A JUMP INVERSION THEOREM FOR THE ENUMERATION JUMP

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ABSTRACT. We prove a jump inversion theorem for the enumeration jump and a minimal pair type theorem for the enumeration reducibilty. As an application some results of Selman, Case and Ash are obtained.

1. INTRODUCTION

Given two sets of natural numbers A and B, we say that A is enumeration reducible to B ($A \leq_e B$) if $A = \Gamma_z(B)$ for some enumeration operator Γ_z. In other words, using the notation D_v for the finite set having canonical code v and $W_0, \ldots, W_z, \ldots$ for the Gödel enumeration of the r.e. sets, we have

$$A \leq_e B \iff \exists z \forall x (x \in A \iff \exists v (\langle v, x \rangle \in W_z \ \& \ D_v \subseteq B)$$

The relation $\leq_e$ is reflexive and transitive and induces an equivalence relation $\equiv_e$ on all subsets of $\mathbb{N}$. The respective equivalence classes are called enumeration degrees. For an introduction to the enumeration degrees the reader might consult COOPER [4].

Given a set A denote by A^+ the set $A \oplus (\mathbb{N} \setminus A)$. The set A is called *total* iff $A \equiv_e A^+$. Clearly A is recursively enumerable in B iff $A \leq_e B^+$ and A is recursive in B iff $A^+ \leq_e B^+$. Notice that the graph of every total function is a total set.

The enumeration jump operator is defined in COOPER [3] and further studied by MCEVOY [5]. Here we shall use the following definition of the e-jump which is m-equivalent to the original one, see [5]:

1.1. Definition. Given a set A, let $K_A^0 = \{\langle x, z \rangle : x \in \Gamma_z(A)\}$. Define the *e-jump* A'_e of A to be the set $(K_A^0)^+$.

Several properties of the e-jump are proved in [5]. Among them it is shown that the e-jump is monotone, agrees with $\equiv_e$ and that for any sets A and B, A is Σ^0_{n+1} relatively to B iff $A \leq_e (B^+)^{(n)}_e$, where for every set B, $B^{(0)}_e = B$ and $B^{(n+1)}_e$ is the e-jump of $B^{(n)}_e$.

Though for total sets the e-jump and the Turing jump are enumeration equivalent, in the general case this is not true. So, for example, the e-jump of Kleene's set K is

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enumeration equivalent to $\emptyset'$ while the Turing jump of K is enumeration equivalent to $\emptyset''$.

Since we are going to consider only e-jumps here, from now on we shall omit the subscript e in the notation of the e-jump. So for any set A by $A^{(n)}$ we shall denote the n-th e-jump of A.

In [5] several analogs of the known jump-inversion theorems for the Turing reducibility are proved, but the relativised versions are not considered. So the following natural question is left open. Given a set B, does there exist a total set F such that $B \leq_e F$ and $B' \equiv_e F'\Gamma$

In the present paper we are going to prove the following result which gives a positive answer to the question above. Given $k+1$ sets $B_0, \dots, B_k$, we define for every $i \leq k$ the set $P(B_0, \dots, B_i)$ by means of the following inductive definition:

(i) $P(B_0) = B_0$;
(ii) If $i < k$, then $P(B_0, \dots, B_{i+1}) = (P(B_0, \dots, B_i))' \oplus B_{i+1}$.

1.2. Theorem. *Let $k \geq 0$ and $B_0, \dots, B_k$ be arbitrary sets of natural numbers. Let Q be a total set and $\mathcal{P}(B_0, \dots, B_k) \leq_e Q$. There exists a total set F having the following properties:*

(i) *For all $i \leq k$, $B_i \in \Sigma^F_{i+1}$;*
(ii) *For all $i, 1 \leq i \leq k$, $F^{(i)} \equiv_e F \oplus \mathcal{P}(B_0, \dots, B_{i-1})'$;*
(iii) *$F^{(k)} \equiv_e Q$.*

Notice that if $B_0 = \dots = B_k = \emptyset$, then $\mathcal{P}(B_0, \dots, B_k) \equiv_e \emptyset^{(k)}$ and hence, since both sets are total, they are Turing equivalent. So Theorem 1.2 is a generalization of Friedberg's jump-inversion theorem.

We shall also prove the following "type omitting" version of the above theorem:

1.3. Theorem. *Let $k > n \geq 0$, $B_0, \dots, B_k$ be arbitrary sets of natural numbers. Let $A \subseteq \mathbb{N}$ and let Q be a total subset of $\mathbb{N}$ such that $\mathcal{P}(B_0, \dots, B_k) \leq_e Q$ and $A^+ \leq_e Q$. Suppose also that $A \not\leq_e \mathcal{P}(B_0, \dots, B_n)$. Then there exists a total set F having the following properties:*

(i) *For all $i \leq k$, $B_i \in \Sigma^F_{i+1}$;*
(ii) *For all $i, 1 \leq i \leq k$, $F^{(i)} \equiv_e F \oplus \mathcal{P}(B_0, \dots, B_{i-1})'$;*
(iii) *$F^{(k)} \equiv_e Q$.*
(iv) *$A \not\leq_e F^{(n)}$.*

In [8] Selman gives the following characterization of the enumeration reducibility in terms of the relation "recursively enumerable in":

$$A \leq_e B \iff \forall X (B \text{ is r.e. in } X \Rightarrow A \text{ is r.e. in } X).$$

As an application of the so far formulated theorems we can get an upper bound of the universal quantifier in the equivalence above:

1.4. Theorem. *$A \leq_e B$ iff for all total X, B is r.e in X and $X' \equiv_e B'$ implies A is r.e. in X.*

Proof. Clearly for total X, B is r.e. in X iff $B \leq_e X$. Now suppose that for all total X, $B \leq_e X$ & $X' \equiv_e B' \Rightarrow A \leq_e X$. First we shall show that $A^+ \leq_e B'$. Indeed, by Theorem 1.2, there exists a total G such that $B \leq_e G$ and $G' \equiv_e B'$. Then $A \leq_e G$ and hence $A' \leq_e G' \leq_e B'$. So since $A^+ \leq_e A'$, $A^+ \leq_e B'$.

Assume that $A \not\leq_e B$. Apply Theorem 1.3 for $k = 1, n = 0, B_0 = B, B_1 = \emptyset$ and $Q = B'$ to get a total F such that $B \leq_e F, F' \equiv_e B'$ and $A \not\leq_e F$. A contradiction. □

Selman's theorem is further generalized in Case [2], where it is shown that for all $n \geq 0$,

$$A \leq_e B \oplus \emptyset^{(n)} \iff \forall X (B \in \Sigma^X_{n+1} \Rightarrow A \in \Sigma^X_{n+1}).$$

Finally Ash [1] studies the general case and characterizes by a certain kind of formally described reducibilities for any given $k+2$ sets $A, B_0, \ldots, B_k$ the relations

$$\mathcal{R}^n_k(A, B_0, \ldots, B_k) \iff \forall X (B_0 \in \Sigma^X_1, \ldots, B_k \in \Sigma^X_{k+1} \Rightarrow A \in \Sigma^X_{n+1}).$$

By an almost direct application of Theorem 1.2 and Theorem 1.3 we obtain the following version of Ash's result:

1.5. Theorem.

(1) *For all* $n < k$, $\mathcal{R}^n_k(A, B_0, \ldots, B_k) \iff A \leq_e \mathcal{P}(B_0, \ldots, B_n)$.
(2) *For all* $n \geq k$, $\mathcal{R}^n_k(A, B_0, \ldots, B_k) \iff A \leq_e \mathcal{P}(B_0, \ldots, B_k)^{(n-k)}$.

Proof. The right to left implications of (1) and (2) are trivial.

Consider the left to right direction of (1). Towards a contradiction suppose that $n < k$, $\mathcal{R}^n_k(A, B_0, \ldots, B_k)$ and $A \not\leq_e \mathcal{P}(B_0, \ldots, B_n)$. By Theorem 1.3, there exists a total F, such that $A \not\leq_e F^{(n)}$ and for all $i \leq k$, $B_i \in \Sigma^F_{i+1}$. Clearly $A \notin \Sigma^F_{n+1}$. A contradiction.

To prove (2) in the non trivial direction assume that $n \geq k, \mathcal{R}^n_k(A, B_0, \ldots, B_k)$ and $A \not\leq_e \mathcal{P}(B_0, \ldots, B_k)^{(n-k)}$. By Selman's theorem, there exists a total Q such that $\mathcal{P}(B_0, \ldots, B_k)^{(n-k)} \leq_e Q$ and $A \not\leq_e Q$. Set $B_{k+1} = \cdots = B_n = \emptyset$. Then $\mathcal{P}(B_0, \ldots, B_n) \equiv_e \mathcal{P}(B_0, \ldots, B_k)^{(n-k)}$. By Theorem 1.2 there exists a total F such that $F^{(n)} \equiv_e Q$ and for all $i \leq k, B_i \in \Sigma^F_{i+1}$. Clearly $A \not\leq_e F^{(n)}$ and hence $A \notin \Sigma^F_{n+1}$. A contradiction. □

A proof very close to that of Theorem 1.4 gives upper bounds of the universal quantifiers in the definitions of the relations $\mathcal{R}^n_k$.

