Computable embeddings for pairs of structures

The First Workshop on Digitalization and Computable Models

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19 December 2019

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Continuous operators

Let us denote by $\mathcal{P}_\omega$ the topological space on $\mathcal{P}(\omega)$, where the basic open sets are $U_v = \{A \subseteq \omega \mid D_v \subseteq A\}$. $\mathcal{P}_\omega$ is known as the Scott topology.

We say that $\Gamma : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is a *generalized* enumeration operator if there exists a set B such that

$$\Gamma(A) = \{x \mid (\exists v)[\langle x, v\rangle \in B \ \&\ D_v \subseteq A]\}.$$

The following proposition is a well-known fact.

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$\Gamma : \mathcal{P}_\omega \to \mathcal{P}_\omega$ is continuous iff Γ is a generalized enumeration operator.

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Recall that the continuous operators are:

- compact, i.e. $x \in \Gamma_e(A)$ iff there is some finite $D \subseteq A$ such that $x \in \Gamma_e(D)$.
- monotone, i.e. $A \subseteq B$ implies $\Gamma_e(A) \subseteq \Gamma_e(B)$.

Enumeration operators

We say that $\Gamma : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is an **enumeration operator** if for some c.e. set W_e,

$$\Gamma(A) = \{x \mid (\exists v)[\langle x, v\rangle \in W_e \ \& \ D_v \subseteq A].$$

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Theorem (Selman)

$B = \Gamma_e(A)$ iff $(\forall X \subseteq \mathbb{N})[A$ is c.e. in $X \implies B$ is c.e. in $X]$.

Computable embeddings

- We work with countable structures with domains **subsets** of ω. This is important!
- We associate with $\mathcal{A}$ the set of **basic** sentences in the language $L \cup \omega$, true in $\mathcal{A}$, which we denote by $D(\mathcal{A})$.
- The class $\mathcal{K}$ is **computably embeddable** in $\mathcal{K}'$,

$$\mathcal{K} \leq_c \mathcal{K}',$$

  if there is an enumeration operator Γ_e such that
  - for each $\mathcal{A} \in \mathcal{K}$,

  $$\Gamma_e(D(\mathcal{A})) = D(\mathcal{B}), \text{ where } \mathcal{B} \in \mathcal{K}';$$

  - Let $\mathcal{A}_1 \mathcal{A}_2 \in \mathcal{K}$, $\Gamma_e(D(\mathcal{A}_1)) = D(\mathcal{B}_1)$ and $\Gamma_e(D(\mathcal{A}_2)) = D(\mathcal{B}_2)$. Then $\mathcal{A}_1 \cong \mathcal{A}_2$ iff $\mathcal{B}_1 \cong \mathcal{B}_2$.

Turing computable embeddings

The class $\mathcal{K}$ is **Turing computably embeddable** in $\mathcal{K}'$,

$$\mathcal{K} \leq_{tc} \mathcal{K}',$$

if there is a Turing operator $\Phi = \varphi_e$ such that

- for each $\mathcal{A} \in \mathcal{K}$,

$$\varphi_e^{D(\mathcal{A})} = \chi_{D(\mathcal{B})}, \text{ where } \mathcal{B} \in \mathcal{K}';$$

- Let $\mathcal{A}_1\mathcal{A}_2 \in \mathcal{K}$, $\varphi_e^{D(\mathcal{A}_1)} = \chi_{D(\mathcal{B}_1)}$ and $\varphi_e^{D(\mathcal{A}_2)} = \chi_{D(\mathcal{B}_2)}$. Then $\mathcal{A}_1 \cong \mathcal{A}_2$ iff $\mathcal{B}_1 \cong \mathcal{B}_2$.

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- Let $\mathcal{A}_1 \mathcal{A}_2 \in \mathcal{K}$, $\varphi_e^{D(\mathcal{A}_1)} = \chi_{D(\mathcal{B}_1)}$ and $\varphi_e^{D(\mathcal{A}_2)} = \chi_{D(\mathcal{B}_2)}$. Then $\mathcal{A}_1 \cong \mathcal{A}_2$ iff $\mathcal{B}_1 \cong \mathcal{B}_2$.

Even though, we allow $D(\mathcal{A}) \subset \omega$, in the Turing case we can find the first element in the domain, the second, and so on, i.e. we work with a fixed enumeration of the domain. We cannot do that in the enumeration case. This is one of the main differences.

