Uniform Operators

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Abstract. We present the definition and a normal form of a class of operators on sets of natural numbers which generalize the enumeration operators.

1 Introduction

In his book [1, p.145] Rogers gives the following intuitive explanation of the notion of enumeration reducibility:

> Let sets A and B be given. ... To put it as briefly as possible: A is *enumeration reducible* to B if there is an effective procedure for getting an enumeration of A from *any* enumeration of B.

On the next page Rogers continues with the formal definition of the enumeration reducibility, where W_z denotes the c.e. set with Gödel number z and D_u denotes the finite set having canonical code u.

Definition 1. A is *enumeration reducible* to B (notation: $A \leq_{\mathrm{e}} B$) if

$$(\exists z)(\forall x)[x \in A \iff (\exists u)[\langle x, u\rangle \in W_z \ \&\ D_u \subseteq B]].$$

A is *enumeration reducible* to B *via* z if

$$(\forall x)[x \in A \iff (\exists u)[\langle x, u\rangle \in W_z \ \&\ D_u \subseteq B]].$$

Finally Rogers defines for every z the enumeration operator $\Phi_z : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$.

Definition 2. $\Phi_z(X) = Y$ if $Y \leq_{\mathrm{e}} X$ via z.

Though the relationship of the intuitive definition with the formal one is well explained in [1] it is tempting to formalize the intuitive definition in a more direct way. Consider again the sets A and B. To get an enumeration of B we need an oracle X and if we have such an enumeration relative to X than B will be c.e. in X, so $B = W_b^X$ for some $b \in \mathbb{N}$, where W_b^X denotes the domain of the b-th Oracle Turing Machine using as oracle the characteristic function of X.

From the intuitive remarks it follows that if $A \leq_e B$, and $B = W_b^X$, then there exists an a such that $A = W_a^X$ and we can obtain such an a from b in a way which does not depend on the oracle X. So it seems reasonable to consider the following definition of a class of operators which we call *uniform operators*.

Definition 3. A mapping $\Gamma : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is called *uniform operator* if there exists a total function γ on the natural numbers such that for all $b \in \mathbb{N}$ and $X \subseteq \mathbb{N}$ we have that $\Gamma(W_b^X) = W_{\gamma(b)}^X$.

The following result shows that the intuitive remarks quoted at the beginning correspond exactly to the formal definition of the enumeration operators.

Theorem 4. *The uniform operators coincide with the enumeration operators.*

The theorem above can be considered as a uniform version of a result of Selman [2].

Theorem 5 (Selman).

$$A \leq_{\mathrm{e}} B \iff \forall X(B \textit{ is c.e. in } X \Rightarrow A \textit{ is c.e. in } X).$$

Selman's theorem is generalized by Case [3] and Ash [4]. Following the same fashion we come to the following definition.

Definition 6. Let $n, k \in \mathbb{N}$. A mapping $\Gamma : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is *uniform operator of type* $(n \to k)$ if there exists a total function γ on the natural numbers such that for all $b \in \mathbb{N}$ and $X \subseteq \mathbb{N}$ we have that $\Gamma(W_b^{X^{(n)}}) = W_{\gamma(b)}^{X^{(k)}}$.

The characterization of the uniform operators of type $(n \to k)$ uses the notion of *enumeration jump* defined in Cooper [5] and further studied by McEvoy [6]. Here we shall use the following definition of the e-jump which is m-equivalent to the original one, see [6]:

Definition 7. Given a set A, let $K_A^0 = \{\langle x, z\rangle : x \in \Phi_z(A)\}$. Define the *e-jump* A'_{e} of A to be the set $K_A^0 \oplus (\mathbb{N} \setminus K_A^0)$.

For any set A by $A_{\mathrm{e}}^{(n)}$ we shall denote the n-th e-jump of A.

Theorem 8. *1. Let $k < n$. Then the uniform operators of type $(n \to k)$ coincide with the constant mappings $\lambda B.S$, where S is some Σ_{k+1}^0 set.*
2. Let $n \leq k$. Then the uniform operators of type $(n \to k)$ are exactly those mappings of $\Gamma : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ for which there exists an enumeration operator Φ such that for all $B \subseteq \mathbb{N}$, $\Gamma(B) = \Phi((B \oplus \emptyset^{(n)})_{\mathrm{e}}^{(k-n)})$.

