Uniform computability of real functions

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Some aspects of certain notions introduced in the paper [1] are studied more systematically in the present one.

1 $\mathcal{F}$-substitutional operators

We will denote by $\mathbb{T}_m$ the set of the m-ary total functions in $\mathbb{N}$. The mappings of $\mathbb{T}_1^k$ into $\mathbb{T}_1$ will be called *k-ary operators*. Let $\mathcal{F} \subseteq \bigcup_{m\in\mathbb{N}} \mathbb{T}_m$.

Definition 1 (of the notion of $\mathcal{F}$-substitutional k-ary operator)

1. *The operator F defined by $F(f_1,\dots,f_k)(n) = n$ is $\mathcal{F}$-substitutional.*

2. *For any $i \in \{1,\dots,k\}$, if F_0 is a $\mathcal{F}$-substitutional k-ary operator, then so is the operator F defined by $F(f_1,\dots,f_k)(n) = f_i(F_0(f_1,\dots,f_k)(n))$.*

3. *For any $m \in \mathbb{N}$ and any $f \in \mathbb{T}_m \cap \mathcal{F}$, if $F_1,\dots,F_m$ are $\mathcal{F}$-substituitional k-ary operators, then so is the operator F defined by*
$$F(f_1,\dots,f_k)(n) = f(F_1(f_1,\dots,f_k)(n),\dots,F_m(f_1,\dots,f_k)(n)).$$

The next five lemmas can be proved by induction on the construction of the operator F.

Lemma 1 (about composition of the obtained functions) *If F and G are $\mathcal{F}$-substitutional k-ary operators, then so is the operator H, defined by*
$$H(f_1,\dots,f_k)(n) = F(f_1,\dots,f_k)(G(f_1,\dots,f_k)(n)).$$

Lemma 2 (about substitution in $\mathcal{F}$-substitutional operators) *Let F be an $\mathcal{F}$-substitutional k-ary operator. If $l \in \mathbb{N}$ and $G_1,\dots,G_k$ are $\mathcal{F}$-substitutional l-ary operators, then so is the operator H, defined by*
$$H(g_1,\dots,g_l) = F(G_1(g_1,\dots,g_l),\dots,G_k(g_1,\dots,g_l)).$$

Lemma 3 (about application to functions from $\mathcal{F}$) *Let the class $\mathcal{F}$ contain the projection functions in $\mathbb{N}$ and be closed under substitution, and F be an $\mathcal{F}$-substitutional k-ary operator. If $m \in \mathbb{N}$ and $f_1,\dots,f_k \in \mathbb{T}_{m+1} \cap \mathcal{F}$, then*
$$\lambda s_1 \dots s_m n.F(\lambda t.f_1(s_1,\dots,s_m,t),\dots,\lambda t.f_k(s_1,\dots,s_m,t))(n) \in \mathcal{F}.$$
In particular, if $f_1,\dots,f_k \in \mathbb{T}_1 \cap \mathcal{F}$, then $F(f_1,\dots,f_k) \in \mathcal{F}$.

Lemma 4 (Domination Lemma) *Let the following conditions be satisfied:*

1. *The class $\mathcal{F}$ contains the projection functions in $\mathbb{N}$ and is closed under substitution.*
2. *For any $m \in \mathbb{N}$ and any function $f \in \mathbb{T}_m \cap \mathcal{F}$ there exists a function from $\mathbb{T}_m \cap \mathcal{F}$, which dominates f and is monotonically increasing with respect to any of its arguments.*

Let F be an $\mathcal{F}$-substitutional k-ary operator and a function $g \in \mathbb{T}_2 \cap \mathcal{F}$ be given. Then there exists a function $h \in \mathbb{T}_2 \cap \mathcal{F}$ such that $F(f_1, \ldots, f_k)$ is dominated by $\lambda n.h(a,n)$, whenever $a \in \mathbb{N}$ and $f_1, \ldots, f_k \in \mathbb{T}_1$ are dominated by $\lambda n.g(a,n)$.