1.6. Corollary.

(1) *Let* $n < k$. *Suppose that* S *is a total subset of* $\mathbb{N}$ *and* $\mathcal{P}(B_0, \ldots, B_k) \leq_e S$. *Then* $\mathcal{R}^n_k(A, B_0, \ldots, B_k)$ *iff for all total* X *such that* $X^{(k)} \equiv_e S$,

$$B_0 \in \Sigma^X_1, \ldots, B_k \in \Sigma^X_{k+1} \Rightarrow A \in \Sigma^X_{n+1}.$$

(2) *Let* $k \leq n$. *Then* $\mathcal{R}^n_k(A, B_0, \ldots, B_k)$ *iff for all total* X *such that* $X^{(n+1)} \equiv_e \mathcal{P}(B_0, \ldots, B_k)^{(n-k+1)}$,

$$B_0 \in \Sigma^X_1, \ldots, B_k \in \Sigma^X_{k+1} \Rightarrow A \in \Sigma^X_{n+1}.$$

Clearly the result of Case can be obtained from Theorem 1.5 by setting $k = n$ and $B_0 = \cdots = B_{n-1} = \emptyset$, $B_n = B$. Another corollary which is worth mentioning is obtained in the case $k = 0$, $n \geq 0$ and $B_0 = B$:

$$A \leq_e B^{(n)} \iff \forall X (B \in \Sigma_1^X \Rightarrow A \in \Sigma_{n+1}^X).$$

We conclude the introduction with a Minimal pair type theorem which generalizes the so far described Selman-Case-Ash results:

1.7. Theorem. *Let $k \geq 0$ and $B_0, \dots, B_k$ be arbitrary sets of natural numbers. There exist total sets F and G such that $F^{(k+2)} \equiv_e \mathcal{P}(B_0 \dots, B_k)''$ and $G^{(k+2)} \equiv_e \mathcal{P}(B_0 \dots, B_k)''$ and*

(i) *For all $n \leq k$, $\mathcal{P}(B_0, \dots, B_n) <_e F^{(n)}$ and $\mathcal{P}(B_0, \dots, B_n) <_e G^{(n)}$.*
(ii) *If $n \leq k$, $A \leq_e F^{(n)}$ and $A \leq_e G^{(n)}$, then $A \leq_e \mathcal{P}(B_0, \dots, B_n)$.*

An immediate corollary of the last Theorem is a result of Rozinas [7] that there exist a minimal pair of total e-degrees **f, g** over every e-degree **b**.

Clearly the minimal pair **f, g** could be constructed below $\mathbf{b}''$. So Theorem 1.7 generalizes Selman's theorem but does not generalize Theorem 1.4. A natural improvement of the last result would be to show that the degrees **f, g** could be constructed below $\mathbf{b}'$. This would give a generalization of the respective result of McEvoy and Cooper [6] where a minimal pair of enumeration degrees below $\mathbf{0}'$ is constructed.

The proofs of our results use of the machinery of the so called regular enumerations, described in the next section. Section 3 contains the final proofs. In the last section 4 a version of Theorem 1.7 involving infinite sequences of sets is presented.

2. Regular Enumerations

Let us fix $k \geq 0$ and subsets $B_0, \dots, B_k$ of $\mathbb{N}$. Since every set B is enumeration equivalent to $B \oplus \mathbb{N} = \{2x : x \in B\} \cup \{2x+1 : x \in \mathbb{N}\}$, we may assume that $B_0, \dots, B_k$ are not empty.

In what follows we shall use the term *finite part* for finite mappings of $\mathbb{N}$ into $\mathbb{N}$ defined on finite segments $[0, q-1]$ of $\mathbb{N}$. Finite parts will be denoted by the letters τ, δ, ρ. If $\mathrm{dom}(\tau) = [0, q-1]$, then let $\mathrm{lh}(\tau) = q$.

We shall suppose that an effective coding of all finite sequences and hence of all finite parts is fixed. Given two finite parts τ and ρ we shall say that τ is less than or equal to ρ if the code of τ is less than or equal to the code of ρ. By $\tau \subseteq \rho$ we shall denote that the partial mapping ρ extends τ and say that ρ is an extension of τ. For any τ, by $\tau \restriction n$ we shall denote the restriction of τ on $[0, n-1]$.

Bellow we define for every $i \leq k$ the i-regular finite parts.

The 0-*regular finite parts* are finite parts τ such that $\mathrm{dom}(\tau) = [0, 2q+1]$ and for all odd $z \in \mathrm{dom}(\tau)$, $\tau(z) \in B_0$.

If $\mathrm{dom}(\tau) = [0, 2q+1]$, then the 0-rank $|\tau|_0$ of τ is equal to the number $q+1$ of the odd elements of $\mathrm{dom}(\tau)$. Notice that if τ and ρ are 0-regular, $\tau \subseteq \rho$ and $|\tau|_0 = |\rho|_0$, then $\tau = \rho$.

For every 0-regular finite part τ, let B_0^τ be the set of the odd elements of $\mathrm{dom}(\tau)$.

Given a 0-regular finite part τ, let

$$\tau \Vdash_0 F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)(\tau((u)_0) \simeq (u)_1))$$

$$\tau \Vdash_0 \neg F_e(x) \iff \forall(0\text{-regular } \rho)(\tau \subseteq \rho \Rightarrow \rho \nVdash_0 F_e(x)).$$

Proceeding by induction, suppose that for some $i < k$ we have defined the i-regular finite parts and for every i-regular τ – the i-rank $|\tau|_i$ of τ, the set B_i^τ and the relations $\tau \Vdash_i F_e(x)$ and $\tau \Vdash_i \neg F_e(x)$. Suppose also that if τ and ρ are i-regular, $\tau \subseteq \rho$ and $|\tau|_i = |\rho|_i$, then $\tau = \rho$.

Set $X_j^i = \{\rho : \rho \text{ is } i\text{-regular } \&\ \rho \Vdash_i F_{(j)_0}((j)_1)\}$.

Given a finite part τ and a set X of i-regular finite parts, let $\mu_i(\tau, X)$ be the least extension of τ belonging to X if any, and $\mu_i(\tau, X)$ be the least i-regular extension of τ otherwise. We shall assume that $\mu_i(\tau, X)$ is undefined if there is no i-regular extension of τ.

2.1. Definition. Let τ be a finite part and $m \geq 0$. Say that δ is an *i-regular m omitting extension* of τ if δ is an i-regular extension of τ, defined on $[0, q-1]$ and there exist natural numbers $q_0 < \cdots < q_m < q_{m+1} = q$ such that:

a) $\delta \restriction q_0 = \tau$.
b) For all $p \leq m$, $\delta \restriction q_{p+1} = \mu_i(\delta \restriction (q_p + 1), X^i_{\langle p, q_p\rangle})$.

Notice that if δ is an i-regular m omitting extension of τ, then there exists a unique sequence of natural numbers $q_0, \ldots, q_{m+1}$ having the properties a) and b) above. We shall denote the sequence $q_0, \ldots, q_m$ by K_τ^δ. Moreover if δ and ρ are two i-regular m omitting extensions of τ and $\delta \subseteq \rho$, then $\delta = \rho$.

Let $\mathcal{R}_i$ denote the set of all i-regular finite parts. Given an index j, by S_j^i we shall denote the intersection $\mathcal{R}_i \cap \Gamma_j(\mathcal{P}(B_0, \ldots, B_i))$, where Γ_j is the j-th enumeration operator.

Let τ be a finite part defined on $[0, q-1]$ and $r \geq 0$. Then τ is $(i+1)$-*regular* with $(i+1)$-*rank* $r+1$ if there exist natural numbers

$$0 < n_0 < l_0 < b_0 < n_1 < l_1 < b_1 \cdots < n_r < l_r < b_r < n_{r+1} = q$$

such that $\tau \restriction n_0$ is an i-regular finite part with i-rank equal to 1 and for all j, $0 \leq j \leq r$, the following conditions are satisfied:

a) $\tau \restriction l_j \simeq \mu_i(\tau \restriction (n_j + 1), S_j^i)$;
b) $\tau \restriction b_j$ is an i-regular j omitting extension of $\tau \restriction l_j$;
c) $\tau(b_j) \in B_{i+1}$;
d) $\tau \restriction n_{j+1}$ is an i-regular extension of $\tau \restriction (b_j + 1)$ with i-rank equal to $|\tau \restriction b_j|_i + 1$

The following Lemma shows that the $(i+1)$-rank is well defined.

