REGULAR ENUMERATIONS

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Abstract. In the paper we introduce and study regular enumerations for arbitrary recursive ordinals. Several applications of the technique are presented.

1. Introduction

Let ζ be a recursive ordinal and let $\{B_\gamma\}_{\gamma\leq\zeta}$ be an arbitrary sequence of sets of natural numbers. Roughly speaking a regular enumeration f is a kind of generic function such that for all $\gamma \leq \zeta$, B_γ is recursively enumerable in $f^{(\gamma)}$ uniformly in γ.

The regular enumerations for finite sequences of sets are introduced by the first author in [13] where several applications to the theory of the enumeration reducibility are presented. In the presented paper, we are concerned with the problem of generalizing the construction from [13] to infinite recursive ordinals ζ. It turned out that this generalization is not as straightforward as might be expected, the main problem being the limit ordinals. To deal with them we introduce in section 3 the so called ordinal approximations. In sections 4 – 6 we define the regular enumerations and study their properties.

Section 7 is devoted to the applications. We prove a general version of the inversion Theorem from [13] and apply it to obtain a characterization of the sets A satisfying the condition:

(*) $\quad (\forall X)[(\forall\gamma\leq\zeta)(B_\gamma \text{ is r.e. in } X^{(\gamma)} \text{ uniformly in } \gamma) \Rightarrow A \text{ is r.e. in } X^{(\alpha)}],$

where α is a recursive ordinal.

Our characterization is in terms of enumeration reducibility giving an alternative version of Ash's Theorem from [1], where the sets A satisfying (*) are described by means of a certain kind of formally described reducibilities.

Other applications are related to the following problem. Let α and β be recursive ordinals. Consider the family

$$\mathcal{S}_{\alpha,\beta} = \{X^{(\alpha)} : (\forall\gamma\leq\beta)(B_\gamma \text{ is r.e. in } X^{(\gamma)} \text{ uniformly in } \gamma)\}.$$

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Now the problem is to determine when the family $\mathcal{S}_{\alpha,\beta}$ possesses an element whose Turing degree is the least among the Turing degrees of the elements of $\mathcal{S}_{\alpha,\beta}$. Problems of that kind were considered in [2] and [8] and recently in [4]. We obtain a characterization of all families $\mathcal{S}_{\alpha,\beta}$ which have an element of least Turing degree from which follow the respective results from [2, 8, 4].

2. Preliminaries

2.1. Ordinal notations. In what follows we shall consider only recursive ordinals α which are below a fixed recursive ordinal η. We shall suppose that a notation $e \in \mathcal{O}$ for η is fixed and the notations for the ordinals $\alpha < \eta$ are elements a of $\mathcal{O}$ such that $a <_o e$. For the definitions of the set $\mathcal{O}$ and the relation "$<_o$" the reader may consult [10] or [11]. We shall identify every ordinal with its notation and denote the ordinals by the letters α, β, γ and δ. In particular we shall write $\alpha < \beta$ instead of $\alpha <_o \beta$. If α is a limit ordinal then by $\{\alpha(p)\}_{p\in\mathbb{N}}$ we shall denote the unique strongly increasing sequence of ordinals with limit α, determined by the notation of α, and write $\alpha = \lim \alpha(p)$.

2.2. The enumeration jump. Given two sets of natural numbers A and B, we say that A is enumeration reducible to B ($A \leq_e B$) if $A = \Gamma_z(B)$ for some enumeration operator Γ_z. In other words, using the notation D_v for the finite set having canonical code v and $W_0, \dots, W_z, \dots$ for the Gödel enumeration of the r.e. sets, we have

$$A \leq_e B \iff \exists z \forall x (x \in A \iff \exists v(\langle v, x\rangle \in W_z \ \&\ D_v \subseteq B)).$$

The relation $\leq_e$ is reflexive and transitive and induces an equivalence relation $\equiv_e$ on all subsets of $\mathbb{N}$. The respective equivalence classes are called enumeration degrees. For an introduction to the enumeration degrees the reader might consult Cooper [6].

Given a set A denote by A^+ the set $A \oplus (\mathbb{N} \setminus A)$. The set A is called *total* iff $A \equiv_e A^+$. Clearly A is recursively enumerable in B iff $A \leq_e B^+$ and A is recursive in B iff $A^+ \leq_e B^+$. Notice that the graph of every total function is a total set.

The enumeration jump operator is defined in Cooper [5] and further studied by McEvoy [9]. Here we shall use the following definition of the e-jump which is m-equivalent to the original one, see [9]:

2.1. Definition. Given a set A, let $K^0_A = \{\langle x, z\rangle : x \in \Gamma_z(A)\}$. Define the *e-jump* A'_e of A to be the set $(K^0_A)^+$.

The following properties of the enumeration jump are proved in [9]:
Let A and B be sets of natural numbers. Set $B^{(0)}_e = B$ and $B^{(n+1)}_e = (B^{(n)}_e)'_e$.

(J1) If $A \leq_e B$, then $A'_e \leq_e B'_e$.
(J2) A is Σ^0_{n+1} relatively to B iff $A \leq_e (B^+)^{(n)}_e$.

Let α be a recursive ordinal. To define the α-th enumeration jump of a set A we are going to use a construction very similar to that used in the definition of the α-th Turing jump. The idea is to modify the definition of the sets H^A_α, see [10] or [11], by taking enumeration jump instead of Turing jump:

2.2. Definition.

(i) $E_0^A = A$.
(ii) $E_{\beta+1}^A = (E_\beta^A)'_e$.
(iii) If $\alpha = \lim \alpha(p)$, then $E_\alpha^A = \{\langle p, x\rangle : x \in E_{\alpha(p)}^A\}$.

From now on $A_e^{(\alpha)}$ will stand for E_α^A.

Of course the definition of the set $A_e^{(\alpha)}$ depends on the fixed notation of the ordinal α. On the other hand, it is easy to see by a minor modification of the proof of the corresponding theorem of Spector for the sets H_α^A, see [10] or [11], that if α_1 and α_2 are two notations of the same recursive ordinal, then $A_e^{(\alpha_1)} \equiv_e A_e^{(\alpha_2)}$.

The following properties of the transfinite iteration of the enumeration jump follow easily from the definition:

(E1) If $\beta \le \alpha$ are recursive ordinals, then $A_e^{(\beta)} \le_e A_e^{(\alpha)}$ uniformly in β and α.
(E2) If $A \le_e B$, then for every recursive ordinal α, $A_e^{(\alpha)} \le_e B_e^{(\alpha)}$.
(E3) If $\alpha > 0$, then $A_e^{(\alpha)}$ is a total set.

Finally, we have that for total sets the α-th enumeration jump and the α-th Turing jump are equivalent. Namely the following is true:

2.3. Proposition. *Let A be a total set of natural numbers. Then for every recursive ordinal α, $E_\alpha^A \equiv_e (H_\alpha^A)^+$ uniformly in α.*

Since we are going to consider only e-jumps here, from now on we shall omit the subscript e in the notation of the enumeration jump. So for every recursive ordinal α by $A^{(\alpha)}$ we shall denote the α-th enumeration jump of A.

2.3. The jump set of a sequence of sets. Let ζ be a recursive ordinal and let $\{B_\gamma\}_{\gamma\le\zeta}$ be a sequence of sets of natural numbers. For every recursive ordinal α we define the *jump set* $\mathcal{P}_\alpha$ *of the sequence* $\{B_\gamma\}$ by means of transfinite recursion on α:

2.4. Definition.

(i) $\mathcal{P}_0 = B_0$.
(ii) Let $\alpha = \beta + 1$. Then let

$$\mathcal{P}_\alpha = \begin{cases} \mathcal{P}'_\beta \oplus B_\alpha & \text{if } \alpha \le \zeta, \\ \mathcal{P}'_\beta & \text{otherwise.} \end{cases}$$

(iii) Let $\alpha = \lim \alpha(p)$. Then set $\mathcal{P}_{<\alpha} = \{\langle p, x\rangle : x \in \mathcal{P}_{\alpha(p)}\}$ and let

$$\mathcal{P}_\alpha = \begin{cases} \mathcal{P}_{<\alpha} \oplus B_\alpha & \text{if } \alpha \le \zeta, \\ \mathcal{P}_{<\alpha} & \text{otherwise.} \end{cases}$$

Notice that if the sequence $\{B_\gamma\}$ contains only one member, i.e $\zeta = 0$, then for every recursive α, $\mathcal{P}_\alpha = B_0^{(\alpha)}$.

The properties of the jump sets $\mathcal{P}_\alpha$ are similar to the properties of the enumeration jumps. Again we have that if α_1 and α_2 are two notations of the same recursive ordinal, then $\mathcal{P}_{\alpha_1} \equiv_e \mathcal{P}_{\alpha_2}$. We shall omit the proof since it is very close to the proof of the corresponding result for the H_α^A sets mentioned above.

We shall use the following properties of the jump sets which follow easily from the definition:

($\mathcal{P}$1) If $\beta \leq \alpha$, then $\mathcal{P}_\beta \leq_e \mathcal{P}_\alpha$ uniformly in β and α.
($\mathcal{P}$2) If $\gamma \leq \min(\alpha, \zeta)$, then $B_\gamma \leq_e \mathcal{P}_\alpha$ uniformly in γ and α.
($\mathcal{P}$3) Let $(\forall \gamma \leq \min(\alpha, \zeta))(B_\gamma \leq_e A^{(\gamma)}$ uniformly in $\gamma)$. Then $\mathcal{P}_\alpha \leq_e A^{(\alpha)}$.
($\mathcal{P}$4) If α is a limit ordinal, then the set $\mathcal{P}_{<\alpha}$ is total.
($\mathcal{P}$5) If $\zeta < \alpha$, then the set $\mathcal{P}_\alpha$ is total.

3. Ordinal approximations

3.1. Definition. Given an ordinal $\alpha > 0$, an *ordinal approximation of* α is a finite sequence $\overline{\alpha} = \alpha_1 < \alpha_2 < \dots \alpha_n < \alpha$ of ordinals, where $n \geq 1$ and $\alpha_1 = 0$.
The only ordinal approximation of 0 is 0.

For every ordinal approximation $\overline{\alpha} = \alpha_1, \alpha_2, \dots, \alpha_n, \alpha$ and every $\beta < \alpha$ we define the *β-predecessor* $\overline{\beta}$ of $\overline{\alpha}$ by means of the following inductive definition:

3.2. Definition.
1) Let $\beta \leq \alpha_n$. Then
 1.1) If $\beta = \alpha_i$ for some $i \in [1, n]$, then $\overline{\beta} = \alpha_1, \dots, \alpha_i$;
 1.2) Otherwise, if α_i is the least element of the sequence $\alpha_1, \dots, \alpha_n$ such that $\beta < \alpha_i$, then $\overline{\beta}$ is the β-predecessor of $\alpha_1, \dots, \alpha_i$;
2) Let $\alpha_n < \beta < \alpha$. Then
 2.1) If $\alpha = \delta + 1$ and $\beta = \delta$, then $\overline{\beta} = \alpha_1, \dots, \alpha_n, \delta$;
 2.2) If $\alpha = \delta + 1$ and $\beta < \delta$, then $\overline{\beta}$ is the β-predecessor of $\alpha_1, \dots, \alpha_n, \delta$;
 2.3) If $\alpha = \lim \alpha(p)$, then $\overline{\beta}$ is the β-predecessor of
$$\alpha_1, \dots, \alpha_n, \alpha(p_0), \alpha(p_0 + 1), \dots, \alpha(p_1), \text{ where}$$
$$p_0 = \mu p[\alpha_n < \alpha(p)] \text{ and } p_1 = \mu p[\beta < \alpha(p)].$$

The following simple lemma can be proved by means of transfinite induction on α.