Lemma 5 (Uniformity Lemma) *Let the conditions 1 and 2 of the Domination Lemma be satisfied, and let there exists a function $j \in \mathbb{T}_2 \cap \mathcal{F}$ such that $j(x,y) \geq x$ and $j(x,y) \geq y$ for all $x, y \in \mathbb{N}$. Let F be an $\mathcal{F}$-substitutional k-ary operator, and a function $g \in \mathbb{T}_2 \cap \mathcal{F}$ be given. Then there exists a function $u \in \mathbb{T}_2 \cap \mathcal{F}$ such that*

$$F(f_1, \ldots, f_k)(e) = F(f'_1, \ldots, f'_k)(e),$$

whenever $f_1, \ldots, f_k, f'_1, \ldots, f'_k \in \mathbb{T}_1$, $e \in \mathbb{N}$ and for some $a \in \mathbb{N}$ any of the functions f_i and f'_i, $i = 1, \ldots, k$, is dominated by the function $\lambda n.g(a,n)$, and $f_1(n) = f'_1(n)$, ..., $f_k(n) = f'_k(n)$ for any natural number $n \leq u(a,e)$.

Obviously if $\mathcal{F} \subseteq \mathcal{F}' \subseteq \bigcup_{m \in \mathbb{N}} \mathbb{T}_m$, then all $\mathcal{F}$-substitutional operators are $\mathcal{F}'$-substitutional. The next lemma indicates an important case when both classes of operators coincide.

Lemma 6 (about the saturation of $\mathcal{F}$) *Let $\mathcal{F}'$ be the least class of total functions in $\mathbb{N}$ which is closed under substitution and contains the functions from $\mathcal{F}$ and the projection functions in $\mathbb{N}$. Then any $\mathcal{F}'$-substitutional operator is $\mathcal{F}$-substitutional.*

The proof of this lemma is based on the fact that the replacement of $\mathbb{T}_m \cap \mathcal{F}$ with $\mathbb{T}_m \cap \mathcal{F}'$ in the third clause of Definition 1 produces a correct statement. The lemma shows that no essential loss of generality would arise if the class $\mathcal{F}$ was required from the very beginning to contain the projection functions in $\mathbb{N}$ and to be closed under substitution.

2 Uniform $\mathcal{F}$-computability of real functions

Definition 2 (of naming of a real number) *For any $f, g, h \in \mathbb{T}_1$, we define a function $\langle f, g, h \rangle : \mathbb{N} \to \mathbb{Q}$ by setting*

$$\langle f, g, h \rangle(n) = \frac{f(n) - g(n)}{h(n) + 1}.$$

A triple $(f, g, h) \in \mathbb{T}_1^3$ will be called to name *a real number ξ if*

$$|\langle f, g, h \rangle(n) - \xi| < \frac{1}{n+1}$$

for any $n \in \mathbb{N}$.

Remark. Let the ternary operator K be defined as follows:

$$K(f,g,h)(n) = \left\lfloor (n+1)\frac{f(2n+1) \dot{-} g(2n+1)}{h(2n+1)+1} + \frac{1}{2} \right\rfloor.$$

Then, for any $f, g, h \in \mathbb{T}_1$ and $n \in \mathbb{N}$ some of the numbers $K(f,g,h)(n)$ and $K(g,f,h)(n)$ is 0 and the inequality

$$|\langle K(f,g,h), K(g,f,h), \mathrm{id}_{\mathbb{N}}\rangle(n) - \langle f,g,h\rangle(2n+1)| \le \frac{1}{2(n+1)}$$

holds. Thanks to this inequality, if the triple (f,g,h) names a real number then this number is named also by the triple $(K(f,g,h), K(g,f,h), \mathrm{id}_{\mathbb{N}})$.