A few examples of previous results

- *PF* - finite prime fields;
- *FLO* - finite linear orders;
- *FVS* - $\mathbb{Q}$-vector spaces of finite dimension;
- *VS* - $\mathbb{Q}$-vector spaces;
- *LO* - linear orders.

Theorem (Calvert-Cummins-Miller-Knight)

$PF <_c FLO <_c FVS <_c VS <_c LO.$

Theorem (Knight-Miller-Vanden Boom)

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$PF <_{tc} FLO <_{tc} FVS <_{tc} VS <_{tc} LO.$

Their motivation was to consider effective versions of Borel embeddings.

Question (Knight-Miller-Vanden Boom)

Which is the better notion, $\leq_c$ or $\leq_{tc}$?

$\leq_c$ implies $\leq_{tc}$

Proposition (Greenberg, Kalimullin)

If $\mathcal{K} \leq_c \mathcal{K}'$, then $\mathcal{K} \leq_{tc} \mathcal{K}'$.

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Suppose that $\mathcal{K} \leq_c \mathcal{K}'$ via the enumeration operator Γ_e. Let $\mathcal{A} \in \mathcal{K}$ and $\Gamma_e(D(\mathcal{A})) = D(\mathcal{B})$, where $\mathcal{B} \in \mathcal{K}'$. It follows that

$$b \in \mathcal{B} \leftrightarrow (\exists s)(\exists v)[\langle \text{"}b = b\text{"}, v\rangle \in W_{e,s} \ \& \ D_v \subset D(\mathcal{A})].$$

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Define $f(b) = \langle b, s\rangle$, where s is the least such stage. Then f is partial computable in $D(\mathcal{A})$.

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Define $f(b) = \langle b, s\rangle$, where s is the least such stage. Then f is partial computable in $D(\mathcal{A})$. Let $\mathcal{B} \cong_f \mathcal{C}$. Then $D(\mathcal{C}) \leq_T D(\mathcal{A})$. This procedure is uniform, so there is such a Turing operator, which produces $D(\mathcal{C})$ given as input $D(\mathcal{A})$.

$\leq_c$ strongly implies $\leq_{tc}$

In general we do not have the converse.

Example (Kalimullin)

- $\{1,2\} \leq_{tc} \{\omega, \omega^\star\}$. Proceed at stages. Start building initial segments of ω. If another element is found in the domain of the input structure, switch to building initial segments of $\omega^\star$. This guess never changes.
- $\{1,2\} \not\leq_c \{\omega, \omega^\star\}$. Trivial - monotonicity of enumeration operators: 1 is a substructure of 2, but ω is not a substructure of $\omega^\star$.

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As usual, $\leq_{tc}$ and $\leq_c$ induce equivalence relations $\equiv_{tc}$ and $\equiv_c$. Given a $\equiv_{tc}$-class, it is natural to ask how it is partitioned in terms of $\equiv_c$.

The Pullback Theorem

Theorem (Knight, Miller, and Vanden Boom)

Suppose that $\mathcal{K}_1 \leq_{tc} \mathcal{K}_2$ via a Turing operator Φ. Then for any computable infinitary sentence ψ_2 in the language of $\mathcal{K}_2$, one can effectively find a computable infinitary sentence ψ_1 in the language of $\mathcal{K}_1$ such that for all $\mathcal{A} \in \mathcal{K}_1$, we have

$$\mathcal{A} \models \psi_1 \ \leftrightarrow \ \Phi(\mathcal{A}) \models \psi_2.$$

Moreover, for a non-zero $\alpha < \omega_1^{CK}$, if ψ_2 is a Σ_α^c sentence, then so is ψ_2.

Since $\leq_c$ implies $\leq_{tc}$, the theorem works for computable embeddings as well.

Motivation

Considering pairs of structures is common in computable structure theory. For example, S is a Δ_2^0 set iff there is a unif. comp. sequence $\{\mathcal{U}_n\}_{n=0}^{\infty}$ such that

$$\mathcal{U}_n \cong \begin{cases} \omega, & n \in S \\ \omega^\star, & n \notin S \end{cases}$$

This kind of encoding is used in a number of jump inversion theorems for structures.
It is natural to ask how the $\equiv_{tc}$-class of $\{\omega, \omega^\star\}$ is partitioned under $\equiv_c$. Surprisingly, this is not so easy to answer.

Characterization of the *tc*-class of $\{\omega, \omega^\star\}$

The Pullback Lemma is used here. Notice that ω and $\omega^\star$ differ by Σ^c_2 sentences.

