in \mathbb{N}$, there exists a set A of at most j natural numbers such that $F(f_1, \dots, f_k)(n) = F(f'_1, \dots, f'_k)(n)$, whenever $f'_1, \dots, f'_k \in \mathcal{T}_1$ and $f_1(t) = f'_1(t), \dots, f_k(t) = f'_k(t)$ for all $t \in A$. On the other hand, the class $\mathcal{F}$ in any of these examples satisfies the condition (a) from Theorem 3.2, hence $(\mathcal{F}, \mathbf{O}_{\mathcal{F}})$ is also an acceptable pair. Therefore the class $\mathcal{F}$ from any of these examples is the first term of at least two different acceptable pairs.

Remark. It is not possible for two different acceptable pairs to have one and the same second term. This follows from the fact that, whenever $(\mathcal{F}, \mathbf{O})$ is an acceptable pair, k is a positive integer and $f \in \mathcal{T}_k$, the function f belongs to $\mathcal{F}$ if and only if there exists a k-ary operator F from $\mathbf{O}$ such that

$$F(\lambda t.n_1, \dots, \lambda t.n_k) = \lambda t.f(n_1, \dots, n_k)$$

for any $n_1, \dots, n_k \in \mathbb{N}$.

4. A Characterization Theorem

The next definition introduces a notion of computability of real functions which is in the spirit of the notion of a real function uniformly in $\mathcal{F}$ introduced by Tent and Ziegler in [6].

Definition 4.1 (Of TZ-style uniform computability). Let $\mathcal{F}$ be a subclass of $\bigcup_{m=1}^{\infty} \mathcal{T}_m$, and let $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^N$. The function θ will be called *TZ-style uniformly $\mathcal{F}$-computable* if there exist $d \in \mathcal{T}_1 \cap \mathcal{F}$ and $f, g, h \in \mathcal{T}_{3N+1} \cap \mathcal{F}$ such that, whenever $(\xi_1, \dots, \xi_N) \in D$, $p_1, q_1, r_1, \dots, p_N, q_N, r_N, e \in \mathbb{N}$ and

$$|\xi_i| \leq e + 1, \quad \left| \frac{p_i - q_i}{r_i + 1} - \xi_i \right| < \frac{1}{d(e) + 1} \quad (i = 1, \dots, N), \tag{2}$$

the numbers

$$p = f(p_1, q_1, r_1, \dots, p_N, q_N, r_N, e), \quad q = g(p_1, q_1, r_1, \dots, p_N, q_N, r_N, e), \tag{3}$$

$$r = h(p_1, q_1, r_1, \dots, p_N, q_N, r_N, e) \tag{4}$$

satisfy the inequality

$$\left| \frac{p - q}{r + 1} - \theta(\xi_1, \dots, \xi_N) \right| < \frac{1}{e + 1}. \tag{5}$$

Remark. Under the assumption that $\mathcal{F}$ is a good class in the sense of [6] and $\theta : D \to \mathbb{R}$, where D is an open subset of $\mathbb{R}^N$, the function θ is TZ-style uniformly $\mathcal{F}$-computable if and only if θ is uniformly in $\mathcal{F}$.

Theorem 4.2 (Characterization Theorem). *Let $(\mathcal{F}, \mathbf{O})$ be an acceptable pair, and let $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^N$. The function θ is uniformly $\mathbf{O}$-computable if and only if θ is TZ-style uniformly $\mathcal{F}$-computable.*

Proof. As in the proof of Theorem 3.2, we see that the constant 0, the addition function and the maximum function of any number of arguments belong to $\mathcal{F}$. For the "if"-part, let us suppose that d, f, g, h are functions satisfying the condition from Definition 4.1. We define $3N$-ary total operators F, G, H by setting

$$F(f_1, g_1, h_1, \dots, f_N, g_N, h_N)(n) = p, \quad G(f_1, g_1, h_1, \dots, f_N, g_N, h_N)(n) = q,$$
$$H(f_1, g_1, h_1, \dots, f_N, g_N, h_N)(n) = r,$$

where the numbers p, q, r are defined by means of the equalities (3–4) with

$$e = \max(f_1(0), g_1(0), \dots, f_N(0), g_N(0), n),$$
$$p_i = f_i(d(e)), \quad q_i = g_i(d(e)), \quad r_i = h_i(d(e)) \quad \text{for } i = 1, \dots, N.$$

By conditions (i)–(v) from Definition 3.1, the operators F, G, H belong to $\mathbf{O}$. Clearly, if an element $(\xi_1, \ldots, \xi_N)$ of D and functions $f_1, g_1, h_1, \ldots, f_N, g_N, h_N$ from $\mathcal{T}_1$ are given such that (f_i, g_i, h_i) names ξ_i for $i = 1, \ldots, N$, then, for any $n \in \mathbb{N}$, the above numbers $p_1, q_1, r_1, \ldots, p_N, q_N, r_N, e$ will satisfy the inequalities (2) and the inequality $e \geq n$, hence the corresponding numbers p, q, r will satisfy the inequality

$$\left|\frac{p-q}{r+1} - \theta(\xi_1, \ldots, \xi_N)\right| < \frac{1}{n+1}\,.$$