Definition 3 (of the notion of computing system for a real function) *Let $N \in \mathbb{N}$ and $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^N$. We will call a* computing system *for θ any triple (F,G,H) of $3N$-ary operators such that, whenever $(\xi_1,\dots,\xi_N) \in D$ and the triples $(f_1,g_1,h_1),\dots,(f_N,g_N,h_N) \in \mathbb{T}_1^3$ name $\xi_1,\dots,\xi_N$, respectively, the operators F,G,H transform the $3N$-tuple $(f_1,g_1,h_1,\dots,f_N,g_N,h_N)$ into the components of a triple, which names $\theta(\xi_1,\dots,\xi_N)$.*

Definition 4 (of uniform $\mathcal{F}$-computability of a real function) *A triple of operators will be called* $\mathcal{F}$-substitutional *if its components are $\mathcal{F}$-substitutional. A function $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^N$, will be called* uniformly $\mathcal{F}$-computable *if there exists an $\mathcal{F}$-substitutional computing system for θ.*

Example 1. The projection functions in $\mathbb{R}$ and the function $\lambda\xi.-\xi$ are uniformly $\varnothing$-computable. For any $N \in \mathbb{N}$, the constant 0, regarded as a function from $\mathbb{R}^N$ to $\mathbb{R}$, is also uniformly $\varnothing$-computable.

Example 2. Let the class $\mathcal{F}$ be closed under substitution and contain the successor and the addition functions in $\mathbb{N}$. Let $k \in \mathbb{Q}$. Then the function $\lambda\xi.k\xi$ in $\mathbb{R}$ is uniformly $\mathcal{F}$-computable.

Example 3. Let the class $\mathcal{F}$ satisfy the assumptions of Example 2, and let $\mathcal{F}$ contain also the multiplication function in $\mathbb{N}$. Then the functions $\lambda\xi_1\xi_2.\xi_1+\xi_2$, $\lambda\xi_1\xi_2.\xi_1-\xi_2$ and $\lambda\xi_1\xi_2.\xi_1\xi_2$ in $\mathbb{R}$ are uniformly $\mathcal{F}$-computable.

Example 4. As seen from [1], if a class $\mathcal{F}$ satisfying the assumptions of Example 3 contains also the function $\lambda xy.x \dot{-} y$ and is closed under the bounded least number operation, then the functions $\lambda\xi.\arctan\xi$, $\lambda\xi.\arcsin\xi$, $\lambda\xi.\arccos\xi$, $\lambda\xi.\sin\xi$ and $\lambda\xi.\cos\xi$ are uniformly $\mathcal{F}$-computable.

Clearly the restriction of functions preserves uniform $\mathcal{F}$-computability.

Theorem 1 (about substitution in $\mathcal{F}$-computable functions) *Let the partial N-ary function θ in $\mathbb{R}$ be defined by an equality of the form*

$$\theta(\xi_1,\dots,\xi_N) = \theta_0(\theta_1(\xi_1,\dots,\xi_N),\dots,\theta_M(\xi_1,\dots,\xi_N)),$$

where $\theta_0,\theta_1,\dots,\theta_M$ are uniformly $\mathcal{F}$-computable partial functions in $\mathbb{R}$ with the appropriate number of arguments. Then θ is also uniformly $\mathcal{F}$-computable.

The proof of this theorem is by applying the lemma about substitution in $\mathcal{F}$-substitutional operators.

Theorem 2 (Local Boundedness Theorem) *Let the class $\mathcal{F}$ satisfy the conditions 1 and 2 of the Domination Lemma and contain the successor and the multiplication functions in $\mathbb{N}$. Let $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^N$, be uniformly $\mathcal{F}$-computable. Then there exists a function $b \in \mathbb{T}_1 \cap \mathcal{F}$ such that, whenever $a \in \mathbb{N}$, $(\xi_1, \ldots, \xi_N) \in D$ and $\max(|\xi_1|, \ldots, |\xi_N|) \leq a$, the inequality $|\theta(\xi_1, \ldots, \xi_N)| \leq b(a)$ holds.*

