SOFIA UNIVERSITY ST. KLIMENT OHRIDSKI

MASTER THESIS

Modal definability: two commuting equivalence relations

Author:
Yana GEORGIEVA,
Faculty number: 26192,
Logic and Algorithms
(Mathematics)

Supervisor:
Prof. Tinko TINCHEV

A thesis submitted in fulfillment of the requirements for the degree of master in Mathematical logic and Algorithms in the
Department of Mathematical Logic and Applications

March 21, 2021

SOFIA UNIVERSITY ST. KLIMENT OHRIDSKI

Abstract

Faculty of Mathematics and Informatics
Department of Mathematical Logic and Applications

master in Mathematical logic and Algorithms

Modal definability: two commuting equivalence relations

by Yana GEORGIEVA

The bimodal logic $S5 \times S5$ ($S5^2$) corresponds to the equality and substitution free fragment of two-variable first-order logic FOL^2, via the standard translation of modal formulae to first-order formulae. This fragment of first-order logic was shown to be decidable a long time ago, rending the logic $S5^2$ to be decidable. The study of the extensions of $S5^2$ is reduced to the fact that this fragment of FOL is decidable as well as to properties of bounded morphisms, rending them also decidable with a better complexity of the satisfiability problem than $S5^2$ itself.

Here the problem of modal definability is really something relating more to the properties of the first-order properties of the theories of the classes of structures, models of $S5^2$. We examine some classes of those classes due to the fact that $S5^2$ is Kripke complete w.r.t. each of them. This is the reason why we examine the properties of different classes of structures in this work and why that is enough.

Those structures in which the equivalence relations commute, but are not "rectangular", are considered as non-standard models of $S5^2$. In the present work we prove that the modal definability problem w.r.t. the class of all structures with two commuting equivalence relations is undecidable. The status of the modal definability problem in the other examined classes will be done in future works.

This is the motivation why we study the first-order theories of these classes of structures. It is inevitable that if we want to know more about the definability problems in this logic, we must know a bit more of the first-order properties of its models.

Contents

Abstract **iii**

1 Preliminaries **1**

1.1 General . . . 1
1.2 First-order logic . . . 4
1.2.1 Syntax . . . 4
1.2.2 Semantics . . . 8
1.2.3 Some foundational theorems of RFOL . . . 11
1.3 Equivalence relations . . . 14
1.3.1 Two commuting equivalence relations . . . 14
1.4 A method to prove a theory undecidable . . . 18
1.4.1 Relative elementary definability . . . 19
1.5 A method to prove a theory decidable . . . 22
1.5.1 Ehrenfeucht–Fraïssé games . . . 22
1.5.2 Decidability and finite model property for first-order logic . . . 24
1.6 Propositional modal logic . . . 26
1.6.1 Syntax . . . 26
1.6.2 Semantics . . . 27
1.7 Correspondence theory . . . 31
1.7.1 Relativization in FOL . . . 32
1.7.2 Stable classes of frames and modal definability . . . 33
1.8 Some history on related theories . . . 34

2 A tale of three theories **35**

2.1 Formulation of the problem . . . 35
2.2 How can we describe the structures? . . . 36
2.3 Do they differ? . . . 38
2.3.1 $Th(\mathcal{K}_{commute})$ is a proper subtheory of $Th(\mathcal{K}_{rectangle})$ . . . 38
2.3.2 $Th(\mathcal{K}_{rectangle})$ is a proper subtheory of $Th(\mathcal{K}_{square})$ . . . 38
2.4 Are the classes axiomatizable? . . . 39
2.5 Undecidability of $Th(\mathcal{K}_{commute})$ . . . 42
2.6 Decidability of $Th(\mathcal{K}_{rectangle})$ and $Th(\mathcal{K}_{square})$ . . . 46
2.6.1 Decidability of $Th(\mathcal{K}_{rectangle})$ . . . 48
2.6.2 Decidability of $Th(\mathcal{K}_{square})$ . . . 54
2.7 $\mathcal{K}_{rectangle}$ and $\mathcal{K}_{square}$ have **FMP** . . . 56
2.7.1 $\mathcal{K}_{rectangle}$ has **FMP** . . . 56
2.7.2 $\mathcal{K}_{square}$ has **FMP** . . . 58
2.7.3 Another way to see that $\mathcal{K}_{rectangle}$ has **FMP** . . . 59
2.7.4 Another way to see that $\mathcal{K}_{square}$ has **FMP** . . . 61

3 Modal definability problem in $\mathcal{K}_{commute}$ **63**

4 Summary and further work **67**

Bibliography **69**

Chapter 1

Preliminaries

1.1 General

For the purposes of talking about mathematical objects and their properties we will have in the metalanguage a vocabulary of symbols $\{\neg, \&, \vee, \Longleftrightarrow, \Rightarrow, \Leftarrow, \exists, \forall, \in, \asymp, \leftrightharpoons, \subseteq, \supseteq\}$, which will aid us to keep our reflections shorter without losing any of their meaning.

- $\neg$ will be an abbreviation for "not ...";
- $\&$ will be an abbreviation for "... and ...";
- $\vee$ will be an abbreviation for "... or ...";
- $\Longleftrightarrow$ will be an abbreviation for "... if and only if ...";
- $\Rightarrow$ will be an abbreviation for "if ... then ...";
- $\exists$ will be an abbreviation for "there exists ...";
- $\forall$ will be an abbreviation for "for all ...";
- $\in$ will be an abbreviation for "... is in ...";
- $\subseteq$ will be an abbreviation for "all elements of ... are elements of ...";
- $\asymp$ will be an abbreviation for "... syntactically matches ...";
- $\leftrightharpoons$ will be an abbreviation for "... is defined as ...";
- WLOG will be an abbreviation for "without loss of generality";
- FTSOC will be an abbreviation for "for the sake of contradiction";
- w.r.t. will be an abbreviation for "with respect to";

The **set of natural numbers** will be denoted with ω and the **set of positive natural numbers** with $\omega^+ \leftrightharpoons \omega \setminus \{0\}$. The cardinal number **card**(A) is the **cardinality of a set** A. We remind that ω is the set of all finite cardinals and afterward we have $\aleph_0, \aleph_1, \ldots$. $max\{\}$ will denote the operation taking in an arbitrary number of cardinal numbers and returning the greatest among them.

The **power set of a set** A will be denoted by $\mathcal{P}(A)$. The **Cartesian product of the sets** A and B is $A \times B$. The elements of $A \times B$ are called **ordered pairs** and are denoted by $\langle a, b\rangle \in A \times B$. An **n-tuple** $\langle a_1, \ldots, a_n\rangle$ is an element of the Cartesian product of the sets $A_1, \ldots, A_n$ denoted $A_1 \times \cdots \times A_n$ and by $pr_i(.)$ we will mean the ***i*-th projection** defined on the elements of the product by $pr_i(\langle a_1, \ldots, a_n\rangle) = a_i$ for $1 \leq i \leq n$.

Remark 1.1.0.1:

For easier notation n-tuples, for some $n \in \omega$ and $a_1, \ldots, a_n \in A$ will be denoted by $\bar{a}$, and we will also write $\bar{a} \in A^n$, where $A^n = A \times \cdots \times A$ $(n-1)$ times. Depending on the context we will distinguish what n-tuple it is.

Let A and B be sets and let $R \subseteq A \times B$ be a binary relation. We will sometimes write aRb and the meaning is the same as that of $\langle a, b\rangle \in R$.

The **domain of** R is $Dom(R)$ and **its range** is $Range(R)$. The **inverse** $R^{-1} \subseteq B \times A$ of R is $R^{-1} = \{\langle b, a\rangle \mid \langle a, b\rangle \in R\}$.

If $A_0 \subseteq A$, the **restriction** $R\restriction_{A_0} \subseteq A_0 \times B$ of R to A_0 is:

$$R\restriction_{A_0} = \{\langle a, b\rangle \mid \langle a, b\rangle \in R \,\&\, a \in A_0\}.$$

If $a \in A$, the R**-successors of** a are $R[a] = \{b \in B \mid \langle a, b\rangle \in R\}$.

Let $R \subseteq B \times C$ and $S \subseteq A \times B$ be relations and A, B and C are sets. Then the composition ("R after S") of R and S we define as:

$$R \circ S = \{\langle a, c\rangle \mid (\exists b \in B)[\langle a, b\rangle \in S \,\&\, \langle b, c\rangle \in R]\}.$$

A **partially ordered set** or **poset** $\mathfrak{P} = \langle P, \leqslant\rangle$ is a set P together with a relation $\leqslant$ on P that is reflexive, transitive, and antisymmetric.

Let $\mathfrak{P} = \langle P, \leqslant\rangle$ be a poset. Given a subset $A \subseteq P$, we say that $a \in P$ is a **lower bound for A** if $(\forall b \in A)[a \leqslant b]$. Define the **infimum** of A, if it exists, to be an element $a = inf(A)$ such that a is a lower bound for A and if a_0 is a lower bound for A, then $a_0 \leqslant a$.

We define **upper bound** and **supremum** analogously.

In the above definition, we use the operators **inf** and **sup** to denote infimum and supremum. The symbols $\curlywedge$ and $\curlyvee$ are used to indicate infimum and supremum. That is to say, $\underset{a\in A}{\curlywedge} a = inf(A)$ and $\underset{a\in A}{\curlyvee} a = sup(A)$. When considering the infimum and supremum of individual elements: $x \curlywedge y$ denotes the **greatest lower bound for a pair of elements** x **and** y or **meet of** x **and** y, and $x \curlyvee y$ denotes the **least upper bound for a pair of elements** x **and** y or **join of** x **and** y.

A poset $\mathfrak{P} = \langle P, \leqslant\rangle$ is called a **lattice** if for all $x, y \in P$, both $x \curlywedge y$ and $x \curlyvee y$ exist.

A **function** $f : A \to B$ is a relation $f \subseteq A \times B$ which is **functional**. $f \subseteq g$ means that g is an **extension of** f. We denote an **injective function** f from A *into* B like this $f : A \rightarrowtail B$, a **surjective function** f from A *onto* B with $f : A \twoheadrightarrow B$, and a **bijective function** f *between* A and B with $f : A \rightarrowtail\!\!\!\!\!\twoheadrightarrow B$. The **identity function** on a set A is Id_A.

If $A_0 \subseteq A$, the **characteristic function** $\Upsilon^A_{A_0} : A \to \{0, 1\}$ of A_0 in A is defined by $\Upsilon^A_{A_0}(a) = 1$ for $a \in A_0$ and $\Upsilon^A_{A_0}(a) = 0$ otherwise.

Let $A \subseteq \omega$. The set A is said to be **decidable (or recursive/solvable/computable)** if there exists an algorithm which takes a number $n \in \omega$ as input and terminates after a finite amount of time, depending on n, with a correct answer whether the number n belongs to the set A or not. A set $A \subseteq \omega$ which is not decidable, is called **undecidable (or not recursive/not solvable/noncomputable)**. A set $A \subseteq \omega$ is called **recursively enumerable or r.e. (computably enumerable/semidecidable/provable/Turing-recognizable)**, if there is an algorithm such that the set of input numbers for which the algorithm halts is exactly A , i.e., there is an algorithm that stops it work only if the input is a member of the set A and will run forever if the input is not an element of the set A.

A set A is called **co-recursively enumerable or co-r.e.** if its complement $\omega \setminus A$ is r.e.

