Definability of the jump operator in the ω-enumeration degrees

Andrey C. Sariev[1] and Hristo Ganchev[2]*

[1] Sofia University, Sofia, Bulgaria
acsariev@gmail.com
[2] Sofia University, Sofia, Bulgaria
ganchev@fmi.uni-sofia.bg

Abstract

We show the first order definability of the jump operator in the upper semi-lattice of the ω-enumeration degrees.

1 The ω-enumeration degrees

The investigation of definability issues in degree structures is a main part of the research in Computability theory. Given a degree structure possessing a jump, naturally arises the question of the first order definability of this operator in the language of the structure order. In the both most explored structures, $\mathcal{D}_T$ of the Turing degrees and $\mathcal{D}_e$ of the enumeration degrees, Shore and Slaman [5] and Kalimullin [4] respectively, show the definability of the jump operator.

This paper concerns the problem of the first order definability of the jump in an extension of $\mathcal{D}_e$ – the ω-enumeration degree structure $\mathcal{D}_\omega$. The both structures are closely related. More precisely, Soskov and Ganchev [7] show that the group $Aut(\mathcal{D}_e)$ of the automorphisms of $\mathcal{D}_e$ and the group $Aut(\mathcal{D}'_\omega)$ of the jump preserving automorphisms of $\mathcal{D}_\omega$, are isomorphic. Again in [7] it is shown also that $\mathcal{D}_e$ is an automorphism base for $\mathcal{D}_\omega$, which is first order definable in $\mathcal{D}_\omega$ in the language of the structure order and the jump operation. Note, that the definability of the jump in $\mathcal{D}_\omega$ guarantees that all automorphisms of $\mathcal{D}_\omega$ preserve the jump, and hence that $\mathcal{D}_e$ and $\mathcal{D}_\omega$ have isomorphic automorphism groups. Thus the rigidity of the enumeration degrees is equivalent to the rigidity of the ω-enumeration degrees.

Unlike the well known structures of the Turing and the enumeration degrees, $\mathcal{D}_\omega$ is induced by a reducibility on the set $\mathcal{S}_\omega$ of the sequences of sets of natural numbers. The study of this degree structure was initiated by Soskov in [6]. He introduces the ω-enumeration reducibility $\leq_\omega$, considering for each sequence $\mathcal{A} = \{A_k\}_{k<\omega}$ of sets of natural numbers its jump-class $J_\mathcal{A}$. This class consists of the Turing degrees of all sets, that can compute, in an uniform way, an enumeration of the n-th element of the considering sequence in their n-th Turing jump:

$$J_\mathcal{A} = \{\deg_T(X) \mid A_k \text{ is c.e. in } X^{(k)} \text{ uniformly in } k\}.$$

Having this, define $\mathcal{A} \leq_\omega \mathcal{B}$ iff $J_\mathcal{B} \subseteq J_\mathcal{A}$. This reducibility is a preorder on $\mathcal{S}_\omega$, and hence it gives rise to a degree structure in the usual way, denoted by $\mathcal{D}_\omega$ – the structure of the ω-enumeration degrees.

The degree of the sequence $\mathcal{A}$ we shall denote by $\deg_\omega(\mathcal{A})$. The relation $\leq$ defined by $\mathbf{a} \leq \mathbf{b} \iff \exists \mathcal{A} \in \mathbf{a} \exists \mathcal{B} \in \mathbf{b}(\mathcal{A} \leq_\omega \mathcal{B})$ is a partial order on the set of all ω-enumeration degrees $\mathbf{D}_\omega$. By $\mathcal{D}_\omega$ we shall denote the structure $(\mathbf{D}_\omega, \leq)$. The ω-enumeration degree $\mathbf{0}_\omega$ of

*The authors were partially supported by an BNSF MON Grant No. DN02/16 and by Sofia University Science Fund, project 80-10-147/21.04.2017

the sequence $\emptyset_\omega = \{\emptyset\}_{k<\omega}$ is the least element in $\mathcal{D}_\omega$. Further, the ω-enumeration degree of the sequence $\mathcal{A} \oplus \mathcal{B} = \{A_k \oplus B_k\}_{k<\omega}$ is the least upper bound $\mathbf{a} \vee \mathbf{b}$ of the pair of degrees $\mathbf{a} = \deg_\omega(\mathcal{A})$ and $\mathbf{b} = \deg_\omega(\mathcal{B})$. Thus $\mathcal{D}_\omega$ is an upper semi-lattice with least element.

Note that for all $A, B \subseteq \omega$, $A \leq_e B \iff (A, \emptyset, \ldots, \emptyset, \ldots) \leq_\omega (B, \emptyset, \ldots, \emptyset, \ldots)$. Hence, the mapping $\kappa : \mathbf{D}_e \to \mathbf{D}_\omega$, defined by $\kappa(\deg_e(A)) = \deg_\omega(A, \emptyset, \ldots, \emptyset, \ldots)$, is an embedding of the enumeration degree structure $\mathcal{D}_e$ into $\mathcal{D}_\omega$. The copy of the enumeration degrees under the embedding κ we shall denote by $\mathbf{D}_1$.