Proof. Let (F, G, H) be an $\mathcal{F}$-substitutional computing system for θ. By applying the Domination Lemma, we can find a function $h \in \mathbb{T}_2 \cap \mathcal{F}$ such that the functions $F(f_1, g_1, h_1, \ldots, f_N, g_N, h_N)$ and $G(f_1, g_1, h_1, \ldots, f_N, g_N, h_N)$ are dominated by $\lambda n.h(a, n)$, whenever $a \in \mathbb{N}$ and $f_1, g_1, h_1, \ldots, f_N, g_N, h_N \in \mathbb{T}_1$ are dominated by $\lambda n.a(n+1)+1$. We set $b(s) = h(s, s)+1$. Let $a \in \mathbb{N}$, $(\xi_1, \ldots, \xi_N) \in D$ and $\max(|\xi_1|, \ldots, |\xi_N|) \leq a$. Then there exist $f_1, g_1, h_1, \ldots, f_N, g_N, h_N \in \mathbb{T}_1$ dominated by $\lambda n.a(n + 1) + 1$ such that (f_i, g_i, h_i) names ξ_i for $i = 1, \ldots, N$, and therefore $|\theta(\xi_1, \ldots, \xi_N)|$ is less than

$$\max(F(f_1, g_1, h_1, \ldots, f_N, g_N, h_N)(a), G(f_1, g_1, h_1, \ldots, f_N, g_N, h_N)(a)) + 1,$$

hence $|\theta(\xi_1, \ldots, \xi_N)| < b(a)$. □

The next theorem gives a characterization of the uniformly $\mathcal{F}$-computable functions which is in the spirit of the definition in [2] of the notion of a real function uniformly in $\mathcal{F}$. It follows from the theorem that the uniformly $\mathcal{F}$-computable functions coincide with the ones uniformly in $\mathcal{F}$ in the case considered in [2].

Theorem 3 (Characterization Theorem) *Let $\mathcal{F}$ satisfy the assumptions of the Local Boundedness Theorem and contain also the functions $\lambda xy.x \dot{-} y$ and $\lambda xy.\left\lfloor \frac{x}{y+1} \right\rfloor$. Let $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^N$. The function θ is uniformly $\mathcal{F}$-computable if and only if there exist $d \in \mathbb{T}_1 \cap \mathcal{F}$ and $f, g, h \in \mathbb{T}_{3N+1} \cap \mathcal{F}$ such that, whenever $(\xi_1, \ldots, \xi_N) \in D$, $p_1, q_1, r_1, \ldots, p_N, q_N, r_N, e \in \mathbb{N}$ and*

$$|\xi_i| \leq e + 1, \quad \left|\frac{p_i - q_i}{r_i + 1} - \xi_i\right| < \frac{1}{d(e) + 1} \quad (i = 1, \ldots, N), \tag{1}$$

the numbers

$$p = f(p_1, q_1, r_1, \ldots, p_N, q_N, r_N, e), \quad q = g(p_1, q_1, r_1, \ldots, p_N, q_N, r_N, e), \tag{2}$$

$$r = h(p_1, q_1, r_1, \ldots, p_N, q_N, r_N, e) \tag{3}$$

satisfy the inequality

$$\left|\frac{p - q}{r + 1} - \theta(\xi_1, \ldots, \xi_N)\right| < \frac{1}{e + 1}\,. \tag{4}$$

Proof. For the "if"-part, let us suppose that d, f, g, h are functions satisfying the above condition. We define $3N$-ary operators F, G, H by setting

$$F(f_1, g_1, h_1, \ldots, f_N, g_N, h_N)(n) = p, \quad G(f_1, g_1, h_1, \ldots, f_N, g_N, h_N)(n) = q,$$
$$H(f_1, g_1, h_1, \ldots, f_N, g_N, h_N)(n) = r,$$

where the numbers p, q, r are defined by means of the equalities (2–3) with

$$e = \max(f_1(0), g_1(0), \ldots, f_N(0), g_N(0), n),$$
$$p_i = f_i(d(e)), \quad q_i = g_i(d(e)), \quad r_i = h_i(d(e)) \text{ for } i = 1, \ldots, N.$$