Finally let us consider the general case.

Definition 9. Let $k_0 < \ldots < k_r$ and k be natural numbers. A mapping $\Gamma : \mathcal{P}(\mathbb{N})^{(r+1)} \to \mathcal{P}(\mathbb{N})$ is a *uniform operator of type* $(k_0, \ldots, k_r \to k)$ if there exists a function $\gamma : \mathbb{N}^{r+1} \to \mathbb{N}$ such that for all $b_0, \ldots, b_r \in \mathbb{N}$ and $X \subseteq \mathbb{N}$,

$$\Gamma(W_{b_0}^{X^{(k_0)}}, \ldots, W_{b_r}^{X^{(k_r)}}) = W_{\gamma(b_0,\ldots,b_r)}^{X^{(k)}}.$$

Let us fix the natural numbers $k_0, \ldots, k_r$. Denote by $\bar{k}$ the sequence $k_0, \ldots, k_r$. Given sets of natural numbers $B_0, \ldots, B_r$, we define the set $\mathcal{P}_{\bar{k}}^{(k)}(B_0, \ldots, B_r)$ by induction on k.

Definition 10. (i) Set

$$\mathcal{P}_{\bar{k}}^{(0)}(B_0, \ldots, B_r) = \begin{cases} B_0, \text{ if } k_0 = 0, \\ \emptyset, \quad \text{otherwise.} \end{cases}$$

(ii) Let

$$\mathcal{P}_{\bar{k}}^{(k+1)}(B_0, \ldots, B_r) = \begin{cases} (\mathcal{P}_{\bar{k}}^{(k)}(B_0, \ldots, B_r))'_{\mathrm{e}}, & \text{if } k+1 \notin \{k_1, \ldots, k_r\}, \\ (\mathcal{P}_{\bar{k}}^{(k)}(B_0, \ldots, B_r))'_{\mathrm{e}} \oplus B_i, & \text{if } k+1 = k_i. \end{cases}$$

For example, for any two natural numbers n and k and any $B \subseteq \mathbb{N}$ we have that

$$\mathcal{P}_n^{(k)}(B) = \begin{cases} \emptyset^{(k)}, & \text{if } k < n, \\ (\emptyset^{(n)} \oplus B)_{\mathrm{e}}^{(k-n)}, & \text{if } n \leq k. \end{cases}$$

The theorem below is our main result.

Theorem 11. *1. The uniform operators of type $(k_0, \ldots, k_r \to k)$ are exactly those mappings $\Gamma : \mathcal{P}(\mathbb{N})^{r+1} \to \mathcal{P}(\mathbb{N})$ for which there exists an enumeration operator Φ such that for all subsets $B_0, \ldots, B_r$ of $\mathbb{N}$,*

$$\Gamma(B_0, \ldots, B_r) = \Phi(\mathcal{P}_{k_0, \ldots, k_r}^{(k)}(B_0, \ldots, B_r)).$$

2. For every uniform operator Γ of type $(k_0, \ldots, k_r \to k)$ there exists a total computable function $\gamma(b_0, \ldots, b_r)$ such that for all $b_0 \ldots, b_r \in \mathbb{N}$ and $X \subseteq \mathbb{N}$,

$$\Gamma(W_{b_0}^{X^{(k_0)}}, \ldots, W_{b_r}^{X^{(k_r)}}) = W_{\gamma(b_0, \ldots, b_r)}^{X^{(k)}}.$$

In the rest of the paper we present a proof of Theorem 11.

2 Regular Enumerations

The proof of Theorem 11 uses the technique of the regular enumerations, presented in [7] and [8].

Let us consider a sequence $\{B_i\}$ of sets of natural numbers.

Roughly speaking a k-regular enumeration f is a kind of generic function such that for all $i \leq k$, B_i is computably enumerable in $f^{(i)}$ uniformly in i.