Let Γ and Δ be disjoint sets. Γ and Δ are **recursively inseparable** if there exists no recursive set Λ such that $\Gamma \subseteq \Lambda$ and $\Delta \cap \Lambda = \emptyset$. Neither Γ, nor Δ is recursive.

A characterization of the property for a set to be decidable is:

Theorem 1.1.0.1 [Complementation Theorem (Post)]**:**
A set A is decidable if and only if both A and the complement of A are semidecidable.

1.2 First-order logic

We are about to introduce what we will mean by a (formal) (first-order logic) language (we may skip the mentioning of "formal" and "first-order logic" at times and substitute "first-order logic" for *FOL*). We will use the letter $\mathfrak{L}$ and variations of it with upper or/and lower indices to denote the languages. This language will have unambiguous syntax and a clear semantics.

1.2.1 Syntax

We will divide a first-order language into two parts: logical and non-logical.

Definition 1.2.1.1 [Logical part]**:**
It consists of the following sets of symbols (we may call them also alphabets):

- an infinite enumerable **alphabet of individual variables** designated $\mathcal{V}ar_{\mathfrak{L}} \leftrightharpoons \{x, y, z, \dots x_1, y_1 z_1, \dots, x', y', z', \dots\}$. We will use lower Latin letters *x, y, z, t, w, u* of the Latin alphabet and variations of them with upper or/and lower indices;
- an **alphabet of quantifiers** $\{\exists\}$;
- an **alphabet of auxiliary symbols** $\{,,(,)\}$;
- an **alphabet of propositional/boolean connectives** $\{\neg, \vee\}$;
- it may or may not contain a symbol $\dot{=}$ which we will call **formal equality**;

Definition 1.2.1.2 [Non-logical part]**:**
It consists of the following changing in size sets of symbols:

- an **alphabet of all individual constant symbols** $Const_{\mathfrak{L}}$. We will mostly use the Latin letters *a, b, c, d, e* and variations of them with upper or/and lower indices;
- an **alphabet of all function symbols** $\mathcal{F}unc_{\mathfrak{L}}$. We will mostly use the Latin letters *f, g, h* and variations of them with upper or/and lower indices;
- an **alphabet of all predicate/relation symbols** $\mathcal{P}red_{\mathfrak{L}}$. Likewise, we will mostly use the Latin letters *p, q, r* and variations of them with upper or/and lower indices;
- We have a function $arity(.) : \mathcal{F}unc_{\mathfrak{L}} \cup \mathcal{P}red_{\mathfrak{L}} \to \omega^+$ called the **arity of the non-logical symbol**, and it gives us the number of arguments that the symbols takes.

Definition 1.2.1.3 [Signature]**:**
The set $Const_{\mathfrak{L}} \cup \mathcal{F}unc_{\mathfrak{L}} \cup \mathcal{P}red_{\mathfrak{L}}(\cup\{\dot{=}\})$ we will call a **signature** for a FOL $\mathfrak{L}$.

Definition 1.2.1.4 [Relational signature]**:**
The set $\mathcal{P}red_{\mathfrak{L}}(\cup\{\dot{=}\})$ we will call a **relational signature** for a FOL $\mathfrak{L}$.

Definition 1.2.1.5 [Cardinality of a language]**:**
The **cardinality of a language** $\mathfrak{L}$, denoted **card**($\mathfrak{L}$) will be the cardinality of its signature without counting the presence of formal equiality.

Remark 1.2.1.1:
In this work we will mainly use only pure relational FOL languages meaning that the sets $Const_{\mathfrak{L}}$ and $\mathcal{F}unc_{\mathfrak{L}}$ are empty (we will call them *RFOL* languages for short). Every individual constant symbol can be represented with a fresh unary relation symbol true only for the interpretation of the individual constant symbol and every *n*-ary function symbol for $n \in \omega^+$ can be represented with a fresh *(n+1)-ary* relation symbol true only for the

arguments and respectful functional values of the interpretation of the function symbol. Thus, we will have only relational signatures. That is why from now on we assume that we work only with RFOL languages and the rest of the definitions will be suited for a RFOL language.

Remark 1.2.1.2:
Let us fix a RFOL language $\mathfrak{L}$ until the end of this subsection. If we need to specify that some property is about a more specific RFOL, we will mention it explicitly.

Definition 1.2.1.6 [Term]**:**
A **term** in $\mathfrak{L}$ is an element of the set $\mathcal{V}ar_{\mathfrak{L}}$. Thus, with $\mathcal{T}erm_{\mathfrak{L}}$ we denote the **set of terms for** $\mathfrak{L}$, $\mathcal{V}ar_{\mathfrak{L}} = \mathcal{T}erm_{\mathfrak{L}}$.

Definition 1.2.1.7 [Atomic formula]**:**
An **atomic formula** of $\mathfrak{L}$ is:

- $p(\tau_1, \dots, \tau_n)$, where $p \in \mathcal{P}red_{\mathfrak{L}}$, $arity(p) = n$ and $\tau_1, \dots, \tau_n \in \mathcal{T}erm_{\mathfrak{L}}$;
- $(\tau \doteq \kappa)$ if $\mathfrak{L}$ has formal equality $\tau, \kappa \in \mathcal{T}erm_{\mathfrak{L}}$;

We will denote the **set of all atomic formulae for** $\mathfrak{L}$ with $\mathcal{A}tomic_{\mathfrak{L}}$.

Definition 1.2.1.8 [Predicate formula]**:**
A **(predicate) formula** of $\mathfrak{L}$ is:

- an atomic formula;
- if ψ is a formula, then $\neg\psi$ is a formula;
- if φ and ψ are formulae, then $(\varphi \vee \psi)$ is a formula;
- if ψ is a formula, then $\exists x\psi$ is a formula, where $x \in \mathcal{V}ar_{\mathfrak{L}}$.

Every formula can be constructed by a finite amount of application of the previous rules or the base case. We will use $\varphi, \psi, \chi, \theta, \mathcal{E}, \mathcal{D}, \dots$ to denote formulae and variations of them with upper or/and lower indices. We will denote the **set of all predicate formulae for** $\mathfrak{L}$ with $\mathcal{F}orm(\mathfrak{L})$.

Remark 1.2.1.3:
If we use only the first three rules of the definition above, we can obtain all **quantifier-free** formulae which means formulae without quantifiers.

Remark 1.2.1.4:
We define the other propositional connectives $\{\wedge, \rightarrow, \leftrightarrow\}$ as usual. The first-order formula $\forall x\varphi$ is obtained as the well-known abbreviation: $\forall x\varphi \leftrightharpoons \neg\exists x\neg\varphi$.

Remark 1.2.1.5:
The propositional connectives are listed in decreasing order of precedence: $\neg, \wedge, \vee, \rightarrow\leftrightarrow$, where $\forall, \exists$ bind as strong as $\neg$.

Also, $\neg$ is a unary connective, $\{\wedge, \vee, \leftrightarrow\}$ are left-associative connectives and $\rightarrow$ is a right-associative connective.

The **set of variables occurring** in φ we will denote with $Var[\varphi]$. The **set of variables freely occurring** in φ is $Var^{free}[\varphi]$ and the **set of variables which are bounded** in φ is $Var^{bound}[\varphi]$. A formula φ is a **sentence** if $Var^{free}[\varphi] = \emptyset$. The **set of all sentences of the language** $\mathfrak{L}$ is denoted by $\mathcal{S}ent(\mathfrak{L})$.

Definition 1.2.1.9 [Quantifier rank of a formula]**:**
Let $\varphi \in \mathcal{Form}(\mathfrak{L})$.
The **quantifier rank** $qr(\varphi) \in \omega$ of φ is defined in the following manner.

- $\varphi \in \mathcal{Atomic}_{\mathfrak{L}}$, then $qr(\varphi) = 0$;
- $\varphi \coloneqq \neg\psi$, then $qr(\varphi) = qr(\psi)$;
- $\varphi \coloneqq (\psi_1 \vee \psi_2)$, then $qr(\varphi) = max\{qr(\psi_1), qr(\psi_2)\}$;
- $\varphi \coloneqq \exists x\psi$ for $x \in \mathcal{Var}_{\mathfrak{L}}$, then $qr(\varphi) = 1 + qr(\psi)$;

A ***k*-rank formula** is a formula having quantifier rank exactly k.

If φ is a formula and $x_1, x_2, \dots, x_n \in \mathcal{Var}_{\mathfrak{L}}$ are distinct variables, we use the notation $\varphi(x_1, x_2, \dots, x_n)$, a **(focused) formula**, to show that we are interested in the free occurrences of the variables x_i in φ.

If $\varphi(x_1, x_2, \dots, x_n)$ is a focused formula and $y_1, y_2, \dots, y_n \in \mathcal{Var}_{\mathfrak{L}}$, then $\varphi(y_1, y_2, \dots, y_n)$ denotes the formula φ where all free occurrences of x_i are replaced by y_i.

Definition 1.2.1.10 [Prenex normal form]**:**
Let $\varphi \in \mathcal{Form}(\mathfrak{L})$.
We say that φ is in **prenex normal form (PNF)** if:

1. $\varphi \coloneqq Q_1x_1Q_2x_2 \dots Q_nx_n\psi(x_1, x_2, \dots, x_n)$;
2. each $Q_i \in \{\forall, \exists\}$ for $1 \leq i \leq n$ and $Q_1x_1Q_2x_2 \dots Q_nx_n$ is called the **quantifier prefix of** φ;
3. $\psi(x_1, x_2, \dots, x_n)$ is a quantifier-free formula and is called **the matrix of** φ.

Remark 1.2.1.6:
A formula may have **many prenex normal forms**.

Definition 1.2.1.11 [Disjunctive normal form]**:**
Let $\varphi \in \mathcal{Form}(\mathfrak{L})$.
We say that φ is in **disjunctive normal form (DNF)** if φ is in prenex normal form and the matrix of φ is a quantifier-free which is a disjunction, where every element of it is a conjunction of atomic formulae or negations of atomic formulae.

Lemma 1.2.1.12:
Every RFOL formula can be written in DNF.

A formula does not change its meaning if a bound variable is changed to another variable.

Definition 1.2.1.13 [Variant]**:**
Let $\varphi, \psi \in \mathcal{Form}(\mathfrak{L})$.
We say that ψ is a **variant** of φ if ψ can be obtained from φ by a sequence of replacements of the type: replace a parts $\exists x\chi$ of φ by $\exists y\chi[x/y]$, where $y \notin Var^{free}[\chi]$ and $\chi[x/y]$ denotes the simultaneous substitution of all free occurrences of the individual variable x in χ by the individual variable y.

Theorem 1.2.1.14 [Variant theorem]**:**
If ψ is a variant of φ, then $\vdash \varphi \leftrightarrow \psi$.

We adopt the standard rules for omission of the parentheses.

Definition 1.2.1.15 [Provability]**:**

Starting from the work of Frege (the Begriffsschrift), Peano, and Whitehead Russell (Principia Mathematica), several equivalent proof/deduction systems (inference rules + axioms and/or axiom schemes) for FOL were formalized by Hilbert and others. We will omit the formulations of a standard framework of predicate calculus where we can precisely formulate the concepts of **proof, deduction, theorem**. We fix one of these FOL proof systems and provability will from now on be stated in terms of it. We need not quibble about the details of the proof system, but there are some properties that all such systems share and that we will invoke as needed. Here is one called the **Closure theorem**:

A formula φ **is provable** $\Longleftrightarrow$ the sentence $\forall x_1 \ldots \forall x_n \varphi$ is provable, where

$$Var^{free}[\varphi] = \{x_1, \ldots, x_n\}.$$

This allows us to use, WLOG just sentences in the following definitions. If Σ is a set of sentences, and ψ a single sentence we write $\Sigma \vdash \psi$ when there exists an FOL proof/deduction of ψ that can use sentences from Σ as additional axioms.