These operators are $\mathcal{F}$-substitutional, and if an element $(\xi_1,\dots,\xi_N)$ of D and functions $f_1,g_1,h_1,\dots,f_N,g_N,h_N \in \mathbb{T}_1$ are given such that (f_i,g_i,h_i) names ξ_i for $i=1,\dots,N$, then, for any $n\in\mathbb{N}$, the above numbers $p_1,q_1,r_1,\dots,$ p_N,q_N,r_N,e will satisfy the inequalities (1) and the inequality $e\geq n$, hence the corresponding numbers p,q,r will satisfy the inequality

$$\left|\frac{p-q}{r+1}-\theta(\xi_1,\dots,\xi_N)\right|<\frac{1}{n+1}\,.$$

For the proof of the "only if"-part, suppose (F,G,H) is an $\mathcal{F}$-substitutional computing system for θ. By applying the Uniformity Lemma, we can find a function $u\in\mathbb{T}_2\cap\mathcal{F}$ such that, whenever $a\in\mathbb{N}$, the functions

$$f_1,g_1,h_1,\dots,f_N,g_N,h_N,f_1',g_1',h_1',\dots,f_N',g_N',h_N'\in\mathbb{T}_1$$

are dominated by $\lambda n.(a+2)(n+1)$ and for a certain $e\in\mathbb{N}$ the functions $f_1,g_1,h_1,\dots,f_N,g_N,h_N$ coincide, respectively, with the functions $f_1',g_1',h_1',\dots,$ f_N',g_N',h_N' at the numbers not exceeding $u(a,e)$, each of the operators F,G,H transforms $(f_1,g_1,h_1,\dots,f_N,g_N,h_N)$ and $(f_1',g_1',h_1',\dots,f_N',g_N',h_N')$ into two functions coinciding at the number e. To define the functions d,f,g,h, we set $d(e)=2u(e,e)+1$, and we take as values of the functions f,g,h at $(p_1,q_1,r_1,\dots,$ $p_N,q_N,r_N,e)$ the values at e, respectively, of the results of applying the operators F,G,H to the $3N$-tuple $(f_1,g_1,\mathrm{id}_{\mathbb{N}},\dots,f_N,g_N,\mathrm{id}_{\mathbb{N}})$, where

$$f_i=\lambda n.\left\lfloor(n+1)\frac{p_i\dot{-}q_i}{r_i+1}+\frac{1}{2}\right\rfloor,\quad g_i=\lambda n.\left\lfloor(n+1)\frac{q_i\dot{-}p_i}{r_i+1}+\frac{1}{2}\right\rfloor \tag{5}$$

for $i=1,\dots,N$. Suppose now $(\xi_1,\dots,\xi_N)\in D$, $p_1,q_1,r_1,\dots,p_N,q_N,r_N,e\in\mathbb{N}$, and the inequalities (1) hold. Let the numbers p,q,r be defined by the equalities (2–3). Besides the functions $f_1,g_1,\dots,f_N,g_N$ defined by (5), we consider functions $f_1',g_1',\dots,f_N',g_N'\in\mathbb{T}_1$ coinciding, respectively, with them at the numbers not exceeding $u(e,e)$ and such that, for all $x>u(e,e)$,

$$\left|\frac{f_i'(x)-g_i'(x)}{x+1}-\xi_i\right|<\frac{1}{x+1}\,,\quad f_i'(x)=0\vee g_i'(x)=0. \tag{6}$$

Making use of (1) and (5), one sees that the conditions (6) will be satisfied also in the case of $x\leq u(e,e)$, hence the triple $(f_i',g_i',\mathrm{id}_{\mathbb{N}})$ names ξ_i for $i=1,\dots,N$. Since the functions $f_i,g_i,f_i',g_i',\mathrm{id}_{\mathbb{N}}$ are dominated by $\lambda n.(e+2)(n+1)$, the values at e of the results of applying the operators F,G,H to the $3N$-tuple $(f_1',g_1',\mathrm{id}_{\mathbb{N}},\dots,f_N',g_N',\mathrm{id}_{\mathbb{N}})$ are equal, respectively, to the numbers p,q,r, and therefore these numbers satisfy the inequality (4). □

References

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[2] Tent, K., Ziegler, M., *Computable functions of reals*, arXiv:0903. 1384v4 [math.LO], March 2009 (Last updated: July 24, 2009) `http://arxiv.org/abs/0903.1384`