Effective embeddings for pairs of linear orders

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Computable embeddings

- We write $\Gamma_e(A) = \{x \mid (\exists v)[\langle x, v\rangle \in W_e \ \&\ D_v \subseteq A]$. We call Γ_e an **enumeration operator**.
- These operators are monotone and compact.
- We say that the set B is enumeration reducible to the set A if $B = \Gamma_e(A)$, for some e.
- We work with countable structures with domains **subsets** of ω. This is important!
- We associate with $\mathcal{A}$ the set of **basic** sentences in the language $L \cup \omega$, true in $\mathcal{A}$, which we denote by $D(\mathcal{A})$.
- The class $\mathcal{K}$ is **computably embeddable** in $\mathcal{K}'$, $\mathcal{K} \leq_c \mathcal{K}'$, if there is Γ_e such that
    - for each $\mathcal{A} \in \mathcal{K}$, $\Gamma_e(D(\mathcal{A})) = D(\mathcal{B})$ and $\mathcal{B} \in \mathcal{K}'$;
    - Let $\mathcal{A}_1\mathcal{A}_2 \in \mathcal{K}$, $D(\mathcal{B}_1) = \Gamma_e(D(\mathcal{A}_1))$ and $D(\mathcal{B}_2) = \Gamma_e(D(\mathcal{A}_2))$. Then $\mathcal{A}_1 \cong \mathcal{A}_2$ iff $\mathcal{B}_1 \cong \mathcal{B}_2$.

Turing computable embeddings

The class $\mathcal{K}$ is **Turing computably embeddable** in $\mathcal{K}'$,

$$\mathcal{K} \leq_{tc} \mathcal{K}',$$

if there is a Turing operator $\Phi = \varphi_e$ such that

- for each $\mathcal{A} \in \mathcal{K}$, $\varphi_e^{D(\mathcal{A})} = \chi_{D(\mathcal{B})}$ and $\mathcal{B} \in \mathcal{K}'$;
- Let $\mathcal{A}_1 \mathcal{A}_2 \in \mathcal{K}$, $\chi_{D(\mathcal{B}_1)} = \varphi_e^{D(\mathcal{A}_1)}$ and $\chi_{D(\mathcal{B}_2)} = \varphi_e^{D(\mathcal{A}_2)}$. Then $\mathcal{A}_1 \cong \mathcal{A}_2$ iff $\mathcal{B}_1 \cong \mathcal{B}_2$.

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If $D(\mathcal{B}) = \Gamma_e(D(\mathcal{A}))$, then any enumeration of $D(\mathcal{A})$ produces an enumeration of $D(\mathcal{B})$. This is the main difference with Turing operators, where we may think that we work with a fixed enumeration of $D(\mathcal{A})$ given by its characteristic function (we can check if an element is in the domain or not). If we are not careful, we may construct an enumeration operator such that for different enumerations of $D(\mathcal{A})$ produces enumerations of different copies of $D(\mathcal{B})$, or something entirely different.

A few examples of previous results

- PF - finite prime fields;
- FLO - finite linear orders;
- FVS - $\mathbb{Q}$-vector spaces of finite dimension;
- VS - $\mathbb{Q}$-vector spaces;
- LO - linear orders.

Theorem (Calvert-Cummins-Miller-Knight)

$PF <_c FLO <_c FVS <_c VS <_c LO.$

Theorem (Knight-Miller-Vanden Boom)

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The difference between $\leq_{tc}$ and $\leq_c$ is not well studied.

$\leq_c$ implies $\leq_{tc}$

Proposition (Greenberg, Kalimullin)

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Suppose that $\mathcal{K} \leq_c \mathcal{K}'$ via the enumeration operator Γ_e. Let $\mathcal{A} \in \mathcal{K}$ and $\Gamma_e(D(\mathcal{A})) = D(\mathcal{B})$, where $\mathcal{B} \in \mathcal{K}'$. It follows that

$$b \in \mathcal{B} \leftrightarrow (\exists s)(\exists \alpha)[\langle "b = b", \alpha\rangle \in W_{e,s} \ \&\ \alpha \subset D(\mathcal{A})].$$

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Define $f(b) = \langle b, s\rangle$, where s is the least such stage. Then f is partial computable in $D(\mathcal{A})$.