2.2. Lemma. *Let τ be an $(i+1)$-regular finite part. Then*

(1) *Let $m_0, q_0, a_0, \ldots, m_p, q_p, a_p, m_{p+1}$ and $n_0, l_0, b_0, \ldots, n_r, l_r, b_r, n_{r+1}$ be two sequences of natural numbers satisfying a)–d). Then $r = p$, $n_{p+1} = m_{p+1}$ and for all $j \leq r$, $n_j = m_j$, $l_j = q_j$ and $b_j = a_j$.*
(2) *If ρ is $(i+1)$-regular, $\tau \subseteq \rho$ and $|\tau|_{i+1} = |\rho|_{i+1}$, then $\tau = \rho$.*
(3) *τ is i-regular and $|\tau|_i > |\tau|_{i+1}$.*

Proof. The proof follows easily from the definition of the $(i+1)$-regular finite parts and from the respective properties of the i-regular finite parts. $\square$

Let τ be $(i+1)$-regular and $n_0, l_0, b_0, \ldots, n_r, l_r, b_r, n_{r+1}$ be the sequence satisfying a)–d). Then let $B^\tau_{i+1} = \{b_0, \ldots, b_r\}$. By K^τ_{i+1} we shall denote the sequence $K^{\tau\lceil b_r}_{\tau\lceil l_r}$. Notice that, since $\tau\lceil b_r$ is an r omitting extension of $\tau\lceil l_r$, the sequence $K^{\tau\lceil b_r}_{\tau\lceil l_r}$ has exactly $r+1$ members.

To conclude with the definition of the regular finite parts, let for every $(i+1)$-regular finite part τ

$$\tau \Vdash_{i+1} F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \ \&\ (\forall u \in D_v)((u = \langle e_u, x_u, 0\rangle \ \&\ \tau \Vdash_i F_{e_u}(x_u))\vee (u = \langle e_u, x_u, 1\rangle \ \&\ \tau \Vdash_i \neg F_{e_u}(x_u)))).$$

$$\tau \Vdash_{i+1} \neg F_e(x) \iff (\forall (i+1)\text{-regular } \rho)(\tau \subseteq \rho \Rightarrow \rho \nVdash_{i+1} F_e(x)).$$

2.3. Definition. Let f be a total mapping of $\mathbb{N}$ in $\mathbb{N}$. Then f is a *regular enumeration* if the following two conditions hold:

(i) For every finite part $\delta \subseteq f$, there exists a k-regular extension τ of δ such that $\tau \subseteq f$.

(ii) If $i \leq k$ and $z \in B_i$, then there exists an i-regular $\tau \subseteq f$, such that $z \in \tau(B^\tau_i)$.

Clearly, if f is a regular enumeration and $i \leq k$, then for every $\delta \subseteq f$, there exists an i-regular $\tau \subseteq f$ such that $\delta \subseteq \tau$. Moreover there exist i-regular finite parts of f of arbitrary large rank.

Given a regular f, let for $i \leq k$, $B^f_i = \{b : (\exists \tau \subseteq f)(\tau \text{ is } i\text{-regular } \&\ b \in B^\tau_i)\}$. Clearly $f(B^f_i) = B_i$.

2.4. Definition. A sequence $A_0, \ldots, A_n, \ldots$ of subsets of $\mathbb{N}$ is *e-reducible* to the set P iff there exists a recursive function h such that for all n, $A_n = \Gamma_{h(n)}(P)$. The sequence $\{A_n\}$ is *T-reducible* to P if there exists a recursive in P function χ such that for all n, $\lambda x.\chi(n, x) = \chi_{A_n}$, where χ_{A_n} denotes the characteristic function of A_n.

2.5. Lemma. *Suppose that the sequence $\{A_n\}$ is e-reducible to P. Then the following assertions hold:*

(1) *The sequence $\{A_n\}$ is T-reducible to P'.*

(2) *If $R \leq_e P$, then either of the following sequences is e-reducible to P:*

  a) $\{R \cap A_n\}$;

  b) $\{C_n\}$, *where* $C_n = \{x : \exists y(\langle x, y\rangle \in R \ \&\ y \in A_n)\}$.

Proof. Let h be a recursive function such that for all n, $A_n = \Gamma_{h(n)}(P)$.

The proof of (1) follows easily from the definition of the e-jump. Indeed,

$$x \in A_n \iff x \in \Gamma_{h(n)}(P) \iff \langle x, h(n)\rangle \in K^0_P \iff 2\langle x, h(n)\rangle \in P'.$$

$$x \notin A_n \iff x \notin \Gamma_{h(n)}(P) \iff \langle x, h(n)\rangle \notin K^0_P \iff 2\langle x, h(n)\rangle + 1 \in P'.$$

To prove the part b) of (2) notice that for every n

$$x \in C_n \iff \exists y(\langle x,y\rangle \in R \;\&\; \exists v(\langle v,y\rangle \in W_{h(n)} \;\&\; D_v \subseteq P)).$$

Let $R = \Gamma_z(P)$. Then $\langle x,y\rangle \in R \iff \exists u(\langle u,\langle x,y\rangle\rangle \in W_z \;\&\; D_u \subseteq P)$.

Clearly there exists a recursive function g such that

$$\langle w,x\rangle \in W_{g(n)} \iff \exists y \exists u \exists v(\langle u,\langle x,y\rangle\rangle \in W_z \;\&\; \langle v,y\rangle \in W_{h(n)} \;\&\; D_w = D_u \cup D_v).$$

Then $x \in C_n \iff \exists w(\langle w,x\rangle \in W_{g(n)} \;\&\; D_w \subseteq P)$. Thus $C_n = \Gamma_{g(n)}(P)$.

The proof of the a) part of (1) is similar. □

Let $i \le k$. Set $P_i = \mathcal{P}(B_0,\dots,B_i)$. Notice that if $i < k$, then $P_{i+1} = P_i' \oplus B_{i+1}$.

For $j \in \mathbb{N}$ let $\mu_i^X(\tau,j) \simeq \mu_i(\tau,X_j^i)$, $\mu_i^S(\tau,j) \simeq \mu_i(\tau,S_j^i)$,

$$Y_j^i = \{\tau : (\exists \rho \supseteq \tau)(\rho \text{ is } i\text{-regular} \;\&\; \rho \Vdash_i F_{(j)_0}((j)_1))\}$$

$$Z_j^i = \{\tau : \tau \text{ is } i\text{-regular} \;\&\; \tau \Vdash_i \neg F_{(j)_0}((j)_1)\} \text{ and}$$

$$O_{\tau,j}^i = \{\rho : \rho \text{ is } i\text{-regular } j \text{ omitting extension of } \tau\}.$$

2.6. Proposition. *For every $i \le k$ the following assertions hold:*

(1) *The set $\mathcal{R}_i$ of all i-regular finite parts is e-reducible to P_i.*
(2) *The function $\lambda\tau.|\tau|_i$ (assumed undefined if $\tau \notin \mathcal{R}_i$) is e-reducible to P_i.*
(3) *The sequences $\{S_j^i\}$, $\{X_j^i\}$ and $\{Y_j^i\}$ are e-reducible to P_i.*
(4) *The sequence $\{Z_j^i\}$ is T-reducible to P_i'.*
(5) *The functions μ_i^X and μ_i^S are partial recursive in P_i'.*
(6) *The sequence $\{O_{\tau,j}^i\}$ is e-reducible to P_i'.*

Proof. The proof is by induction on i. Suppose that $i = 0$. The validity of (1)–(6) follows easily from the definitions of the 0-regular finite parts and the relation "$\Vdash_0$" and Lemma 2.5.

Suppose that for some $i < k$ the assertions (1)–(6) hold. Now the validity of (1) and (2) for $i+1$ follows directly from the definition of the $(i+1)$-regular finite parts. Since $\mathcal{R}_{i+1} \le_e P_{i+1}$, by Lemma 2.5 the sequence $\{S_j^{i+1}\}$ is e-reducible to P_{i+1}. Further, by induction and by Lemma 2.5 the sequence $\{X_j^i\}$ is T-reducible to P_i'. By induction $\{Z_j^i\}$ is also T-reducible to P_i'. From here it follows that the sets $\{\tau : \tau \Vdash_{i+1} F_e(x)\}$ are uniformly in e and x r. e. in P_i' and therefore these sets are uniformly e-reducible to P_i'. We have that

$$\tau \in X_j^{i+1} \iff \tau \in \mathcal{R}_{i+1} \;\&\; \tau \Vdash_{i+1} F_{(j)_0}((j)_1).$$

Hence the sequence $\{X_j^{i+1}\}$ is e-reducible to P_{i+1}. Then by Lemma 2.5 the sequence $\{Y_j^{i+1}\}$ is e-reducible to P_{i+1} and hence it is uniformly T-reducible to P_{i+1}'. From here, since $Z_j^{i+1} = \mathcal{R}_{i+1} \setminus Y_j^{i+1}$, we get the validity of (4) for $i+1$. Now the truth of (5) and (6) for $i+1$ follows directly from the respective definitions. □

2.7. Corollary. *For every $i \le k$ and every j, X_j^i is a member of the sequence $\{S_j^i\}$.*

2.8. Proposition. *Suppose that f is a regular enumeration. Then*

(1) *$B_0 \le_e f$.*

(2) *If* $i < k$, *then* $B_{i+1} \leq_e f \oplus P'_i$.
(3) *If* $i \leq k$, *then* $P_i \leq_e f^{(i)}$.

Proof. Since f is regular, $B_0 = f(B_0^f)$. Clearly B_0^f is equal to the set of all odd natural numbers. So, $B_0 \leq_e f$.