Let f be a total mapping on $\mathbb{N}$. We define for every i, e, x the relation $f \models_i F_e(x)$ by induction on i:

Definition 12. (i) $f \models_0 F_e(x) \iff \exists v(\langle x, v\rangle \in W_e \ \& \ (\forall u \in D_v)(f((u)_0) = (u)_1))$;

(ii)

$$f \models_{i+1} F_e(x) \iff \exists v(\langle x, v\rangle \in W_e \ \&\ (\forall u \in D_v)((u = \langle e_u, x_u, 0\rangle \ \&\ f \models_i F_{e_u}(x_u)) \vee (u = \langle e_u, x_u, 1\rangle \ \&\ f \not\models_i F_{e_u}(x_u)))).$$

Set $f \models_i \neg F_e(x) \iff f \not\models_i F_e(x)$.

The following lemma can be easily proved by induction on i:

Lemma 13. *For every i there exists a total computable function $h_i(a)$ such that for all a,*

$$W_a^{f^{(i)}} = \{x : f \models_i F_{h_i(a)}(x)\}.$$

In what follows we shall use the term *finite part* for finite mappings of $\mathbb{N}$ into $\mathbb{N}$ defined on finite segments $[0, q-1]$ of $\mathbb{N}$. Finite parts will be denoted by the letters τ, δ, ρ. If $\mathrm{dom}(\tau) = [0, q-1]$, then let $\mathrm{lh}(\tau) = q$.

We shall suppose that an effective coding of all finite sequences and hence of all finite parts is fixed. Given two finite parts τ and ρ we shall say that τ is less than or equal to ρ if the code of τ is less than or equal to the code of ρ. By $\tau \subseteq \rho$ we shall denote that the partial mapping ρ extends τ and say that ρ is an extension of τ. For any τ, by $\tau \restriction n$ we shall denote the restriction of τ on $[0, n-1]$.

Set for every i, $\overline{B}_i = \mathbb{N} \oplus B_i$.

Below we define for every i the i-regular finite parts.

The 0-*regular finite parts* are finite parts τ such that $\mathrm{dom}(\tau) = [0, 2q+1]$ and for all odd $z \in \mathrm{dom}(\tau), \tau(z) \in \overline{B}_0$.

If $\mathrm{dom}(\tau) = [0, 2q+1]$, then the 0-rank $|\tau|_0$ of τ is equal to the number $q+1$ of the odd elements of $\mathrm{dom}(\tau)$. Notice that if τ and ρ are 0-regular, $\tau \subseteq \rho$ and $|\tau|_0 = |\rho|_0$, then $\tau = \rho$.

For every 0-regular finite part τ, let B_0^τ be the set of the odd elements of $\mathrm{dom}(\tau)$.

Given a 0-regular finite part τ, let

$$\tau \Vdash_0 F_e(x) \iff \exists v(\langle x, v\rangle \in W_e \ \&\ (\forall u \in D_v)(\tau((u)_0) \simeq (u)_1))$$

$$\tau \Vdash_0 \neg F_e(x) \iff \forall (\text{0-regular } \rho)(\tau \subseteq \rho \Rightarrow \rho \not\Vdash_0 F_e(x)).$$

Proceeding by induction, suppose that for some i we have defined the i-regular finite parts and for every i-regular τ – the i-rank $|\tau|_i$ of τ, the set B_i^τ and the relations $\tau \Vdash_i F_e(x)$ and $\tau \Vdash_i \neg F_e(x)$. Suppose also that if τ and ρ are i-regular, $\tau \subseteq \rho$ and $|\tau|_i = |\rho|_i$, then $\tau = \rho$.

Set $X_j^i = \{\rho : \rho \text{ is } i\text{-regular } \&\ \rho \Vdash_i F_{(j)_0}((j)_1)\}$.

Given a finite part τ and a set X of i-regular finite parts, let $\mu_i(\tau, X)$ be the least extension of τ belonging to X if any, and $\mu_i(\tau, X)$ be the least i-regular extension of τ otherwise. We shall assume that $\mu_i(\tau, X)$ is undefined if there is no i-regular extension of τ.

A *normal i-regular extension* of an i-regular finite part τ is any i-regular finite part $\rho \supseteq \tau$ such that $|\rho|_i = |\tau|_i + 1$.

