Complexity of some real numbers and functions with respect to the subrecursive class $\mathcal{M}^2$

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Given a real function $\theta : [a, b] \to \mathbb{R}$ *and real numbers* a, b*, which are efficiently computable, is it true that the real number*

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Our framework for complexity is subrecursive, that is we are interested in inductively defined classes of total functions in $\mathbb{N}$, contained in the low levels of Grzegorczyk's hierarchy.

The classes $\mathcal{M}^2, \mathcal{L}^2, \mathcal{E}^2$

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The functions $\lambda x_1 \dots x_n . x_m (1 \leq m \leq n)$, $\lambda x . x + 1$, $\lambda xy . x \dot{-} y$, $\lambda xy . xy$, belonging to $\mathcal{T}$, will be called the initial functions.

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Definition

The class $\mathcal{M}^2$ is the smallest subclass of $\mathcal{T}$, which contains the initial functions and is closed under substitution and bounded minimization ($f \mapsto \lambda \vec{x} y . \mu_{z \leq y} [f(\vec{x}, z) = 0]$).

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The same for the class $\mathcal{E}^2$, where bounded minimization is replaced by limited primitive recursion.

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We have $\mathcal{M}^2 \subseteq \mathcal{L}^2 \subseteq \mathcal{E}^2$ and whether each of these inclusions is proper is an open question.

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Nevertheless, we have the following:

Theorem ([1])

For any $k, m \in \mathbb{N}$ and any function $f \in \mathcal{T}_{m+1} \cap \mathcal{M}^2$, the function $g \in \mathcal{T}_{m+1}$ defined by

$$g(\vec{x}, y) = \sum_{z \leq \log_2^k(y+1)} f(\vec{x}, z)$$

also belongs to $\mathcal{M}^2$.

Relative computability of real numbers

Definition

The triple of functions $(f, g, h) \in \mathcal{T}_1^3$ is a name of the real number ξ iff for all $n \in \mathbb{N}$,

$$\left| \frac{f(n) - g(n)}{h(n) + 1} - \xi \right| < \frac{1}{n+1}.$$

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A function $S : D \to \mathbb{R}, D \subseteq \mathbb{N}^k$ is $\mathcal{F}$-computable, if there exist $f, g, h \in \mathcal{T}_{k+1} \cap \mathcal{F}$, such that for all $\vec{s} \in D$

$$(\lambda n.f(\vec{s}, n), \lambda n.g(\vec{s}, n), \lambda n.h(\vec{s}, n))$$

is a name for the real number $S(\vec{s})$.

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Let $k \in \mathbb{N}$ and θ be a real function, $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^k$. The triple (F, G, H), where F, G, H are $(3k, 1)$-operators, is called a computing system for θ if for all $(\xi_1, \xi_2, \ldots, \xi_k) \in D$ and triples (f_i, g_i, h_i) that name ξ_i for $i = 1, 2, \ldots, k$, the triple

$$(F(f_1, g_1, h_1, f_2, g_2, h_2, \ldots, f_k, g_k, h_k),$$

$$G(f_1, g_1, h_1, f_2, g_2, h_2, \ldots, f_k, g_k, h_k),$$

$$H(f_1, g_1, h_1, f_2, g_2, h_2, \ldots, f_k, g_k, h_k))$$

names the real number $\theta(\xi_1, \xi_2, \ldots, \xi_k)$.

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For a class $\mathbf{O}$ of operators, the function θ is uniformly $\mathbf{O}$-computable, if there exists a computing system (F, G, H) for θ, such that $F, G, H \in \mathbf{O}$.

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3. For all n, m, k, if F_0 is an (n, k)-operator and $F_1, \ldots, F_k$ are (n, m)-operators all belonging to **RO**, then the (n, m)-operator F defined by

$$F(\vec{f})(\vec{x}) = F_0(\vec{f})(F_1(\vec{f})(\vec{x}), \ldots, F_k(\vec{f})(\vec{x}))$$

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also belongs to **RO**.
4. For all m, n, if F_0 is an $(n, m+1)$-operator which belongs to **RO**, then so is the operator F defined by
$$F(\vec{f})(\vec{x}, y) = \mu_{z \leq y}[F_0(\vec{f})(\vec{x}, z) = 0].$$

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If there is a uniform definition of log-bounded summation for the class $\mathcal{M}^2$, then the same definition, easily modified for operators, will show that **RO** $=$ **logRO**.

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3. For any m, n, k and $a \in \mathcal{T}_k \cap \mathcal{M}^2$, if $F_1, \ldots, F_k$ are (n, m)-operators which belong to **MSO**, then so is the operator F defined by
$$F(\vec{f})(\vec{x}) = a(F_1(\vec{f})(\vec{x}), \ldots, F_k(\vec{f})(\vec{x})).$$

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The pairs $(\mathcal{M}^2, \mathbf{MSO})$, $(\mathcal{M}^2, \mathbf{RO})$ and $(\mathcal{M}^2, \mathbf{LogRO})$ are acceptable in the sense of [2]. Therefore, by the characterization theorem of Skordev in [2], the following three conditions are equivalent for a real function θ:

- θ is uniformly **MSO**-computable;
- θ is uniformly **RO**-computable;
- θ is uniformly **LogRO**-computable.