3.3. Lemma. *For every ordinal approximation $\overline{\alpha}$ and every $\beta < \alpha$, there exists exactly one β-predecessor $\overline{\beta}$ of $\overline{\alpha}$.*

From the definition it follows immediately that there exists a recursive function π such that if $\overline{\alpha}$ is an ordinal approximation and $\beta < \alpha$, then $\pi(\overline{\alpha}, \beta)$ yields the β-predecessor of $\overline{\alpha}$.
By $\overline{\beta} \prec \overline{\alpha}$ we shall denote that $\overline{\beta}$ is the β-predecessor of $\overline{\alpha}$. As usual $\overline{\beta} \preceq \overline{\alpha}$ will stand for $\overline{\beta} \prec \overline{\alpha}$ or $\overline{\beta} = \overline{\alpha}$.
Let us point out some useful properties of the predecessor relation which follow directly from the definition.

3.4. Lemma. *Let $\overline{\alpha} = \alpha_1, \dots, \alpha_n, \alpha$ be an ordinal approximation of α. Then the following assertions hold:*
(1) *If $\beta \leq \alpha_k$, $1 \leq k \leq n$, then $\overline{\beta} \prec \overline{\alpha} \iff \overline{\beta} \preceq \alpha_1, \dots, \alpha_k$.*

(2) *If for some* $k \in [1,n]$, $\alpha_k \leq \beta < \alpha$ *and* $\beta_1, \ldots, \beta_l$ *is the* β*-predecessor of* α, *then* $k \leq l$ *and* $\alpha_i = \beta_i, i = 1, \ldots, k$.
(3) *Let* $\alpha = \delta + 1$, $\alpha_n < \delta$ *and* $\beta \leq \delta$. *Then* $\overline{\beta} \prec \overline{\alpha} \iff \overline{\beta} \preceq \alpha_1, \ldots, \alpha_n, \delta$.
(4) *Let* $\alpha = \lim \alpha(p)$ *be a limit ordinal and* $p_0 = \mu p[\alpha_n < \alpha(p)]$. *Let* $\beta < \alpha$, $p_1 \geq p_0$ *and* $\alpha(p_1) \geq \beta$. *Then*

$$\overline{\beta} \prec \alpha \iff \overline{\beta} \preceq \alpha_1, \ldots, \alpha_n, \alpha(p_0), \alpha(p_0+1), \ldots, \alpha(p_1).$$

3.5. Lemma. *Let* $\gamma < \beta < \alpha$ *be ordinals,* $\overline{\gamma} \prec \overline{\beta}$ *and* $\overline{\beta} \prec \overline{\alpha}$. *Then* $\overline{\gamma} \prec \overline{\alpha}$.

Proof. Transfinite induction on α. Suppose that $\overline{\alpha} = \alpha_1, \ldots, \alpha_n, \alpha$.

Let $\beta \leq \alpha_n$. Then $\overline{\beta} \preceq \alpha_1, \ldots, \alpha_n$. By the induction hypothesis, $\overline{\gamma} \prec \alpha_1, \ldots, \alpha_n$. Therefore by Lemma 3.4 $\overline{\gamma} \prec \overline{\alpha}$.

Suppose now that $\alpha_n < \beta$. Let $\alpha = \delta + 1$. Set $\overline{\delta} = \alpha_1, \ldots, \alpha_n, \delta$. Since $\beta \leq \delta$, $\alpha_n < \delta$. By Lemma 3.4 $\overline{\beta} \preceq \overline{\delta}$. By the induction hypothesis $\overline{\gamma} \prec \overline{\delta}$. From here again by Lemma 3.4 it follows that $\overline{\gamma} \prec \overline{\alpha}$.

It remains to consider the case $\alpha_n < \beta$ and $\alpha = \lim \alpha(p)$. Let

$$p_0 = \mu p[\alpha_n < \alpha(p)] \text{ and } p_\beta = \mu p[\beta < \alpha(p)].$$

Set $\overline{\alpha(p_\beta)} = \alpha_1, \ldots, \alpha_n, \alpha(p_0), \alpha(p_0+1), \ldots, \alpha(p_\beta)$. Now we have that $\overline{\beta} \prec \overline{\alpha(p_\beta)}$. By induction $\overline{\gamma} \prec \overline{\alpha(p_\beta)}$ and hence by Lemma 3.4 $\overline{\gamma} \prec \overline{\alpha}$. □

From the last lemma it follows that if we fix an ordinal approximation $\overline{\alpha}$ and consider the set of all ordinal approximations $\overline{\beta} \prec \overline{\alpha}$, then this set is well ordered by the relation " $\prec$ " and its order type is α.

4. Regular finite parts

Let us fix a sequence $\{B_\alpha\}, \alpha \leq \zeta$, of subsets of $\mathbb{N}$. Since every set B is enumeration equivalent to $B \oplus \mathbb{N} = \{2x : x \in B\} \cup \{2x+1 : x \in \mathbb{N}\}$, we may assume that $B_\alpha \neq \emptyset$ for all $\alpha \leq \zeta$.

In what follows we shall use the term *finite part* for finite mappings of $\mathbb{N}$ into $\mathbb{N}$ defined on finite segments $[0, q-1]$ of $\mathbb{N}$. Finite parts will be denoted by the letters τ, ρ. If $\text{dom}(\tau) = [0, q-1]$, then let $\text{lh}(\tau) = q$.

We shall suppose that an effective coding of all finite sequences and hence of all finite parts is fixed. Given two finite parts τ and ρ we shall say that τ is less than or equal to ρ if the code of τ is less than or equal to the code of ρ. By $\tau \subseteq \rho$ we shall denote that the partial mapping ρ extends τ and say that ρ is an extension of τ. For any τ, by $\tau \restriction n$ we shall denote the restriction of τ on $[0, n-1]$.

Below we define for every $\alpha \leq \zeta$ and every ordinal approximation $\overline{\alpha}$ of α the $\overline{\alpha}$-regular finite parts. The definition is by transfinite recursion on α.

Let $\alpha \leq \zeta$. Suppose that for all $\beta < \alpha$ we have defined the $\overline{\beta}$-regular finite parts and for every $\overline{\beta}$-regular τ we have defined the $\overline{\beta}$-rank $|\tau|_{\overline{\beta}}$ of τ. Suppose also that for all finite parts ρ and for all $e, x \in \mathbb{N}$ we have defined the forcing relations $\rho \Vdash_{\overline{\beta}} F_e(x)$ and $\rho \Vdash_{\overline{\beta}} \neg F_e(x)$

Let us fix an ordinal approximation $\overline{\alpha}$ of α.

1) $\alpha = 0$. Then $\overline{\alpha} = 0$. The 0-regular finite parts are finite parts τ such that $\mathrm{dom}(\tau) = [0, 2q+1]$ and for all odd $z \in \mathrm{dom}(\tau), \tau(z) \in B_0$.

If $\mathrm{dom}(\tau) = [0, 2q+1]$, then the 0-rank $|\tau|_0$ of τ is equal to the number $q+1$ of the odd elements of $\mathrm{dom}(\tau)$.

Given a finite part ρ, let

$$\rho \Vdash_0 F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(\rho((u)_0) \simeq (u)_1))$$

$$\rho \Vdash_0 \neg F_e(x) \iff \forall(0\text{-regular } \tau)(\rho \subseteq \tau \Rightarrow \tau \nVdash_0 F_e(x)).$$

2) $\alpha = \beta + 1$. Let $\overline{\beta}$ be the β-predecessor of $\overline{\alpha}$.

Set $X_j^{\overline{\beta}} = \{\rho : \rho \text{ is } \overline{\beta}\text{-regular} \ \&\ \rho \Vdash_{\overline{\beta}} F_{(j)_0}((j)_1)\}$.

Given a finite part τ and a set X of $\overline{\beta}$-regular finite parts, let $\mu_{\overline{\beta}}(\tau, X)$ be the least extension of τ belonging to X if any, and $\mu_{\overline{\beta}}(\tau, X)$ be the least $\overline{\beta}$-regular extension of τ otherwise. We shall assume that $\mu_{\overline{\beta}}(\tau, X)$ is undefined if there is no $\overline{\beta}$-regular extension of τ.

4.1. Definition. Let τ be a finite part and $m \geq 0$. Say that ρ is a *$\overline{\beta}$-regular m omitting extension* of τ if ρ is a $\overline{\beta}$-regular extension of τ, defined on $[0, q-1]$ and there exist natural numbers $q_0 < \cdots < q_m < q_{m+1} = q$ such that:

a) $\rho \restriction q_0 = \tau$.

b) For all $p \leq m$, $\rho \restriction q_{p+1} = \mu_{\overline{\beta}}(\rho \restriction (q_p + 1), X^{\overline{\beta}}_{\langle p, q_p\rangle})$.

Notice that if ρ is a $\overline{\beta}$-regular m omitting extension of τ, then there exists a unique sequence of natural numbers $q_0, \ldots, q_{m+1}$ having the properties a) and b) above. Moreover if ρ_1 and ρ_2 are two $\overline{\beta}$-regular m omitting extensions of τ and $\rho_1 \subseteq \rho_2$, then $\rho_1 = \rho_2$.

Let $\mathcal{R}_{\overline{\beta}}$ denote the set of all $\overline{\beta}$-regular finite parts. Given an index j, by $S_j^{\overline{\beta}}$ we shall denote the intersection $\mathcal{R}_{\overline{\beta}} \cap \Gamma_j(\mathcal{P}_\beta)$, where Γ_j is the j-th enumeration operator.

Let τ be a finite part defined on $[0, q-1]$ and $r \geq 0$. Then τ is *$\overline{\alpha}$-regular* of *$\overline{\alpha}$-rank* $r+1$ if there exist natural numbers

$$0 < n_0 < l_0 < b_0 < n_1 < l_1 < b_1 \cdots < n_r < l_r < b_r < n_{r+1} = q$$

such that $\tau \restriction n_0$ is a $\overline{\beta}$-regular finite part of rank 1 and for all j, $0 \leq j \leq r$, the following conditions are satisfied:

s_a) $\tau \restriction l_j \simeq \mu_{\overline{\beta}}(\tau \restriction (n_j + 1), S_j^{\overline{\beta}})$;

s_b) $\tau \restriction b_j$ is a $\overline{\beta}$-regular j omitting extension of $\tau \restriction l_j$;

s_c) $\tau(b_j) \in B_\alpha$;

s_d) $\tau \restriction n_{j+1}$ is a $\overline{\beta}$-regular extension of $\tau \restriction (b_j + 1)$ of rank $|\tau \restriction b_j|_{\overline{\beta}} + 1$

To conclude with the definition of the $\overline{\alpha}$-regular finite parts in this case, let for every ρ, e and x

$$\rho \Vdash_{\overline{\alpha}} F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)((u = \langle e_u, x_u, 0\rangle \ \&\ \rho \Vdash_{\overline{\beta}} F_{e_u}(x_u)) \vee$$
$$(u = \langle e_u, x_u, 1\rangle \ \&\ \rho \Vdash_{\overline{\beta}} \neg F_{e_u}(x_u)))).$$

$$\rho \Vdash_{\overline{\alpha}} \neg F_e(x) \iff \forall(\overline{\alpha}\text{-regular } \tau)(\rho \subseteq \tau \Rightarrow \tau \nVdash_{\overline{\alpha}} F_e(x)).$$

3) $\alpha = \lim \alpha(p)$ is a limit ordinal. Let $\overline{\alpha} = \alpha_1, \dots, \alpha_n, \alpha$. Set $p_0 = \mu p[\alpha(p) > \alpha_n]$. For every p denote by $\overline{\alpha(p)}$ the $\alpha(p)$-predecessor of $\overline{\alpha}$. Notice that for every $p \geq p_0$

$$\overline{\alpha(p)} = \alpha_1, \dots, \alpha_n, \alpha(p_0), \alpha(p_0+1), \dots, \alpha(p).$$

A finite part τ defined on $[0, q-1]$ is $\overline{\alpha}$-regular with $\overline{\alpha}$-rank $r+1$ if there exists natural numbers

$$0 < n_0 < b_0 < n_1 < b_1 < \dots < n_r < b_r < n_{r+1} = q,$$

such that $\tau \restriction n_0$ is an $\alpha_1, \dots, \alpha_n$-regular finite part of rank 1 and for all $j, 0 \leq j \leq r$, the following conditions are satisfied:

l_a) $\tau \restriction b_j$ is an $\overline{\alpha(p_0+2j)}$-regular finite part of rank 1;
l_b) $\tau(b_j) \in B_\alpha$;
l_c) $\tau \restriction n_{j+1}$ is an $\overline{\alpha(p_0+2j+1)}$-regular finite part of rank 1.