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Define $f(b) = \langle b, s \rangle$, where s is the least such stage. Then f is partial computable in $D(\mathcal{A})$. Let $\mathcal{B} \cong_f \mathcal{C}$. Then $D(\mathcal{C}) \leq_T D(\mathcal{A})$. This procedure is uniform, so there is such Turing operator, which produces $D(\mathcal{C})$ given as input $D(\mathcal{A})$.

$\leq_{tc}$ does not imply $\leq_c$

Here 1 and 2 are linear orders.

Example (Kalimullin)

- $\{1, 2\} \leq_{tc} \{\omega, \omega^\star\}$. Proceed at stages. Start building initial segments of ω. If another element is found in the domain of the input structure, switch to building initial segments of $\omega^\star$. This guess never changes.
- $\{1, 2\} \not\leq_c \{\omega, \omega^\star\}$. Trivial - monotonicity of enumeration operators: 1 is a substructure of 2, but ω is not a substructure of $\omega^\star$.

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As usual, $\leq_{tc}$ and $\leq_c$ induce equicalence relations $\equiv_{tc}$ and $\equiv_c$. Given an $\equiv_{tc}$-class, we ask how it is partitioned in terms of $\equiv_c$. In particular, how is the $\equiv_{tc}$-class of $\{\omega, \omega^\star\}$ partitioned? It turns out that this is not so easy to answer.

The Pullback Theorem

Theorem (Knight, Miller, and Vanden Boom)

Suppose that $\mathcal{K}_1 \leq_{tc} \mathcal{K}_2$ via a Turing operator Φ. Then for any computable infinitary sentence ψ_2 in the language of $\mathcal{K}_2$, one can effectively find a computable infinitary sentence ψ_1 in the language of $\mathcal{K}_1$ such that for all $\mathcal{A} \in \mathcal{K}_1$, we have

$$\mathcal{A} \models \psi_1 \ \leftrightarrow \ \Phi(\mathcal{A}) \models \psi_2.$$

Moreover, for a non-zero $\alpha < \omega_1^{CK}$, if ψ_2 is a Σ^c_α sentence, then so is ψ_2.

Since $\leq_c$ implies $\leq_{tc}$, the theorem works for computable embeddings as well.

Characterization of the *tc*-class of $\{\omega, \omega^\star\}$

The Pullback Lemma is important here.

Proposition

If $\{\omega, \omega^\star\} \equiv_{tc} \{\mathcal{A}, \mathcal{B}\}$, then $\mathcal{A}$ and $\mathcal{B}$ have computable copies, $\mathcal{A} \equiv_1 \mathcal{B}$ and they differ by Σ_2 sentences.

Proposition

If $\mathcal{A}$ and $\mathcal{B}$ differ by Σ_2^c sentences, then $\{\mathcal{A}, \mathcal{B}\} \leq_{tc} \{\omega, \omega^\star\}$.

Proposition

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are non-isomorphic L-structures with computable copies. If $\mathcal{A} \equiv_1 \mathcal{B}$, then $\{\omega, \omega^\star\} \leq_{tc} \{\mathcal{A}, \mathcal{B}\}$.

Characterization of the tc-class of $\{\omega, \omega^\star\}$

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Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be non-isomorphic L-structures. T.F.A.E.

(1) $\{\omega, \omega^\star\} \equiv_{tc} \{\mathcal{A}, \mathcal{B}\}$;

(2) *$\mathcal{A}$ and $\mathcal{B}$ have computable copies, $\mathcal{A} \equiv_1 \mathcal{B}$, and they differ by Σ_2^c sentences.*

We conclude the Turing case

It follows that all pairs of the form $\{\omega \cdot k, \omega^\star \cdot k\}$, for any $k > 0$ are equivalent under Turing computable embeddings.
What about computable embeddings (enumeration operators) ?

The top pair among linear orderings

We want to study the pairs of linear orderings inside the *tc*-degree of $\{\omega, \omega^\star\}$ relative to $\leq_c$. It turns out that we have a top pair.