When $\Sigma = \emptyset$ we just write $\vdash \psi$. An important property that FOL provability inherits from propositional logic is the following:

$$\text{If } \Sigma \text{ is a finite set of sentences then } \Sigma \vdash \psi \Longleftrightarrow \vdash \bigwedge \Sigma \to \psi,$$
$$\text{where } \bigwedge \Sigma \text{ is the conjunction of all the sentences in } \Sigma.$$

Further, we introduce notations for the **set of sentences provable in FOL** and for the **provable/deductive consequences of a set of sentences**.

$$\mathcal{P}rovable \leftrightharpoons \{\varphi \mid \ \vdash \varphi\}$$
$$\mathcal{D}educible(\Sigma) \leftrightharpoons \{\varphi \mid \Sigma \vdash \varphi\}$$

Remark that $\mathcal{P}rovable = \mathcal{D}educible(\emptyset)$.

We say that a set Σ of sentences is **inconsistent** if $\Sigma \vdash \varphi$ and $\Sigma \vdash \neg\varphi$ for some sentence φ and **consistent** otherwise. The consistency of $\emptyset$ is the "consistency" of the FOL proof system (that we fixed) itself.

Because proofs are finite and because it is decidable when a finite object is a proof as well as what formula it proves, the concept of FOL provability is "computational" in the following sense:

Theorem 1.2.1.16:

$\mathcal{P}rovable$ is semidecidable.

FOL is just a framework for specifying mathematical theories and the **theorems** of such a theory. This can be done both syntactically and semantically. Here we will see the syntactical definition:

Definition 1.2.1.17 [First-order theory]**:**

A set of sentences $T \subseteq Sent(\mathfrak{L})$ is called a **first-order theory** if it is closed w.r.t. the logical operations of deduction , i.e., $\mathcal{D}educible(T) = T$. The **theorems** of T are simply the sentences in T.

Definition 1.2.1.18 [Axiomatized theory]**:**

Let $\Sigma \subseteq Sent(\mathfrak{L})$ and T be a first-order theory for $\mathfrak{L}$.

Σ **axiomatizes** T if and only if $\mathcal{D}educible(\Sigma) = T$. In this case we say that Σ is a set of **(non-logical) axioms** for T.

T is called **axiomatizable** if there exists a semidecidable set of sentences Σ, which when closed w.r.t. the logical operations of deduction, equals T , i.e., $\mathcal{D}educible(\Sigma) = T$.

Some examples are first-order Peano arithmetic PA and algebraic theories like the theory of groups.

Definition 1.2.1.19 [Recursively axiomatizable theory, finitely axiomatizable theory]**:**
Let $\Sigma \subseteq \mathcal{S}ent(\mathfrak{L})$ and T be a first-order theory for $\mathfrak{L}$.

The theory T is called **recursively axiomatizable** if it has a decidable set of non-logical axioms. If this set of non-logical axioms is finite, then T is called **finitely axiomatizable**.

Σ is a **finite/recursive axiomatization** for T if and only if Σ axiomatizes T and Σ is a finite/decidable set.

Theorem 1.2.1.20 [Craig's theorem]**:**
Every theory that admits a semidecidable set of axioms can be recursively axiomatized.

Theorem 1.2.1.21:
If the theory T is axiomatizable, then the set of syntactically derived theorems of T is semidecidable , i.e., if we have a set of sentences Σ and Σ is decidable, then $\mathcal{D}educible(\Sigma)$ is semidecidable

1.2.2 Semantics

Now we will discuss briefly how we can give a clear semantic of a relational first-order logic language given some universe. Most importantly, we must talk about how we interpret the non-logical symbols of the RFOL language in this universe.

Again, let us fix a RFOL language $\mathfrak{L}$ until the end of this subsection.

Definition 1.2.2.1 [Structure]**:**
A **structure for** $\mathfrak{L}$ will be an ordered pair $\mathfrak{A} = \langle A, I \rangle$ such that:

- A is a non-empty set called a **universe** or **domain of the structure**;
- I is a mapping, which we call an **interpretation of the non-logical symbols of** $\mathfrak{L}$ **in the universe** A; thus, for $p \in \mathcal{P}red_{\mathfrak{L}}$, then $I(p) = p^{\mathfrak{A}} \subseteq A^{arity(p)}$; thus, the predicate symbols are interpreted with relations on the universe;

We will use the letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}$ to denote structures and variations of them with upper or/and lower indices. With A, B, C, F we will denote the universes of the structures and variations of them with upper or/and lower indices. A structure is **finite** if its universe is finite, otherwise it is called **infinite**.

Definition 1.2.2.2 [Truth]**:**
An **assignment** on a structure $\mathfrak{A}$ for the language $\mathfrak{L}$ is a function v assigning to each individual variable $x \in \mathcal{V}ar_{\mathfrak{L}}$ an individual $v(x)$ in the universe A.

The **modified assignment** v on a structure $\mathfrak{A}$ w.r.t. an individual $a \in A$ and an individual variable x, denoted v^x_a, is the assignment v^x_a on $\mathfrak{A}$ such that $v^x_a(x) = a$ and for all individual variables $y \in \mathcal{V}ar_{\mathfrak{L}} \setminus \{x\}$, $v^x_a(y) = v(y)$.

The **satisfiability of a first-order formula** φ **of** $\mathfrak{L}$ **w.r.t. an assignment** v **in a structure** $\mathfrak{A}$, denoted $\mathfrak{A} \overset{v}{\models} \varphi$, is inductively defined as follows.

- If $\varphi \asymp p(x_1, \ldots, x_n)$ for $p \in \mathcal{P}red_{\mathfrak{L}}$ and $arity(p) = n$, then $\mathfrak{A} \overset{v}{\models} p(x_1, \ldots, x_n) \iff \langle v(x_1), \ldots v(x_n) \rangle \in p^{\mathfrak{A}}$;
- If $\varphi \asymp (x \doteq y)$ and if $\mathfrak{L}$ has formal equality and $x, y \in \mathcal{V}ar_{\mathfrak{L}}$, then $\mathfrak{A} \overset{v}{\models} (x \doteq y) \iff v(x) = v(y)$;

- If $\varphi \eqcirc \neg\psi$, then $\mathfrak{A} \overset{v}{\models} \neg\psi \iff \mathfrak{A} \overset{v}{\not\models} \psi$;
- If $\varphi \eqcirc (\psi_1 \vee \psi_2)$, then $\mathfrak{A} \overset{v}{\models} (\psi_1 \vee \psi_2) \iff \mathfrak{A} \overset{v}{\models} \psi_1 \,\mathbb{V}\, \mathfrak{A} \overset{v}{\models} \psi_2$;
- If $\varphi \eqcirc \exists x\psi$ for $x \in \mathcal{V}ar_{\mathfrak{L}}$, then $\mathfrak{A} \overset{v}{\models} \exists x\psi \iff (\exists a \in A)[\mathfrak{A} \overset{v^x_a}{\models} \psi]$.

As a result, $\mathfrak{A} \overset{v}{\models} \forall x\psi \iff (\forall a \in A)[\mathfrak{A} \overset{v^x_a}{\models} \psi]$.

Let φ and v, v' be two assignments in $\mathfrak{A}$ such that $(\forall x \in Var^{free}[\varphi])[v(x) = v'(x)]$. Then:

$$\mathfrak{A} \overset{v}{\models} \varphi \iff \mathfrak{A} \overset{v'}{\models} \varphi$$

Let $\varphi = \varphi(x_1, \dots, x_n)$ and v be an assignment in $\mathfrak{A}$ such that $v(x_1) = a_1, \dots, v(x_n) = a_n$ for $a_1, \dots, a_n \in A$. By writing $\mathfrak{A} \models \varphi[\![a_1, \dots, a_n]\!]$ we mean $\mathfrak{A} \overset{v}{\models} \varphi$.

A first-order formula φ is **valid in a structure** $\mathfrak{A}$, denoted $\mathfrak{A} \models \varphi$, if φ is satisfied w.r.t. all assignments in $\mathfrak{A}$.

A set of formulae Σ is **valid in a structure** $\mathfrak{A}$, we denote it $\mathfrak{A} \models \Sigma$, when each of the formulae in Σ is valid in $\mathfrak{A}$, read "**$\mathfrak{A}$ is a model of** Σ".

Here is the semantic equivalent to the **Closure theorem**:

$$\mathfrak{A} \overset{v}{\models} \varphi \text{ for all valuations } v \iff \mathfrak{A} \text{ is a model of the sentence } \forall x_1 \dots \forall x_n\varphi, \text{ where } Var^{free}[\varphi] = \{x_1, \dots, x_n\}.$$

I.e., WLOG we can use just sentences in the definitions that follow unless we want to state a more peculiar property.

A sentence is **satisfiable** if it has a model; therefore,, a set Σ of sentences is **satisfiable** if $\mathfrak{A} \models \Sigma$ for some structure $\mathfrak{A}$.

Let $\varphi, \psi \in Sent(\mathfrak{L})$. φ and ψ are called **logically equivalent**, denoted $\varphi \models\!\mid \psi$ if they have the same models.

If ψ is a prenex normal form of the formula φ, then $\varphi \models\!\mid \psi$.

φ **is valid in a class of structures** $\mathcal{K}$, denoted $\mathcal{K} \models \varphi$, if φ is valid in all structures in $\mathcal{K}$.

We also define the **set of valid** FOL sentences: $\mathcal{V}alid \leftrightharpoons \{\varphi \mid \forall\mathfrak{A}[\mathfrak{A} \models \varphi]\}$ as well as the notion of **logical consequence**: $Consequences(\Sigma) \leftrightharpoons \{\varphi \mid \forall\mathfrak{A}[\mathfrak{A} \models \Sigma \Rightarrow \mathfrak{A} \models \varphi]\}$.

Note that $\mathcal{V}alid = Consequences(\emptyset)$, and, thus, for any structure $\mathfrak{A}$ we have $\mathfrak{A} \models \mathcal{V}alid$.

Remark 1.2.2.1:

There are many other important notions and properties which are not noted here and one may consult (Shoenfield, 1967).

Remark 1.2.2.2:

With $\mathcal{K}^{fin}$ we will denote the class of all the structure of a class of structures $\mathcal{K}$ having a **finite universe**.

Definition 1.2.2.3 [Axiomatized class of structures]**:**

Let $\Sigma \subseteq Sent(\mathfrak{L})$ and $\mathcal{K}$ be a class of structures for $\mathfrak{L}$.

Σ **axiomatizes the class of structures** $\mathcal{K}$ if for all structures $\mathfrak{A}$ for $\mathfrak{L}$ $[\mathfrak{A} \models \Sigma \iff \mathfrak{A} \in \mathcal{K}]$.

Definition 1.2.2.4 [Finitely axiomatized class of structures]**:**

Let $\varphi \in Sent(\mathfrak{L})$ and $\mathcal{K}$ be a class of structures for $\mathfrak{L}$.

φ **finitely axiomatizes the class of structures** $\mathcal{K}$ if for all structures $\mathfrak{A}$ for $\mathfrak{L}$ $[\mathfrak{A} \models \varphi \iff \mathfrak{A} \in \mathcal{K}]$.