Let τ be a finite part defined on $[0, q-1]$ and $r \geq 0$. Then τ is $(i+1)$-*regular* with $(i+1)$-*rank* $r+1$ if there exist natural numbers

$$0 < n_0 < l_0 < m_0 < b_0 < n_1 < l_1 < m_1 < b_1 \ldots < n_r < l_r < m_r < b_r < n_{r+1} = q$$

such that $\tau \upharpoonright n_0$ is an i-regular finite part with i-rank equal to 1 and for all j, $0 \leq j \leq r$, the following conditions are satisfied:

a) $\tau \upharpoonright l_j$ is a normal i-regular extension of $\tau \upharpoonright n_j$;
b)

$$\tau \upharpoonright m_j = \begin{cases} \mu_i(\tau \upharpoonright (l_j+1), X^i_{\langle p, l_j \rangle}), & \text{if } \tau(n_j) \simeq \langle i+1, p\rangle + 1, \\ \text{a normal } i\text{-regular extension of } \tau \upharpoonright l_j, & \text{otherwise;} \end{cases}$$

c)

$$\tau \upharpoonright b_j = \begin{cases} \mu_i(\tau \upharpoonright (m_j+1), X^i_{\langle p, q \rangle}), & \text{if } \tau(m_j) \simeq \langle p, q\rangle + 1, \\ \text{a normal } i\text{-regular extension of } \tau \upharpoonright m_j, & \text{if } \tau(m_j) \simeq 0; \end{cases}$$

d) $\tau(b_j) \in \overline{B}_{i+1}$;
e) $\tau \upharpoonright n_{j+1}$ is a normal i-regular extension of $\tau \upharpoonright b_j$.

The following lemma shows that the $(i+1)$-rank is well defined.

Lemma 14. *Let τ be an $(i+1)$-regular finite part. Then*

1. *Let $n'_0, l'_0, m'_0, b'_0, \ldots, n'_p, l'_p, m'_p, b'_p, n'_{p+1}$ and $n_0, l_0, m_0, b_0, \ldots, n_r, l_r, m_r, b_r, n_{r+1}$ be two sequences of natural numbers satisfying a)–e). Then $r = p, n_{p+1} = n'_{p+1}$ and for all $j \leq r, n_j = n'_j, l_j = l'_j, m_j = m'_j$ and $b_j = b'_j$.*
2. *If ρ is $(i+1)$-regular, $\tau \subseteq \rho$ and $|\tau|_{i+1} = |\rho|_{i+1}$, then $\tau = \rho$.*
3. *τ is i-regular and $|\tau|_i > |\tau|_{i+1}$.*

Let τ be $(i+1)$-regular and $n_0, l_0, m_0, b_0, \ldots, n_r, l_r, m_r, b_r, n_{r+1}$ be the sequence satisfying a)–e). Then let $B^\tau_{i+1} = \{b_0, \ldots, b_r\}$ and $M^\tau_{i+1} = \{m_0, \ldots, m_r\}$.

To conclude with the definition of the regular finite parts, let for every $(i+1)$-regular finite part τ

$$\tau \Vdash_{i+1} F_e(x) \iff \exists v(\langle x, v\rangle \in W_e \;\&\; (\forall u \in D_v)((u = \langle e_u, x_u, 0\rangle \;\&\; \tau \Vdash_i F_{e_u}(x_u)) \vee (u = \langle e_u, x_u, 1\rangle \;\&\; \tau \Vdash_i \neg F_{e_u}(x_u)))).$$

$$\tau \Vdash_{i+1} \neg F_e(x) \iff (\forall (i+1)\text{-regular } \rho)(\tau \subseteq \rho \Rightarrow \rho \nVdash_{i+1} F_e(x)).$$

Definition 15. Let f be a total mapping of $\mathbb{N}$ in $\mathbb{N}$. Then f is a *k-regular enumeration* (with respect to $\{B_i\}$) if the following conditions hold:

(i) For every finite part $\delta \subseteq f$, there exists a k-regular extension τ of δ such that $\tau \subseteq f$.
(ii) If $i \leq k$ and $z \in \overline{B}_i$, then there exists an i-regular $\tau \subseteq f$ such that $z \in \tau(B^\tau_i)$.
(iv) If $i < k$, then for every pair $\langle p, q\rangle$ of natural numbers, there exists an $i+1$-regular finite part $\tau \subseteq f$ such that for some $m \in M^\tau_{i+1}$, $\tau(m) \simeq \langle p, q\rangle + 1$.