For every finite part ρ and every $e, x \in \mathbb{N}$ set:

$$\rho \Vdash_{\overline{\alpha}} F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \& \ (\forall u \in D_v)(u = \langle p_u, e_u, x_u\rangle \ \& \ \rho \Vdash_{\overline{\alpha(p_u)}} F_{e_u}(x_u)).$$

$$\rho \Vdash_{\overline{\alpha}} \neg F_e(x) \iff \forall(\overline{\alpha}\text{-regular } \tau)(\rho \subseteq \tau \Rightarrow \tau \nVdash_{\overline{\alpha}} F_e(x)).$$

The following lemma shows that the $\overline{\alpha}$-rank is well defined.

4.2. Lemma. *Let $\alpha \leq \zeta$ and let τ be an $\overline{\alpha}$-regular finite part. Then the following assertions hold:*

(1) *Suppose that $\alpha = \beta + 1$. Let $m_0, q_0, a_0, \dots, m_p, q_p, a_p, m_{p+1}$ and $n_0, l_0, b_0, \dots, n_r, l_r, b_r, n_{r+1}$ be two sequences of natural numbers satisfying s_a)–s_d) from the definition above. Then $r = p, n_{p+1} = m_{p+1}$ and for all $j \leq r, n_j = m_j, l_j = q_j$ and $b_j = a_j$.*
(2) *Suppose that $\alpha = \lim \alpha(p)$ is a limit ordinal and let $m_0, a_0, \dots, m_p, a_p, m_{p+1}$ and $n_0, b_0, \dots, n_r, b_r, n_{r+1}$ be two sequences of natural numbers satisfying the conditions l_a) – l_c). Then $r = p, n_{p+1} = m_{p+1}$ and for all $j \leq r, m_j = n_j$ and $a_j = b_j$.*
(3) *If ρ is $\overline{\alpha}$-regular, $\tau \subseteq \rho$ and $|\tau|_{\overline{\alpha}} = |\rho|_{\overline{\alpha}}$, then $\tau = \rho$.*

Proof. The proof follows easily from the definition of the $\overline{\alpha}$-regular finite parts by transfinite induction on α. □

4.3. Corollary. *Let $\alpha = \beta+1$, $\overline{\alpha}$ be an ordinal approximation of α and let $\overline{\beta}$ be the β-predecessor of $\overline{\alpha}$. Then every $\overline{\alpha}$-regular finite part τ is $\overline{\beta}$-regular and $|\tau|_{\overline{\beta}} > |\tau|_{\overline{\alpha}}$.*

4.4. Lemma. *Let $1 \leq \alpha \leq \zeta$ and $\overline{\alpha} = \alpha_1, \dots, \alpha_n, \alpha$. Then every $\overline{\alpha}$-regular finite part τ is $\alpha_1, \dots, \alpha_n$-regular and the $\alpha_1, \dots, \alpha_n$-rank of τ is strictly greater than $|\tau|_{\overline{\alpha}}$.*

Proof. The proof is by transfinite induction on α. Let τ be an $\overline{\alpha}$-regular finite part.

Suppose that $\alpha = 1$. Clearly $\overline{\alpha} = 0, 1$. We have to show that τ is also 0-regular and $|\tau|_0 > |\tau|_1$. Both assertions follow from Corollary 4.3.

Now let $\alpha = \beta + 1$ and let $\overline{\beta}$ be the β-predecessor of $\overline{\alpha}$. Clearly τ is $\overline{\beta}$-regular and $|\tau|_{\overline{\beta}} > |\tau|_{\overline{\alpha}}$. By Lemma 3.4 $\overline{\beta}$ is in the form $\alpha_1, \dots, \alpha_n, \beta_{n+1}, \dots, \beta_{n+i}$, where $i \geq 0$.

By induction τ is $\alpha_1, \ldots, \alpha_n$-regular and the $\alpha_1, \ldots, \alpha_n$-rank of τ is greater than or equal to $|\tau|_{\overline{\beta}}$.

It remains to consider the case $\alpha = \lim \alpha(p)$. Suppose that $|\tau|_{\overline{\alpha}} = r + 1$. Set $p_0 = \mu p[\alpha(p) > \alpha_n]$. By definition τ is $\alpha_1, \ldots, \alpha_n, \alpha(p_0), \ldots, \alpha(p_0 + 2r + 1)$-regular finite part of rank 1. By induction τ is $\alpha_1, \ldots, \alpha_n, \alpha(p_0), \ldots, \alpha(p_0 + 2r)$-regular of rank at least 2, ... , τ is $\alpha_1, \ldots, \alpha_n, \alpha(p_0)$ - regular of rank at least $2r+2$ and hence τ is $\alpha_1, \ldots, \alpha_n$-regular of rank greater than $r+1$. $\square$

4.5. Lemma. *Let $\alpha \leq \zeta$ and let $\overline{\alpha}$ be an approximation of α. Suppose that $\overline{\beta} \preceq \overline{\alpha}$. Then there exists a natural number $k(\overline{\alpha}, \overline{\beta})$ such that every $\overline{\alpha}$-regular finite part of rank greater than or equal to $k(\overline{\alpha}, \overline{\beta})$ is $\overline{\beta}$-regular.*

Proof. We shall use transfinite induction on α. The assertion is obviously true for $\alpha = 0$.

Suppose that $\alpha = \delta + 1$ and $\overline{\delta}$ is the δ-predecessor of $\overline{\alpha}$. Let $\overline{\beta} \prec \overline{\alpha}$. Then $\overline{\beta} \preceq \overline{\delta}$. By induction every $\overline{\delta}$-regular finite part of rank at least $k(\overline{\delta}, \overline{\beta})$ is $\overline{\beta}$-regular. Set $k(\overline{\alpha}, \overline{\beta}) = k(\overline{\delta}, \overline{\beta}) = k$. Consider an $\overline{\alpha}$-regular finite part τ of rank at least k. Then τ is $\overline{\delta}$-regular of rank greater than k and hence τ is $\overline{\beta}$-regular.

Let $\alpha = \lim \alpha(p)$, $\overline{\alpha} = \alpha_1, \ldots, \alpha_n, \alpha$ and $\overline{\beta} \prec \overline{\alpha}$. Set $p_0 = \mu p[\alpha(p) > \alpha(n)]$ and fix a $p_1 \geq p_0$ and such that $\alpha(p_1) \geq \beta$. Denote by $\overline{\alpha(p)}$ the $\alpha(p)$-predecessor of $\overline{\alpha}$, $p = 0, 1, \ldots$.

By Lemma 3.4 $\overline{\beta} \preceq \overline{\alpha(p_1)}$. Hence, by induction, every $\overline{\alpha(p_1)}$-regular finite part of rank at least $k(\overline{\alpha(p_1)}, \overline{\beta})$ is $\overline{\beta}$-regular. On the other hand, as we saw in the proof of the previous lemma, there exists a natural number r such that every $\overline{\alpha}$-regular finite part of rank at least $r+1$ is $\overline{\alpha(p_1)}$-regular and of rank greater than $k(\overline{\alpha(p_1)}, \overline{\beta})$. Set $k(\overline{\alpha}, \overline{\beta}) = r + 1$. $\square$

4.6. Corollary. *Let $\alpha \leq \zeta$, $\overline{\alpha}$ is an ordinal approximation of α and $\overline{\beta} \preceq \overline{\alpha}$. Suppose that τ is an $\overline{\alpha}$-regular finite part of rank greater than $k(\overline{\alpha}, \overline{\beta}) + s$. Then $|\tau|_{\overline{\beta}} > s$.*

Proof. From the definition of the $\overline{\alpha}$-regular finite parts it follows that there exist natural numbers $q_0 < \cdots < q_s$ such that $\tau \restriction q_s = \tau$ and for all $j \leq s$ we have that $\tau_j = \tau \restriction q_j$ is a $\overline{\alpha}$ regular finite part of rank greater than $k(\overline{\alpha}, \overline{\beta})$. Hence every τ_j is $\overline{\beta}$ regular. Clearly $\tau_0 \subset \tau_1 \subset \cdots \subset \tau_s = \tau$ and $|\tau_0|_{\overline{\beta}} \geq 1$. By Lemma 4.2 $|\tau_s|_{\overline{\beta}} > s$. $\square$

4.7. Lemma. *Let $\alpha = \lim \alpha(p)$ be a limit ordinal. Let $\overline{\alpha} = \alpha_1, \ldots, \alpha_n, \alpha$ and $p_0 = \mu p[\alpha(p) > \alpha_n]$. Suppose that $p_1 \geq p_0$ and τ is an $\alpha_1, \ldots, \alpha_n, \alpha(p_0), \ldots, \alpha(p_1)$-regular finite part of rank 1. Then for every $\overline{\beta} \prec \overline{\alpha}$ if τ is $\overline{\beta}$-regular, then $\beta \leq \alpha(p_1)$.*

Proof. Towards a contradiction assume that τ is $\overline{\beta}$-regular for some β such that $\overline{\beta} \prec \overline{\alpha}$ and $\alpha(p_1) < \beta < \alpha$. Then $\overline{\beta}$ is a β-predecessor of

$$\alpha_1, \ldots, \alpha_n, \alpha(p_0), \ldots, \alpha(p_1), \ldots, \alpha(p_1 + k),$$

where $k \geq 1$. By Lemma 3.4 $\overline{\beta}$ is in the form:

$$\alpha_1, \ldots, \alpha_n, \alpha(p_0), \ldots, \alpha(p_1), ..., \beta.$$

Hence, since $|\tau|_{\overline{\beta}} \geq 1$, using Lemma 4.4, we get that the $\alpha_1, \ldots, \alpha_n, \alpha(p_0), \ldots, \alpha(p_1)$-rank of τ is greater than 1. A contradiction. □

4.8. Definition. For every finite part τ and every ordinal approximation $\overline{\alpha}$ let

$$Reg(\tau, \overline{\alpha}) = \{\overline{\beta} : \overline{\beta} \preceq \overline{\alpha} \text{ and } \tau \text{ is } \overline{\beta}\text{-regular}\}.$$

4.9. Lemma. *Let $\alpha \leq \zeta$, let $\overline{\alpha} = \alpha_1, \ldots, \alpha_n, \alpha$ be an ordinal approximation of α and let τ be an $\overline{\alpha}$-regular finite part. Then the following assertions are true:*

(1) *If $\alpha = \delta + 1$ and $\overline{\delta}$ is the δ-predecessor of $\overline{\alpha}$, then*

$$\overline{\beta} \in Reg(\tau, \overline{\alpha}) \iff \overline{\beta} = \overline{\alpha} \ \vee \ \overline{\beta} \in Reg(\tau, \overline{\delta}).$$

(2) *Let $\alpha = \lim \alpha(p)$. Set $p_0 = \mu p[\alpha(p) > \alpha_n]$ and for every $p \geq p_0$ let $\overline{\alpha(p)}$ be the $\alpha(p)$-predecessor of $\overline{\alpha}$. Suppose that $p_1 \geq p_0$ and τ is an $\overline{\alpha(p_1)}$-regular finite part of rank 1. Then*

$$\overline{\beta} \in Reg(\tau, \overline{\alpha}) \iff \overline{\beta} = \overline{\alpha} \ \vee \ \overline{\beta} \in Reg(\tau, \overline{\alpha(p_1)}).$$

Proof. The assertion (1) is obvious, (2) follows from the previous lemma. □

We conclude this section by a technical proposition which can be proved in a way very similar to the proof of the respective proposition in [13].