Definition 1.2.2.5 [Finitely axiomatized class of finite structures]**:**
Let $\varphi \in \mathcal{Sent}(\mathfrak{L})$ and $\mathcal{K}$ be a class of **finite** structures for $\mathfrak{L}$.
φ **finitely axiomatizes the class of finite structures** $\mathcal{K}$ if for all structures $\mathfrak{A}$ for $\mathfrak{L}$ $[\mathfrak{A}$ is finite $\Rightarrow [\mathfrak{A} \models \varphi \iff \mathfrak{A} \in \mathcal{K}]]$.

Definition 1.2.2.6 [Theory of a class of structures]**:**
Let $\mathcal{K}$ be a class of structures for $\mathfrak{L}$.
We call the **theory of the class of structures** $\mathcal{K}$ the set of all sentences of the language $\mathfrak{L}$ which are valid in $\mathcal{K}$, and we denote it by $Th(\mathcal{K}) \leftrightharpoons \{\varphi \mid (\forall \mathfrak{A} \in \mathcal{K})[\mathfrak{A} \models \varphi]\}$.
The most common way in which we use this definition is to talk about the theory defined by a single model , i.e., $\mathcal{K} = \{\mathfrak{A}\}$, written just as $Th(\mathfrak{A})$.
Some examples are Number theory and Presburger arithmetic.

Definition 1.2.2.7 [k-equivalent structures]**:**
Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathfrak{L}$.
The structures $\mathfrak{A}$ and $\mathfrak{B}$ are called ***k*-equivalent**, denoted $\mathfrak{A} \equiv_k \mathfrak{B}$, if they satisfy the same i-rank first-order sentences for $0 \leq i \leq k$.

Lemma 1.2.2.8:
Let $\mathfrak{L}'$ be a **finite RFOL language** and $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathfrak{L}'$.
For all $n \in \omega$, variables $\{x_1, \dots, x_k\} \subseteq \mathcal{Var}_{\mathfrak{L}}$ there exist a finite number of formulae with quantifier rank less to equal to n and free variables among $x_1, \dots, x_k$ which are **not logically equivalent**.

Proof. We can prove it using double induction on $n \in \omega$ and k, and using the property that every formula has a disjunctive normal form lemma 1.2.1.12. ∎

Definition 1.2.2.9 [Elementarily equivalent structures]**:**
Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathfrak{L}$.
$\mathfrak{A}$ and $\mathfrak{B}$ are called **elementarily equivalent**, denoted $\mathfrak{A} \equiv \mathfrak{B}$, if they satisfy the same first-order sentences, i.e., $Th(\mathfrak{A}) = Th(\mathfrak{B})$.

Remark 1.2.2.3:
Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathfrak{L}$ and $\mathbf{card}(\mathfrak{L}) \leq \aleph_0$.
If $\mathfrak{A} \equiv_k \mathfrak{B}$ for all $k \in \omega$, then $\mathfrak{A} \equiv \mathfrak{B}$.

Definition 1.2.2.10 [Substructure]**:**
Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathfrak{L}$.
$\mathfrak{A}$ is a **substructure or reduct** of $\mathfrak{B}$, denoted $\mathfrak{A} \sqsubseteq \mathfrak{B}$, if $A \subseteq B$ and each n-ary relation $p^{\mathfrak{A}}$ of $\mathfrak{A}$ is the restriction to A of the corresponding relation $p^{\mathfrak{B}}$ of $\mathfrak{B}$, i.e., $p^{\mathfrak{A}} = p^{\mathfrak{B}}\restriction_A$.
$\sqsubseteq$ is a partial-order relation and if $\mathfrak{A} \sqsubseteq \mathfrak{B}$, then $\mathbf{card}(A) \leq \mathbf{card}(B)$. We say that $\mathfrak{B}$ is an **extension** of $\mathfrak{A}$ if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$.

Definition 1.2.2.11 [Isomorphic structures]**:**
Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathfrak{L}$.
$\mathfrak{A}$ is **isomorphic** to $\mathfrak{B}$, denoted $\mathfrak{A} \cong \mathfrak{B}$, if there is a bijective mapping $f : A \rightarrowtail\!\!\!\!\twoheadrightarrow B$ such that for each n-ary relation symbol $p \in \mathcal{Pred}_{\mathfrak{L}}$ and for every $a_1, \dots, a_n \in A$:

$$\langle a_1, \dots, a_n \rangle \in p^{\mathfrak{A}} \iff \langle f(a_1), \dots, f(a_n) \rangle \in p^{\mathfrak{B}}.$$

A function f that satisfies the above is called an **isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$**, or an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.
We use the notation $f : \mathfrak{A} \cong \mathfrak{B}$ to denote that f is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$.
$\cong$ is an equivalence relation and furthermore, it preserves powers, that is, if $\mathfrak{A} \cong \mathfrak{B}$, then $\mathbf{card}(A) = \mathbf{card}(B)$.

Combining the above two notions:

Definition 1.2.2.12 [Isomorphically embedded structures]**:**

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathfrak{L}$.

We say that $\mathfrak{A}$ is **isomorphically embedded** in $\mathfrak{B}$ if there is a structure $\mathfrak{C}$ and an isomorphism f such that $f : \mathfrak{A} \cong \mathfrak{C}$ and $\mathfrak{C} \sqsubseteq \mathfrak{B}$.

In this case we call the function f an isomorphic embedding of $\mathfrak{A}$ in $\mathfrak{B}$. If $\mathfrak{A}$ is isomorphically embedded in $\mathfrak{B}$, then $\mathfrak{B}$ is isomorphic to an extension of $\mathfrak{A}$.

Definition 1.2.2.13 [Elementary extension]**:**

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathfrak{L}$.

We say that $\mathfrak{B}$ is an **elementary extension** of $\mathfrak{A}$ if $\mathfrak{B}$ is an extension of $\mathfrak{A}$ and for any formula $\varphi(x_1, \dots, x_n) \in \mathcal{F}orm(\mathfrak{L})$ and any $a_1, \dots, a_n \in A$:

$$\mathfrak{A} \models \varphi[\![a_1, \dots, a_n]\!] \Longleftrightarrow \mathfrak{B} \models \varphi[\![a_1, \dots, a_n]\!].$$

We denote it by $\mathfrak{A} \preccurlyeq \mathfrak{B}$. When $\mathfrak{B}$ is an **elementary extension** of $\mathfrak{A}$, we also say that $\mathfrak{A}$ is an **elementary substructure** of $\mathfrak{B}$.

Definition 1.2.2.14 [Elementary embedding]**:**

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathfrak{L}$.

A mapping $f : A \to B$ is said to be an **elementary embedding** of $\mathfrak{A}$ into $\mathfrak{B}$, denoted $f : \mathfrak{A} \preccurlyeq \mathfrak{B}$, if and only if for all formulae $\varphi(x_1, \dots, x_n) \in \mathcal{F}orm(\mathfrak{L})$ and any individuals $a_1, \dots, a_n \in A$, we have:

$$\mathfrak{A} \models \varphi[\![a_1, \dots, a_n]\!] \Longleftrightarrow \mathfrak{B} \models \varphi[\![f(a_1), \dots, f(a_n)]\!].$$

Remark 1.2.2.4:

An elementary embedding of $\mathfrak{A}$ into $\mathfrak{B}$ is the same thing as an isomorphism of $\mathfrak{A}$ onto an elementary substructure of $\mathfrak{B}$.

Definition 1.2.2.15 [Direct product of two structures]**:**

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathfrak{L}$.

We call $\mathfrak{A} \times \mathfrak{B}$ the **direct product** of $\mathfrak{A}$ and $\mathfrak{B}$, which is a structures for $\mathfrak{L}$ and is defined as following:

- The universe is $A \times B$;

- For every k-ary relation symbol $p \in \mathcal{P}red_{\mathfrak{L}}$ and every $c_1, \dots, c_k \in A \times B$ we have that:

$$\langle c_1, \dots, c_k \rangle \in p^{A\times B} \Longleftrightarrow [\langle pr_1(c_1), \dots, pr_1(c_k)\rangle \in p^{\mathfrak{A}} \mathbin{\&} \langle pr_2(c_1), \dots, pr_2(c_k)\rangle \in p^{\mathfrak{B}}].$$

Remark 1.2.2.5:

All definitions and properties are valid for FOL having individual constant symbols or/and function symbols with or without some changes.

1.2.3 Some foundational theorems of RFOL

Let $\mathfrak{L}$ be a RFOL language and let $\varphi \in \mathcal{S}ent(\mathfrak{L})$.

Proposition 1.2.3.1:

φ is satisfiable if and only if $\neg\varphi$ is not valid and φ is valid if and only if $\neg\varphi$ is not satisfiable.

Theorem 1.2.3.2 [Soundness]**:**

$\Sigma \vdash \varphi \Rightarrow \Sigma \models \varphi$.

Corollary 1.2.3.2.1:
$\mathcal{P}rovable$ is consistent.

The next property to worry about is whether the proof system is powerful enough to match logical consequence.

Theorem 1.2.3.3 [Gödel's Completeness theorem]**:**
$\Sigma \models \varphi \Rightarrow \Sigma \vdash \varphi$.
In view of soundness, for all Σ we have $\mathcal{C}onsequences(\Sigma) = \mathcal{D}educible(\Sigma)$.

What is the computational nature of $\mathcal{V}alid$? Going by its definition, how can we "check" truth in all models? Gödel's Completeness theorem tells that this highly complex concept with the following consequence:

Corollary 1.2.3.3.1:
$\mathcal{V}alid$ is semidecidable.

Corollary 1.2.3.3.2:
The set of satisfiable FOL sentences is co-semidecidable.

Can we actually decide first-order provability (hence logical consequence)? This question, known as the Entscheidungsproblem (Halting problem) has a negative answer:

Theorem 1.2.3.4 [Turing/Church's Undecidability Theorem]**:**
Let $\mathfrak{L}'$ be a **RFOL language with one at least binary relation symbol**.
$\mathcal{V}alid_{\mathfrak{L}'}$ is undecidable , i.e., there does not exist an algorithm such that given a sentence $\varphi \in \mathcal{S}ent(\mathfrak{L}')$ can effectively determine if φ is satisfiable.

But:

Theorem 1.2.3.5:
There is an algorithm which given a sentence ends its execution if and only if the sentence is not satisfiable and continues to work infinitely long if the sentence is satisfiable.

Also:

Theorem 1.2.3.6 [Löwenheim, 1915]**:**
Let $\mathfrak{L}'$ be a **RFOL language with only unary relation symbols, with or without formal equality**.
There is an algorithm which decides whether a sentence of $\mathfrak{L}'$ is satisfiable or not.

Remark 1.2.3.1:
If a FOL language $\mathfrak{L}'$ has at least one function symbol, then the decision problem of validity is an undecidable problem via Turing/Church's Undecidability Theorem (functions are relations).

Other properties of FOL we will use are that of Compactness theorem and the Downward Löwenheim–Skolem theorem.

Theorem 1.2.3.7 [Compactness theorem]**:**
Let Σ be a set of sentences in $\mathfrak{L}$.
Σ is called **finitely satisfiable** if and only if every finite subset Σ_0 of Σ is satisfiable. Therefore, Σ is satisfiable if and only if it is finitely satisfiable.

Theorem 1.2.3.8 [Downward Löwenheim–Skolem theorem]**:**
Let $\mathfrak{B}$ be an infinite structure for $\mathfrak{L}$ and let μ be an infinite cardinal number such that $\mathbf{card}(\mathfrak{L}) \leq \mu \leq \mathbf{card}(\mathfrak{B})$.
Then for any $X \subseteq B$ with $\mathbf{card}(X) \leq \mu$ there exists a structure $\mathfrak{A}$ such that $X \subseteq A$, $\mathbf{card}(A) = \mu$ and $\mathfrak{A} \preccurlyeq \mathfrak{B}$.