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be non-isomorphic L-structures. T.F.A.E.

(1) $\{\omega, \omega^\star\} \equiv_{tc} \{\mathcal{A}, \mathcal{B}\}$;

(2) *$\mathcal{A}$ and $\mathcal{B}$ have computable copies, $\mathcal{A} \equiv_1 \mathcal{B}$, and they differ by Σ^c_2 sentences.*

Characterization of the tc-class of $\{\omega, \omega^\star\}$

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It follows that all pairs of the form $\{\omega \cdot k, \omega^\star \cdot k\}$, for any $k > 0$ are equivalent under Turing computable embeddings.
What about computable embeddings (enumeration operators) ?

The top pair among linear orderings

We want to study the pairs of linear orderings inside the *tc*-degree of $\{\omega, \omega^\star\}$ relative to $\leq_c$. It turns out that we have a top pair.

Theorem

For any pair $\{\mathcal{A}, \mathcal{B}\} \equiv_{tc} \{\omega, \omega^\star\}$, we have that

$$\{\mathcal{A}, \mathcal{B}\} \leq_c \{1 + \eta, \eta + 1\},$$

where η is the order type of the rationals.

Corollary

For any natural number $k > 0$,

$$\{\omega \cdot k, \omega^\star \cdot k\} <_c \{1 + \eta, \eta + 1\}.$$

(The strictness comes from monotonicity.)

Infinite chain of pairs (1)

Our second step is to show that for any natural number $k \geq 1$,

$$\{\omega, \omega^\star\} <_c \cdots <_c \{\omega \cdot 2^k, \omega^\star \cdot 2^k\} <_c \cdots <_c \{1+\eta, \eta+1\}.$$

We clearly have the following:

$$m \leq k \implies \{\omega \cdot 2^m, \omega^\star \cdot 2^m\} \leq_c \{\omega \cdot 2^k, \omega^\star \cdot 2^k\}.$$

Since 2^m divides 2^k, the enumeration operator just copies its input a fixed number of times.

Infinite chain of pairs (2)

- We denote by α, β, γ finite linear orderings.
- Define $\alpha \Vdash_\Gamma x < y$ iff
  $x, y \in \Gamma(\alpha)$ & $\neg(\exists \beta \supseteq \alpha)[\Gamma(\beta) \models y \leq x]$.

Proposition

Let $x, y \in \Gamma(\alpha)$ be distinct elements. Then

$$\alpha \Vdash_\Gamma x < y \text{ or } \alpha \Vdash_\Gamma y < x.$$

Moreover,

$$\alpha \Vdash_\Gamma x < y \text{ iff } \alpha \nVdash_\Gamma y < x.$$

(Easy proof: monotonicity and compactness are used.)

Proposition

For distinct elements $x_0, x_1, \dots, x_n \in \Gamma(\alpha)$, there is exactly one permutation π of $\{0, 1, \dots, n\}$ such that

$$\alpha \Vdash_\Gamma x_{\pi(0)} < x_{\pi(1)} < \cdots < x_{\pi(n)}.$$

Infinite chain of pairs (3)

Notice that in general, for finite α, $\Gamma(\alpha)$ might be infinite.
Moreover, in general $\alpha \cap \beta = \emptyset$ does not imply $\Gamma(\alpha) \cap \Gamma(\beta) = \emptyset$.

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Notice that in general, for finite α, $\Gamma(\alpha)$ might be infinite.
Moreover, in general $\alpha \cap \beta = \emptyset$ does not imply $\Gamma(\alpha) \cap \Gamma(\beta) = \emptyset$.
Let $\{\mathcal{A}, \mathcal{B}\} \leq_c \{\mathcal{C}, \mathcal{D}\}$, where $\mathcal{A}, \mathcal{C}$ has no infinite descending chains and $\mathcal{B}, \mathcal{D}$ have no infinite ascending chains.

Proposition

$\Gamma(\alpha)$ is finite for **all** finite α.

Proposition

Let $\alpha \cap \beta = \emptyset$ and $x, y \in \Gamma(\alpha) \cap \Gamma(\beta)$ be distinct elements. Then

$$\alpha \Vdash_\Gamma x < y \;\leftrightarrow\; \beta \Vdash_\Gamma x < y.$$

It follows that there are at most **finitely many** elements x with the property that $x \in \Gamma(\alpha) \cap \Gamma(\beta)$ for some α and β with $\alpha \cap \beta = \emptyset$.