Let us turn to the proof of (2). Fix an $i < k$. Since f is regular, for every finite part δ of f there exists an $(i+1)$-regular $\tau \subseteq f$ such that $\delta \subseteq \tau$. Hence there exist natural numbers

$$0 < n_0 < l_0 < b_0 < n_1 < l_1 < b_1 < \cdots < n_r < l_r < b_r < \ldots,$$

such that for every $r \geq 0$, the finite part $\tau_r = f\restriction n_{r+1}$ is $(i+1)$-regular and $n_0, l_0, b_0, \ldots, n_r, l_r, b_r, n_{r+1}$ are the numbers satisfying the conditions a)–d) from the definition of the $(i+1)$-regular finite part τ_r. Clearly $B_{i+1}^f = \{b_0, b_1 \ldots\}$. We shall show that there exists a recursive in $f \oplus P'_i$ procedure which lists $n_0, l_0, b_0, \ldots$ in an increasing order.

Clearly $f\restriction n_0$ is i-regular and $|f\restriction n_0|_i = 1$. By Lemma 2.6 $\mathcal{R}_i$ is recursive in P'_i. Using f we can generate consecutively the finite parts $f\restriction q$ for $q = 1, 2 \ldots$. By Lemma 2.2 $f\restriction n_0$ is the first element of this sequence which belongs to $\mathcal{R}_i$. Clearly $n_0 = \mathrm{lh}(f\restriction n_0)$.

Suppose that $r \geq -1$ and $n_0, l_0, b_0, \ldots, n_r, l_r, b_r, n_{r+1}$ have already been listed. Since $f\restriction l_{r+1} \simeq \mu_i(f\restriction(n_{r+1}+1), S^i_{r+1})$, we can find recursively in $f \oplus P'_i$ the finite part $f\restriction l_{r+1}$. Then $l_{r+1} = \mathrm{lh}(f\restriction l_{r+1})$. Next we have that $f\restriction b_{r+1}$ is an i-regular $(r+1)$ omitting extension of $f\restriction l_{r+1}$. So there exist natural numbers $l_{r+1} = q_0 < \cdots < q_{r+1} < q_{r+2} = b_{r+1}$ such that for $p \leq r+1$,

$$f\restriction q_{p+1} \simeq \mu_i(f\restriction(q_p+1), X^i_{\langle p, q_p\rangle}).$$

Using the oracle $f \oplus P'_i$ we can find consecutively the numbers q_p and the finite parts $f\restriction(q_p+1)$, $p = 0, \ldots, r+2$. By the end of this procedure we reach b_{r+1}. It remains to show that we can find the number n_{r+2}. By definition $f\restriction n_{r+2}$ is an i-regular extension of $f\restriction(b_{r+1}+1)$ having i-rank equal to $|f\restriction b_{r+1}|_i + 1$. Using f we can generate consecutively the finite parts $f\restriction(b_{r+1}+1+q)$, $q = 0, 1, \ldots$. By Lemma 2.2 $f\restriction n_{r+2}$ is the first element of this sequence which belongs to $\mathcal{R}_i$.

So B^f_{i+1} is recursive in $f \oplus P'_i$. Hence, since $B_{i+1} = f(B^f_{i+1})$, $B_{i+1} \leq_e f \oplus P'_i$.

We shall prove (3) by induction on i. Clearly $P_0 = B_0 \leq_e f$. Suppose that $i < k$ and $P_i \leq_e f^{(i)}$. Then $B_{i+1} \leq_e f \oplus P'_i \leq_e f^{(i+1)}$. Therefore $P_{i+1} = P'_i \oplus B_{i+1} \leq_e f^{(i+1)}$. □

Let f be a total mapping on $\mathbb{N}$. We define for every $i \leq k, e, x$ the relation $f \models_i F_e(x)$ by induction on i:

2.9. Definition.

(i) $f \models_0 F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \;\&\; (\forall u \in D_v)(f((u)_0) = (u)_1))$;

(ii) $$f \models_{i+1} F_e(x) \iff \exists v(\langle v, x\rangle \in W_e \;\&\; (\forall u \in D_v)((u = \langle e_u, x_u, 0\rangle \;\&\; f \models_i F_{e_u}(x_u)) \vee (u = \langle e_u, x_u, 1\rangle \;\&\; f \not\models_i F_{e_u}(x_u)))).$$

Set $f \models_i \neg F_e(x) \iff f \not\models_i F_e(x)$.

The following Lemma can be proved by induction on i.

2.10. Lemma. *Let f be a total mapping on $\mathbb{N}$ and $i \leq k$. Then $A \in \Sigma^f_{i+1}$ iff there exists e such that for all x, $x \in A \iff f \models_i F_e(x)$.*

Our next goal is the proof of the Truth Lemma. Notice that for all $i \leq k$ the relation $\Vdash_i$ is monotone, i.e. if $\tau \subseteq \rho$ are i-regular and $\tau \Vdash_i (\neg)F_e(x)$, then $\rho \Vdash_i (\neg)F_e(x)$.

2.11. Lemma. *Let f be a regular enumeration. Then*

(1) *For all $i \leq k$, $f \models_i F_e(x) \iff (\exists \tau \subseteq f)(\tau$ is i-regular & $\tau \Vdash_i F_e(x))$.*
(2) *For all $i < k$, $f \models_i \neg F_e(x) \iff (\exists \tau \subseteq f)(\tau$ is i-regular & $\tau \Vdash_i \neg F_e(x))$.*

Proof. We shall use induction on i. The condition (1) is obviously true for $i = 0$. Suppose that $i < k$ and (1) is true for i.

First we shall show the validity of (2) for i. Assume that $f \models_i \neg F_e(x)$ and for all i-regular $\tau \subseteq f$, $\tau \not\Vdash_i \neg F_e(x)$. Then for all i-regular finite parts τ of f there exists an i-regular $\rho \supseteq \tau$ such that $\rho \Vdash_i F_e(x)$. Fix a $j \in \mathbb{N}$ such that

$$S^i_j = \{\rho : \rho \in \mathcal{R}_i \ \&\ \rho \Vdash_i F_e(x)\}.$$

Let δ be an $(i+1)$-regular finite part of f such that $|\delta|_{i+1} > j$. By the definition of the $(i+1)$-regular finite parts, there exists an i-regular $\rho' \subseteq \delta$ such that $\rho' \in S^i_j$. By (1), Since $\rho' \subseteq f$, $f \models_i F_e(x)$. A contradiction. Assume now that $\tau \subseteq f$ is i-regular, $\tau \Vdash_i \neg F_e(x)$ and $f \models_i F_e(x)$. By induction there exists an i-regular $\rho \subseteq f$ such that $\rho \Vdash_i F_e(x)$. Using the monotonicity of $\Vdash_i$, we can assume that $\tau \subseteq \rho$ and get a contradiction.

Now having (1) and (2) for i one can easily obtain the validity of (1) for $i+1$. □

2.12. Proposition. *Let f be a regular enumeration and $1 \leq i \leq k$. Then $f^{(i)} \equiv_e f \oplus P'_{i-1}$.*

Proof. Let $1 \leq i \leq k$. By Proposition 2.8 it is sufficient to show that $f^{(i)} \leq_e f \oplus P'_{i-1}$. Recall that $f^{(i)} = K^0_{f^{(i-1)}} \oplus (\mathbb{N} \setminus K^0_{f^{(i-1)}})$, where $K^0_{f^{(i-1)}} = \{\langle y, z\rangle : y \in \Gamma_z(f^{(i-1)})\}$. Clearly $K^0_{f^{(i-1)}}$ is Σ^0_i in f and hence there exists an e such that $f \models_{i-1} F_e(x) \iff x \in K^0_{f^{(i-1)}}$. From here, using Lemma 2.11, we obtain that

$$x \in K^0_{f^{(i-1)}} \iff (\exists \tau \subseteq f)(\tau \text{ is } (i-1)\text{-regular } \& \ \tau \Vdash_{i-1} F_e(x)) \text{ and}$$

$$x \in (\mathbb{N} \setminus K^0_{f^{(i-1)}}) \iff (\exists \tau \subseteq f)(\tau \text{ is } (i-1)\text{-regular } \& \ \tau \Vdash_{i-1} \neg F_e(x)).$$

So, by Proposition 2.6 $K^0_{f^{(i-1)}}$ and $(\mathbb{N} \setminus K^0_{f^{(i-1)}})$ are e-reducible to $f \oplus P'_{i-1}$. Hence $f^{(i)} \leq_e f \oplus P'_{i-1}$. □

3. Constructions of regular enumerations

Given a finite mapping τ defined on $[0,q-1]$, by $\tau*z$ we shall denote the extension ρ of τ defined on $[0,q]$ and such that $\rho(q)\simeq z$. If $\vec{k}=q_0,\dots,q_p$ is a sequence of elements of $\mathrm{dom}(\tau)$, then by $\tau(\vec{k})$ we shall denote the sequence $\tau(q_0),\dots,\tau(q_p)$.