4.10. Definition. A sequence $A_0, \ldots, A_n, \ldots$ of subsets of $\mathbb{N}$ is *e-reducible* to the set P iff there exists a recursive function h such that for all n, $A_n = \Gamma_{h(n)}(P)$. The sequence $\{A_n\}$ is *T-reducible* to P if there exists a recursive in P function χ such that for all n, $\lambda x.\chi(n, x) = \chi_{A_n}$, where χ_{A_n} denotes the characteristic function of A_n.

From the definition of the enumeration jump it follows immediately that if the sequence $\{A_n\}$ is e-reducible to P, then it is T-reducible to P'.

Let $\alpha \leq \zeta$ and let $\overline{\alpha}$ be an ordinal approximation of α.

For $j \in \mathbb{N}$ let $\mu^X_{\overline{\alpha}}(\tau, j) \simeq \mu_{\overline{\alpha}}(\tau, X^{\overline{\alpha}}_j)$, $\mu^S_{\overline{\alpha}}(\tau, j) \simeq \mu_{\overline{\alpha}}(\tau, S^{\overline{\alpha}}_j)$,

$$Z^{\overline{\alpha}}_j = \{\tau : \tau \text{ is } \overline{\alpha}\text{-regular } \& \ \tau \Vdash_{\overline{\alpha}} \neg F_{(j)_0}((j)_1)\} \text{ and}$$

$$O^{\overline{\alpha}}_{\tau,j} = \{\rho : \rho \text{ is } \overline{\alpha}\text{-regular } j \text{ omitting extension of } \tau\}.$$

4.11. Proposition. *For every ordinal approximation $\overline{\alpha}$, $\alpha \leq \zeta$, the following assertions are true:*

(1) *The set $\mathcal{R}_{\overline{\alpha}}$ of all $\overline{\alpha}$-regular finite parts is e-reducible to $\mathcal{P}_\alpha$ uniformly in $\overline{\alpha}$.*
(2) *The function $\lambda\tau.|\tau|_{\overline{\alpha}}$ (assumed undefined if $\tau \notin \mathcal{R}_{\overline{\alpha}}$) is partial recursive in $\mathcal{P}'_\beta$, if $\alpha = \beta + 1$ (in $\mathcal{P}_{<\alpha}$, if α is a limit ordinal), uniformly in $\overline{\alpha}$.*
(3) *The sequences $\{S^{\overline{\alpha}}_j\}$ and $\{X^{\overline{\alpha}}_j\}$ are e-reducible to $\mathcal{P}_\alpha$ uniformly in $\overline{\alpha}$.*
(4) *The sequence $\{Z^{\overline{\alpha}}_j\}$ is T-reducible to $\mathcal{P}'_\alpha$ uniformly in $\overline{\alpha}$.*
(5) *The functions $\mu^X_{\overline{\alpha}}$ and $\mu^S_{\overline{\alpha}}$ are partial recursive in $\mathcal{P}'_\alpha$ uniformly in $\overline{\alpha}$.*
(6) *The sequence $\{O^{\overline{\alpha}}_{\tau,j}\}$ is e-reducible to $\mathcal{P}'_\alpha$ uniformly in $\overline{\alpha}$.*

5. Regular enumerations

For every $\overline{\alpha}$-regular finite part τ of rank $r+1$ we define the subset $B^{\tau}_{\overline{\alpha}}$ of $\mathrm{dom}(\tau)$ as follows.

5.1. Definition.

a) If $\alpha = 0$, then let $B^{\tau}_{\overline{\alpha}} = \{b : b \in \mathrm{dom}(\tau) \ \&\ b \text{ is odd}\}$.

a) Let $\alpha = \beta + 1$ and let $n_0, l_0, b_0, \dots, n_r, l_r, b_r, n_{r+1}$ be the elements of $\mathrm{dom}(\tau)$ satisfying the conditions s_a)–s_d) from the definition of the regular finite parts. Set $B^{\tau}_{\overline{\alpha}} = \{b_0, \dots, b_r\}$.

b) Let $\alpha = \lim \alpha(p)$ and $n_0, b_0, \dots, n_r, b_r, n_{r+1}$ be the the elements of $\mathrm{dom}(\tau)$ satisfying the conditions l_a)–l_c) from the definition of the regular finite parts. Set $B^{\tau}_{\overline{\alpha}} = \{b_0, \dots, b_r\}$.

5.2. Definition. Let $\overline{\zeta}$ be an ordinal approximation of ζ. A a total mapping f of $\mathbb{N}$ in $\mathbb{N}$ is called *regular enumeration* (with respect to $\overline{\zeta}$) if the following two conditions hold:

(i) For every finite part $\rho \subseteq f$, there exists a $\overline{\zeta}$-regular extension τ of ρ such that $\tau \subseteq f$.

(ii) If $\overline{\alpha} \preceq \overline{\zeta}$ and $z \in B_\alpha$, then there exists an $\overline{\alpha}$-regular $\tau \subseteq f$, such that $z \in \tau(B^{\tau}_{\overline{\alpha}})$.

Clearly, if f is a regular enumeration and $\overline{\alpha} \preceq \overline{\zeta}$, then for every $\rho \subseteq f$, there exists an $\overline{\alpha}$-regular $\tau \subseteq f$ such that $\rho \subseteq \tau$. Moreover there exist $\overline{\alpha}$-regular finite parts of f of arbitrary large rank.

Given a regular f and $\overline{\alpha} \preceq \overline{\zeta}$, let $B^{f}_{\overline{\alpha}} = \{b : (\exists \tau \subseteq f)(\tau \text{ is } \overline{\alpha}\text{-regular } \&\ b \in B^{\tau}_{\overline{\alpha}})\}$. Evidently $f(B^{f}_{\overline{\alpha}}) = B_\alpha$.

For every function f on $\mathbb{N}$ and every recursive ordinal α by $f^{(\alpha)}$ we shall denote the α-th enumeration jump of the graph of f.

5.3. Proposition. *Suppose that f is a regular enumeration. Then*

(1) $B_0 \leq_e f$.

(2) *If $\alpha = \beta + 1 \leq \zeta$, then $B_\alpha \leq_e f \oplus \mathcal{P}'_\beta$ uniformly in α.*

(3) *If $\alpha \preceq \zeta$ is a limit ordinal, then $B_\alpha \leq_e f \oplus \mathcal{P}_{<\alpha}$ uniformly in α.*

(4) $\mathcal{P}_\alpha \leq_e f^{(\alpha)}$ *uniformly in α.*

Proof. Since f is regular, $B_0 = f(B^{f}_{0})$. Clearly B^{f}_{0} is equal to the set of all odd natural numbers. So, $B_0 \leq_e f$.

Let us turn to the proof of (2) and (3). We shall describe an effective procedure satisfying the requirements of (2) and (3) by means of effective transfinite recursion on α.

Let $\alpha = \beta+1$. Suppose that $\overline{\alpha}$ is the α-predecessor of $\overline{\zeta}$ and $\overline{\beta}$ is the β-predecessor of $\overline{\alpha}$.

Since f is regular, for every finite part ρ of f there exists an $\overline{\alpha}$-regular $\tau \subseteq f$ such that $\rho \subseteq \tau$. Hence there exist natural numbers

$$0 < n_0 < l_0 < b_0 < n_1 < l_1 < b_1 < \dots < n_r < l_r < b_r < \dots,$$

such that for every $r \geq 0$, the finite part $\tau_r = f\restriction n_{r+1}$ is $\overline{\alpha}$-regular and $n_0, l_0, b_0, \dots, n_r, l_r, b_r, n_{r+1}$ are the numbers satisfying the conditions s_a)–s_d) from the definition of the $\overline{\alpha}$-regular finite part τ_r. Clearly $B^f_{\overline{\alpha}} = \{b_0, b_1 \dots\}$. We shall show that there exists a recursive in $f \oplus \mathcal{P}'_\beta$ way to list $n_0, l_0, b_0, \dots$ in an increasing order.

Clearly $f\restriction n_0$ is $\overline{\beta}$-regular and $|f\restriction n_0|_{\overline{\beta}} = 1$. By Proposition 4.11 $\mathcal{R}_{\overline{\beta}}$ is uniformly recursive in $\mathcal{P}'_\beta$. Using f we can generate consecutively the finite parts $f\restriction q$ for $q = 1, 2 \dots$. By Lemma 4.2 $f\restriction n_0$ is the first element of this sequence which belongs to $\mathcal{R}_{\overline{\beta}}$. Clearly $n_0 = \mathrm{lh}(f\restriction n_0)$.

Suppose that $r \geq -1$ and $n_0, l_0, b_0, \dots, n_r, l_r, b_r, n_{r+1}$ have already been listed. Since $f\restriction l_{r+1} \simeq \mu_{\overline{\beta}}(f\restriction(n_{r+1}+1), S^{\overline{\beta}}_{r+1})$, we can find recursively in $f \oplus \mathcal{P}'_\beta$ the finite part $f\restriction l_{r+1}$. Then $l_{r+1} = \mathrm{lh}(f\restriction l_{r+1})$. Next we have that $f\restriction b_{r+1}$ is a $\overline{\beta}$-regular $(r+1)$ omitting extension of $f\restriction l_{r+1}$. So there exist natural numbers $l_{r+1} = q_0 < \dots < q_{r+1} < q_{r+2} = b_{r+1}$ such that for $p \leq r+1$,

$$f\restriction q_{p+1} \simeq \mu_{\overline{\beta}}(f\restriction(q_p+1), X^{\overline{\beta}}_{\langle p, q_p\rangle}).$$

Using the oracle $f \oplus \mathcal{P}'_\beta$ we can find consecutively the numbers q_p and the finite parts $f\restriction(q_p+1)$, $p = 0, \dots, r+2$. By the end of this procedure we reach b_{r+1}. It remains to show that we can find the number n_{r+2}. By definition $f\restriction n_{r+2}$ is an $\overline{\beta}$-regular extension of $f\restriction(b_{r+1}+1)$ having $\overline{\beta}$-rank $|f\restriction b_{r+1}|_{\overline{\beta}} + 1$. Using f we can generate consecutively the finite parts $f\restriction(b_{r+1}+1+q)$, $q = 0, 1, \dots$. By Lemma 4.2 $f\restriction n_{r+2}$ is the first element of this sequence which belongs to $\mathcal{R}_{\overline{\beta}}$.

So $B^f_{\overline{\alpha}}$ is recursive in $f \oplus \mathcal{P}'_\beta$. Hence, since $B_\alpha = f(B^f_{\overline{\alpha}})$, $B_\alpha \leq_e f \oplus \mathcal{P}'_\beta$.

Suppose now that $\alpha = \lim \alpha(p)$ is a limit ordinal. Clearly the sequence $\{\mathcal{P}_{\alpha(p)}\}$ is uniformly e-reducible to $\mathcal{P}_{<\alpha}$. Let $\overline{\alpha}$ be the α-predecessor of $\overline{\zeta}$ and for every p let $\overline{\alpha(p)}$ be the $\alpha(p)$-predecessor of $\overline{\alpha}$. We may think that f is an infinite union of $\overline{\alpha}$-regular finite parts. So there exist an infinite sequence of natural numbers

$$n_0 < b_0 < n_1 < b_1 < ... < n_r < b_r < n_{r+1} < ...$$

s.t. for every r the finite part $f\restriction n_{r+1}$ is $\overline{\alpha}$-regular of rank $r+1$ and $n_0, b_0, \dots, n_r, b_r$ are the elements of $\mathrm{dom}(\tau)$ satisfying the conditions l_a)–l_c) from the definition of the $\overline{\alpha}$-regular finite part. As in the previous case there exists an recursive in $f \oplus \mathcal{P}_{<\alpha}$ way to list the numbers $n_0, b_0, \dots$ in an increasing order. To show this we need to know only that for every p the set $\mathcal{R}_{\overline{\alpha(p)}}$ is uniformly recursive $\mathcal{P}'_{\alpha(p)}$ and hence, it is uniformly recursive in $\mathcal{P}_{<\alpha}$.