Clearly, if f is a k-regular enumeration and $i \leq k$, then for every $\delta \subseteq f$, there exists an i-regular $\tau \subseteq f$ such that $\delta \subseteq \tau$. Moreover there exist i-regular finite parts of f of arbitrary large rank.

Given a regular f, let for $i \leq k$, $B_i^f = \{b : (\exists \tau \subseteq f)(\tau \text{ is } i\text{-regular } \& \ b \in B_i^\tau)\}$. Clearly $f(B_i^f) = B_i$.

Now let us turn to the properties of the regular finite parts and of the regular enumerations.

3 Properties of the regular enumerations

First of all, notice that the clause (iv) of the definition of the regular enumerations ensures that a k-regular enumeration f is generic with respect to the family $\{X_j^i : i < k, j \in \mathbb{N}\}$. So we have the following Truth Lemma:

Lemma 16. *Let f be a k-regular enumeration. Then*

1. *For all $i \leq k$, $f \models_i F_e(x) \iff (\exists \tau \subseteq f)(\tau$ is i-regular & $\tau \Vdash_i F_e(x))$.*
2. *For all $i < k$, $f \models_i \neg F_e(x) \iff (\exists \tau \subseteq f)(\tau$ is i-regular & $\tau \Vdash_i \neg F_e(x))$.*

Let us define for every natural k the set P_k by induction on k:

Definition 17. (i) $P_0 = \overline{B}_0$;
(ii) $P_{k+1} = (P_k)'_e \oplus \overline{B}_{k+1}$.

Denote by $\mathcal{R}_i$ the set of all i-regular finite parts.

For $j \in \mathbb{N}$ let $\mu_i(\tau, j) \simeq \mu_i(\tau, X_j^i)$,

$$Y_j^i = \{\tau : (\exists \rho \supseteq \tau)(\rho \text{ is } i\text{-regular } \& \ \rho \Vdash_i F_{(j)_0}((j)_1))\}$$

$$Z_j^i = \{\tau : \tau \text{ is } i\text{-regular } \& \ \tau \Vdash_i \neg F_{(j)_0}((j)_1)\}.$$

Proposition 18. *For every $i \in \mathbb{N}$ the following assertions hold:*

1. *There exists an enumeration operators R_i such that for every sequence $\{B_i\}$ of sets of natural numbers, $\mathcal{R}_i = R_i(P_i)$.*
2. *There exist computable functions $x_i(j)$ and $y_i(j)$ such that for every j and every sequence $\{B_i\}$ of sets of natural numbers,*

$$X_j^i = \Phi_{x_i(j)}(P_i) \text{ and } Y_j^i = \Phi_{y_i(j)}(P_i).$$

3. *There exists a computable function $z_i(j)$ such that for every j and every sequence $\{B_i\}$ of sets of natural numbers,*

$$Z_j^i = \{z_i(j)\}^{P_i'}.$$

4. *There exists an Oracle Turing Machine m_i such that for every sequence $\{B_i\}$ of sets of natural numbers,*

$$\mu_i = \{m_i\}^{P_i'}.$$

The following proposition is important for the proof of Theorem 11.

Proposition 19. *For every $i \in \mathbb{N}$ there exists an Oracle Turing Machine b_i such that for every sequence $\{B_i\}$ of sets of natural numbers and every k-regular with respect to $\{B_i\}$ enumeration f,*

$$(\forall i \leq k)(B_i = W_{b_i}^{f^{(i)}}).$$

Proof. We shall define the machines b_i by induction on i. Clearly for every sequence $\{B_i\}$ of sets of natural numbers and every k-regular with respect to $\{B_i\}$ enumeration f, $B_0 = \{x : 2x+1 \in \overline{B}_0\}$, $\overline{B}_0 = f(B_0^f)$ and B_0^f is equal to the set of all odd numbers.