Infinite chain of pairs (4)

Proposition

Suppose $\{\omega \cdot 2, \omega^\star \cdot 2\} \leq_c \{\mathcal{C}, \mathcal{D}\}$ via Γ, where $\mathcal{C}$ has no infinite descending chains and $\mathcal{D}$ has no infinite ascending chains. Let $\mathcal{A}, \hat{\mathcal{A}}$ and $\mathcal{B}, \hat{\mathcal{B}}$ be copies of ω such that $\Gamma(\mathcal{A}) \supseteq \hat{\mathcal{A}}$ and $\Gamma(\mathcal{B}) \supseteq \hat{\mathcal{B}}$. Then we have (up to finite difference) the following:

$$\Gamma(\mathcal{A} + \mathcal{B}) \supseteq \hat{\mathcal{A}} + \hat{\mathcal{B}}$$

or

$$\Gamma(\mathcal{A} + \mathcal{B}) \supseteq \hat{\mathcal{B}} + \hat{\mathcal{A}}.$$

It other words, the output copies of ω cannot be merged.

Corollary

Suppose $\{\omega \cdot 2, \omega^\star \cdot 2\} \leq_c \{\mathcal{C}, \mathcal{D}\}$ via Γ, where $\mathcal{C}$ and $\mathcal{D}$ are as before. Then $\mathcal{C}$ includes $\omega \cdot 2$ and $\mathcal{D}$ includes $\omega^\star \cdot 2$.

Infinite chain of pairs (5)

We can generalize the previous proposition in the following way:

Theorem

Fix some $k \geq 2$ *and suppose* $\{\omega \cdot k, \omega^\star \cdot k\} \leq_c \{\mathcal{C}, \mathcal{D}\}$ *via* Γ, *where* $\mathcal{C}$ *and* $\mathcal{D}$ *are as before. Then* $\mathcal{C}$ *includes* $\omega \cdot k$ *and* $\mathcal{D}$ *includes* $\omega^\star \cdot k$.

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It follows that we have the following chain:

$$\{\omega, \omega^\star\} <_c \{\omega{\cdot}2, \omega^\star{\cdot}2\} <_c \cdots <_c \{\omega{\cdot}2^k, \omega^\star{\cdot}2^k\} <_c \cdots <_c \{1{+}\eta, \eta{+}1\}.$$

Recall that all of these pairs are equivalent relative to Turing computable embeddings.

The main result

In this context, the enumeration operators can only copy their input structure a fixed number of times and do nothing else. More formally,

Theorem

For any two non-zero natural nummbers n and k,

$$n \textit{ divides } k \;\leftrightarrow\; \{\omega \cdot n, \omega^\star \cdot n\} \leq_c \{\omega \cdot k, \omega^\star \cdot k\}.$$

A sample case $(2 \mapsto 3)$

We already know that $\{\omega \cdot 3, \omega^\star \cdot 3\} \not\leq_c \{\omega \cdot 2, \omega^\star \cdot 2\}$. Now assume $\{\omega \cdot 2, \omega^\star \cdot 2\} \leq_c \{\omega \cdot 3, \omega^\star \cdot 3\}$ via Γ.

- If $\mathcal{A}$ is a copy of ω, then $\Gamma(\mathcal{A})$ is a copy of ω.
- Let $\mathcal{A}$ and $\mathcal{B}$ be copies of ω. Then

$$\Gamma(\mathcal{A} + \mathcal{B}) \supseteq \Gamma(\mathcal{A}) + \Gamma(\mathcal{B}).$$

- Then we have one of the following cases:
  - $\Gamma(\mathcal{A} + \mathcal{B}) = \Gamma(\mathcal{A}) + \Gamma(\mathcal{B}) + \mathcal{C}$;
  - $\Gamma(\mathcal{A} + \mathcal{B}) = \mathcal{C} + \Gamma(\mathcal{A}) + \Gamma(\mathcal{B})$;
  - $\Gamma(\mathcal{A} + \mathcal{B}) = \Gamma(\mathcal{A}) + \mathcal{C} + \Gamma(\mathcal{B})$.

  We prove that none of these cases are possible and thus, $\{\omega \cdot 2, \omega^\star \cdot 2\} \not\leq_c \{\omega \cdot 3, \omega^\star \cdot 3\}$.