The assertion (4) follows easily from (1), (2) and (3). □

Since every B_α is uniformly in α e-reducible to $\mathcal{P}_\alpha$ we get immediately the following corollary:

5.4. Corollary. *Let f be a regular enumeration then $B_\alpha \leq_e f^{(\alpha)}$ uniformly in α.*

Let f be a total mapping on $\mathbb{N}$. We define for every recursive ordinal α, $e, x \in \mathbb{N}$ the relations $f \models_\alpha F_e(x)$ and $f \models \neg F_e(x)$ by means of transfinite recursion on α:

5.5. Definition.

(i) Let $\alpha = 0$. Then

$$f \models_0 F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(f((u)_0) = (u)_1)).$$

(ii) Let $\alpha = \beta + 1$. Then

$$f \models_\alpha F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)((u = \langle e_u, x_u, 0\rangle \ \&\ f \models_\beta F_{e_u}(x_u)) \vee (u = \langle e_u, x_u, 1\rangle \ \&\ f \models_\beta \neg F_{e_u}(x_u)))).$$

(iii) Let $\alpha = \lim \alpha(p)$ be a limit ordinal. Then

$$f \models_\alpha F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(u = \langle p_u, e_u, x_u\rangle \ \&\ f \models_{\alpha(p_u)} F_{e_u}(x_u))).$$

(iv) $f \models_\alpha \neg F_e(x) \iff f \not\models_\alpha F_e(x)$.

Following the definition of the enumeration jump and the definition above, we can define a recursive function h such that for every recursive ordinal α and every enumeration operator Γ_z the following equivalence is true:

$$x \in \Gamma_z(f^{(\alpha)}) \iff f \models_\alpha F_{h(\alpha,z)}(x).$$

Therefore we have the following lemma:

5.6. Lemma. *Let f be a total mapping on $\mathbb{N}$ and let α be a recursive ordinal. Then $A \leq_e f^{(\alpha)}$ iff there exists an e such that for all x, $x \in A \iff f \models_\alpha F_e(x)$.*

Our next goal is the proof of the Truth Lemma. Notice that for all $\overline{\alpha} \preceq \overline{\zeta}$ the relation $\Vdash_{\overline{\alpha}}$ is monotone, i.e. if $\tau \subseteq \rho$ are $\overline{\alpha}$-regular and $\tau \Vdash_{\overline{\alpha}} (\neg)F_e(x)$, then $\rho \Vdash_{\overline{\alpha}} (\neg)F_e(x)$.

5.7. Lemma. *Let $\alpha < \zeta$ and let $\overline{\alpha}$ be the α-predecessor of $\overline{\zeta}$. Assume also that*

$$f \models_\alpha F_e(x) \iff (\exists \tau \subseteq f)(\tau \textit{ is } \overline{\alpha}\textit{-regular} \ \&\ \tau \Vdash_{\overline{\alpha}} F_e(x)).$$

Then

$$f \models_\alpha \neg F_e(x) \iff (\exists \tau \subseteq f)(\tau \textit{ is } \overline{\alpha}\textit{-regular} \ \&\ \tau \Vdash_{\overline{\alpha}} \neg F_e(x)).$$

Proof. Assume that $f \models_\alpha \neg F_e(x)$ and for all $\overline{\alpha}$-regular $\tau \subseteq f$, $\tau \nVdash_{\overline{\alpha}} \neg F_e(x)$. Then for all $\overline{\alpha}$-regular finite parts τ of f there exists an $\overline{\alpha}$-regular $\rho \supseteq \tau$ such that $\rho \Vdash_{\overline{\alpha}} F_e(x)$. Fix a $j \in \mathbb{N}$ such that

$$S_j^{\overline{\alpha}} = \{\rho : \rho \in \mathcal{R}_{\overline{\alpha}} \ \&\ \rho \Vdash_{\overline{\alpha}} F_e(x)\}.$$

Let μ be an $\overline{\alpha+1}$-regular finite part of f such that $|\mu|_{\overline{\alpha+1}} > j$. By the definition of the $\overline{\alpha+1}$-regular finite parts, there exists an $\overline{\alpha}$-regular $\rho' \subseteq \mu$ such that $\rho' \in S_j^{\overline{\alpha}}$. Since $\rho' \subseteq f$, $f \models_\alpha F_e(x)$. A contradiction.

Assume now that $\tau \subseteq f$ is $\overline{\alpha}$-regular, $\tau \Vdash_{\overline{\alpha}} \neg F_e(x)$ and $f \models_\alpha F_e(x)$. Then there exists a $\overline{\alpha}$-regular $\rho \subseteq f$ such that $\rho \Vdash_{\overline{\alpha}} F_e(x)$. Using the monotonicity of $\Vdash_{\overline{\alpha}}$, we can assume that $\tau \subseteq \rho$ and get a contradiction. $\square$

5.8. Lemma. *Let f be a regular enumeration. Then*

(1) *For all $\overline{\alpha} \preceq \overline{\zeta}$, $f \models_\alpha F_e(x) \iff (\exists \tau \subseteq f)(\tau$ is $\overline{\alpha}$-regular $\&$ $\tau \Vdash_{\overline{\alpha}} F_e(x))$.*

(2) *For all* $\overline{\alpha} \prec \overline{\zeta}$, $f \models_\alpha \neg F_e(x) \iff (\exists \tau \subseteq f)(\tau$ *is* $\overline{\alpha}$*-regular* & $\tau \Vdash_{\overline{\alpha}} \neg F_e(x))$.

Proof. We shall use transfinite induction on α. The condition (1) is obviously true for $\alpha = 0$ and hence according to the Lemma above (2) is also true in this case.

Let $\alpha = \beta + 1$. The truth of (1) for α follows easily from the induction hypothesis. The truth of (2) follows from the Lemma above.

Suppose that $\alpha \preceq \zeta$ and $\alpha = \lim \alpha(p)$ is limit ordinal. It is sufficient to show that (1) is true for α. Assume that $f \models_\alpha F_e(x)$. Then there exists a pair $\langle v, x\rangle \in W_e$ such that if $u \in D_v$, then $u = \langle p_u, e_u, x_u\rangle$ and $f \models_{\alpha(p_u)} F_{e_u}(x_u)$. By induction for every $u \in D_v$ there exists a $\overline{\alpha(p_u)}$-regular finite part $\tau_u \subseteq f$ such that $\tau_u \Vdash_{\overline{\alpha(p_u)}} F_{e_u}(x_u)$. Clearly there exists a $\overline{\alpha}$-regular finite part τ of f such that for all $u \in D_v$, $\tau_u \subseteq \tau$ and τ is $\overline{\alpha(p_u)}$-regular. Then $\tau \Vdash_{\overline{\alpha}} F_e(x)$.

To prove (1) in the reverse direction assume that $\tau \subseteq f$ and $\tau \Vdash_{\overline{\alpha}} F_e(x)$. Again there exists an element $\langle v, x\rangle$ of W_e such that for all $u \in D_v$, $u = \langle \underline{p_u, e_u}, x_u\rangle$ and $\tau \Vdash_{\overline{\alpha(p_u)}} F_{e_u}(x_u)$. Without loss of generality, we may assume that τ is $\overline{\alpha(p_u)}$-regular for every $u \in D_v$. By induction $f \models_{\alpha(p_u)} F_{e_u}(x_u)$ for all $u \in D_v$. So $f \models_\alpha F_e(x)$. $\square$

5.9. Proposition. *Let f be a regular enumeration and $\alpha \le \zeta$. Then the following assertions hold:*

(1) *If $\alpha = \beta + 1$, then $f^{(\alpha)} \le_e f \oplus \mathcal{P}'_\beta$ uniformly in α.*
(2) *If α is a limit ordinal, then $f^{(\alpha)} \le_e f \oplus \mathcal{P}_{<\alpha}$ uniformly in α.*

Proof. Suppose that $\alpha = \beta + 1$. Recall that $f^{(\alpha)} = K^0_{f^{(\beta)}} \oplus (\mathbb{N} \setminus K^0_{f^{(\beta)}})$, where $K^0_{f^{(\beta)}} = \{\langle y, z\rangle : y \in \Gamma_z(f^{(\beta)})\}$. Clearly there exists a z_0 which does not depend on β and such that $K^0_{f^{(\beta)}} = \Gamma_{z_0}(f^{(\beta)})$. Therefore

$$f \models_\beta F_{h(\beta,z_0)}(x) \iff x \in K^0_{f^{(\beta)}}.$$

From here, using Lemma 5.8, we obtain that

$$x \in K^0_{f^{(\beta)}} \iff (\exists \tau \subseteq f)(\tau \text{ is } \overline{\beta}\text{-regular } \& \ \tau \Vdash_{\overline{\beta}} F_{h(\beta,z_0)}(x)) \text{ and}$$

$$x \in (\mathbb{N} \setminus K^0_{f^{(\beta)}}) \iff (\exists \tau \subseteq f)(\tau \text{ is } \overline{\beta}\text{-regular } \& \ \tau \Vdash_{\overline{\beta}} \neg F_{h(\beta,z_0)}(x)).$$

So, by Proposition 4.11 $K^0_{f^{(\beta)}}$ and $(\mathbb{N} \setminus K^0_{f^{(\beta)}})$ are uniformly e-reducible to $f \oplus P'_\beta$. Hence $f^{(\alpha)} \le_e f \oplus \mathcal{P}'_\beta$.

Suppose now that α is a limit ordinal. There exists an z_0 which does not depend on α and such that $f^{(\alpha)} = \Gamma_{z_0}(f^{(\alpha)})$ and hence

$$x \in f^{(\alpha)} \iff f \models_\alpha F_{h(\alpha,z_0)}(x).$$

Using Lemma 5.8, we obtain that

$$x \in f^{(\alpha)} \iff (\exists \tau \subseteq f)(\tau \text{ is } \overline{\alpha}\text{-regular } \& \ \tau \Vdash_{\overline{\alpha}} F_{h(\alpha,z_0)}(x)).$$

Hence, by Proposition 4.11, $f^{(\alpha)} \le_e f \oplus \mathcal{P}_\alpha$. On the other hand, according Proposition 5.3, $\mathcal{P}_\alpha \le_e f \oplus \mathcal{P}_{<\alpha}$. So, $f^{(\alpha)} \le_e f \oplus \mathcal{P}_{<\alpha}$. $\square$

5.10. Corollary. *Let f be a regular enumeration and $\alpha \le \zeta$. Then*

(1) *If $\alpha = \beta + 1$, then $f^{(\alpha)} \equiv_e f \oplus \mathcal{P}'_\beta$ uniformly in α.*

(2) *If α is a limit ordinal, then $f^{(\alpha)} \equiv_e f \oplus \mathcal{P}_{<\alpha}$ uniformly in α.*

6. Regular extensions

Let us fix a total function σ on $\mathbb{N}$ such that for every $\alpha \leq \zeta$, $\sigma(\alpha) \in B_\alpha$.

6.1. Definition. Let $\alpha \leq \zeta$ and let $\overline{\alpha}$ be an ordinal approximation of α. A finite part τ is $\overline{\alpha}$ *complete* (with respect to σ) if

$$\overline{\beta} \in Reg(\tau, \overline{\alpha}) \Rightarrow \sigma(\beta) \in \tau(B^{\tau}_{\overline{\beta}}).$$

Let $\{A_\gamma\}_{\gamma<\zeta}$ be a sequence of subsets of $\mathbb{N}$ such that $(\forall \gamma < \zeta)(A_\gamma \not\leq_e \mathcal{P}_\gamma)$.