3.1. Lemma. *Let $i\le k$ and τ be an i-regular finite part defined on $[0,q-1]$.*

(1) *For every $y\in\mathbb{N}, z_0\in B_0,\dots,z_i\in B_i$, there exists an i-regular extension ρ of τ s.t. $|\rho|_i=|\tau|_i+1$ and $\rho(q)\simeq y, z_0\in\rho(B_0^\rho),\dots,z_i\in\rho(B_i^\rho)$.*

(2) *For every sequence $\vec{a}=a_0,\dots,a_m$ of natural numbers there exists an i-regular m omitting extension δ of τ such that $\delta(K_\tau^\delta)=\vec{a}$.*

Proof. We shall prove simultaneously (1) and (2) by induction on i. Clearly (1) is true for $i=0$. Now suppose that (1) holds for some $i<k$. First we shall prove (2). Notice that from (1) it follows that $\mu_i(\delta*a,X_j^i)$ is defined for all $a,j\in\mathbb{N}$ and $\delta\in\mathcal{R}_i$. Next we define recursively the i-regular finite parts δ_p for $p\le m+1$. Let $\delta_0=\tau$. For $p\le m$ let $q_p=\mathrm{lh}(\delta_p)$ and $\delta_{p+1}=\mu_i(\delta_p*a_p,X_{\langle p,q_p\rangle})$. Let $q_{m+1}=\mathrm{lh}(\delta_{m+1})$. Clearly δ_{m+1} satisfies the requirements of Definition 2.1 with respect to $q_0,\dots,q_{m+1}$ and $\delta_{m+1}(q_0,\dots,q_m)=a_0,\dots,a_m$.

Now we turn to the proof of (1) for $i+1$. Let τ be an $(i+1)$-regular finite part s.t. $\mathrm{dom}(\tau)=[0,q-1]$. Let $y\in\mathbb{N}, z_0\in B_0,\dots,z_{i+1}\in B_{i+1}$ be given. Suppose that $|\tau|_{i+1}=r+1$ and $n_0,l_0,b_0,\dots,n_r,l_r,b_r,n_{r+1}$ are the natural numbers satisfying the conditions a)–d) from the definition of the $(i+1)$-regular finite parts. Notice that $n_{r+1}=q$. Since τ is i-regular, by the induction hypothesis there exists an i-regular extension of $\tau*y$. Therefore $\rho_0\simeq\mu_i(\tau*y,S_{r+1}^i)$ is defined. Let $l_{r+1}=\mathrm{lh}(\rho_0)$. By (2) there exists an i-regular $r+1$ omitting extension δ of ρ_0. Let $b_{r+1}=\mathrm{lh}(\delta)$. By induction there exists an i-regular finite part $\rho\supseteq\delta$ such that $|\rho|_i=|\delta|_i+1$, $\rho(b_{r+1})\simeq z_{i+1}$ and $z_0\in\rho(B_0^\rho),\dots,z_i\in\rho(B_i^\rho)$. Set $n_{r+2}=\mathrm{lh}(\rho)$. Clearly ρ satisfies the conditions a)–d) from definition of the $(i+1)$-regular finite parts with respect to $n_0,l_0,b_0,\dots,n_{r+1},l_{r+1},b_{r+1},n_{r+2}$. □

Remark. From the proof above it follows that the i-regular extension ρ satisfying (1) can be constructed recursively for $i=0$ and recursively in P'_{i-1} if $i>0$. The construction of δ from (2) is recursive in P'_i.

3.2. Corollary. *For every $i\le k$ there exists an i-regular finite part having i-rank equal to 1.*

As an application of Lemma 3.1 we obtain the following property of the regular enumerations which will be used in the proof of Theorem 1.7:

3.3. Lemma. *Let f be a regular enumeration and $i<k$. Then $f\not\le_e P_i$.*

Proof. A standard forcing argument. Assume that $f\le_e P_i$. Then the set

$$S=\{\tau:\tau\in\mathcal{R}_i\ \&\ (\exists x\in\mathrm{dom}(\tau))(\tau(x)\not\simeq f(x))\}$$

is e-reducible to P_i. Let $S=S_j^i$ and δ be an $(i+1)$-regular finite part of f such that $|\delta|_{i+1}\ge j+1$. From the definition of the $(i+1)$-regular finite parts it follows that either there exists a $\rho \subseteq \delta$ such that $\rho \in S$ or for all i-regular $\rho \supseteq \delta$, $\rho \notin S$. Clearly the first is impossible. Let $\mathrm{lh}(\delta) = q$ and $f(q) \simeq y$. By Lemma 3.1 there exists an i-regular $\rho \supseteq \delta$ such that $\rho(q) \not\simeq y$ and hence $\rho \in S$. A contradiction. □

3.4. Corollary. *If f is a regular enumeration, then for all $i < k$, $P_i <_e f^{(i)}$.*

Let δ be a k-regular finite part and $1 \leq i \leq k$. By definition the sequence K_i^δ has exactly $|\delta|_i$ members. So, by Lemma 2.2 if $1 \leq i \leq k$, then the length of K_i^δ is greater than or equal to $|\delta|_k + (k - i)$.

3.5. Lemma. *Let $i < k$, $A \not\leq_e P_i$ and let τ be an $(i+1)$-regular finite part, defined on $[0, q-1]$. Suppose that $|\tau|_{i+1} = r + 1$, $y \in \mathbb{N}$, $z_0 \in B_0, \ldots, z_{i+1} \in B_{i+1}$ and $s \leq r + 1$. Then one can construct recursively in $P_i' \oplus A^+$ an $(i+1)$-regular extension ρ of τ such that*

(i) $|\rho|_{i+1} = r + 2$;
(ii) $\rho(q) \simeq y$, $z_0 \in \rho(B_0^\rho), \ldots, z_{i+1} \in \rho(B_{i+1}^\rho)$;
(iii) *if* $K_{i+1}^\rho = q_0, \ldots, q_s, \ldots, q_{r+1}$, *then*
  a) $\rho(q_s) \in A \Rightarrow \rho \Vdash_i \neg F_s(q_s)$;
  b) $\rho(q_s) \notin A \Rightarrow \rho \Vdash_i F_s(q_s)$.

Proof. Let $0 < n_0 < l_0 < b_0, \cdots < n_r < l_r < b_r < n_{r+1} = q$ be the natural numbers satisfying the conditions a)–d) from the definition of the $(i+1)$-regular finite part τ. Set $\rho_0 \simeq \mu_i(\tau * y, S_{r+1}^i)$ and $l_{r+1} = \mathrm{lh}(\rho_0)$. Let $\delta_0 = \rho_0$. Suppose that $p < s$ and δ_p is defined. Then let $q_p = \mathrm{lh}(\delta_p)$ and $\delta_{p+1} \simeq \mu_i(\delta_p * 0, X_{\langle p, q_p \rangle}^i)$. Now let $q_s = \mathrm{lh}(\delta_s)$. Clearly the set

$$C = \{x : (\exists \delta \supseteq \delta_s)(\delta \in \mathcal{R}_i \ \&\ \delta(q_s) \simeq x \ \&\ \delta \Vdash_i F_s(q_s))\}.$$

is e-reducible to P_i. Since $A \not\leq_e P_i$, there exists an a such that $a \in C \ \&\ a \notin A$ or $a \notin C \ \&\ a \in A$. Denote by a_0 the least such a. Notice that a_0 can be found recursively in $P_i' \oplus A^+$. Set $\delta_{s+1} \simeq \mu_i(\delta_s * a_0, X_{\langle s, q_s \rangle}^i)$. By the definition of the function μ_i we have that either $a_0 \in A$ and $\delta_{s+1} \Vdash_i \neg F_s(q_s)$ or $a_0 \notin A$ and $\delta_{s+1} \Vdash_i F_s(q_s)$. Next we extend δ_{s+1} to an i-regular $r+1$ omitting extension ρ_1 of ρ_0 in the usual way. Let $b_{r+1} = \mathrm{lh}(\rho_1)$. Using Lemma 3.1, we can extend ρ_1 to an i-regular finite part ρ such that $|\rho|_i = |\rho_1|_i + 1$, $\rho(b_{r+1}) \simeq z_{i+1}$ and $z_j \in \rho(B_j^\rho)$ for $j \leq i$. Let $n_{r+2} = \mathrm{lh}(\rho)$. Clearly $n_0, l_0, b_0, \ldots, n_{r+1}, l_{r+1}, b_{r+1}, n_{r+2}$ satisfy the conditions a)-d) from the definition of the $(i+1)$-regular finite parts. So, ρ is $(i+1)$-regular and $|\rho|_{i+1} = r + 2$. Clearly q_s is the $s+1$-th member of K_{i+1}^ρ and since $\rho \supseteq \delta_{s+1}$, (iii) holds. □

3.6. Lemma. *Let $k > i \geq 0$, $A \not\leq_e P_i$ and let τ be a k-regular finite part, defined on $[0, q-1]$ and $|\tau|_k = r + 1$. Suppose that $y \in \mathbb{N}$, $z_0 \in B_0, \ldots, z_k \in B_k$ and $s \leq r + (k - i)$. Then one can construct recursively in $P_{k-1}' \oplus A^+$ a k-regular extension ρ of τ such that*

(i) $|\rho|_k = r + 2$;
(ii) $\rho(q) \simeq y$, $z_0 \in \rho(B_0^\rho), \ldots, z_k \in \rho(B_k^\rho)$;
(iii) *if* $K_{i+1}^\rho = q_0, \ldots, q_s, \ldots, q_{m_i}$, *then*

a) $\rho(q_s) \in A \Rightarrow \rho \Vdash_i \neg F_s(q_s)$;
b) $\rho(q_s) \notin A \Rightarrow \rho \Vdash_i F_s(q_s)$.