In computer science we are concerned mostly with finite structures so this is very natural to ask about whether the logical consequence is decidable.

Let $\mathcal{V}alid^{fin} \risingdotseq \{\varphi \mid \mathfrak{A} \models \varphi \text{ for all finite } \mathfrak{A}\}$.

Unfortunately, finite validity is also undecidable (not semidecidable), it is **co-semidecidable-complete**. I.e., we cannot axiomatize finite validity.

Theorem 1.2.3.9 [Trakhtenbrot's Theorem]**:**

Let $\mathfrak{L}'$ be a **RFOL language with one at least binary relation symbol**.

The set of $\mathcal{V}alid^{fin}_{\mathfrak{L}'}$ is not semidecidable.

A stronger result (rephrasing Turing/Church's Undecidability Theorem and Trakhtenbrot's Theorem). Note that $\mathcal{V}alid \subseteq \mathcal{V}alid^{fin}$.

Theorem 1.2.3.10:

Let $\mathfrak{L}'$ be a **RFOL language with one at least binary relation symbol**.

There is no recursive set X such that $\mathcal{V}alid_{\mathfrak{L}'} \subseteq X \subseteq \mathcal{V}alid^{fin}_{\mathfrak{L}'}$.

Corollary 1.2.3.10.1:

Neither $\mathcal{V}alid_{\mathfrak{L}'}$ or $\mathcal{V}alid^{fin}_{\mathfrak{L}'}$ are decidable.

Remark 1.2.3.2:

By Gödel's Completeness theorem, $\mathcal{V}alid_{\mathfrak{L}'}$ is semidecidable and it can be shown that $\mathcal{V}alid^{fin}_{\mathfrak{L}'}$ is co-semidecidable. Thus,the previous theorem implies that $\mathcal{V}alid_{\mathfrak{L}'}$ and the complement of $\mathcal{V}alid^{fin}_{\mathfrak{L}'}$ form a recursively inseparable pair of recursively enumerable sets.

We can conclude that:

1. Gödel's Completeness theorem fails in the finite since completeness implies recursive enumerability.
2. Compactness theorem also fails in the finite.
3. There is no recursive function f such that if φ has a finite model, then it has a model of size at most $f(\varphi)$. In other words, there is no effective analogue to the Downward Löwenheim–Skolem theorem in the finite.

1.3 Equivalence relations

A **binary equivalence relation** R on a set A is a subset of $A \times A$ such that it is **reflexive**, **symmetric** and **transitive**.

Let the **set of all equivalence relations on a set** A be denoted with $\mathcal{E}quiv(A)$.

If $a \in A$ then the **equivalence class of the element *a* modulo *R*** is denoted $[a]_R \risingdotseq \{b \mid b \in A \,\&\, \langle a, b\rangle \in R\}$. For phonetic reasons we will also call the equivalence classes of R **blocks**.

With $\#_R$ we will denote the **cardinality of the set** $\{[a]_R \mid a \in A\}$.

A **partition** P of a set A is a subset of $P \subseteq \mathcal{P}(A) \setminus \{\emptyset\}$ such that $\bigcup P = A$ and $(\forall C_1 \in P)(\forall C_2 \in P)[C_1 \neq C_2 \Rightarrow C_1 \cap C_2 = \emptyset]$.

Let the **set of all partitions on a set** A be denoted with $\mathcal{P}artit(A)$.

The elements of a partition will be called **blocks**.

The first theorem is a very basic one, but it is essential for this work:

Theorem 1.3.0.1:

Let A be a set. Then:

- If $R \in \mathcal{E}quiv(A)$ then $\{[a]_R \mid a \in A\}$ the set of all equivalence classes form a partition of A;
- If $P \in \mathcal{P}artit(A)$, then the relation

$$R \risingdotseq \{\langle a, b\rangle \mid a \in A \,\&\, b \in A \,\&\, (\exists C \in P)[a \in C \,\&\, b \in C]\}$$

is an equivalence relation on A.

I.e., a partition of a set and an equivalence relation on a set are the same mathematical object, described from different view points.

We denote by R_P the **equivalence relation associated to the partition** P and P_R the **partition associated to the equivalence relation** R.

1.3.1 Two commuting equivalence relations

Definition 1.3.1.1 [Commuting equivalence relations]**:**

Let A be a set and $R, S \in \mathcal{E}quiv(A)$.

We say that two relations R and S **commute** when $R{\circ}S = S{\circ}R$.

We will be interested in proving some properties of such relations. For further reading one may consult section 3 of The Logic of Commuting Equivalence Relations (Finberg, Mainetti, and Rota, 1996).

Lemma 1.3.1.2:

Let A be a set and $R, S \in \mathcal{E}quiv(A)$.

R and S commute if and only if $R{\circ}S \in \mathcal{E}quiv(A)$.

Proof. ($\Rightarrow$): Let $R{\circ}S = S{\circ}R$.

Let $a \in A$. Since R and S are reflexive, then $\langle a, a\rangle \in R$ and $\langle a, a\rangle \in S$. By the definition of composition, then $\langle a, a\rangle \in R{\circ}S$ meaning $R{\circ}S$ is reflexive.

Let $a, b \in A$ such that $\langle a, b\rangle \in R{\circ}S$. Then by the definition of composition of two relations $(\exists c \in A)[\langle c, b\rangle \in R \,\&\, \langle a, c\rangle \in S]$.

Let $c_0 \in A$ be a witness. R and S are symmetric; therefore, $\langle b, c_0\rangle \in R$ and $\langle c_0, a\rangle \in S$ which fits the definition for membership of $\langle b, a\rangle$ in $S{\circ}R$. $R{\circ}S = S{\circ}R$; therefore, $R{\circ}S$ is symmetric.

Let $a, b, c \in A$ and let $\langle a, b\rangle, \langle b, c\rangle \in R\circ S$. Then by the definition of composition of two relations $(\exists d \in A)[\langle d, b\rangle \in R \,\&\, \langle a, d\rangle \in S]$ and $(\exists d \in A)[\langle d, c\rangle \in R \,\&\, \langle b, d\rangle \in S]$.

Let $d_a \in A$ and $d_c \in A$ be a witnesses such that $[\langle d_a, b\rangle \in R \,\&\, \langle a, d_a\rangle \in S]$ and $[\langle d_c, c\rangle \in R \,\&\, \langle b, d_c\rangle \in S]$.

Then $\langle d_c, d_a\rangle \in R\circ S$ from $\langle d_c, b\rangle \in S$ and $\langle b, d_a\rangle \in R$, and R and S being symmetric. But $R\circ S = S\circ R$; hence, $(\exists e \in A)[\langle e, d_a\rangle \in S \,\&\, \langle d_c, e\rangle \in R]$ and let $e_0 \in A$ be a witness.

Then R and S are symmetric, so we have $\langle d_a, e_0\rangle \in S$ and $\langle e_0, d_c\rangle \in R$.

From the fact that S is transitive and $\langle a, d_a\rangle \in S$ and $\langle d_a, e_0\rangle \in S$ we have $\langle a, e_0\rangle \in S$.

Since R is transitive and $\langle d_c, c\rangle \in R$ and $\langle e_0, d_c\rangle \in R$ we have $\langle e_0, c\rangle \in R$.

Finally, from the last two memberships we obtain $\langle a, c\rangle \in R\circ S$; therefore, $R\circ S$ is transitive.

We conclude that $R\circ S \in \mathcal{E}quiv(A)$.

$(\Leftarrow)$: Let $R\circ S \in \mathcal{E}quiv(A)$.

Let $\langle a, c\rangle \in R\circ S$. Then $\langle c, a\rangle \in R\circ S$ because $R\circ S$ is symmetric.

Let $b_0 \in A$ be such that $\langle b_0, a\rangle \in R$ and $\langle c, b_0\rangle \in S$. S and R are symmetric, so we have $\langle a, b_0\rangle \in R$ and $\langle b_0, c\rangle \in S$, and we can conclude that $\langle a, c\rangle \in S\circ R$.

The other direction is analogous, and so we obtain that $R\circ S = S\circ R$, i.e., R and S commute. ■

Lemma 1.3.1.3:

Let A be a set and $R, S \in \mathcal{E}quiv(A)$.

If $R\circ S \in \mathcal{E}quiv(A)$, then $\bigcap\{T \mid T \in \mathcal{E}quiv(A) \,\&\, R \subseteq T \,\&\, S \subseteq T\}$ which is the least equivalence relation on A containing both R and S is equal to $R\circ S$.

Proof. Let $H \leftrightharpoons \{T \mid T \in \mathcal{E}quiv(A) \,\&\, R \subseteq T \,\&\, S \subseteq T\} = \{T \mid T \in \mathcal{E}quiv(A) \,\&\, R \cup S \subseteq T\}$.

$(\Rightarrow)$: Let $\langle a, c\rangle \in R\circ S$. Then by the definition of composition of two relations $(\exists b \in A)[\langle b, c\rangle \in R \,\&\, \langle a, b\rangle \in S]$. Let $b_0 \in A$ be a witness.

Let T be an arbitrary element of H. Since by the definition of H, $R\cup S \subseteq T$, then $\langle b_0, a\rangle \in T$ and $\langle c, b_0\rangle \in T$. But for $T \in \mathcal{E}quiv(A)$, then T is transitive and symmetric; hence, $\langle a, c\rangle \in T$.

Since $T \in H$ was arbitrary, then we have $\langle a, c\rangle \in \bigcap H$.

$(\Leftarrow)$: We will show that $R\circ S \in H$. We must show that $S \subseteq R\circ S$ and $R \subseteq R\circ S$.

Let $\langle a, b\rangle \in S$. Since R is reflexive, then $\langle b, b\rangle \in R$; hence, $\langle a, b\rangle \in R\circ S$. Analogously $R \subseteq R\circ S$. From the assumption we have $R\circ S \in \mathcal{E}quiv(A)$, so we can conclude that $R\circ S \in H$.

Since $R\circ S \in H$, then $\bigcap H \subseteq R\circ S$.

From $(\Rightarrow)$ and $(\Leftarrow)$ we conclude that $R\circ S = \bigcap H$. ■

Proposition 1.3.1.4:

Let A be a set and $R, S \in \mathcal{E}quiv(A)$.

If $R \cup S \in \mathcal{E}quiv(A)$, then $R\circ S = R \cup S$.

Proof. $(\Rightarrow)$: Let $\langle a, c\rangle \in R\circ S$. Let $b_0 \in A$ be a witness for $(\exists b \in A)[\langle b, c\rangle \in R \,\&\, \langle a, b\rangle \in S]$. So $\langle b, c\rangle \in R \cup S$ and $\langle a, b\rangle \in R \cup S$ which implies $\langle a, c\rangle \in R \cup S$, because $R \cup S$ is transitive.

$(\Leftarrow)$: Let $\langle a, c\rangle \in R\cup S$. WLOG let $\langle a, c\rangle \in S$. Since $\langle c, c\rangle \in R$, because R is reflexive, then this implies that $\langle a, c\rangle \in R\circ S$.