6.2. Definition. Let $\alpha \leq \zeta$ and let $\overline{\alpha}$ be an ordinal approximation of α. A finite part τ is $\overline{\alpha}$ *omitting* (with respect to the sequence $\{A_\gamma\}$) if the following *omitting condition* is true for all $\overline{\beta} \in Reg(\tau, \overline{\alpha})$:

If $\beta = \delta + 1$, $\overline{\delta}$ is the δ-predecessor of $\overline{\beta}$ and $|\tau|_{\overline{\beta}} = r + 1$, then for each $p \leq r$ there exist an element q_p of $\mathrm{dom}(\tau)$ and a $\overline{\delta}$-regular finite part $\mu_{p+1} \subseteq \tau$ such that one of the following is true:

a) $\mu_{p+1} \Vdash_{\overline{\delta}} F_p(q_p)$ & $\tau(q_p) \not\in A_\delta$.
b) $\mu_{p+1} \Vdash_{\overline{\delta}} \neg F_p(q_p)$ & $\tau(q_p) \in A_\delta$.

Given a finite mapping τ defined on $[0, q-1]$, by $\tau * z$ we shall denote the extension ρ of τ defined on $[0, q]$ and such that $\rho(q) \simeq z$.

6.3. Proposition. *Let $\alpha \leq \zeta$ and let $\overline{\alpha}$ be an ordinal approximation of α. Then the following assertions are true:*

(1) *For every $\overline{\alpha}$-regular finite part τ and every $y \in \mathbb{N}$ there exists an $\overline{\alpha}$-regular extension ρ of τ such that $|\rho|_{\overline{\alpha}} = |\tau|_{\overline{\alpha}} + 1$, $\rho(lh(\tau)) \simeq y$, ρ is $\overline{\alpha}$ omitting and ρ is $\overline{\alpha}$ complete.*
(2) *For every $\overline{\delta} \prec \overline{\alpha}$, every $\overline{\delta}$-regular τ of rank 1 and every $y \in \mathbb{N}$ there exists a $\overline{\delta}, \alpha$-regular extension ρ of τ of rank 1 and such that $\rho(lh(\tau)) \simeq y$, ρ is $\overline{\delta}, \alpha$ omitting and $\overline{\delta}, \alpha$ complete.*

Proof. We shall prove simultaneously (1) and (2) by means of transfinite induction on α.

a) $\alpha = 0$. In this case (2) is trivial. To prove (1) suppose that τ is 0-regular and $y \in \mathbb{N}$. Define ρ as follows

$$\rho(x) \simeq \begin{cases} \tau(x), & \text{if } x < \mathrm{lh}(\tau), \\ y, & \text{if } x = \mathrm{lh}(\tau), \\ \sigma(0), & \text{if } x = \mathrm{lh}(\tau) + 1, \\ \text{undefined}, & \text{if } x > \mathrm{lh}(\tau) + 1. \end{cases}$$

b) Let $\alpha = \beta + 1$ and let $\overline{\beta}$ be the β-predecessor of $\overline{\alpha}$.

We start with the proof of (1). Suppose that we are given an $\overline{\alpha}$-regular τ and $y \in \mathbb{N}$. Let $\mathrm{dom}(\tau) = [0, q-1]$ and $|\tau|_{\overline{\alpha}} = r + 1$. Set $n_{r+1} = q$. Since τ is $\overline{\beta}$-regular, by induction there exists a $\overline{\beta}$-regular extension of $\tau * y$. Therefore $\rho_0 \simeq \mu_{\overline{\beta}}(\tau * y, S^{\overline{\beta}}_{r+1})$

is defined. Let $l_{r+1} = \mathrm{lh}(\rho_0)$. Next we construct a $\overline{\beta}$-regular $r+1$ omitting extension ρ_1 of ρ_0. For we define the numbers q_p and the finite parts μ_p, $p \leq r+2$, by means of induction on p. Set $q_0 = l_{r+1}$ and $\mu_0 = \rho_0$. Now suppose that for some $p < r+2$ we have defined q_p and μ_p. Consider the set

$$C = \{x : (\exists \mu \supseteq \mu_p)(\mu \text{ is } \overline{\beta}\text{-regular } \ \& \ \mu(q_p) \simeq x \ \& \ \mu \Vdash F_p(q_p))\}.$$

It follows from Proposition 4.11 that $C \leq_e \mathcal{P}_\beta$ and hence $C \neq A_\beta$. Let x_0 be the least natural number such that

$$x_0 \in A_\beta \ \& \ x_0 \notin C \vee x_0 \notin A_\beta \ \& \ x_0 \in C.$$

Set $\mu_{p+1} \simeq \mu_{\overline{\beta}}(\mu_p * x_0, X^{\beta}_{\langle p, q_p\rangle})$ and $q_{p+1} = \mathrm{lh}(q_p)$. Clearly $\rho_1 = \mu_{r+2}$ is a $r+1$ omitting extension of ρ_0. Set $b_{r+1} = \mathrm{lh}(\rho_1)$. Finally let ρ be a $\overline{\beta}$ regular extension of ρ_1 such that $|\rho|_{\overline{\beta}} = |\rho_1|_{\overline{\beta}} + 1$, $\rho(b_{r+1}) \simeq \sigma(\alpha)$, ρ is $\overline{\beta}$ omitting and ρ is $\overline{\beta}$ complete. Clearly ρ satisfies the requirements of (1).

Let us turn to the proof of (2). Let $\overline{\delta} \prec \overline{\alpha}$ and let τ be a $\overline{\delta}$-regular finite part of rank 1 and $y \in \mathbb{N}$.

Suppose that $\delta = \beta$. Then $\overline{\beta} = \overline{\delta}$. Notice that the β-predecessor of $\overline{\delta}, \alpha$ is $\overline{\beta}$.

Let $n_0 = \mathrm{lh}(\tau)$, $\rho_0 \simeq \mu_{\overline{\beta}}(\tau * y, S^{\overline{\beta}}_0)$, ρ_1 is a 0 omitting $\overline{\beta}$-regular extension of ρ_0, constructed as above, $b_1 = \mathrm{lh}(\rho_1)$ and ρ is a $\overline{\beta}$ complete and $\overline{\beta}$ omitting $\overline{\beta}$-regular extension of ρ_1 such that $\rho(b_1) \simeq \sigma(\alpha)$ and $|\rho|_{\overline{\beta}} = |\rho_1|_{\overline{\beta}} + 1$. Clearly ρ is $\overline{\delta}, \alpha$-regular of rank 1, ρ is $\overline{\delta}, \alpha$ complete and $\overline{\delta}, \alpha$ omitting.

Suppose that $\delta < \beta$. Then the β-predecessor of $\overline{\delta}, \alpha$ is $\overline{\delta}, \beta$ and $\overline{\delta} \prec \overline{\beta}$. Using the induction hypothesis, we extend τ to a $\overline{\delta}, \beta$-regular finite part ρ_1 of rank 1 such that $\rho_1(\mathrm{lh}(\tau)) \simeq y$. After that we extend ρ_1 to an $\overline{\delta}, \alpha$ complete and $\overline{\delta}, \alpha$ omitting $\overline{\delta}, \alpha$-regular finite part ρ of rank 1 in the same way as in the previous case.

c) Let $\alpha = \lim \alpha(p)$ be a limit ordinal. Let $\overline{\alpha} = \alpha_1, \dots, \alpha_n, \alpha$ and $p_0 = \mu p[\alpha_n < \alpha(p)]$. For every p by $\overline{\alpha(p)}$ we shall denote the $\alpha(p)$-predecessor of $\overline{\alpha}$.

To prove (1) suppose that τ is an α-regular finite part of rank $r+1$ and $y \in \mathbb{N}$. Clearly τ is a $\overline{\alpha(p_0 + 2r + 1)}$-regular finite part of rank 1. By induction there exists a $\overline{\alpha(p_0 + 2r + 1)}, \alpha(p_0 + 2r + 2)$-regular extension ρ_0 of τ of rank 1 and such that $\rho_0(\mathrm{lh}(\tau)) \simeq y$. Set $b_{r+1} = \mathrm{lh}(\rho_0)$. Applying again the induction hypothesis we obtain a $\overline{\alpha(p_0 + 2r + 1)}, \alpha(p_0 + 2r + 2), \alpha(p_0 + 2r + 3)$-regular extension ρ of ρ_0 which is of rank 1, $\overline{\alpha(p_0 + 2r + 1)}, \alpha(p_0 + 2r + 2), \alpha(p_0 + 2r + 3)$ complete, $\overline{\alpha(p_0 + 2r + 1)}, \alpha(p_0 + 2r + 2), \alpha(p_0 + 2r + 3)$ omitting and such that $\rho(b_{r+1}) \simeq \sigma(\alpha)$. Clearly $\overline{\alpha(p_0 + 2r + 1)}, \alpha(p_0 + 2r + 2), \alpha(p_0 + 2r + 3) = \overline{\alpha(p_0 + 2r + 3)}$. So ρ is an $\overline{\alpha}$-regular finite part of rank $r+2$. It remains to show that ρ is $\overline{\alpha}$ complete and $\overline{\alpha}$ omitting. Indeed, let $\overline{\beta} \in \mathit{Reg}(\rho, \overline{\alpha})$. Then $\overline{\beta} = \overline{\alpha}$ or $\overline{\beta} \in \mathit{Reg}(\rho, \overline{\alpha(p_0 + 2r + 3)})$. By the construction in both cases $\sigma(\beta) \in \rho(B^{\rho}_{\overline{\beta}})$. Suppose that $\beta = \delta + 1$. Then $\beta \neq \alpha$ and hence $\overline{\beta} \in \mathit{Reg}(\rho, \overline{\alpha(p_0 + 2r + 3)})$. Since ρ is $\overline{\alpha(p_0 + 2r + 3)}$ omitting, it satisfies the omitting condition with respect to β.

Let us turn to the proof of (2). Let $\overline{\delta} \prec \overline{\alpha}$, let τ be a $\overline{\delta}$-regular finite part of rank 1 and $y \in \mathbb{N}$. Let $p_\delta = \mu p[\delta < \alpha(p)]$. By induction there exists a $\overline{\delta}, \alpha(p_\delta)$-regular extension ρ_1 of τ which is of rank 1 and such that $\rho_1(\mathrm{lh}(\tau)) \simeq y$. Set $b_0 = \mathrm{lh}(\rho_1)$. Applying again the induction hypothesis, we get an $\overline{\delta}, \alpha(p_\delta), \alpha(p_\delta+1)$-regular extension ρ of ρ_1 which is of rank 1; $\overline{\delta}, \alpha(p_\delta), \alpha(p_\delta+1)$ complete, $\overline{\delta}, \alpha(p_\delta), \alpha(p_\delta+1)$ omitting and $\rho(b_0) \simeq \sigma(\alpha)$. Clearly ρ is a $\overline{\delta}, \alpha$-regular extension of τ which is of rank 1; $\overline{\delta}, \alpha$ complete and $\overline{\delta}, \alpha$ omitting. $\square$

Set $A = \bigoplus_{\gamma<\zeta} A^+_\gamma$.

Remark. From the proof above, it follows immediately that the constructions of the finite parts satisfying (1) and (2) are uniformly recursive in $\mathcal{P}'_\beta \oplus \sigma \oplus A$, if $\alpha = \beta + 1$, and in $\mathcal{P}_{<\alpha} \oplus \sigma \oplus A$, if α is a limit ordinal.