Proof. We shall use induction on $k-(i+1)$. The previous Lemma settles the case $k = i+1$. Now suppose that $k > i+1$. Let $\rho_0 \simeq \mu_{k-1}(\tau * y, S^{k-1}_{r+1})$ and let ρ_1 be a $(k-1)$-regular $r+1$ omitting extension of ρ_0, such that $\rho_1(K^{\rho_1}_{\rho_0}) = 0, 0, \ldots, 0$. Let $b_{r+1} = \mathrm{lh}(\rho_1)$. Suppose that $|\rho_1|_{k-1} = r_1 + 1$. Since $|\rho_1|_{k-1} > |\tau|_{k-1} > |\tau|_k$, $s \leq r_1 + (k-1-i)$. By induction there exists a $(k-1)$-regular extension ρ of ρ_1 such that $|\rho|_{k-1} = |\rho_1|_{k-1} + 1$, $\rho(b_{r+1}) \simeq z_k$, $z_0 \in \rho(B^\rho_0), \ldots, z_{k-1} \in \rho(B^\rho_{k-1})$ and such that (iii) holds. Clearly ρ is a k-regular extension of τ with k-rank equal to $r+2$. □

The following lemma can be proved in a similar way:

3.7. Lemma. *Let $k \geq 1$ and $A_0, \ldots, A_{k-1}$ be subsets of $\mathbb{N}$ such that $A_i \not\leq_e P_i$. Let τ be a k-regular finite part, defined on $[0, q-1]$. Suppose that $|\tau|_k = r+1$, $y \in \mathbb{N}$, $z_0 \in B_0, \ldots, z_k \in B_k$ and $s \leq r+1$. Then one can construct recursively in $P'_{k-1} \oplus A^+_0 \cdots \oplus A^+_{k-1}$ a k-regular extension ρ of τ such that*

(i) $|\rho|_k = r+2$;
(ii) $\rho(q) \simeq y$, $z_0 \in \rho(B^\rho_0), \ldots, z_k \in \rho(B^\rho_k)$;
(iii) *if $i < k$ and $K^\rho_{i+1} = q^i_0, \ldots, q^i_s, \ldots q^i_{m_i}$, then*
 a) $\rho(q^i_s) \in A_i \Rightarrow \rho \Vdash_i \neg F_s(q^i_s)$;
 b) $\rho(q^i_s) \notin A_i \Rightarrow \rho \Vdash_i F_s(q^i_s)$.

Now we turn to the proofs of the formulated in the introduction theorems. Let a total set $Q \geq_e P_k$ be given. Clearly the sets $B_0, \ldots, B_k$ are r. e. in Q. Let us fix some recursive in Q functions $\sigma_0, \ldots, \sigma_k$ which enumerate $B_0, \ldots, B_k$, respectively. Let $y_0, \ldots, y_r, \ldots$ be a recursive in Q enumeration of the elements of Q.

Proof of Theorem 1.2. By Proposition 2.8 and Proposition 2.12 it is sufficient to show that there exists a regular enumeration f such that $f^{(k)} \equiv_e Q$.

We shall construct f as a recursive in Q union of k-regular finite parts δ_s such that for all s, $\delta_s \subseteq \delta_{s+1}$ and $|\delta_s|_k = s+1$.

Let δ_0 be an arbitrary finite part such that $|\delta_0|_k = 1$. Suppose that δ_s is defined. Set $z_0 = \sigma_0(s), \ldots, z_k = \sigma_k(s)$. Using Lemma 3.1 construct recursively (in P'_{k-1}, if $k \geq 1$) a k-regular $\rho \supseteq \delta_s$ such that $|\rho|_k = |\delta_s|_k + 1$, $\rho(\mathrm{lh}(\delta_s)) = y_s$ and $z_0 \in \rho(B^\rho_0), \ldots, z_k \in \rho(B^\rho_k)$. Set $\delta_{s+1} = \rho$.

Clearly the obtained this way enumeration f is regular and $f \leq_e Q$. Therefore by Proposition 2.12 $f^{(k)} \leq_e Q$. On the other hand, using the oracle f (and P'_{k-1}, if $k \geq 1$) we can generate as in the proof of Proposition 2.8 consecutively the sequence $n_1, \ldots, n_s, \ldots$ such that $f\restriction n_{s+1} = \delta_s$. By the construction $y \in Q \iff \exists s(f(n_{s+1}) = y)$. Hence $Q \leq_e f \oplus P'_{k-1} \leq_e f^{(k)}$. □

Suppose that $k > i \geq 0$ and A is a subset of $\mathbb{N}$ such that $A^+ \leq_e Q$ and $A \not\leq_e P_i$.

Proof of Theorem 1.3. We shall construct a regular enumeration f such that $f^{(k)} \equiv_e Q$ and $A \not\leq_e f^{(i)}$. The construction of f will be carried out again by steps. At each step s we shall define a k-regular finite part δ_s having k-rank equal to $s+1$.

Compared to the previous proof, we shall ensure in addition that at each step $s+1$, if $K_{i+1}^{\delta_{s+1}} = q_0, \dots, q_s, \dots, q_{m_i}$, then

$$(\delta_{s+1}(q_s) \in A \Rightarrow \delta_{s+1} \Vdash_i \neg F_s(q_s)) \ \& \ (\delta_{s+1}(q_s) \notin A \Rightarrow \delta_{s+1} \Vdash_i F_s(q_s)) \tag{3.1}$$

We start by an arbitrary k-regular finite part δ_0 having k-rank equal to 1. Suppose that δ_s is defined. Set $z_0 = \sigma_0(s), \dots, z_k = \sigma_k(s)$. Using Lemma 3.6, construct recursively in Q a k-regular $\delta_{s+1} \supseteq \delta_s$ such that $|\delta_{s+1}|_k = |\delta_s|_k + 1$, $\delta_{s+1}(\mathrm{lh}(\delta_s)) = y_s$, and $z_0 \in \rho(B_0^\rho), \dots, z_k \in \rho(B_k^\rho)$ and if $K_{i+1}^{\delta_{s+1}} = q_0, \dots, q_s, \dots, q_{m_i}$, then (3.1) holds.

Clearly the whole construction is recursive in Q and hence $f \leq_e Q$. Then $f^{(k)} \equiv_e f \oplus P'_{k-1} \leq_e Q$. The inequality $Q \leq_e f^{(k)}$ can be proved exactly as in the previous proof. It remains to show that $A \not\leq_e f^{(i)}$. Indeed, assume that $A \leq_e f^{(i)}$. Then the set $C = \{x : f(x) \in A\}$ is also e-reducible to $f^{(i)}$. Let s be an index such that $\forall x (x \in C \iff f \models_i F_s(x))$. Then for all x

$$f(x) \in A \iff f \models_i F_s(x) \tag{3.2}$$

Consider δ_{s+1} and q_s. Clearly $\delta_{s+1}(q_s) \simeq f(q_s)$. Now assume that $f(q_s) \in A$. Then $\delta_{s+1} \Vdash_i \neg F_s(q_s)$. Hence $f \models_i \neg F_s(q_s)$ which is impossible. It remains that $f(q_s) \notin A$. In this case $\delta_{s+1} \Vdash_i F_s(q_s)$ and hence $f \models_i F_s(q_s)$. The last again contradicts (3.2). So $A \not\leq_e f^{(i)}$. □

Now we turn to the proof of Theorem 1.7. Set $B_{k+1} = \mathbb{N}$ and $Q = P_{k+1} = P'_k \oplus B_{k+1}$. Clearly $Q \equiv_e P'_k$. From now on an enumeration f will be called regular if it is regular with respect to $B_0, \dots, B_k, B_{k+1}$.

Proof of Theorem 1.7. Since Q is a total set, by Theorem 1.2 there exists a regular enumeration g such that $g^{(k+1)} \equiv_e Q$. By Corollary 3.4 for all $i \leq k$, $P_i <_e g^{(i)}$. Finally notice that $g^{(k+2)} \equiv_e Q' \equiv_e P''_k$.

For $i \leq k$, set $G_z^i = \Gamma_z(g^{(i)})$, where Γ_z is the z-th enumeration operator. We shall construct recursively in Q' a regular enumeration f so that

(1) $f^{(k+2)} \equiv_e Q'$;
(2) if $i \leq k$ and $G_z^i \not\leq_e P_i$, then $G_z^i \not\leq_e f^{(i)}$.

The construction of f will be carried out by steps. At each step s we shall construct a $(k+1)$-regular finite part δ_s so that $|\delta_s|_{k+1} \geq s+1$ and $\delta_s \subseteq \delta_{s+1}$. On the even steps we shall ensure (1), on the odd steps – (2).

Let $\mathcal{R}_{k+1}$ be the set of all $(k+1)$-regular finite parts and $S_j^{k+1} = \mathcal{R}_{k+1} \cap \Gamma_j(Q)$. By Lemma 2.6 the sequence $\{S_j^{k+1}\}$ is T-reducible to Q'. Let $\sigma_0, \dots, \sigma_{k+1}$ be recursive in Q enumerations of the sets $B_0, \dots, B_{k+1}$, respectively.