From $(\Rightarrow)$ and $(\Leftarrow)$ we conclude that $R\circ S = R \cup S$. ■

The set of partitions of a set A $\mathcal{P}artit(A)$, is endowed with the **partial order of refinement**: for $P, Q \in \mathcal{P}artit(A)$ we say that $P \leqslant Q$ when every block of P is contained in a block of Q. The refinement partial order has a unique maximal element $\hat{1}$, namely, the partition having only one block, and a unique minimal element $\hat{0}$, namely, the partition for which every block has exactly one element. The partially ordered set $\mathcal{P}artit(A)$ is a lattice (check the definition). Lattice meets and joins, denoted by $P \curlyvee Q$ and $P \curlywedge Q$, can be described by using the equivalence relations R_P and R_Q as follows:

Lemma 1.3.1.5:

(1) $R_{P \curlywedge Q} = R_P \cap R_Q$

(2) $R_{P \curlyvee Q} = R_P \cup R_P \circ R_Q \cup R_P \circ R_Q \circ R_P \cup \cdots \cup$
$\cup\, R_Q \cup R_Q \circ R_P \cup R_Q \circ R_P \circ R_Q \cup \ldots$

Proof. The proof of lemma 1.3.1.5.(1) is immediate. For lemma 1.3.1.5.(2) we use the definition of $P \curlyvee Q$ that it is the smallest partition containing both P and Q:

$$P \curlyvee Q = \bigcap \{T \mid T \in \mathcal{P}artit(A) \,\&\, P \subseteq T \,\&\, Q \subseteq T\}.$$

Now using transitivity of $R_{P \curlyvee Q}$ we obtain the right-hand side of the equality. ■

Theorem 1.3.1.6:

$R_{P \curlyvee Q} = R_P \circ R_Q \iff R_P \circ R_Q = R_Q \circ R_P.$

Proof. ($\Rightarrow$): Let $R_{P \curlyvee Q} = R_P \circ R_Q$.

Then by lemma 1.3.1.5.(2) we have $R_P \circ R_Q \subseteq R_Q \circ R_P$. Now by taking the inverses and applying equivalent transformations, and R_P, R_Q being equivalence relations, we conclude $R_P \circ R_Q \supseteq R_Q \circ R_P$:

$$R_Q \circ R_P = R_Q^{-1} \circ R_P^{-1} = (R_P \circ R_Q)^{-1} \subseteq (R_Q \circ R_P)^{-1} = R_P^{-1} \circ R_Q^{-1} = R_P \circ R_Q$$

($\Leftarrow$): Let $R_P \circ R_Q = R_Q \circ R_P$.

Then by transitivity and $R_P, R_Q \subseteq R_R \circ R_Q$ (easily proved) the right-hand side of lemma 1.3.1.5.(2) is reduced to only $R_{P \curlyvee Q} = R_P \circ R_Q$.

■

Let A be a set and $P, Q \in \mathcal{P}artit(A)$.

Two equivalence relations R_P, R_Q or, equivalently, two partitions P and Q are said to be **independent** when, for any two blocks $p \in P,\ q \in Q$, we have $p \cap q \neq \emptyset$.

Remark 1.3.1.1:

Independent relations commute, since $R_{P \curlyvee Q} = R_{\hat{1}} = R_Q \circ R_Q$.

If $A_0 \subseteq A$, then $P\restriction_{A_0}$ means **restriction of the partition** P **to the set** A_0, that is, the partition whose blocks are the intersections of the blocks of P with the set A_0, whenever such an intersection is not empty.

Lemma 1.3.1.7:

Let A be a set and $P, Q \in \mathcal{P}artit(A)$.

Two equivalence relations R_P and R_Q commute if and only if, for any elements $a, b \in A$ such that $\langle a, b\rangle \in R_{P \curlyvee Q}$, there exist elements $c, d \in A$ such that:

$$\langle c, b\rangle \in R_P \text{ and } \langle a, c\rangle \in R_Q \text{ and } \langle d, b\rangle \in R_Q \text{ and } \langle a, d\rangle \in R_P.$$

Proof. ($\Rightarrow$): Let $R_P \circ R_Q = R_Q \circ R_P$ and let $a, b \in A$ such that $\langle a, b\rangle \in R_{P \curlyvee Q}$. From theorem 1.3.1.6 and the assumption we have that $R_{P \curlyvee Q} = R_P \circ R_Q = R_Q \circ R_P$; therefore, the existence of the elements with the desired properties is immediate from the definition of the composition of two relations.

($\Leftarrow$): Let the right-hand side of the "if and only if" be true.

Let $\langle a, b\rangle \in R_P \circ R_Q$. From lemma 1.3.1.5.(2) we have that $R_P \circ R_Q \subseteq R_{P \curlyvee Q}$ and $a, b \in A$. Therefore, we can apply the right-hand side and let $c_0, d_0 \in A$ be witnesses such that:

$$\langle c_0, b\rangle \in R_P \text{ and } \langle a, c_0\rangle \in R_Q \text{ and } \langle d_0, b\rangle \in R_Q \text{ and } \langle a, d_0\rangle \in R_P$$

From $\langle d_0, b\rangle \in R_Q$ and $\langle a, d_0\rangle \in R_P$ we have $\langle a, b\rangle \in R_Q \circ R_P$.

The other direction is analogous, and so we obtain that $R_P \circ R_Q = R_Q \circ R_P$. ■

Lemma 1.3.1.8:

If R_P and R_Q commute, and $P \curlyvee Q = \hat{1}$, then the equivalence relations R_P and R_Q are independent.

Proof. Let $p \in P$ and $q \in Q$ be blocks and let $a \in p$ and $b \in q$ be elements. Since $\hat{1} = P \curlyvee Q$, then $\langle a, b\rangle \in R_{P \curlyvee Q}$. By assumption R_P and R_Q commute, then apply 1.3.1.7 and obtain a witness $c_0 \in A$ such that $\langle a, c_0\rangle \in R_P$ and $\langle c_0, b\rangle \in R_Q$, i.e., $c_0 \in p \cap q$; therefore, $p \cap q \neq \emptyset$. ■

Theorem 1.3.1.9 [Dubreil-Jacotin theorem]**:**

Two equivalence relations R_P and R_Q associated with partitions P and Q commute if and only if for every block C of the partition $P \curlyvee Q$, the restrictions $P\restriction_C, Q\restriction_C$ are independent partitions.

Proof. Suppose R_P and R_Q commute. Then $R_P\restriction_C$ and $R_Q\restriction_C$ commute too.

Moreover, in the lattice $\mathcal{P}artit(C)$ of partitions of the block C, we have $P\restriction_C \curlyvee Q\restriction_C = (P \curlyvee Q)\restriction_C = \hat{1}_C$ by definition of the join of partitions, where $\hat{1}_C$ is the maximum element of the partition lattice $\mathcal{P}artit(C)$. By lemma 1.3.1.8, the equivalence relations R_P and R_Q are independent.

The converse is that independent relations commute, since we have remark 1.3.1.1 and lemma 1.3.1.6. ■

1.4 A method to prove a theory undecidable

Before describing one of the many methods used to prove that a first-order theory is undecidable, we will introduce some definitions.

We have attached a precise meaning to the notion of decidable theory, axiomatizable theory and other similar notions; thus, we can now assign to each formula of a RFOL language a certain number. In what follows we shall use only the **numbering of the set of all formulae of a given enumerable relational signature σ of a RFOL language $\mathfrak{L}$**. Also, if σ_1 and σ_2 are two such relational signatures for RFOL language $\mathfrak{L}_1$ and $\mathfrak{L}_2$, it will be convenient to have the formulae of signature $\sigma_1 \cup \sigma_2$ numbered to extend the numbering of the formulae of the signature σ_1 as well as that of formulae of the signature σ_2 (, i.e., such that the number of any formula of signature σ_2 , in the numbering of all formulae of signature σ_2 , coincides with its number in the numbering of all formulae of signature $\sigma_1 \cup \sigma_2$ and by $\mathfrak{L}_1 \cup \mathfrak{L}_2$ we mean the $\sigma_1 \cup \sigma_2$). An example numbering can be found in (Ershov, Lavrov, Taimanov, and Taitslin, 1965) chapter 1, section 2.

In this way to every formula φ of a given RFOL language $\mathfrak{L}$ a number is assigned which we shall write as $\ulcorner\varphi\urcorner$. It is clear that any natural number can be the number of not more than one formula.

If T is a theory for $\mathfrak{L}$, then let $\ulcorner T\urcorner \risingdotseq \{\ulcorner\varphi\urcorner \mid \varphi \in T\}$.

Definition 1.4.0.1 [Effective mapping on formulae]**:**

Let $\ulcorner . \urcorner$ be a numbering of the formulae of a RFOL language $\mathfrak{L}_0$. Suppose that to each formula φ of $\mathfrak{L}_0$ there corresponds a formula φ^* a RFOL language $\mathfrak{L}_1$. Let:

$$f(n) = \begin{cases} \ulcorner\varphi^*\urcorner, & if\ n = \ulcorner\varphi\urcorner\ for\ some\ \varphi \in Sent(\mathfrak{L}_0) \\ 0, otherwise & \end{cases}$$

We say that the **correspondence** * **is effective** if the function f is decidable.

Theorem 1.4.0.2:

Let $\ulcorner . \urcorner$ be a numbering of the formulae of a RFOL language $\mathfrak{L}_0 \cup \mathfrak{L}_1$.

Suppose that the theory T in $\mathfrak{L}_0$ is undecidable and that each sentence $\varphi \in Sent(\mathfrak{L}_0)$ is effectively associated with a sentence $\varphi^* \in Sent(\mathfrak{L}_1)$.

If T_1 is a theory in $\mathfrak{L}_1$ and

$$\varphi \in T \Longleftrightarrow \varphi^* \in T_1,$$

then the theory T_1 is undecidable.

Proof. If the characteristic function $\Upsilon_{T_1}(n)$ of the set $\ulcorner T_1\urcorner$ were recursive, then $\Upsilon_{T_1}(f(n))$, the characteristic function of the set $\ulcorner T\urcorner$ would also be recursive by the second assumption, but this contradicts the undecidability of T. ■

Definition 1.4.0.3 [Hereditarily undecidable theory]**:**

Let T be a first-order theory for a RFOL language $\mathfrak{L}$.

Then T is called **hereditarily undecidable** if every subtheory of T for the same language is also undecidable.

Definition 1.4.0.4 [Essentially undecidable theory]**:**

Let T be a first-order theory for a RFOL language $\mathfrak{L}$.

Then T is called **essentially undecidable** if every theory for which T is a subtheory for the same language is also undecidable.

Lemma 1.4.0.5:
Let T be a theory for a RFOL language $\mathfrak{L}$, $\varphi \in \mathcal{S}ent(\mathfrak{L})$ and suppose that the theory T' with added non-logical φ is undecidable. Then T is also undecidable.

Proof. We have by the Deduction theorem that:

$$\psi \in T' \iff \varphi \to \psi \in T$$

for every $\psi \in \mathcal{S}ent(\mathfrak{L})$. We can form $\varphi \to \psi$ effectively so by theorem 1.4.0.2 the lemma follows.

■

Remark 1.4.0.1:
If a theory T_0 is hereditarily undecidable and the theory T_1 is a subtheory of T_0, then T_1 is also hereditarily undecidable.

Corollary 1.4.0.5.1:
Let $\mathfrak{L}$ be a RFOL language.
Every finitely axiomatizable undecidable theory $T \subseteq \mathcal{S}ent(\mathfrak{L})$ is hereditarily undecidable.