6.4. Corollary. *For every $\alpha \leq \zeta$ and every ordinal approximation $\overline{\alpha}$ of α there exists an $\overline{\alpha}$-regular finite part of rank 1.*

7. Applications

In this section we are going to present several applications of the technique developed so far.

Let us fix a sequence $\{B_\gamma\}_{\gamma\leq\zeta}$ of sets of natural numbers. As usual by $\mathcal{P}_\alpha$ we shall demote the α-th jump set of the sequence $\{B_\gamma\}_{\gamma\leq\zeta}$.

We start with a general version of the inversion theorem from [13].

7.1. The jump inversion theorem.

7.1. Theorem. *Let $\{A_\gamma\}_{\gamma<\zeta}$, be a sequence of sets such that $(\forall\gamma < \zeta)(A_\gamma \not\leq_e \mathcal{P}_\gamma)$. Let Q be a total subset of $\mathbb{N}$ such that $\mathcal{P}_\zeta \leq_e Q$ and $\bigoplus_{\gamma<\zeta} A^+_\gamma \leq_e Q$. Then there exists a total set F having the following properties:*

1. *For all $\gamma \leq \zeta$, $B_\gamma \leq_e F^{(\gamma)}$ uniformly in γ;*
2. *For all $\gamma \leq \zeta$ if $\gamma = \beta + 1$, then $F^{(\gamma)} \equiv_e F \oplus \mathcal{P}'_\beta$ uniformly in γ;*
3. *For all limit $\gamma \leq \zeta$, $F^{(\gamma)} \equiv_e F \oplus \mathcal{P}_{<\gamma}$ uniformly in γ;*
4. *$F^{(\zeta)} \equiv_e Q$.*
5. *For all $\gamma < \zeta$, $A_\gamma \not\leq_e F^{(\gamma)}$.*

Proof. Let us fix an ordinal approximation $\overline{\zeta}$ of ζ. We are going to construct F as the graph of a regular enumeration f such that $f^{(\zeta)} \equiv_e Q$ and $(\forall\gamma < \zeta)(A_\gamma \not\leq_e f^{(\gamma)})$. Let us fix a function $\sigma(\gamma, i)$, recursive in Q, such that if $\gamma \leq \zeta$, then $\lambda i.\sigma(\gamma, i)$ enumerates B_γ.

The construction of f will be carried out in steps. At each step s we shall define a $\overline{\zeta}$-regular finite part τ_s having $\overline{\zeta}$-rank equal to $s + 1$. We shall ensure that $\tau_s \subseteq \tau_{s+1}$ and that every τ_{s+1} is $\overline{\zeta}$ omitting with respect to the sequence $\{A_\gamma\}$ and $\overline{\zeta}$ complete with respect to the function $\sigma_s = \lambda\gamma.\sigma(\gamma, (s)_0)$. Finally we shall define $f = \bigcup_s \tau_s$.

Let $y_0, \ldots, y_s, \ldots$ be a recursive in Q enumeration of Q. We start with an arbitrary $\overline{\zeta}$-regular finite part τ_0 having $\overline{\zeta}$-rank equal to 1. Suppose that τ_s is defined. Using Proposition 6.3, construct recursively in Q a $\overline{\zeta}$-regular extension τ_{s+1} of τ_s

such that $|\tau_{s+1}|_{\overline{\zeta}} = |\tau_s|_{\overline{\zeta}} + 1$, $\tau_{s+1}(\mathrm{lh}(\tau_s)) = y_s$, τ_{s+1} is $\overline{\zeta}$ omitting and $\overline{\zeta}$ complete with respect to σ_s.

First we shall show that f is a regular enumeration. Notice that for every s, τ_{s+1} is a proper extension of τ_s. Hence f is a total function and for every finite part ρ of f there exists a $\overline{\zeta}$ regular finite part τ of f such that $\rho \subseteq \tau$. Let $\overline{\gamma} \preceq \overline{\zeta}$ and let $z \in B_\gamma$. Consider a s so large that every $\overline{\zeta}$-regular finite part of rank greater than s is also $\overline{\gamma}$-regular and such that $z = \sigma(\gamma, (s)_0)$. By the construction, τ_{s+1} is σ_s complete and of rank $s+2$. Hence $z = \sigma_s(\gamma) \in \tau_{s+1}(B_{\overline{\gamma}}^{\tau_{s+1}})$.

Clearly the whole construction is recursive in Q and hence $f \leq_e Q$. Since f is regular, $f^{(\zeta)} \leq_e f \oplus \mathcal{P}_\zeta \leq_e Q$.

To see that $Q \leq_e f^{(\zeta)}$ notice that as in the proof of Proposition 5.3 we have a procedure recursive in $f \oplus \mathcal{P}_\zeta$ which lists the sequence $q_s = \mathrm{lh}(\tau_s)$ in a increasing order. Clearly $y \in Q \iff \exists s(y = f(q_s))$.

It remains to show that for all $\gamma < \zeta$, $A_\gamma \not\leq_e f^{(\gamma)}$. Towards a contradiction, assume that for some $\gamma < \zeta$, $A_\gamma \leq_e f^{(\gamma)}$. Then $C = f^{-1}(A_\gamma)$ is also e-reducible to $f^{(\gamma)}$. Then there exists an e such that for all $x \in \mathbb{N}$,

$$x \in C \iff f \models_\gamma F_e(x).$$

Let $\overline{\gamma+1}$ be the $\gamma+1$-predecessor of $\overline{\zeta}$ and $\overline{\gamma}$ be the γ-predecessor of $\overline{\gamma+1}$. Consider a s so large that every $\overline{\zeta}$-regular finite part of rank greater than s is also $\overline{\gamma+1}$-regular of rank greater than e. By the choice of s, τ_{s+1} is $\overline{\gamma+1}$-regular and $|\tau_{s+1}|_{\overline{\gamma+1}} > e$. Since τ_{s+1} is a $\overline{\zeta}$-omitting finite part, there exist a $q \in \mathrm{dom}(\tau)$ and a $\overline{\gamma}$-regular $\mu \subseteq \tau$ such that:

$$\mu \Vdash_{\overline{\gamma}} F_e(q) \ \&\ \tau_{s+1}(q) \notin A_\gamma \ \vee\ \mu \Vdash_{\overline{\gamma}} \neg F_e(q) \ \&\ \tau_{s+1}(q) \in A_\gamma.$$

Clearly $\overline{\gamma} \prec \overline{\zeta}$. Applying Lemma 5.8 we get $f \models_\gamma F_e(q) \iff q \notin C$. A contradiction. $\square$

By varying the sequences $\{B_\gamma\}_{\gamma \leq \zeta}$ and $\{A_\gamma\}_{\gamma < \zeta}$ we can get several corollaries of the theorem above.

7.2. Theorem. *Let Q be a total set such that $\mathcal{P}_\zeta \leq_e Q$. There exists a total set F such that the following assertions hold:*

(1) *For all $\gamma \leq \zeta$, $B_\gamma \leq_e F^{(\gamma)}$ uniformly in γ;*
(2) *For all $\gamma \leq \zeta$ if $\gamma = \beta + 1$, then $F^{(\gamma)} \equiv_e F \oplus \mathcal{P}'_\beta$ uniformly in γ;*
(3) *For all limit $\gamma \leq \zeta$, $F^{(\gamma)} \equiv_e F \oplus \mathcal{P}_{<\gamma}$ uniformly in γ;*
(4) *$F^{(\zeta)} \equiv_e Q$.*

Proof. For $\gamma < \zeta$ set $A_\gamma = \mathcal{P}'_\gamma$. Clearly for every $\gamma < \zeta$, $A_\gamma \not\leq_e \mathcal{P}_\gamma$ and $\bigoplus_{\gamma<\zeta} A_\gamma \leq_e Q$. Apply the previous theorem. $\square$

If we take $B_\gamma = \emptyset$ for all $\gamma \leq \zeta$, then $\mathcal{P}_\zeta \equiv_e \emptyset^{(\zeta)}$. So the last theorem is a generalization of Friedberg's jump inversion theorem.

7.3. Theorem. *Let Q be a total set such that $\mathcal{P}_\zeta \le_e Q$. Let $\alpha < \zeta$ and let A be a total set such that $A \not\le_e \mathcal{P}_\alpha$ and $A^+ \le_e Q$. There exists a total set satisfying the following conditions:*

(1) *For all $\gamma \le \zeta$, $B_\gamma \le_e F^{(\gamma)}$ uniformly in γ;*
(2) *For all $\gamma \le \zeta$ if $\gamma = \beta + 1$, then $F^{(\gamma)} \equiv_e F \oplus \mathcal{P}'_\beta$ uniformly in γ;*
(3) *For all limit $\gamma \le \zeta$, $F^{(\gamma)} \equiv_e F \oplus \mathcal{P}_{<\gamma}$ uniformly in γ;*
(4) *$F^{(\zeta)} \equiv_e Q$.*
(5) *$A \not\le_e F^{(\alpha)}$.*

Proof. For every $\gamma < \zeta$ let $A_\gamma = \mathcal{P}'_\gamma$, if $\gamma \ne \alpha$ and let $A_\alpha = A$. Apply Theorem 7.1. □

Using the last two theorems we obtain a version of Ash's Theorem [1].

7.2. A version of Ash's Theorem.

7.4. Theorem. *Let α be a recursive ordinal and $A \subseteq \mathbb{N}$. Suppose that for all total sets X such that $(\forall\gamma \le \zeta)(B_\gamma \le_e X^{(\gamma)})$ uniformly in γ we have that $A \le_e X^{(\alpha)}$. Then $A \le_e \mathcal{P}_\alpha$.*

Proof. Consider first the case $\alpha < \zeta$. Assume that $A \not\le_e \mathcal{P}_\alpha$. Use the previous theorem to get a contradiction.

Suppose now that $\zeta \le \alpha$. Extend the sequence B_γ by setting for all γ, $\zeta < \gamma \le \alpha$, $B_\gamma = \emptyset$. Clearly for every recursive γ the jump set of the extend sequence is equal to $\mathcal{P}_\gamma$.

Assume that $A \not\le_e \mathcal{P}_\alpha$. According Selman's Theorem [12], there exists a total set Q such that $\mathcal{P}_\alpha \le_e Q$ and $A \not\le_e Q$. From Theorem 7.2 it follows that there exists a total set F such that for all $\gamma \le \zeta$, $B_\gamma \le_e F^{(\gamma)}$ uniformly in γ and $F^{(\alpha)} \equiv_e Q$. Clearly $A \not\le_e F^{(\alpha)}$. A contradiction. □

The following result is proved in [3] for finite ordinals.

7.5. Corollary. *Let α be a recursive ordinal and A and B be subsets of $\mathbb{N}$. Suppose that for all total X,*

$$B \le_e X^{(\alpha)} \Rightarrow A \le_e X^{(\alpha)}.$$

Then $A \le_e \emptyset^{(\alpha)} \oplus B$.

Proof. Consider the sequence $\{B_\gamma\}_{\gamma \le \alpha}$, where $B_\gamma = \emptyset$ if $\gamma < \alpha$ and $B_\alpha = B$. Notice that $\mathcal{P}_\alpha = \emptyset^{(\alpha)} \oplus B$. □

Finally we have and the following variation of the results above:

7.6. Theorem. *Suppose that $\{A_\gamma\}_{\gamma<\zeta}$ is a sequence of sets such that for all total $X \subseteq \mathbb{N}$ satisfying*

$$(\forall\gamma \le \zeta)(B_\gamma \le_e X^{(\gamma)} \text{ uniformly in } \gamma), \tag{7.1}$$

we have that there exists at least one $\alpha < \zeta$ such that $A_\alpha \le_e X^{(\alpha)}$.