So we may define the machine b_0 as follows:

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input X;
 Y:= 0;
 while (2X + 1 =\= f(2Y+1)) do
 Y := Y+1;
end.
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Suppose that $i < k$ and the machines $b_0, \ldots, b_i$ are defined. Following the definition of P_i we can define an oracle machine p' which given a sequence $\{B_i\}$ and a k-regular f computes the characteristic function of P_i' using $f^{(i+1)}$ as an oracle. So it is sufficient to show that we can enumerate the set $\overline{B}_i$ by means of P_i' and f, uniformly in P_i' and f.

Since f is $(i+1)$-regular, for every finite part δ of f there exists an $(i+1)$-regular $\tau \subseteq f$ such that $\delta \subseteq \tau$. Hence there exist natural numbers

$$0 < n_0 < l_0 < m_0 < b_0 < n_1 < l_1 < m_1 < b_1 < \ldots < n_r < l_r < m_r < b_r < \ldots,$$

such that for every $r \geq 0$, the finite part $\tau_r = f \upharpoonright n_{r+1}$ is $(i+1)$-regular and $n_0, l_0, m_0, b_0, \ldots, n_r, l_r, m_r, b_r, n_{r+1}$ are the numbers satisfying the conditions a)–e) from the definition of the $(i+1)$-regular finite part τ_r. Clearly $B_{i+1}^f = \{b_0, b_1 \ldots\}$. We shall describe a procedure which lists $n_0, l_0, m_0, b_0, \ldots$ in an increasing order using the oracles P_i' and f.

Clearly $f \upharpoonright n_0$ is i-regular and $|f \upharpoonright n_0|_i = 1$. By Lemma 18 $\mathcal{R}_i$ is uniformly computable in P_i'. Using f we can generate consecutively the finite parts $f \upharpoonright q$ for $q = 1, 2 \ldots$. By Lemma 14 $f \upharpoonright n_0$ is the first element of this sequence which belongs to $\mathcal{R}_i$. Clearly $n_0 = \mathrm{lh}(f \upharpoonright n_0)$.

Suppose that $r \geq -1$ and $n_0, l_0, m_0, b_0, \ldots, n_r, l_r, m_r, b_r, n_{r+1}$ have already been listed. Since $f \upharpoonright l_{r+1}$ is a normal i-regular extension of $f \upharpoonright n_{r+1}$ it is the shortest finite part of f which extends $f \upharpoonright n_{r+1}$ and belongs to $\mathcal{R}_i$. So we can find the number l_{r+1}. Now, we have to consider two cases:

a) $f(n_{r+1}) = 0$ or $f(n_{r+1}) = \langle j, p \rangle + 1$, where $j \neq i+1$. Then again $f \upharpoonright m_{r+1}$ is the shortest finite part of f which belongs to $\mathcal{R}_i$ and extends $f \upharpoonright l_{r+1}$.

b) $f(n_{r+1}) = \langle i+1, p \rangle + 1$. Then $f \upharpoonright m_{r+1} = \mu_i(f \upharpoonright (l_{r+1}+1), X^i_{\langle p, l_{r+1} \rangle})$.

In both cases we can find $f \upharpoonright m_{r+1}$ effectively in f and P'_i. Clearly $m_{r+1} = \mathrm{lh}(f \upharpoonright m_{r+1})$. From m_{r+1} we reach b_{r+1} in a way similar to the previous one. Finally, from b_{r+1} we reach n_{r+2} using the fact that $f \upharpoonright n_{r+2}$ is a normal i-regular extension of $f \upharpoonright b_{r+1}$. Now we have a machine which decides the set B^f_{i+1} using the oracle $f^{(i+1)}$. From here, since $\overline{B} = f(B^f_{i+1})$ we can easily obtain the machine b_{i+1}. □

4 Constructions of regular enumerations

Suppose that a sequence $\{B_i\}$ of sets of natural numbers is fixed.

Given a finite mapping τ defined on $[0, q-1]$, by $\tau * z$ we shall denote the extension ρ of τ defined on $[0, q]$ and such that $\rho(q) \simeq z$.