Let δ_0 be an arbitrary $(k+1)$-regular finite part with $(k+1)$-rank equal to 1. Suppose that δ_s is defined.

Case $s = 2m$. Check whether there exists a $\rho \in S_m^{k+1}$ such that $\delta_s \subset \rho$. If so let δ_{s+1} be the least such ρ. Otherwise let δ_{s+1} be the least $(k+1)$-regular extension of δ_s with $(k+1)$-rank equal to $|\delta_s|_{k+1} + 1$.

Case $s = 2m+1$. Let $|\delta_s|_{k+1} = r + 1 \geq s + 1$. Let $m = \langle z, e \rangle$. We may assume that the recursive coding $\langle .,. \rangle$ is chosen so that $e \leq m$. Then $e < r + 1$. Let $\sigma_0(m) \simeq z_0, \dots, \sigma_{k+1}(m) \simeq z_{k+1}$. Set $\tau_0 \simeq \mu_k(\delta_s * z_{k+1}, S_{r+1}^k)$. Set $l_{r+1} = \mathrm{lh}(\tau_0)$ and

$q_0^k = l_{r+1}$. For $j < e$, let $\tau_{j+1} = \mu_k(\tau_j * 0, X^k_{(j,q_j^k)})$ and $q^k_{j+1} = \mathrm{lh}(\tau_{j+1})$. Now we have defined τ_e and q_e^k. Let

$$C = \{x : (\exists \tau \supseteq \tau_e)(\tau \in \mathcal{R}_k \;\&\; \tau(q_e^k) \simeq x \;\&\; \tau \Vdash_k F_e(q_e^k))\}.$$

Clearly C is recursive in Q. Since $G_z^k = \Gamma_z(g^{(k)})$ and $g^{(k+1)} \equiv_e Q$, we can check recursively in Q' whether there exists an a such that

$$a \in C \;\&\; a \notin G_z^k \vee a \notin C \;\&\; a \in G_z^k. \tag{3.3}$$

If the answer is positive, then let a_0 be the least a satisfying (3.3). If the answer is negative, then let $a_0 = 0$. Notice that we can find a_0 recursively in Q'. Next we extend recursively in Q' the finite part $\tau_e * a_0$ to a finite part τ so that τ is a k-regular $r+1$ omitting extension of τ_0. Set $b_{r+1} = \mathrm{lh}(\tau)$.

Now consider the sets G_z^i, $i < k$. Notice that $g^{(i+3)}$ is recursive in Q'. Since $P_i \leq_e g^{(i)}$ and

$$G_z^i \leq_e P_i \iff \exists u \forall x (x \in \Gamma_z(g^{(i)}) \iff x \in \Gamma_u(P_i)),$$

we can check recursively in $g^{(i+3)}$ for each i whether $G_z^i \leq_e P_i$. Set $A_i = G_z^i$, if $G_z^i \not\leq_e P_i$ and $A_i = P_i'$, otherwise. Clearly $A_i \not\leq_e P_i$ and $A_0^+ \oplus \cdots \oplus A_{k-1}^+ \leq_e Q'$. By Lemma 3.7 we can construct recursively in Q' a k-regular extension ρ of τ such that

(i) $|\rho|_k = |\tau|_k + 1$;
(ii) $\rho(b_{r+1}) \simeq z_{k+1}$ and $z_0 \in \rho(B_0^\rho), \dots, z_k \in \rho(B_k^\rho)$;
(iii) if $i < k$ and $K_{i+1}^\rho = q_0^i, \dots, q_e^i, \dots q_{h_i}^i$, then
  a) $\rho(q_e^i) \in A_i \Rightarrow \rho \Vdash_i \neg F_e(q_e^i)$;
  b) $\rho(q_e^i) \notin A_i \Rightarrow \rho \Vdash_i F_e(q_e^i)$.

Set $\delta_{s+1} = \rho$.

Let $f = \bigcup \delta_s$. Clearly f is a regular enumeration and $f \leq_e Q'$. First we shall show that $f^{(k+2)} \equiv_e Q'$. Since f is regular, $P_{k+1} \leq_e f^{(k+1)}$. Therefore $Q' = P'_{k+1} \leq_e f^{(k+2)}$. Clearly for every z, x, $\{\tau : \tau \in \mathcal{R}_{k+1} \;\&\; \tau \Vdash_{k+1} F_z(x)\}$ is e-reducible to Q. From here, by the even stages of the construction, it follows that for all z, x,

$$f \models_{k+1} (\neg)F_z(x) \iff (\exists \tau \subseteq f)(\tau \in \mathcal{R}_{k+1} \;\&\; \tau \Vdash_{k+1} (\neg)F_z(x)).$$

Using the last equivalence we may conclude as in the proof of Proposition 2.12 that $f^{(k+2)} \leq_e f \oplus Q'$. Hence $f^{(k+2)} \equiv_e Q'$.

Let us turn to the proof of the condition (ii) of the Theorem. Since f is regular we have that if $i \leq k$, then for all e and x,

$$f \models_i (\neg)F_e(x) \iff (\exists \tau \subseteq f)(\tau \in \mathcal{R}_i \;\& \tau \Vdash_i (\neg)F_e(x)).$$

Now suppose that $i \leq k$, $A \leq_e g^{(i)}$ and $A \leq_e f^{(i)}$. Assume that $A \not\leq_e P_i$. Fix z and e such that $A = \Gamma_z(g^{(i)})$ and for all x,

$$f(x) \in A \iff f \models_i F_e(x).$$

Consider the step $s = 2\langle z, e\rangle + 1$. By the construction, there exists a $q_e^i \in \mathrm{dom}(\delta_{s+1})$ such that

$$(f(q_e^i) \in A \Rightarrow f \models \neg F_e(q_e^i)) \ \&\ (f(q_e^i) \notin A \Rightarrow f \models F_e(q_e^i)).$$

A contradiction. □

4. ω-REGULAR ENUMERATIONS

Let $B_0, \dots, B_k, \dots$ be a sequence of subsets of $\mathbb{N}$. We shall call a finite part or an enumeration k-regular if it is regular with respect to $B_0, \dots, B_k$.

4.1. Definition. A finite part τ defined on $[0, q-1]$ is called *ω-regular* if there exist natural numbers $0 < n_0 < \dots < n_k = q$ such that for every $j \leq k$, $\tau\lceil n_j$ is a j-regular finite part and $|\tau\lceil n_j|_j = 1$.

4.2. Definition. A total mapping f of $\mathbb{N}$ in $\mathbb{N}$ is called an *ω-regular* enumeration if the following two conditions are satisfied:

(i) For every $\delta \subseteq f$ there exists an ω-regular $\tau \subseteq f$ such that $\delta \subseteq \tau$.
(ii) For every k and $z \in B_k$ there exists a k-regular $\tau \subseteq f$ such that $z \in \tau(B_k^\tau)$.

Let $P_k = \mathcal{P}(B_0, \dots, B_k)$ and $P_\omega = \{\langle k, x\rangle : x \in P_k\}$. The set P_ω is total. Indeed, fix z_0 so that for all sets A, $\Gamma_{z_0}(A) = A$. Then

$$\begin{aligned}&\langle k, x\rangle \notin P_\omega \iff x \notin P_k \iff x \notin \Gamma_{z_0}(P_k) \iff \\ &2\langle x, z_0\rangle + 1 \in P_k' \iff 2(2\langle x, z_0\rangle + 1) \in P_{k+1} = P_k' \oplus B_{k+1} \iff \\ &\langle k+1, 2(2\langle x, z_0\rangle + 1)\rangle \in P_\omega.\end{aligned}$$

So, $\omega \setminus P_\omega \leq_e P_\omega$.

Using Lemma 2.2 we obtain immediately the following:

4.3. Lemma. *If f is ω-regular, then f is k-regular for every k.*

4.4. Corollary. *If f is ω-regular, then $(\forall k \geq 1)(f^{(k)} \equiv_e f \oplus P_{k-1}')$.*

An examination of the proofs of Proposition 2.6 and Proposition 2.8 shows the truth of the following uniform versions:

4.5. Proposition.

(1) *The sets $\mathcal{R}_k$ of all k-regular finite parts are uniformly in k e-reducible to P_k and hence the sequence $\{\mathcal{R}_k\}$ is T-reducible to P_ω.*
(2) *The sequences $\{S_j^k\}$ and $\{X_j^k\}$ are uniformly in k e-reducible to P_k and hence these sequences are uniformly in k T-reducible to P_ω.*
(3) *The functions μ_k^S and μ_k^X are uniformly in k partial recursive in P_k' and hence they are uniformly partial recursive in P_ω.*

4.6. Proposition. *If f is an ω-regular enumeration, then the sets B_k and P_k are uniformly in k e-reducible to $f^{(k)}$.*

4.7. Corollary. *If f is an ω-regular enumeration, then $f^{(\omega)} \equiv_e f \oplus P_\omega$.*

4.8. Theorem. *Let Q be a total set and $P_\omega \leq_e Q$. There exists an ω-regular enumeration f such that $f^{(\omega)} \equiv_e Q$.*

Proof. The construction of f will be carried out by steps. At each step we shall define a s-regular finite part δ_s with s-rank 1. We shall ensure that $\delta_s \subseteq \delta_{s+1}$ and define $f = \bigcup \delta_s$.