Theorem

For any pair $\{\mathcal{A}, \mathcal{B}\} \equiv_{tc} \{\omega, \omega^\star\}$, *we have that*

$$\{\mathcal{A}, \mathcal{B}\} \leq_c \{1 + \eta, \eta + 1\},$$

where η *is the order type of the rationals.*

Corollary

For any $k > 0$,

$$\{\omega \cdot k, \omega^\star \cdot k\} <_c \{1 + \eta, \eta + 1\}.$$

Infinite chain of pairs

We will show that for any $k \in \mathbb{N}$,

$$\{\omega, \omega^\star\} <_c \cdots <_c \{\omega \cdot 2^k, \omega^\star \cdot 2^k\} <_c \cdots <_c \{1 + \eta, \eta + 1\}.$$

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We clearly have the following:

$$m \leq k \implies \{\omega \cdot 2^m, \omega^\star \cdot 2^m\} \leq_c \{\omega \cdot 2^k, \omega^\star \cdot 2^k\}.$$

The enumeration operator just copies its input a fixed number of times, since 2^m divides 2^k.

Infinite chain of pairs (1)

- We denote by α, β, γ finite linear orderings.
- Define $\alpha \Vdash_\Gamma x < y$ iff $x, y \in \Gamma(\alpha)$ & $\neg(\exists \beta \supseteq \alpha)[\Gamma(\beta) \models y \leq x]$.

Proposition

Let $x, y \in \Gamma(\alpha)$ be distinct elements. Then

$$\alpha \Vdash_\Gamma x < y \text{ or } \alpha \Vdash_\Gamma y < x.$$

Moreover,

$$\alpha \Vdash_\Gamma x < y \text{ iff } \alpha \nVdash_\Gamma y < x.$$

Easy proof: monotonicity and compactness are used.

Proposition

For distinct elements $x_0, x_1, \ldots, x_n \in \Gamma(\alpha)$, there is exactly one permutation π of $\{0, 1, \ldots, n\}$ such that

$$\alpha \Vdash_\Gamma x_{\pi(0)} < x_{\pi(1)} < \cdots < x_{\pi(n)}.$$

Infinite chain of pairs (2)

Notice that in general, for finite α, $\Gamma(\alpha)$ might be infinite.

Proposition

Let $\{\mathcal{A}, \mathcal{B}\} \leq_c \{\mathcal{C}, \mathcal{D}\}$, where $\mathcal{C}$ has no infinite descending chains and $\mathcal{D}$ has no infinite ascending chains. Then $\Gamma(\alpha)$ is finite for all finite α.

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Notice that in general, $\alpha \cap \beta = \emptyset$ does not imply $\Gamma(\alpha) \cap \Gamma(\beta) = \emptyset$.

Proposition

Let $\alpha \cap \beta = \emptyset$ and $x, y \in \Gamma(\alpha) \cap \Gamma(\beta)$ be distinct elements. Then

$$\alpha \Vdash_\Gamma x < y \ \leftrightarrow\ \beta \Vdash_\Gamma x < y.$$

Proposition

Let $\{\mathcal{A}, \mathcal{B}\} \leq_c \{\mathcal{C}, \mathcal{D}\}$, where $\mathcal{A}, \mathcal{C}$ and $\mathcal{B}, \mathcal{D}$ are as above. There are at most **finitely many** elements x with the property that there exist α and β with disjoint domains and $x \in \Gamma(\alpha) \cap \Gamma(\beta)$.

Infinite chain of pairs (3)

We call (x, α) a Γ-pair if $x \in \Gamma(\alpha)$. Now we can work with infinite chains of Γ-pairs $(x_i, \alpha_i)_{i<\omega}$ of distinct elements x_i and mutually disjoint α_i.

Proposition

Suppose $\{\omega \cdot 2, \omega^\star \cdot 2\} \leq_c \{\mathcal{C}, \mathcal{D}\}$ via Γ, where $\mathcal{C}$ and $\mathcal{D}$ are as before. For any two sequences of Γ-pairs $(x_i, \alpha_i)_{i\in\omega}$ and $(y_i, \beta_i)_{i\in\omega}$, there is a number q such that either

$$\Gamma(\sum_{i\in\omega} \alpha_i + \sum_{i\in\omega} \beta_i) \models \bigwedge_{i,j>q} x_i < y_j \text{ or } \Gamma(\sum_{i\in\omega} \alpha_i + \sum_{i\in\omega} \beta_i) \models \bigwedge_{i,j>q} y_j < x_i.$$

In other words, the $\bar{x}$ and $\bar{y}$ are not merged, with the exception of finitely many elements.