Proof. Let T' be a theory such that $T' \subseteq T$. Let $\varphi_T \in T$ finitely axiomatizes T. Let T_0 be the theory of $T' \cup \{\varphi_T\}$. Then $T \subseteq T_0$ (because φ_T axiomatizes T) and $T_0 \subseteq T$ (because $T' \subseteq T$ and $\varphi_T \in T$). Therefore, $T = T_0$ rending T_0 hereditarily undecidable. By remark 1.4.0.1 T is hereditarily undecidable. ■

1.4.1 Relative elementary definability

Relative elementary definability introduced by Ershov is derived from Tarski's *method of interpretations* which is one of the methods for proving undecidability, but it differs slightly. You can find the original work in (Ershov, 1980) that we closely follow.

Let $\mathfrak{L}_0$ be a RFOL language with formal equality and $(k+1)$ predicate symbols $p_0, p_1, \ldots, p_k$ with arities $arity(p_0) = n_0, arity(p_1) = n_1, \ldots, arity(p_k) = n_k$. Let $\mathfrak{L}_1$ be a RFOL language with formal equality. Let $\mathcal{K}_0$ be a class of structures for the language $\mathfrak{L}_0$ and $\mathcal{K}_1$ be a class of structures for the language $\mathfrak{L}_1$.

We say that the class $\mathcal{K}_0$ is ***relatively elementary definable*** in the class $\mathcal{K}_1$ if there exist such formulae:

$$\mathcal{U}(\bar{x}; \bar{y});$$
$$\mathcal{E}(\bar{x}^1; \bar{x}^2; \bar{y});$$
$$\chi_0(\bar{x}^1; \bar{x}^2; \ldots; \bar{x}^{n_0}; \bar{y}), \chi_1(\bar{x}^1; \bar{x}^2; \ldots; \bar{x}^{n_1}; \bar{y}), \ldots, \chi_k(\bar{x}^1; \bar{x}^2; \ldots; \bar{x}^{n_k}; \bar{y})$$

of the RFOL language $\mathfrak{L}_1$ (where hereinafter $\bar{x}^i \leftrightharpoons \langle x^i_1, x^i_2, \ldots, x^i_m \rangle$ and $\bar{y} \leftrightharpoons \langle y_1, y_2, \ldots, y_n \rangle$) such that for any structure $\mathfrak{A} \in \mathcal{K}_0$ there is a structure $\mathfrak{B} \in \mathcal{K}_1$ and elements $b_1, b_2, \ldots, b_n \in B$, satisfying the conditions:

(1) the set $C \leftrightharpoons \{\bar{a} \mid \bar{a} \in B^m \,\&\, \mathfrak{B} \models \mathcal{U}(\bar{a}; \bar{b})\}$ is not empty;

(2) the formula $\mathcal{E}(\bar{x}^1; \bar{x}^2; \bar{b})$ defines a congruence relation η of structures $\mathfrak{C}$ of the RFOL language $\mathfrak{L}_0$, the universe of which is C, and the interpretation of the predicate symbol p_i is defined by the formula $\chi_i(\bar{x}^1; \bar{x}^2; \ldots; \bar{x}^{n_i}; \bar{b})$ for $i \in \{0, 1, \ldots, k\}$. We say that $\chi_i(\bar{x}^1; \bar{x}^2; \ldots; \bar{x}^{n_i}; \bar{b})$ is a **possible definition** for p_i;

(3) the factor structure $\mathfrak{C}/\eta \cong \mathfrak{A}$.

Remark 1.4.1.1:

If a class of structures $\mathcal{K}'_1$ for the language $\mathfrak{L}_1$ and $\mathcal{K}'_1 \supseteq \mathcal{K}_1$ and the class of structures $\mathcal{K}_0$ for the language $\mathfrak{L}_0$ is relatively elementary definable in the class $\mathcal{K}_1$, then $\mathcal{K}_0$ is relatively elementary definable in the class $\mathcal{K}'_1$.

Theorem 1.4.1.1:

If the class of structures $\mathcal{K}_0$ is relatively elementary definable in the class of structures $\mathcal{K}_1$ and the theory $Th(\mathcal{K}_0)$ is hereditarily undecidable, then the theory $Th(\mathcal{K}_1)$ is also hereditarily undecidable.

Proof. For every formula $\varphi(x_1, x_2, \dots, x_n) \in \mathcal{F}orm(\mathfrak{L}_1)$ we will effectively produce a formula $\bar{\varphi}(\bar{x}^1; \bar{x}^2; \dots; \bar{x}^n; \bar{y}) \in \mathcal{F}orm(\mathfrak{L}_0)$ using the following recursive rules.

- If $\varphi(x_1, x_2, \dots, x_n) \eqcirc (x_i \doteq x_j)$ for some $1 \leq i, j \leq n$, then:

$$\bar{\varphi}(\bar{x}^1; \bar{x}^2; \dots; \bar{x}^n; \bar{y}) \leftrightharpoons \mathcal{E}(\bar{x}^i; \bar{x}^j; \bar{y});$$

- If $\varphi(x_1, x_2, \dots, x_n) \eqcirc p_i(x_{j_1}, \dots x_{j_{n_i}})$ for some indices $\{j_1, \dots j_{n_i}\} \subseteq \{1, \dots, n\}$ and n_i-ary predicate symbol p_i of $\mathfrak{L}_1$, $0 \leq i \leq k$, then:

$$\bar{\varphi}(\bar{x}^1; \bar{x}^2; \dots; \bar{x}^n; \bar{y}) \leftrightharpoons \chi_i(\bar{x}^1; \bar{x}^2; \dots \bar{x}^{n_i}; \bar{y});$$

- If $\varphi(x_1, x_2, \dots, x_n) \eqcirc (\varphi_1(x_1, x_2, \dots, x_n)\, \sigma\, \varphi_2(x_1, x_2, \dots, x_n))$ for $\sigma \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$ and we have $\bar{\varphi}_1, \bar{\varphi}_2$ by the induction hypothesis, then:

$$\bar{\varphi}(\bar{x}^1; \bar{x}^2; \dots; \bar{x}^n; \bar{y}) \leftrightharpoons (\bar{\varphi}_1(\bar{x}^1; \bar{x}^2; \dots; \bar{x}^n; \bar{y})\, \sigma\, \bar{\varphi}_2(\bar{x}^1; \bar{x}^2; \dots; \bar{x}^n; \bar{y}));$$

- If $\varphi(x_1, x_2, \dots, x_n) \eqcirc \neg\psi(x_1, x_2, \dots, x_n)$ and we have $\bar{\psi}(\bar{x}^1; \bar{x}^2; \dots; \bar{x}^n; \bar{y})$ by the induction hypothesis, then:

$$\bar{\varphi}(\bar{x}^1; \bar{x}^2; \dots; \bar{x}^n; \bar{y}) \leftrightharpoons \neg\bar{\psi}(\bar{x}^1; \bar{x}^2; \dots; \bar{x}^n; \bar{y});$$

- If $\varphi(x_1, x_2, \dots, x_n) \eqcirc \exists x_{n+1}\psi(x_1, x_2, \dots, x_n, x_{n+1})$ and we have $\bar{\psi}(\bar{x}^1; \bar{x}^2; \dots; \bar{x}^n; \bar{x}^{n+1}; \bar{y})$ by the induction hypothesis, then:

$$\bar{\varphi}(\bar{x}^1; \bar{x}^2; \dots; \bar{x}^n; \bar{y}) \leftrightharpoons \exists x_1^{n+1} \dots \exists x_m^{n+1}(\mathcal{U}(\bar{x}^{n+1}; \bar{y}) \wedge \bar{\psi}(\bar{x}^1; \bar{x}^2; \dots; \bar{x}^n; \bar{x}^{n+1}; \bar{y}));$$

- If $\varphi(x_1, x_2, \dots, x_n) \eqcirc \forall x_{n+1}\psi(x_1, x_2, \dots, x_n, x_{n+1})$ and we have $\bar{\psi}(\bar{x}^1; \bar{x}^2; \dots; \bar{x}^n; \bar{x}^{n+1}; \bar{y})$ by the induction hypothesis, then:

$$\bar{\varphi}(\bar{x}^1; \bar{x}^2; \dots; \bar{x}^n; \bar{y}) \leftrightharpoons \forall x_1^{n+1} \dots \forall x_m^{n+1}(\mathcal{U}(\bar{x}^{n+1}; \bar{y}) \rightarrow \bar{\psi}(\bar{x}^1; \bar{x}^2; \dots; \bar{x}^n; \bar{x}^{n+1}; \bar{y})).$$

Let $\mathcal{D}(\bar{y}) = \mathcal{D}(y_1, y_2, \dots, y_n)$ be the following formula:

$$\exists \bar{x} \mathcal{U}(\bar{x}; \bar{y}) \wedge (\forall \bar{x}^0 \forall \bar{x}^1 \forall \bar{x}^2 (\bigwedge_{0 \le i \le 2} \mathcal{U}(\bar{x}^i; \bar{y}) \rightarrow$$
$$\mathcal{E}(\bar{x}^0; \bar{x}^0; \bar{y}) \wedge$$
$$(\mathcal{E}(\bar{x}^0; \bar{x}^1; \bar{y}) \rightarrow \mathcal{E}(\bar{x}^1; \bar{x}^0; \bar{y})) \wedge$$
$$(\mathcal{E}(\bar{x}^0; \bar{x}^1; \bar{y}) \wedge \mathcal{E}(\bar{x}^1; \bar{x}^2; \bar{y}) \rightarrow \mathcal{E}(\bar{x}^0; \bar{x}^2; \bar{y})))) \wedge$$
$$\bigwedge_{0 \le i \le k} (\forall \bar{x}^1 \dots \forall \bar{x}^{n_i} \forall \bar{z}^1 \dots \forall \bar{z}^{n_i} (\bigwedge_{0 \le j \le n_i} (\mathcal{U}(\bar{x}^j; \bar{y}) \wedge \mathcal{U}(\bar{z}^j; \bar{y}) \wedge \mathcal{E}(\bar{x}^j; \bar{z}^j; \bar{y}))$$
$$\wedge \chi_i(\bar{x}^1; \bar{x}^2; \dots; \bar{x}^{n_i}; \bar{y}) \rightarrow \chi_i(\bar{z}^1; \bar{z}^2; \dots; \bar{z}^{n_i}; \bar{y})))),$$

where $Q\bar{y}$ means $Qy_1 \dots Qy_m$ for $Q \in \{\forall, \exists\}$.

The last formula describes that the universe is non-empty, $\mathcal{E}$ is a congruence relation and $\chi_1, \dots \chi_k$ are invariant w.r.t. $\mathcal{E}$.

Finally, for every formula $\varphi \in \mathcal{S}ent(\mathfrak{L}_0)$ let:

$$\varphi^* \leftrightharpoons \forall y_1 \forall y_2 \dots \forall y_n (\mathcal{D}(y_1, y_2, \dots, y_n) \rightarrow \bar{\varphi}(y_1, y_2, \dots, y_n)).$$

Let us establish the following fact: the set $T^* \leftrightharpoons \{\varphi \mid \varphi \in \mathcal{S}ent(\mathfrak{L}_0) \,\&\, \varphi^* \in Th(\mathcal{K}_1)\}$ is a theory for the language $\mathfrak{L}_0$ such that $T^* \subseteq Th(\mathcal{K}_0)$.