Let $\sigma(k,s)$ be a recursive in Q function such that for all k, $\lambda s.\sigma(k,s)$ enumerates B_k. Let $y_0, y_1, \dots$ be a recursive in Q enumeration of Q.

Define δ_0 on $[0,1]$ so that $\delta_0(0) \simeq y_0$ and $\delta_0(1) \simeq \sigma(0,0)$.

Suppose that δ_s is defined. Let $n_0 = \mathrm{lh}(\delta_s)$, $\tau_0 = \mu_s(\delta_s * y_s, S_0^s)$ and $l_0 = \mathrm{lh}(\tau_0)$. Next set $\tau = \mu_s(\tau_0 * 0, X^s_{\langle 0, l_0\rangle})$ and $b_0 = \mathrm{lh}(\tau)$. Notice that τ is a s-regular 0 omitting extension of τ_0. Using Lemma 3.1, construct a s-regular extension ρ of τ such that $|\rho_s|_s = |\tau|_s + 1$, $\rho(b_0) \simeq \sigma(s+1,0)$ and $\sigma(s,1) \in \rho(B_s^\rho), \dots, \sigma(0,s+1) \in \rho(B_0^\rho)$. Set $\delta_{s+1} = \rho$.

Clearly the obtained by the construction above enumeration f is ω-regular. Since the whole construction is recursive in Q, we have that $f \leq_e Q$ and hence $f^{(\omega)} \equiv_e f \oplus P_\omega \leq_e Q$. It remains to show that $Q \leq_e f \oplus P_\omega$. Indeed, let $n^0 = 0$ and $n^{s+1} = \mathrm{lh}(\delta_s)$. Clearly we have a recursive in $f \oplus P_\omega$ procedure which generates consecutively the finite parts δ_s, $s = 0, 1, \dots$. Therefore the set $\{n^s : s \in \mathbb{N}\}$ is recursive in $f \oplus P_\omega$. Since $y \in Q \iff \exists s(f(n^s) \simeq y)$, $Q \leq_e f \oplus P_\omega$. $\square$

We shall need the following version of Lemma 3.7 which can be proved in a way similar to the proof of Lemma 3.6:

4.9. Lemma. *Let $k \geq 1$, and let τ be a k-regular finite part, defined on $[0, q-1]$. Suppose that $|\tau|_k = r+1$. Let $s_{k-1} \leq r+1, s_{k-2} \leq r+2, \dots, s_0 \leq r+k$. Let for $i < k$ and $j \leq s_i$, $A_j^i \not\leq_e P_i$. Finally let $y \in \mathbb{N}$, $z_0 \in B_0, \dots, z_k \in B_k$. Denote by A the set $\bigoplus_{i<k, j\leq s_i} (A_j^i)^+$. Then one can construct recursively in $P'_{k-1} \oplus A$ a k-regular extension ρ of τ such that*

(i) *$|\rho|_k = r+2$;*
(ii) *$\rho(q) \simeq y$, $z_0 \in \rho(B_0^\rho), \dots, z_k \in \rho(B_k^\rho)$;*
(iii) *if $i < k$ and $K_{i+1}^\rho = q_0^i, \dots, q_{s_i}^i, \dots q_{m_i}^i$, then for $j \leq s_i$:*
 a) *$\rho(q_j^i) \in A_j^i \Rightarrow \rho \Vdash_i \neg F_j(q_j^i)$;*
 b) *$\rho(q_j^i) \notin A_j^i \Rightarrow \rho \Vdash_i F_j(q_j^i)$.*

Now we are ready for the main result of this section:

4.10. Theorem. *There exist total sets F and G such that $F^{(\omega)} \equiv_e G^{(\omega)} \equiv_e P_\omega$ and such that for all k the following conditions hold:*

(i) *P_k is uniformly e-reducible to $F^{(k)}$ and to $G^{(k)}$, $F^{(k)} \not\leq_e P_k$ and $G^{(k)} \not\leq_e P_k$.*
(ii) *If $A \leq_e F^{(k)}$ and $A \leq_e G^{(k)}$, then $A \leq_e P_k$.*

Proof. We shall construct F and G as graphs of ω-regular enumerations f and g. This will ensure by Proposition 4.6 and Lemma 3.4 the condition (i).

Let g be an arbitrary ω-regular enumeration such that $g^{(\omega)} \equiv_e P_\omega$.

The construction of f is similar to that in the proof of Theorem 1.7. Let $\sigma(k,s)$ be a recursive in P_ω function such that for all k, $\lambda s.\sigma(k,s)$ enumerates B_k. For every

k and z, set $G^k_z = \Gamma_z(g^{(k)})$. We start the construction of f by putting $\delta_0(0) \simeq 0$ and $\delta_0(1) \simeq \sigma(0,0)$. Suppose that δ_s is defined and δ_s is a s-regular finite part with s-rank 1. Consider the sets $G^s_0, G^{s-1}_1, G^{s-1}_0, \dots, G^0_s, \dots, G^0_0$. For $i \leq s$ and $j \leq s-i$ set $A^i_j = G^i_{s-i-j}$ if $G^i_{s-i-j} \not\leq_e P_i$ and $A^i_j = P'_i$, otherwise. Clearly this assignment can be done recursively in P_ω. Notice that $A^i_j \not\leq_e P_i$ and $(A^i_j)^+ \leq_e P_\omega$

Let $\tau_0 = \mu_s(\delta_s * 0, S^s_0)$ and $l_0 = \mathrm{lh}(\tau_0)$. Next let a_0 be the least a such that $a \in A^s_0$ is not equivalent to $(\exists \tau \supseteq \tau_0)(\tau \in \mathcal{R}_s \ \&\ \tau(l_0) \simeq a_0 \ \&\ \tau \Vdash_s F_0(l_0))$. Set $\tau = \mu_s(\tau_0 * a_0, X^s_{\langle 0,l_0\rangle})$ and $b_0 = \mathrm{lh}(\tau)$. Using Lemma 4.9, construct a s-regular extension ρ of τ such that $|\rho|_s = |\tau|_s + 1$, $\rho(b_0) \simeq \sigma(s+1,0)$ and $\sigma(s,1) \in \rho(B^\rho_{s-1}), \dots, \sigma(0,s+1) \in \rho(B^\rho_0)$ and if $i < s$ and $K^\rho_{i+1} = q^i_0, \dots, q^i_{s-i}, \dots q^i_{m_i}$, then for all $j \leq s-i$

a) $\rho(q^i_j) \in A^i_j \Rightarrow \rho \Vdash_i \neg F_j(q^i_j)$;
b) $\rho(q^i_j) \notin A^i_j \Rightarrow \rho \Vdash_i F_j(q^i_j)$.

Set $\delta_{s+1} = \rho$.

Let $f = \bigcup \delta_s$. Clearly f is ω-regular, $f \leq_e P_\omega$ and hence $f^{(\omega)} \equiv_e P_\omega$. It remains to show the validity of (ii). Fix a k and assume that $A = G'^k_z$ and $A \not\leq_e P_k$. We shall show that $A \not\leq_e f^{(k)}$. Assume that $A \leq_e f^{(k)}$. Then the set $C = \{x : f(x) \in A\}$ is also e-reducible to $f^{(k)}$. Let p be such that for all x, $f \models_k F_p(x) \iff x \in C$. Then for all x

$$f(x) \in A \iff f \models_k F_p(x). \tag{4.1}$$

Consider the step $s = k + z + p$. Then $A^k_p = G^k_z = A$. By the construction there exists a $q \in \mathrm{dom}(\delta_{s+1})$ such that

$$(\delta_{s+1}(q) \in A \ \&\ \delta_{s+1} \Vdash_k \neg F_p(q)) \vee (\delta_{s+1}(q) \notin A \ \&\ \delta_{s+1} \Vdash_k F_p(q)).$$

Since f is $(k+1)$-regular, by Lemma 2.11 $f(q) \in A \Rightarrow f \not\models_k F_p(q)$ and $f(q) \notin A \Rightarrow f \models_k F_p(q)$. The last contradicts (4.1). □

The following corollary should be compared with the respective result in [1]:

4.11. Corollary. *Let $A \subseteq \mathbb{N}$, then $A \leq_e P_k$ iff $A \in \Sigma^X_{k+1}$ for all total X such that $X^{(\omega)} \equiv_e P_\omega$ and $\forall i(B_i \in \Sigma^X_{i+1})$ uniformly in i.*

4.12. Definition. The set A is *arithmetical* in the sequence $\{B_k\}$ if for some k, $A \leq_e P_k$. The sequence $\{B_k\}$ is *arithmetical* in X if there exist recursive functions g, h such that $B_k = \Gamma_{g(k)}((X^+)^{(h(k))})$.

4.13. Corollary. *The following assertions are equivalent:*

(1) *A is arithmetical in $\{B_k\}$.*
(2) *A is arithmetical in all X such that $X^{(\omega)} \equiv_e P_\omega$ and $\{B_k\}$ is arithmetical in X.*
(3) *A is arithmetical in all X such that $X^{(\omega)} \equiv_e P_\omega$ and for all k, B_k is arithmetical in X.*

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