For the proof of the "only if"-part, suppose (F, G, H) is a computing system for θ such that $F, G, H \in \mathbf{O}$. Let $F^{\nabla}, G^{\nabla}, H^{\nabla}$ be unary operators from $\mathbf{O}$ related to F, G, H, respectively, in the way from condition (vii). Let the function $v \in \mathcal{T}_2$ be defined as follows:

$$v(s, e) = \max(F^{\nabla}(\lambda t.u(t, s))(e), G^{\nabla}(\lambda t.u(t, s))(e), H^{\nabla}(\lambda t.u(t, s))(e)),$$

where $u(t, s) = (t+1)(s+2)$. By applying conditions (i), (ii) and (vi), we see that $v \in \mathcal{F}$. Clearly, whenever $f_1, g_1, h_1, \ldots, f_N, g_N, h_N, f'_1, g'_1, h'_1, \ldots, f'_N, g'_N, h'_N$ are functions from $\mathcal{T}_1$ dominated by a function of the form $\lambda t.(t+1)(s+2)$, and for a certain natural number e the functions $f_1, g_1, h_1, \ldots, f_N, g_N, h_N$ coincide, respectively, with the functions $f'_1, g'_1, h'_1, \ldots, f'_N, g'_N, h'_N$ at the numbers not exceeding $v(s, e)$, each of the operators F, G, H transforms the $3N$-tuples $(f_1, g_1, h_1, \ldots, f_N, g_N, h_N)$ and $(f'_1, g'_1, h'_1, \ldots, f'_N, g'_N, h'_N)$ into two functions coinciding at the number e. To define the functions d, f, g, h, we set $d(e) = 2v(e, e) + 1$, and we take as values of the functions f, g, h at the $(3N+1)$-tuple $(p_1, q_1, r_1, \ldots, p_N, q_N, r_N, e)$ the values at e, respectively, of the results of applying the operators F, G, H to the $3N$-tuple $(f_1, g_1, \mathrm{id}_{\mathbb{N}}, \ldots, f_N, g_N, \mathrm{id}_{\mathbb{N}})$, where

$$f_i = \lambda t.\left\lfloor (t+1)\frac{p_i \dot{-} q_i}{r_i+1} + \frac{1}{2}\right\rfloor, \quad g_i = \lambda t.\left\lfloor (t+1)\frac{q_i \dot{-} p_i}{r_i+1} + \frac{1}{2}\right\rfloor$$

for $i = 1, \ldots, N$. The functions d, f, g, h belong to $\mathcal{F}$ due to conditions (i), (ii) and (vi). Suppose now $(\xi_1, \ldots, \xi_N) \in D$, $p_1, q_1, r_1, \ldots, p_N, q_N, r_N, e \in \mathbb{N}$, and the inequalities (2) hold. Let the numbers p, q, r be defined by the equalities (3–4). Besides the functions $f_1, g_1, \ldots, f_N, g_N$ defined above, let us consider functions $f'_1, g'_1, \ldots, f'_N, g'_N \in \mathcal{T}_1$ which coincide, respectively, with them at the numbers not exceeding $v(e, e)$ and which satisfy for all $t > v(e, e)$ the conditions

$$\left|\frac{f'_i(t) - g'_i(t)}{t+1} - \xi_i\right| < \frac{1}{t+1}\,,\ f'_i(t) = 0 \vee g'_i(t) = 0. \tag{6}$$

Making use of (2) and of the fact that always

$$\left|\frac{f_i(t) - g_i(t)}{t+1} - \frac{p_i - q_i}{r_i+1}\right| \leq \frac{1}{2(t+1)}\,,\ f_i(t) = 0 \vee g_i(t) = 0\,, \tag{7}$$

one sees that conditions (6) will be satisfied also in the case of $t \leq v(e, e)$. The validity of (6) and (7) for all $t \in \mathbb{N}$ shows that the triple $(f'_i, g'_i, \mathrm{id}_{\mathbb{N}})$ names ξ_i for $i = 1, \ldots, N$, and allows concluding that $f_i, g_i, f'_i, g'_i, \mathrm{id}_{\mathbb{N}}$ are dominated by $\lambda t.(t+1)(e+2)$. Hence the values at e of the results of applying the operators F, G, H to the $3N$-tuple $(f'_1, g'_1, \mathrm{id}_{\mathbb{N}}, \ldots, f'_N, g'_N, \mathrm{id}_{\mathbb{N}})$ are equal, respectively, to the numbers p, q, r, and therefore these numbers satisfy the inequality (5). □

Corollary 4.3. *A real function is computable in the sense of [1] if and only if it is TZ-style uniformly $\mathcal{C}$-computable, where $\mathcal{C}$ is the class of all computable functions from $\bigcup_{m=1}^{\infty}\mathcal{T}_m$. Thus a real function with an open domain is computable in the sense of [1] if and only if it is uniformly in $\mathcal{C}$.*

Corollary 4.4 (The Characterization Theorem from [4]). *Under the assumptions of Theorem 3.2, a real function is uniformly $\mathbf{O}_{\mathcal{F}}$-computable if and only if it is TZ-style uniformly $\mathcal{F}$-computable.*

Corollary 4.5. *Under the assumptions of Theorem 4.2, if $\mathcal{F}$ satisfies the condition (a) of Theorem 3.2 then the following three conditions are equivalent:*

1. *θ is uniformly $\mathbf{O}$-computable;*
2. *θ is uniformly $\mathbf{O}_{\mathcal{F}}$-computable;*
3. *θ is TZ-style uniformly $\mathcal{F}$-computable.*

Let us mention that the uniformly $\mathbf{O}_{\mathcal{F}}$-computable real functions are actually the uniformly $\mathcal{F}$-computable functions considered in [5].

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