Then there exists an $\alpha < \zeta$ such that for all total X satisfying (7.1) *we have that $A_\alpha \le_e X^{(\alpha)}$.*

Proof. It is sufficient to show that there exists an $\alpha < \zeta$ such that $A_\alpha \leq_e \mathcal{P}_\alpha$. Towards a contradiction, assume that there is no such α. Apply Theorem 7.1 to get a contradiction. □

7.3. Elements of least Turing degree of families of sets. In their investigations of the jump degrees of linear orderings Ash, Jockush and Knight [2] and Downey and Knight [8] used the following result which shows that there exist families of sets of natural numbers which do not posses elements of least Turing degree:

7.7. Theorem. *Let α be a recursive ordinal and let S be an $(\alpha+1)$-generic set. Then the family*

$$\mathcal{S} = \{X^{(\alpha)} : X \textit{ is a total subset of } \mathbb{N} \textit{ and } S \leq_e X^{(\alpha)}\}$$

has no element of least Turing degree.

In contrast to this result stands the following Theorem, proved by Coles, Downey and Slaman in [4]:

7.8. Theorem. *Let $n < \omega$ and let $A \subseteq \mathbb{N}$. Then the family*

$$\mathcal{C}^{(n+1)}(A) = \{X^{(n+1)} : X \textit{ is a total subset of } \mathbb{N} \textit{ and } A \leq_e X^{(n)}\}$$

has an element of least Turing degree.

Both Theorems give partial solutions of the following more general problem. Let α and β be recursive ordinals and let $\{B_\gamma\}_{\gamma \leq \beta}$ be a sequence of sets of natural numbers. Set

$$\mathcal{S}_{\alpha,\beta} = \{X^{(\alpha)} : X \text{ is a total subset of } \mathbb{N} \text{ and } (\forall \gamma \leq \beta)(B_\gamma \leq_e X^{(\gamma)} \text{ uniformly in } \gamma)\}.$$

Now the problem is to determine when the family $S_{\alpha,\beta}$ possesses an element whose Turing degree is the least among the Turing degrees of the elements of $\mathcal{S}_{\alpha,\beta}$.

It turns out that if $\alpha > \beta$, then $\mathcal{S}_{\alpha,\beta}$ always has an element of least Turing degree.

7.9. Theorem. *Let $\alpha > \beta$. Then the Turing degree of the α-th jump set $\mathcal{P}_\alpha$ of the sequence $\{B_\gamma\}_{\gamma \leq \beta}$ is the least among the Turing degrees of the elements of $\mathcal{S}_{\alpha,\beta}$.*

Proof. Notice that since $\beta < \alpha$, the set $\mathcal{P}_\alpha$ is total. Consider and element $X^{(\alpha)}$ of $\mathcal{S}_{\alpha,\beta}$. Clearly $\mathcal{P}_\alpha \leq_e X^{(\alpha)}$ and hence, since both sets are total, $\mathcal{P}_\alpha \leq_T X^{(\alpha)}$. So to finish the proof it is sufficient to show that there exists an element Y of $\mathcal{S}_{\alpha,\beta}$ such that $Y \equiv_e \mathcal{P}_\alpha$.

Indeed, set for every γ such that $\beta < \gamma \leq \alpha$, $B_\gamma = \emptyset$. Clearly the α-th jump set of the extended sequence is equal to $\mathcal{P}_\alpha$. It follows from Theorem 7.2 that there exists a total set F such that $(\forall \gamma \leq \alpha)(B_\gamma \leq_e F^{(\gamma)}$ uniformly in $\gamma)$ and $F^{(\alpha)} \equiv_e \mathcal{P}_\alpha$. Obviously $F^{(\alpha)} \in \mathcal{S}_{\alpha,\beta}$. □

7.10. Corollary. *The Turing degree of $(A \oplus \emptyset^{(n)})'$ is the least among the Turing degrees of the elements of $\mathcal{C}^{(n+1)}(A)$.*

Proof. Let $B_k = \emptyset$, if $k < n$, and $B_n = A$. Then $\mathcal{C}^{(n+1)}(A) = \mathcal{S}_{n+1,n}$. Clearly $\mathcal{P}_{n+1} = (A \oplus \emptyset^{(n)})'$. □

The case $\alpha \le \beta$ is a little bit more complicated.

For every recursive ordinal ξ by $\mathcal{P}^*_{\alpha,\xi}$ denote the ξ-th jump set of the sequence $\{B_\gamma\}_{\gamma\le\alpha}$. As usual by $\mathcal{P}_\xi$ we shall denote the ξ-th jump set of the sequence $\{B_\gamma\}_{\gamma\le\beta}$.

Clearly $\mathcal{P}^*_{\alpha,\xi} = \mathcal{P}_\xi$ if $\xi \le \alpha$ and $\mathcal{P}^*_{\alpha,\xi} \le_e \mathcal{P}_\xi$ uniformly in ξ if $\xi > \alpha$

7.11. Theorem. *Let $\alpha \le \beta$. Then $\mathcal{S}_{\alpha,\beta}$ possesses an element of least Turing degree if and only if the following two conditions hold:*

(1) *The enumeration degree of $\mathcal{P}_\alpha$ is total, i.e. it contains a total set.*
(2) $(\forall\gamma \le \beta)(B_\gamma \le_e \mathcal{P}^*_{\alpha,\gamma}$ *uniformly in* $\gamma)$.

If there exists an element of $\mathcal{S}_{\alpha,\beta}$ of least Turing degree, then its enumeration degree is equal to the enumeration degree of $\mathcal{P}_\alpha$.

Proof. Suppose that $Y = X_0^{(\alpha)}$ is an element of $\mathcal{S}_{\alpha,\beta}$ of least Turing degree. Clearly $\mathcal{P}_\alpha \le_e Y$. Assume that $Y \not\le_e \mathcal{P}_\alpha$.

Suppose first that $\alpha < \beta$. Then according Theorem 7.3 there exists an element $F^{(\alpha)}$ of $\mathcal{S}_{\alpha,\beta}$ such that $Y \not\le_e F^{(\alpha)}$ and hence $Y \not\le_T F^{(\alpha)}$. A contradiction.

Let $\alpha = \beta$. From Selman's Theorem [12] it follows that there exists a total set Q such that $\mathcal{P}_\alpha \le_e Q$ and $Y \not\le_e Q$. From Theorem 7.2 it follows that there exists an element $F^{(\alpha)}$ of $\mathcal{S}_{\alpha,\beta}$ such that $F^{(\alpha)} \equiv_e Q$. From here we get again that $Y \not\le_T F^{(\alpha)}$ which is a contradiction.

So $Y \le_e \mathcal{P}_\alpha$ and hence $Y \equiv_e \mathcal{P}_\alpha$. Since Y is a total set, the enumeration degree of $\mathcal{P}_\alpha$ is total. This proves (1).

To prove (2) notice that from $X_0^{(\alpha)} \equiv_e \mathcal{P}_\alpha$ it follows that for every $\gamma \ge \alpha$ we have that $\mathcal{P}^*_{\alpha,\gamma} \equiv_e X_0^{(\gamma)}$ uniformly in γ.

Suppose now that (1) and (2) hold. Let Q be a total set such that $Q \equiv_e \mathcal{P}_\alpha$. According Theorem 7.2 there exists a total F such that for all $\gamma \le \alpha$ we have that $B_\gamma \le_e F^{(\gamma)}$ uniformly in γ and $F^{(\alpha)} \equiv_e Q$. Clearly for every $\gamma \ge \alpha$ we have that $F^{(\gamma)} \equiv_e \mathcal{P}^*_{\alpha,\gamma}$ uniformly in γ. Combining this with (2) we get that $F^{(\alpha)} \in \mathcal{S}_{\alpha,\beta}$.

Consider an element Y of $\mathcal{S}_{\alpha,\beta}$. Obviously $\mathcal{P}_\alpha \le_e Y$ and hence $F^{(\alpha)} \le_e Y$. Since both sets are total, $F^{(\alpha)} \le_T Y$. $\square$

7.12. Corollary. *If for some α_0 the family $\mathcal{S}_{\alpha_0,\beta}$ possesses an element of least Turing degree, then for every $\alpha \ge \alpha_0$ the family $\mathcal{S}_{\alpha,\beta}$ possesses an element of least Turing degree.*

Proof. Let $\alpha_0 < \alpha$. If $\beta < \alpha$, then the family $\mathcal{S}_{\alpha,\beta}$ contains an element of least degree according Theorem 7.9. Suppose that $\alpha \le \beta$. Then $\mathcal{P}_\alpha \equiv_e \mathcal{P}^*_{\alpha_0,\alpha}$. Since $\alpha > \alpha_0$, the set $\mathcal{P}^*_{\alpha_0,\alpha}$ is total. Hence the enumeration degree of $\mathcal{P}_\alpha$ is total. So the condition (1) from the Theorem above is true for α. The truth of (2) for α follows from the fact that for all γ, $\mathcal{P}^*_{\alpha_0,\gamma} \le_e \mathcal{P}^*_{\alpha,\gamma}$ uniformly in γ. $\square$

7.13. Corollary. *If $\alpha = \beta$, then the family $\mathcal{S}_{\alpha,\beta}$ contains an element of least degree if and only if the enumeration degree of $\mathcal{P}_\alpha$ is total.*

So to obtain families $\mathcal{S}_{\alpha,\alpha}$ without an element of least Turing degree it is sufficient to construct a sequence $\{B_\gamma\}_{\gamma\leq\alpha}$ such that the enumeration degree of $\mathcal{P}_\alpha$ does not contain a total set. This can be done by several methods known from the theory of the enumeration degrees. Here we are going to use generic sets following the ideas and results of Copestake [7].

By τ, ρ we shall denote two-valued strings, i.e. finite mappings from initial segments of $\mathbb{N}$ into $\{0,1\}$. Given a set S and a string τ, by $\tau \subseteq S$ we shall denote that $\tau \subseteq \chi_S$, where χ_S is the characteristic function of S. As usual $\tau \subseteq \rho$ will denote that the string ρ extends the string τ.

7.14. Definition. Let $P \subseteq \mathbb{N}$. A set S is P-generic if for every set $W \leq_e P$ of strings the following condition holds:

$$(\exists \tau \subseteq S)(\tau \in W \vee (\forall \rho \supseteq \tau)(\rho \notin W)).$$

In particular if $P = \emptyset^{(\alpha)}$ then the P-generic sets are the well known $(\alpha+1)$-generic sets. In [7] Copestake studies the enumeration degrees of the 1-generic sets. The properties of the P-generic sets are very similar to the properties of the 1-generic sets. Here we shall list some of them omitting the proofs which are either obvious or a straightforward generalization of the respective proofs from [7].

Let S be a P-generic set. Then the following assertions are true:

(G1) $S \not\leq_e P$ and hence $P <_e S \oplus P$.

(G2) (see Theorem 3.12 of [7]) Let φ be a partial function and $\varphi \leq_e S \oplus P$. Then φ has an extension ψ such that $\psi \leq_e P$.

(G3) Suppose that B is a total set and $B \leq_e S \oplus P$. Then $B \leq_e P$.

Combining (G1) and (G3) we get that the enumeration degree of $S \oplus P$ is not total.

Now consider a recursive ordinal α. Let $\{B_\gamma\}_{\gamma<\alpha}$ be an arbitrary sequence of subsets of $\mathbb{N}$. Let

$$P = \begin{cases} \emptyset & \text{if } \alpha = 0, \\ \mathcal{P}'_\beta & \text{if } \alpha = \beta + 1, \\ \mathcal{P}_{<\alpha} & \text{if } \alpha \text{ is a limit ordinal} \end{cases}$$

and let $B_\alpha = S$, where S is a P-generic set.

Clearly $\mathcal{P}_\alpha \equiv_e P \oplus S$ and hence the respective family $\mathcal{S}_{\alpha,\alpha}$ has no element of least Turing degree.

In particular, if we set $B_\gamma = \emptyset$ for all $\gamma < \alpha$, then we obtain the Theorem from [2] and [8], formulated at the beginning of the subsection.

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