Infinite chain of pairs (4)

By the choice of $\mathcal{C}$ and $\mathcal{D}$, the last result shows that the output structures have limit points.

Corollary

Suppose $\{\omega \cdot 2, \omega^\star \cdot 2\} \leq_c \{\mathcal{C}, \mathcal{D}\}$ *via* Γ, *where* $\mathcal{C}$ *and* $\mathcal{D}$ *are as before. Then* $\mathcal{C}$ *includes* $\omega \cdot 2$ *and* $\mathcal{D}$ *includes* $\omega^\star \cdot 2$.

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It can be generalized in the following way:

Theorem

Fix some $k \geq 2$ *and suppose* $\{\omega \cdot k, \omega^\star \cdot k\} \leq_c \{\mathcal{C}, \mathcal{D}\}$ *via* Γ, *where* $\mathcal{C}$ *and* $\mathcal{D}$ *are as before. Then* $\mathcal{C}$ *includes* $\omega \cdot k$ *and* $\mathcal{D}$ *includes* $\omega^\star \cdot k$.

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Corollary

For any $k < \omega$, $\{\omega \cdot 2^k, \omega^\star \cdot 2^k\} <_c \{\omega \cdot 2^{k+1}, \omega^\star \cdot 2^{k+1}\}$.

It follows that we have the following chain above $\{\omega, \omega^\star\}$:

$$\{\omega, \omega^\star\} <_c \{\omega\cdot 2, \omega^\star\cdot 2\} <_c \cdots <_c \{\omega\cdot 2^k, \omega^\star\cdot 2^k\} <_c \cdots <_c \{1+\eta, \eta+1\}.$$

Current work

In this context, the enumeration operators are so "stupid" that they can only copy and do nothing else. More formally,

Theorem

For any two natural nummbers n and k,

$$n \mid k \ \leftrightarrow \ \{\omega \cdot n, \omega^\star \cdot n\} \leq_c \{\omega \cdot k, \omega^\star \cdot k\}.$$

A sample case $(2 \mapsto 3)$

We already know that $\{\omega \cdot 3, \omega^\star \cdot 3\} \not\leq_c \{\omega \cdot 2, \omega^\star \cdot 2\}$. Assume that

$$\{\omega \cdot 2, \omega^\star \cdot 2\} \leq_c \{\omega \cdot 3, \omega^\star \cdot 3\} \text{ via } \Gamma.$$

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$$\{\omega \cdot 2, \omega^\star \cdot 2\} \leq_c \{\omega \cdot 3, \omega^\star \cdot 3\} \text{ via } \Gamma.$$

- If $\mathcal{A}$ is a copy of ω, then $\Gamma(\mathcal{A})$ is a copy of ω.
- For any two sequences of Γ-pairs $(x_i, \alpha_i)_{i\in\omega}$ and $(y_i, \beta_i)_{i\in\omega}$, there is a number q such that

$$\Gamma(\sum_{i\in\omega} \alpha_i + \sum_{i\in\omega} \beta_i) \models \bigwedge_{i,j>q} x_i < y_j.$$

- Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ denote copies of ω. Then we have one of the following cases:
  - $\Gamma(\mathcal{A}+\mathcal{B}) \cong \Gamma(\mathcal{A}) + \Gamma(\mathcal{B}) + \mathcal{C}$;
  - $\Gamma(\mathcal{A}+\mathcal{B}) \cong \mathcal{C} + \Gamma(\mathcal{A}) + \Gamma(\mathcal{B})$;
  - $\Gamma(\mathcal{A}+\mathcal{B}) \cong \Gamma(\mathcal{A}) + \mathcal{C} + \Gamma(\mathcal{B})$.

  We prove that none of these cases is possible. Thus, $\{\omega \cdot 2, \omega^\star \cdot 2\} \not\leq_c \{\omega \cdot 3, \omega^\star \cdot 3\}$.

The end

Thank you for your attention!