We need the following definition, in order to characterize ω-enumeration reducibility. Given a sequence $\mathcal{A} \in \mathcal{S}_\omega$ we define the *jump-sequence* $\mathcal{P}(\mathcal{A})$ of $\mathcal{A}$ as the sequence $\{P_k(\mathcal{A})\}_{k<\omega}$ such that $P_0(\mathcal{A}) = A_0$ and for each k, $P_{k+1}(\mathcal{A}) = P_k(\mathcal{A})' \oplus A_{k+1}$, [1].

Now, according to [6], $\mathcal{A} \leq_\omega \mathcal{B} \iff A_n \leq_e P_n(\mathcal{B})$ uniformly in n. From here, one can show that each sequence is ω-enumeration equivalent with its jump-sequence, i.e. for all $\mathcal{A} \in \mathcal{S}_\omega$, $\mathcal{A} \equiv_\omega \mathcal{P}(\mathcal{A})$.

Following the lines of [7], the ω-enumeration jump $\mathcal{A}'$ of $\mathcal{A} \in \mathcal{S}_\omega$ is defined as the sequence $\mathcal{A}' = (P_1(\mathcal{A}), A_2, A_3, \ldots, A_k, \ldots)$. This operator is defined in such a way, that the jump-class $J_{\mathcal{A}'}$ of $\mathcal{A}'$ contains exactly the jumps of the degrees in the jump-class $J_{\mathcal{A}}$ of $\mathcal{A}$. Note also, that for each k, $P_k(\mathcal{A}') = P_{1+k}(\mathcal{A})$, so $\mathcal{A}' \equiv_\omega \{P_{k+1}(\mathcal{A})\}$. The jump operator is strictly monotone, i.e. $\mathcal{A} \lneq_\omega \mathcal{A}'$ and $\mathcal{A} \leq_\omega \mathcal{B} \Rightarrow \mathcal{A}' \leq_\omega \mathcal{B}'$. This allows to define a jump operation on the ω-enumeration degrees by setting $\mathbf{a}' = \deg_\omega(\mathcal{A}')$, where $\mathcal{A} \in \mathbf{a}$ is an arbitrary. Clearly $\mathbf{a} < \mathbf{a}'$ and $\mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{a}' \leq \mathbf{b}'$. Let us note that ${\emptyset_\omega}' \equiv_\omega \{\emptyset^{(k+1)}\}_{k<\omega}$.

A partial result concerning the definability of the jump is achieved in [2]. Namely, it is shown that the jump $\mathbf{0}'_\omega$ of the least element is first order definable in $\mathcal{D}_\omega$. We use this result as one of the bases of the definition of the jump.

2 Basic steps of the proof

We start with a result, concerning the structure of the enumeration degrees:

Lemma 1. *The only enumeration degree* $\mathbf{x}$ *satisfying* $(\forall \mathbf{y})[\mathbf{x} \vee \mathbf{y} \geq \mathbf{0}'_e \to \mathbf{y} \geq \mathbf{0}'_e]$ *is the least degree* $\mathbf{0}_e$.

The main tool we use to prove the above Lemma are the $\mathcal{K}$-pairs. They are introduced by Kalimullin [4] in order to define the enumeration jump operator. One equivalent definition is the following.

Definition 2. *Let* $\mathbf{a}, \mathbf{b}$ *be enumeration degrees. Then* $\{\mathbf{a}, \mathbf{b}\}$ *is a* $\mathcal{K}$*-pair iff for every enumeration degree* $\mathbf{x}$, $\mathbf{x} = (\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x})$.

We shall call a $\mathcal{K}$-pair *nontrivial*, if the both its elements are nonzero. Let $\{\mathbf{a}, \mathbf{b}\}$ be a nontrivial $\mathcal{K}$-pair and $A \in \mathbf{a}, B \in \mathbf{b}$. Then the following holds:

- $A \leq_e \overline{B}$;
- the set of all degrees, which form a $\mathcal{K}$-pair with $\mathbf{a}$ is an ideal;
- the degrees $\mathbf{a}$ and $\mathbf{b}$ are incomparable and quasiminimal.

A proof of all these properties can be found in [4].

[1] unless otherwise stated, if A is a set, then A' will denote the enumeration jump of A

Ganchev and M. Soskova [3] find a much simpler definition of the jump than that given by Kalimullin. Namely, for every nonzero enumeration degree $\mathbf{u} \in \mathbf{D}_e$, $\mathbf{u}'$ is the greatest among the all lest upper bounds $\mathbf{a} \vee \mathbf{b}$ of nontrivial $\mathcal{K}$-pairs $\{\mathbf{a}, \mathbf{b}\}$, such that $\mathbf{a} \leq_e \mathbf{u}$.