Going higher to powers of ω

Notice that the pair $\{\omega^2, (\omega^2)^\star\}$ is *tc*-equivalent to $\{\omega, \omega^\star\}$. Now this should be clear:

$$\{\omega, \omega^\star\} <_c \{\omega^2, (\omega^2)^\star\}.$$

The following result was surprising:

Theorem

$$\{\omega \cdot 2, \omega^\star \cdot 2\} <_c \{\omega^2, (\omega^2)^\star\},$$

but

$$\{\omega \cdot 3, \omega^\star \cdot 3\} \not<_c \{\omega^2, (\omega^2)^\star\}.$$

Intuition: enumeration operators can "guess" whether an element is finitely or infinitely far from the beginning (respectively, the end).

$\{\omega \cdot 2, \omega^\star \cdot 2\} <_c \{\omega^2, (\omega^2)^\star\}$

For a linear ordering $\mathcal{L}$ and an element a, we define

$$\begin{aligned}
\mathtt{left}_{\mathcal{L}}(a) &= |\{b \in \mathtt{dom}(\mathcal{L}) \mid b \leq_{\mathcal{L}} a\}| \\
\mathtt{right}_{\mathcal{L}}(a) &= |\{b \in \mathtt{dom}(\mathcal{L}) \mid b \geq_{\mathcal{L}} a\}| \\
\mathtt{rad}_{\mathcal{L}}(a) &= \min\{\mathtt{left}_{\mathcal{L}}(a), \mathtt{right}_{\mathcal{L}}(a)\}.
\end{aligned}$$

Suppose we have as input the finite linear ordering $\mathcal{L} = a_0 < a_1 < a_2 < \cdots < a_n$. For each i such that $0 \leq i \leq n$, Γ outputs the pairs of the form (a_i, a_j), where

$$a_j \leq_{\mathbb{N}} \mathtt{rad}_{\mathcal{L}}(a_i).$$

All pairs in the output diagram are ordered in lexicographic order.

$\{\omega \cdot 2, \omega^\star \cdot 2\} <_c \{\omega^2, (\omega^2)^\star\}$

- Suppose that $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$, where $\mathcal{A}_{1,2}$ are copies of ω. Then

$$\Gamma(\mathcal{A}) \cong \sum_{i \in \omega} i + \sum_{i \in \omega} \omega \cdot 2 \cong \omega + \omega^2 = \omega^2.$$

- Suppose that $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$, where $\mathcal{A}_{1,2}$ are copies of $\omega^\star$. Then

$$\Gamma(\mathcal{A}) \cong \sum_{i \in \omega^\star} \omega^\star \cdot 2 + \sum_{i \in \omega^\star} i \cong (\omega^2)^\star + \omega^\star = (\omega^2)^\star.$$

Corollary

For any natural number $n \geq 1$, we have the following:

$$\{\omega \cdot (n+1), \omega^\star \cdot (n+1)\} \leq_c \{\omega^2 \cdot n, (\omega^2)^\star \cdot n\}.$$

$\{\omega \cdot 3, \omega^\star \cdot 3\} \not<_c \{\omega^2, (\omega^2)^\star\}$

- Assume that $\{\omega \cdot 3, \omega^\star \cdot 3\} <_c \{\omega^2, (\omega^2)^\star\}$ via Γ.
- For any copy $\mathcal{A}$ of ω, $\Gamma(\mathcal{A})$ is a copy of ω.

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- Assume that $\{\omega \cdot 3, \omega^\star \cdot 3\} <_c \{\omega^2, (\omega^2)^\star\}$ via Γ.
- For any copy $\mathcal{A}$ of ω, $\Gamma(\mathcal{A})$ is a copy of ω.
- If $\mathcal{M}$ is a copy of $\omega \cdot 3$ and $\Gamma(\mathcal{M}) \cong \omega^2$, then there is $\mathcal{N}$, a copy of $\omega \cdot 2$ with $dom(\mathcal{N}) = dom(\mathcal{M})$ such that $\Gamma(\mathcal{N}) \cong \omega^2$.

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- If $\mathcal{N}_0$ and $\mathcal{N}_1$ are copies of $\omega \cdot 2$ such that $\Gamma(\mathcal{N}_0) \cong \omega^2$ and $\Gamma(\mathcal{N}_1) \cong \omega^2$, then there is a copy $\mathcal{M}$ of $\omega \cdot 3$ such that $\Gamma(\mathcal{M}) \cong \omega^2 \cdot 2$.

Final remark

In all negative results, we actually prove that there is no generalized enumeration operator, in other words, no continuous operator in the Scott topology.

The end

Thank you for your attention!