Lemma 20. *Let τ be an i-regular finite part defined on $[0, q-1]$. Let $x, y_1, \ldots y_i \in \mathbb{N}$ and $z_0 \in \overline{B}_0, \ldots, z_i \in \overline{B}_i$. There exists a normal i-regular extension ρ of τ such that:*

1. *$\rho(q) \simeq x$;*
2. *$(\forall j < i)(y_{j+1} \in \rho(M^\rho_{j+1}))$.*
3. *$(\forall j \le i)(z_j \in \rho(B^\rho_j))$.*

Proof. Induction on i. The assertion is obvious for $i = 0$. Let τ be an $(i+1)$-regular finite part s.t. $\mathrm{dom}(\tau) = [0, q-1]$. Let $x, y_1, \ldots, y_{i+1} \in \mathbb{N}, z_0 \in \overline{B}_0, \ldots, z_{i+1} \in \overline{B}_{i+1}$ be given. Suppose that $|\tau|_{i+1} = r+1$ and $n_0, l_0, m_0, b_0, \ldots, n_r, l_r, m_r, b_r, n_{r+1}$ are the natural numbers satisfying the conditions a)–e) from the definition of the $(i+1)$-regular finite parts. Notice that $n_{r+1} = q$. Since τ is also i-regular, by the induction hypothesis there exists a normal i-regular extension ρ_0 of $\tau * x$ such that $(\forall j < i)(y_{j+1} \in \rho(M^\rho_{j+1}))$ and $(\forall j \le i)(z_j \in \rho(B^\rho_j))$. Let $l_{r+1} = \mathrm{lh}(\rho_0)$. Clearly there exists a normal i-regular extension δ of $\rho_0 * 0$ and hence the function $\mu_i(\rho_0 * 0, X^i_p)$ is defined for all $p \in \mathbb{N}$. Set

$$\rho_1 = \begin{cases} \delta, & \text{if } x = 0 \vee (\exists j)(x = \langle j, p\rangle + 1 \ \&\ j \neq i+1), \\ \mu_i(\rho_0 * 0, X^i_p), & \text{if } x = \langle i+1, p\rangle. \end{cases}$$

Set $m_{r+1} = \mathrm{lh}(\rho_1)$. Let ν be a normal extension of $\rho_1 * y_{i+1}$ and set

$$\rho_2 = \begin{cases} \nu, & \text{if } y_{i+1} = 0, \\ \mu_i(\rho_0 * 0, X^i_{y_{i+1}-1}), & \text{if } y_{i+1} > 0. \end{cases}$$

Set $b_{r+1} = \mathrm{lh}(\rho_2)$ and let ρ be a normal i-regular extension of $\rho_2 * z_{i+1}$. □

Corollary 21. *If $i \le k$, then every i-regular finite part of rank 1 can be extended to a k-regular finite part of rank 1 and to a k-regular enumeration.*

Using similar arguments we may prove and the following proposition.

Proposition 22. *Let δ be an i-regular finite part. Let $y = 0$ or $y = \langle j, p\rangle + 1$ for some $j > i$. There exists a normal i-regular extension ρ of $\delta * y$ such that $(\forall x \in dom(\rho))(x > lh(\delta) \Rightarrow \rho(x) \simeq 0)$.*

Corollary 23. *For every* $i \in \mathbb{N}$ *there exists a* canonical *i-regular finite part* δ_i *of rank* 1 *such that* $(\forall x \in dom(\delta_i))(\delta_i(x) \simeq 0)$*.*

Now we are ready to present a proof of Theorem 11.

Proof (of Theorem 11).
Let us fix natural numbers $k_0 < \ldots < k_r$ and k. Given sets $A_0, \ldots, A_r$ of natural numbers, we define the sequence $\{B_i\}$ by setting

$$B_i = \begin{cases} A_j, \text{ if } i = k_j, \\ \emptyset, \quad \text{if } i \notin \{k_0, \ldots, k_r\}. \end{cases}$$

We call a finite part or an enumeration i-regular with respect to $A_0, \ldots, A_r$ if it is i-regular with respect to the sequence $\{B_i\}$.

