Uniform Limits of Conditionally Computable Real Functions

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We discuss some properties common for both classes and we also point out some principal differences between them.

The main point is that the uniform notion is preserved by certain kind of uniform limits, but the conditional notion is not.

This leads to a broader complexity class of real functions.

The classes $\mathcal{M}^2, \mathcal{L}^2, \mathcal{E}^2$

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The functions $\lambda x_1 \ldots x_n . x_m (1 \leq m \leq n)$, $\lambda x . x + 1$, $\lambda xy . \max(x - y, 0)$, $\lambda xy . xy$, belonging to $\mathcal{T}$, will be called the initial functions.

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The class $\mathcal{M}^2$ is the smallest subclass of $\mathcal{T}$, which contains the initial functions and is closed under substitution and bounded minimization ($f \mapsto \lambda \vec{x} y . \mu_{z \leq y}[f(\vec{x}, z) = 0]$).

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The same for the class $\mathcal{E}^2$, where bounded minimization is replaced by limited primitive recursion.

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The same for the class $\mathcal{E}^2$, where bounded minimization is replaced by limited primitive recursion.
We have $\mathcal{M}^2 \subseteq \mathcal{L}^2 \subseteq \mathcal{E}^2$ and whether each of these inclusions is proper is an open question.

Log-bounded sums

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Nevertheless, we have the following:

Theorem ([2])

For any $k, m \in \mathbb{N}$ *and any function* $f \in \mathcal{T}_{m+1} \cap \mathcal{M}^2$*, the function* $g \in \mathcal{T}_{m+1}$ *defined by*

$$g(\vec{x}, y) = \sum_{z \leq \lfloor \log_2(y+1) \rfloor^k} f(\vec{x}, z)$$

also belongs to $\mathcal{M}^2$*.*

Relative computability of real numbers

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$$\left| \frac{f(n) - g(n)}{h(n) + 1} - \xi \right| < \frac{1}{n+1}.$$

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- For any $r \in \mathbb{N}$ and function $f \in \mathcal{T}_r \cap \mathcal{F}$, if $F_1, \ldots, F_r$ are $\mathcal{F}$-substitutional k-operators, then so is the operator F, defined by
$$F(f_1, \ldots, f_k)(n) = f(F_1(f_1, \ldots, f_k)(n), \ldots, F_r(f_1, \ldots, f_k)(n)).$$

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$$(F(f_1, g_1, h_1, f_2, g_2, h_2, \ldots, f_k, g_k, h_k),$$

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For $\mathcal{F} \subseteq \mathcal{T}$, the real function θ will be called *uniformly $\mathcal{F}$-computable*, if there exists a uniform realiser (F, G, H) for θ, such that F, G, H are $\mathcal{F}$-substitutional.

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As shown in [4], all elementary functions of calculus are uniformly $\mathcal{M}^2$-computable on the compact subsets of their domains.

Conditional computability of real functions

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Let $k \in \mathbb{N}$ and θ be a real function, $\theta : D \to \mathbb{R}, D \subseteq \mathbb{R}^k$.
The quadruple (E, F, G, H), where E is a $3k$-operator and F, G, H are $(3k+1)$-operators, will be called a *conditional realiser* for θ if for all $(\xi_1, \ldots, \xi_k) \in D$ and all triples (f_i, g_i, h_i) that name ξ_i for $i = 1, 2, \ldots, k$, the following two hold:

- There exists a natural number s satisfying the equality
$$E(f_1, g_1, h_1, \ldots, f_k, g_k, h_k)(s) = 0.$$

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- There exists a natural number s satisfying the equality
$$E(f_1, g_1, h_1, \ldots, f_k, g_k, h_k)(s) = 0.$$
- For any natural number s satisfying the above equality, the triple $(\tilde{f}, \tilde{g}, \tilde{h})$ names the real number $\theta(\xi_1, \ldots, \xi_k)$, where
$$\tilde{f} = F(f_1, g_1, h_1, \ldots, f_k, g_k, h_k, \lambda x.s),$$
$$\tilde{g} = G(f_1, g_1, h_1, \ldots, f_k, g_k, h_k, \lambda x.s),$$
$$\tilde{h} = H(f_1, g_1, h_1, \ldots, f_k, g_k, h_k, \lambda x.s).$$

Conditional computability of real functions (continued)

For $\mathcal{F} \subseteq \mathcal{T}$, the real function θ will be called *conditionally $\mathcal{F}$-computable*, if there exists a conditional realiser (E, F, G, H) for θ, such that E, F, G, H are $\mathcal{F}$-substitutional.

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Any uniformly $\mathcal{F}$-computable real function is also conditionally $\mathcal{F}$-computable.

Conditional computability of real functions (continued)

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Any uniformly $\mathcal{F}$-computable real function is also conditionally $\mathcal{F}$-computable.
All elementary functions of calculus are conditionally $\mathcal{F}$-computable on their whole domains, see [3].