From here, one can easily derive that if $\mathbf{x}$ in a nonzero enumeration degree, then there is a degree $\mathbf{y}$ such that $\mathbf{x} \vee \mathbf{y} \geq_e \mathbf{0}_e'$, but $\mathbf{y}$ is not above $\mathbf{0}_e'$. Indeed, let $\mathbf{x} \in \mathbf{D}_e$ be a nonzero. Let $\{\mathbf{a}, \mathbf{b}\}$ be a nontrivial $\mathcal{K}$-pair, such that $\mathbf{a} \leq_e \mathbf{x}$, which realizes $\mathbf{x}'$, i. e. $\mathbf{a} \vee \mathbf{b} = \mathbf{x}'$. Since $\mathbf{a} \leq_e \mathbf{x}$, we have that $\mathbf{x} \vee \mathbf{b} \geq_e \mathbf{a} \vee \mathbf{b} = \mathbf{x}' \geq_e \mathbf{0}_e'$. Suppose now that $\mathbf{0}_e' \leq_e \mathbf{b}$. Then $\{\mathbf{a}, \mathbf{0}_e'\}$ must be a nontrivial $\mathcal{K}$-pair. If $A \in \mathbf{a}$, then $A \leq_e \overline{\emptyset'} \equiv_e \emptyset'$ by the fact that $\emptyset'$ is a total set. Hence $\mathbf{a} \leq_e \mathbf{0}_e' \leq_e \mathbf{b}$. This is a contradiction with the third $\mathcal{K}$-pair property, that were mentioned above. Thus $\mathbf{0}_e' \not\leq_e \mathbf{b}$.

Next, let us consider the set of ω-enumeration degrees defined by this formula in $\mathcal{D}_\omega$. For the purpose, let $\mathcal{X} = \{X_k\}_{k<\omega}$ be a sequence such that for each sequence $\mathcal{Y} = \{Y_k\}_{k<\omega}$ if ${\emptyset_\omega}' \leq_\omega \mathcal{X} \oplus \mathcal{Y}$ then ${\emptyset_\omega}' \leq_\omega \mathcal{Y}$. Noting that for each sequence $\mathcal{A} = \{A_k\}_{k<\omega}$, ${\emptyset_\omega}' \leq_\omega \mathcal{A}$ is equivalent to $\emptyset' \leq_e A_0$, and then using Lemma 1, we conclude that $X_0 \equiv_e \emptyset$.

Now, let $\mathcal{X} = \{X_k\}_{k<\omega}$ be such that $X_0 \equiv_e \emptyset$ and the sequence $\mathcal{Y} = \{Y_k\}_{k<\omega}$ be such that ${\emptyset_\omega}' \leq_\omega \mathcal{X} \oplus \mathcal{Y}$. Then we have that $\emptyset' \leq_e X_0 \oplus Y_0 \equiv_e Y_0$, hence ${\emptyset_\omega}' \leq_\omega \mathcal{Y}$.

Thus, the degrees in $\mathcal{D}_\omega$, which satisfy the formula mentioned above, are exactly these that contain a sequence whose zeroth element is the empty set. We shall denote the set of all these degrees by $\widetilde{\mathbf{D}_1}$, $\widetilde{\mathbf{D}_1} = \{\mathbf{x} \in \mathbf{D}_\omega \mid (\exists \{A_k\}_{k<\omega} \in \mathbf{x})[\ A_0 = \emptyset\ ]\}$.

Here is the moment when we use the first-order definability of $\mathbf{0}_\omega'$, proved in [2]. By this result, we now have the first-order definability of the set $\widetilde{\mathbf{D}_1}$.

Further, for each $\mathbf{a} \in \mathbf{D}_\omega$, denote by $\mu(\mathbf{a})$ the least (ω-enumeration) degree $\mathbf{x}$, for which exists degree $\mathbf{y} \in \widetilde{\mathbf{D}_1}$ such that $\mathbf{x} \vee \mathbf{y} = \mathbf{a}$. It is not difficult to see that the operation μ is correctly defined. Moreover, for each $\mathbf{a}$, if $\{A_k\}_{k<\omega} \in \mathbf{a}$ then $\mu(\mathbf{a})$ contains the sequence $(A_0, \emptyset, \ldots, \emptyset, \ldots)$. Hence, the range of μ is exactly the copy $\mathbf{D}_1$ of the enumeration degrees under the embedding κ:

$$\mathbf{D}_1 = \{\mu(\mathbf{a}) \mid \mathbf{a} \in \mathbf{D}_\omega\}.$$

Combining all together, we conclude the following.

Lemma 3. *The copy $\mathbf{D}_1$ of the enumeration degrees under the embedding κ is first-order definable in $\mathcal{D}_\omega$.*

The final step in the proof is a result of Ganchev, [1]. Namely, the result states that the set $\mathbf{D}_1$ is first-order definable in $\mathcal{D}_\omega$ if and only if the jump operation is first-order definable in $\mathcal{D}_\omega$.

Thus, we have the definability of the jump operation.

Theorem 4. *The jump operator is first-order definable in the structure $\mathcal{D}_\omega$ of the ω-enumeration degrees.*

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