First theorem on integration

Theorem

Let a, b be $\mathcal{M}^2$-computable real numbers and $\theta : [a, b] \to \mathbb{R}$ be uniformly **MSO***-computable and analytic real function. Then the definite integral $\int_a^b \theta(x)dx$ is an $\mathcal{M}^2$-computable real number.*

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$$\int_{-1}^{1} \theta(x)dx = \int_{-\infty}^{\infty} \theta(tanh(t)).\frac{1}{cosh^2(t)}dt$$

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$$\approx h \sum_{k=-n}^{n} \theta(tanh(kh)).\frac{1}{cosh^2(kh)} = I_{h,n}.$$

First theorem on integration (continued)

By a careful choice h (depending on n) and using the analyticity of g we can obtain

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for some positive real constants A, M and therefore

$$\left| I_{\log_2^2(n+1)} - \int_{-1}^{1} \theta(x) dx \right| \leq \frac{M}{(n+1)^A - 1}.$$

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Theorem

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$$I(y) = \int_a^b \theta(x, y) dx$$

is uniformly **MSO***-computable.*

Second theorem on integration

Theorem

Let a, b be $\mathcal{M}^2$-computable real numbers and $\theta : [a, b] \times D \to \mathbb{R}$ be uniformly **MSO**-*computable. Let there exist a real constant $A > 0$, such that for any fixed $y \in D$, θ has an analytic continuation defined in $[a, b] \times [-A, A] \subseteq \mathbb{C}$. Let there also exist a polynomial P, such that $|\theta(x \pm Bi, y)| \leq P(|y|)$ for all $y \in D, x \in [a, b], B \leq A$. Then the real function $I : D \to \mathbb{R}$, defined by*

$$I(y) = \int_a^b \theta(x, y) dx$$

is uniformly **MSO**-*computable.*

The proof follows the same argument. The important thing to show is that the log-bounded sum of a uniformly **MSO**-computable real function is again uniformly **MSO**-computable. This requires the use of log-rudimentary operators.

Third theorem on integration

Theorem

Let a be an $\mathcal{M}^2$*-computable real number. Let* $\theta : [a, +\infty) \to \mathbb{R}$ *be uniformly* **MSO***-computable real function, which has an analytic continuation defined in the half-plane* $Re(z) \geq a$*.*

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Let a be an $\mathcal{M}^2$-computable real number. Let $\theta : [a, +\infty) \to \mathbb{R}$ be uniformly **MSO***-computable real function, which has an analytic continuation defined in the half-plane $Re(z) \geq a$. Let there exist a real constant $A > 0$ and a polynomial P, such that*
$|\theta(x + yi)| \leq P(|\xi|)$ for all $\xi \geq a, x \in [a, \xi], |y| \leq A(\xi - a)$.

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By the linear change $x = \frac{\xi - a}{2} t + \frac{\xi + a}{2}$ we have

$$I(\xi) = \int_a^\xi \theta(x) dx = \frac{\xi - a}{2} \int_{-1}^1 \theta\left(\frac{\xi - a}{2} t + \frac{\xi + a}{2}\right) dt$$

and we can apply the second theorem.

Euler-Mascheroni constant

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This integral is the sum of the following two integrals:

$$I_1 = \int_1^\infty e^{-x} \ln x \, dx,$$

$$I_2 = \int_0^1 e^{-x} \ln x \, dx = \int_1^\infty e^{-\frac{1}{t}} \ln t \frac{1}{t^2} \, dt,$$

which are easily seen to be $\mathcal{M}^2$-computable by using the third theorem.

Euler-Mascheroni constant (continued)

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$$A(n) = \frac{\sqrt{e}^{\sqrt{n}} - 1}{2\sqrt{n}} \sum_{k=-n}^{n} \theta\left(\tanh(\frac{k}{\sqrt{n}}), \sqrt{e}^{\sqrt{n}} - 1\right) \frac{1}{\cosh^2(\frac{k}{\sqrt{n}})},$$

where $\theta(u, \xi) = \phi(\frac{\xi}{2}.u + \frac{\xi+2}{2}) - \psi(\frac{\xi}{2}.u + \frac{\xi+2}{2})$.

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where $\theta(u,\xi) = \phi(\frac{\xi}{2}.u + \frac{\xi+2}{2}) - \psi(\frac{\xi}{2}.u + \frac{\xi+2}{2})$. Then

$$|A(n) - \gamma| \leq \frac{(\pi+3)\sqrt{n} + 7\pi + 16}{\sqrt{e}^{\sqrt{n}}}$$

for all $n > 0$.

Conditional computability of real functions

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- There exists a natural number s satisfying the equality
$$E(f_1, g_1, h_1, \ldots, f_k, g_k, h_k)(s) = 0.$$
- For any natural number s satisfying the above equality, the triple $(\tilde{f}, \tilde{g}, \tilde{h})$ names the real number $\theta(\xi_1, \ldots, \xi_k)$, where
$$\tilde{f} = \lambda t.F(f_1, g_1, h_1, \ldots, f_k, g_k, h_k)(s, t),$$
$$\tilde{g} = \lambda t.G(f_1, g_1, h_1, \ldots, f_k, g_k, h_k)(s, t),$$
$$\tilde{h} = \lambda t.H(f_1, g_1, h_1, \ldots, f_k, g_k, h_k)(s, t).$$

Properties of conditional computability

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Moreover, the characterization theorem can be extended to show that for a real function θ:

- θ is conditionally **MSO**-computable;
- θ is conditionally **RO**-computable;
- θ is conditionally **LogRO**-computable.

Gamma function and Riemann zeta function

By using the results on integration, we can prove that the gamma function

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \, dx$$

is conditionally **MSO**-computable for $s > 0$.

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Thank you for your attention!