As in the previous sections by $\overline{B}_i$ we denote $\mathbb{N} \oplus B_i$ and by P_i we denote the set

$$(...((\overline{B}_0)'_{\mathrm{e}} \oplus \overline{B}_1)'_{\mathrm{e}} \oplus \ldots \oplus \overline{B}i - 1)'_{\mathrm{e}} \oplus \overline{B}_i.$$

Clearly there exist computable functions $p_1(i)$ and $p_2(i)$ which do not depend on the choice of the sets $A_0, \ldots, A_r$ and such that

$$P_i = \Phi_{p_1(i)}(\mathcal{P}^{(i)}_{k_0,\ldots,k_r}(A_0, \ldots, A_r)) \text{ and } \mathcal{P}^{(i)}_{k_0,\ldots,k_r}(A_0, \ldots, A_r) = \Phi_{p_2(i)}(P_i).$$

Now let us consider a uniform operator Γ of type $(k_0, \ldots, k_r \to k)$. Let γ be the respective index function of Γ. By Proposition 19 there exist Oracle Turing Machines $b_{k_0}, \ldots, b_{k_r}$ such that for every $i \geq k_r$, every sequence of sets $A_0, \ldots, A_r$ and every i-regular enumeration f,

$$A_0 = W^{f^{(k_0)}}_{b_{k_0}}, \ldots, A_r = W^{f^{(k_r)}}_{b_{k_r}}.$$

Let $b = \gamma(b_{k_0}, \ldots, b_{k_r})$. Clearly for every sequence $A_0, \ldots, A_r$ of sets, for every $i \geq k_r$ and every i-regular enumeration f we have that

$$\Gamma(A_0, \ldots, A_r) = W^{f^{(k)}}_b.$$

Therefore there exists a c such that for every sequence $A_0, \ldots, A_r$ of sets, every $i \geq k_r$ and every i-regular enumeration f,

$$(\forall n)(f(n) \in \Gamma(A_0, \ldots, A_r) \iff f \models_k F_c(n)).$$

Consider the canonical k-regular finite part δ_k of rank 1. Set $n_0 = \mathrm{lh}(\delta_k)$. Let δ be a normal k-regular extension of $\delta_k * (\langle k+1, c\rangle + 1)$ such that $(\forall x \in \mathrm{dom}(\delta))(x > \mathrm{lh}(\delta_k) \Rightarrow \delta(x) \simeq 0)$. Let $\mathrm{lh}(\delta) = l_0$. We shall show that

$$x \in \Gamma(A_0, \ldots, A_r) \iff (\exists \tau \supseteq \delta)(\tau \text{ is } k\text{-regular with respect to } A_0, \ldots, A_r \ \& \ \tau(l_0) \simeq x \ \& \ \tau \Vdash_k F_c(l_0)).$$

Indeed, suppose that there exist a $\tau \supseteq \delta$ which is k-regular with respect to $A_0, \ldots. A_r$, $\tau(l_0) \simeq x$ and $\tau \Vdash_k F_c(l_0)$. Then there exists a $\max(k_r, k+1)$-regular with respect to $A_0, \ldots, A_r$ enumeration f which extends the least such τ. Clearly $f \models_k F_c(l_0)$ and $f(l_0) \simeq x$. So, $x \in \Gamma(A_0, \ldots, A_r)$.

Suppose now that $x \in \Gamma(A_0, \ldots, A_r)$. Consider a $\max(k_r, k+1)$-regular enumeration f which extends $\delta * x$. Then $f(l_0) \simeq x$ and hence $f \models_k F_c(l_0)$. Then there exists a $\tau \subseteq f$ such that $\tau \Vdash_k F_c(l_0)$. Clearly we may assume that $\delta \subseteq \tau$ and $\tau(l_0) \simeq x$.

From here using Proposition 18 one can find easily an enumeration operator Φ such that for all $A_1, \ldots, A_r$,

$$\Gamma(A_0, \ldots, A_r) = \Phi(\mathcal{P}^{(k)}_{k_0,\ldots,k_r}(A_0, \ldots, A_r)).$$

By this we have proved the nontrivial part of the theorem. The proof of the rest is routine. □

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