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3. Every uniformly (conditionally) $\mathcal{F}$-computable real function maps tuples of $\mathcal{F}$-computable real numbers into an $\mathcal{F}$-computable real number.
4. (*gluing property*) For an $\mathcal{F}$-computable real number r and a real function $\theta : D \to \mathbb{R}$, $D \subseteq \mathbb{R}$, if the restrictions of θ to $D \cap (-\infty, r]$ and to $D \cap [r, +\infty)$ are uniformly (conditionally) $\mathcal{F}$-computable, then θ is uniformly (conditionally) $\mathcal{F}$-computable on its whole domain D.

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Let $\mathcal{F} \in \{\mathcal{M}^2, \mathcal{L}^2, \mathcal{E}^2\}$.

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2. Any uniformly $\mathcal{F}$-computable real function is uniformly continuous (with modulus of continuity in $\mathcal{F}$) on the bounded subsets of its domain. Thus the reciprocal and the logarithmic function are not uniformly $\mathcal{E}^2$-computable, but they are conditionally $\mathcal{M}^2$-computable.

Uniform limits of uniformly computable real functions

Theorem

Let $\mathcal{F} \in \{\mathcal{M}^2, \mathcal{L}^2, \mathcal{E}^2\}$, $k \in \mathbb{N}$, $U \subseteq \mathbb{R}^k$ *and* $\theta : \mathbb{N} \times U \to \mathbb{R}$ *be a real function, which is uniformly* $\mathcal{F}$*-computable, such that the limit* $\rho(\vec{\eta}) = \lim_{n\to\infty} \theta(n, \vec{\eta})$ *exists for any* $\vec{\eta} \in U$*.*

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Let $\mathcal{F} \in \{\mathcal{M}^2, \mathcal{L}^2, \mathcal{E}^2\}$, $k \in \mathbb{N}$, $U \subseteq \mathbb{R}^k$ and $\theta : \mathbb{N} \times U \to \mathbb{R}$ be a real function, which is uniformly $\mathcal{F}$-computable, such that the limit $\rho(\vec{\eta}) = \lim_{n\to\infty} \theta(n, \vec{\eta})$ exists for any $\vec{\eta} \in U$. Let there also exist an $\mathcal{F}$-substitutional $3k$-operator R, such that for any $\vec{\eta} \in U$ and any triples (f_i, g_i, h_i) naming η_i for $i = 1, \ldots, k$, we have the inequality

$$|\rho(\vec{\eta}) - \theta(n, \vec{\eta})| \leq \frac{1}{t+1}$$

for all $t \in \mathbb{N}$ and $n = R(f_1, g_1, h_1, \ldots, f_k, g_k, h_k)(t)$.

Uniform limits of uniformly computable real functions

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for all $t \in \mathbb{N}$ and $n = R(f_1, g_1, h_1, \ldots, f_k, g_k, h_k)(t)$. Then the real function $\rho : U \to \mathbb{R}$ is uniformly $\mathcal{F}$-computable.

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Let $D = \mathbb{R} \setminus \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ and the real function $\theta : D \to \mathbb{R}$ be defined by

$$\theta(\xi) = \sum_{k=1}^{\infty} \frac{1}{2^k} \chi\left(\xi - \frac{1}{k}\right).$$

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Question. Does there exist a real function, which is computable in the usual sense, but which is not the uniform limit of a conditionally $\mathcal{M}^2$-computable sequence?

Complexity of integration

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The following theorem is proven in [1].

Theorem

Let α, β *be* $\mathcal{M}^2$*-computable real numbers,* $D \subseteq \mathbb{R}^k$ *be a set for some* $k \in \mathbb{N}$ *and* $\theta : [\alpha, \beta] \times U \to \mathbb{R}$ *be a real function, which is uniformly* $\mathcal{M}^2$*-computable. Let there exist* $A \in \mathbb{R}$*,* $A > 0$*, such that for every fixed* $\vec{\eta} \in U$*,* $\theta(x, \vec{\eta})$ *(as a real function of* x*) has an analytic continuation defined in the set* $[\alpha, \beta] \times [-A, A] \subseteq \mathbb{C}$*. Let there also exist a polynomial* P*, such that* $|\theta(x + Bi, \vec{\eta})| \leq P(|\vec{\eta}|)$ *for all* $\vec{\eta} \in U, x \in [\alpha, \beta], B \in [-A, A]$*. Then the real function* $I : U \to \mathbb{R}$ *defined by*

$$I(\vec{\eta}) = \int_{\alpha}^{\beta} \theta(x, \vec{\eta})\, dx$$

is uniformly $\mathcal{M}^2$*-computable.*

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For $\mathcal{F} \in \{\mathcal{L}^2, \mathcal{E}^2\}$ we can relax the analyticity condition.

Integration of conditionally computable real functions

For conditional computability we have the following result: (retaining all other assumptions) if θ is conditionally $\mathcal{M}^2$-computable, then the integral I is the uniform limit of a conditionally $\mathcal{M}^2$-computable sequence.

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Question. Can we generalise these results for uniformly or conditionally $\mathcal{M}^2$-computable real functions, which are not analytic?

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Thank you for your attention!