Let $\mathcal{K}_0^*$ be the class of all structures $\mathfrak{A}$ for the language $\mathfrak{L}_0$ such that there is a structure $\mathfrak{B} \in \mathcal{K}_1$ and elements $b_1, b_2 \dots, b_n \in B$ satisfying the conditions (1), (2) and (3) defined above. Then by the hypothesis of the theorem we have that $\mathcal{K}_0^* \supseteq \mathcal{K}_0$. From the definition of the effective mapping $\varphi \rightarrow \varphi^*$ it follows that $\varphi \in Th(\mathcal{K}_0^*) \iff \varphi^* \in Th(\mathcal{K}_1)$ for any sentence $\varphi \in \mathcal{S}ent(\mathfrak{L}_0)$; therefore, $T^* = Th(\mathcal{K}_0^*)$ and since $\mathcal{K}_0 \subseteq \mathcal{K}_0^*$, then $T^* \subseteq Th(\mathcal{K}_0)$.

If the theory $Th(\mathcal{K}_1)$ is decidable, then having the equivalence $\varphi \in T^* \iff \varphi^* \in Th(\mathcal{K}_1)$ and the effective mapping $\varphi \rightarrow \varphi^*$ gives us a decision procedure for the theory T^*. Since the theory $Th(\mathcal{K}_0)$ is hereditarily undecidable, then the theory T^* is undecidable. Therefore, the theory $Th(\mathcal{K}_1)$ is also undecidable. It is clear that if we take a subtheory $T' \subseteq Th(\mathcal{K}_1)$, then the class of structures $\mathcal{K}_1' \leftrightharpoons \{\mathfrak{B} \mid \mathfrak{B} \models T'\}$, $\mathcal{K}_1' \supseteq \mathcal{K}_1$. By remark 1.4.1.1 the class $\mathcal{K}_1'$ also satisfies the condition of the theorem; therefore, $Th(\mathcal{K}_1') = T^*$ is undecidable. We can conclude that $Th(\mathcal{K}_1)$ is hereditarily undecidable. ■

1.5 A method to prove a theory decidable

1.5.1 Ehrenfeucht–Fraïssé games

Mostly the definitions and formulations in this book (Ebbinghaus and Flum, 1995) will be used.

The Ehrenfeucht–Fraïssé games present a purely game theoretic characterization of the relation $\equiv_k$,for some $k \in \omega$. It helps us to understand the expressive power of first-order logic, capture structure equivalence, etc. One of the central ingredients of the characterization are **partial isomorphisms**.

Until the end of this subsection, let be a $\mathfrak{L}$ finite FOL language such that it has no function symbols.

Definition 1.5.1.1 [Partial isomorphism]**:**

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathfrak{L}$.

Let h be a mapping such that $Dom(h) \subseteq A$ and $Range(h) \subseteq B$. h is called a **partial isomorphism** from $\mathfrak{A}$ to $\mathfrak{B}$ if:

- it is injective;
- we have $Dom(h) \subseteq A$ and $Range(h) \subseteq B$, and $(\forall c \in Const_{\mathfrak{L}})[h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}]$;
- for all n-ary relation symbol $p \in \mathcal{P}red_{\mathfrak{L}}$ and for every $a_1, \dots, a_n \in Dom(h)$:

$$\langle a_1, \dots, a_n \rangle \in p^{\mathfrak{A}} \iff \langle h(a_1), \dots, h(a_n) \rangle \in p^{\mathfrak{B}}.$$

We will denote the **set of all partial isomorphisms** from $\mathfrak{A}$ to $\mathfrak{B}$ with $\mathcal{P}art(\mathfrak{A}, \mathfrak{B})$.

Remark 1.5.1.1:

If $Const_{\mathfrak{L}} = \emptyset$, then $\emptyset \in \mathcal{P}art(\mathfrak{A}, \mathfrak{B})$.

Proposition 1.5.1.2:

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathfrak{L}$.

Then for all m-tuples $\bar{a} \in A^m$ and $\bar{b} \in B^m$ the following are equivalent:

(1) The mapping h having the properties $h(a_i) = b_i$ for $1 \le i \le m$ and $h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$ for all $c \in Const_{\mathfrak{L}}$ is a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ (we will denote this with $\bar{a} \mapsto \bar{b} \in \mathcal{P}art(\mathfrak{A}, \mathfrak{B})$ omitting the constants);

(2) for all quantifier-free formulae of $\mathfrak{L}$ $\varphi(x_1, \dots, x_m)$:

$$\mathfrak{A} \models \varphi[\![a_1, \dots, a_m]\!] \iff \mathfrak{B} \models \varphi[\![b_1, \dots, b_m]\!];$$

(3) for all atomic formulae of $\mathfrak{L}$ $\varphi(x_1, \dots, x_m)$:

$$\mathfrak{A} \models \varphi[\![a_1, \dots, a_m]\!] \iff \mathfrak{B} \models \varphi[\![b_1, \dots, b_m]\!];$$

The basic idea behind the algebraic characterization of $\equiv_k$ we have in mind is that the k-equivalence of structures amounts to the existence of partial isomorphisms that can be extended k times.

Definition 1.5.1.3 [Ehrenfeucht–Fraïssé games]**:**

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathfrak{L}$ and $k \in \omega$.

The **Ehrenfeucht–Fraïssé game** $G_k(\mathfrak{A}, \mathfrak{B})$ is played by two players called the $\mathcal{S}poiler$ and the $\mathcal{D}uplicator$. Each player has to make k moves in the course of a play. The players take turns. In his i-th move the $\mathcal{S}poiler$ first selects a structure, $\mathfrak{A}$ or $\mathfrak{B}$, and an element in this structure. If the $\mathcal{S}poiler$ chooses $s_i \in A$ then the $\mathcal{D}uplicator$ in his i-th move must choose an element $d_i \in B$. If the *Spoiler* chooses $d_i \in B$ then the *Duplicator* must choose an element $s_i \in A$.

Let $\{\langle s_i, d_i\rangle \mid 1 \leq i \leq k\}$ be the corresponding choices for all rounds. The *Duplicator* **wins** if and only if $\bar{s} \mapsto \bar{d} \in \mathcal{P}art(\mathfrak{A}, \mathfrak{B})$. If $k = 0$, then we need a mapping h such that $Dom(h) = \{c^{\mathfrak{A}} \mid c \in Const_{\mathfrak{L}}\}$, $Range(h) = \{c^{\mathfrak{B}} \mid c \in Const_{\mathfrak{L}}\}$ and $h \in \mathcal{P}art(\mathfrak{A}, \mathfrak{B})$. Otherwise, the *Spoiler* wins.

Equivalently, the *Spoiler* wins if, after some $i < k$, $s_1 \dots s_i \mapsto d_1 \dots d_i \notin \mathcal{P}art(\mathfrak{A}, \mathfrak{B})$.

A **strategy** is a system of rules which tells the player what move to make, depending on the history of the game up to the current moment.

We say that a player has a **winning strategy** in $G_k(\mathfrak{A}, \mathfrak{B})$, or shortly, a player wins $G_k(\mathfrak{A}, \mathfrak{B})$, if it is guaranteed that he is always the winner of the game (following mindlessly the strategy).

The proof of the items of the following proposition is immediate from the definition of the Ehrenfeucht–Fraïssé games. $\emptyset \in \mathcal{P}art(\mathfrak{A}, \mathfrak{B})$

Remark 1.5.1.2:

Let $\mathfrak{A}$ be a structure for $\mathfrak{L}$. Let $a \in A$. By $(\mathfrak{A}, a)$ we will denote the structure which is for an extension of the language $\mathfrak{L}$ with one new individual constant symbol c_a, such that $c_a^{\mathfrak{A}} \leftrightharpoons a$.

Proposition 1.5.1.4:

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathfrak{L}$ and $k \in \omega$.

(1) The *Duplicator* wins $G_0(\mathfrak{A}, \mathfrak{B}) \iff$ there exists a mapping h such that $Dom(h) = \{c^{\mathfrak{A}} \mid c \in Const_{\mathfrak{L}}\}$, $Range(h) = \{c^{\mathfrak{B}} \mid c \in Const_{\mathfrak{L}}\}$ and $h \in \mathcal{P}art(\mathfrak{A}, \mathfrak{B})$;

(2) Splitting lemma : for $k > 0$ the following are equivalent:

  (i) The *Duplicator* wins $G_k(\mathfrak{A}, \mathfrak{B})$

  (ii) The following two properties hold:

$$\text{(forth): } (\forall a \in A)(\exists b \in B)[\text{the } \mathcal{D}uplicator \text{ wins } G_{k-1}((\mathfrak{A}, a), (\mathfrak{B}, b))]$$
$$\text{(back): } (\forall b \in B)(\exists a \in A)[\text{the } \mathcal{D}uplicator \text{ wins } G_{k-1}((\mathfrak{A}, a), (\mathfrak{B}, b))]$$

(3) If the *Duplicator* wins $G_k(\mathfrak{A}, \mathfrak{B})$ and $t \in \omega$, $t < k$, then the *Duplicator* wins $G_t(\mathfrak{A}, \mathfrak{B})$.

Now one of the main results:

Theorem 1.5.1.5 [Fraïssé–Hintikka theorem]**:**

For all $k \in \omega$, for all finite FOL languages without function symbols $\mathfrak{L}$ and for all structures $\mathfrak{A}$ and $\mathfrak{B}$ for $\mathfrak{L}$ the following are equivalent:

(i) The *Duplicator* has a winning strategy for $G_k(\mathfrak{A}, \mathfrak{B})$;

(ii) $\mathfrak{A} \equiv_k \mathfrak{B}$.

Lemma 1.5.1.6:

Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathfrak{L}$ and let $k \in \omega$ is a natural number.

If the $\mathfrak{A}_1 \equiv_k \mathfrak{B}_1$ and $\mathfrak{A}_2 \equiv_k \mathfrak{B}_2$, then $\mathfrak{A}_1 \times \mathfrak{A}_2 \equiv_k \mathfrak{B}_1 \times \mathfrak{B}_2$.

Proof. Suppose $\mathfrak{A}_1 \equiv_k \mathfrak{B}_1$ and $\mathfrak{A}_2 \equiv_k \mathfrak{B}_2$. By Fraïssé–Hintikka theorem there are winning strategies for the *Duplicator* has winning strategies for $G_k(\mathfrak{A}_1, \mathfrak{B}_1)$ and $G_k(\mathfrak{A}_2, \mathfrak{B}_2)$. Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be winning strategies for the games $G_k(\mathfrak{A}_1, \mathfrak{B}_1)$ and $G_k(\mathfrak{A}_2, \mathfrak{B}_2)$ respectively.

We will create a winning strategy for the *Duplicator* for the game $G_k(\mathfrak{A}_1 \times \mathfrak{A}_2, \mathfrak{B}_1 \times \mathfrak{B}_2)$. The *Spoiler* and the *Duplicator* play the game $G_k(\mathfrak{A}_1 \times \mathfrak{A}_2, \mathfrak{B}_1 \times \mathfrak{B}_2)$, but the *Duplicator* also hiddenly simulates the games $G_k(\mathfrak{A}_1, \mathfrak{B}_1)$ and $G_k(\mathfrak{A}_2, \mathfrak{B}_2)$.

Suppose that in his i-th move the *Spoiler* chooses, say, $\langle a_1, a_2\rangle \in A_1 \times A_2$ for the game $G_k(\mathfrak{A}_1 \times \mathfrak{A}_2, \mathfrak{B}_1 \times \mathfrak{B}_2)$. Then the *Duplicator* hiddenly applies the strategy $\mathfrak{S}_1$ for a_1 w.r.t. the history of the game $G_k(\mathfrak{A}_1, \mathfrak{B}_1)$ up to now to get an element $b_1 \in B_1$. Also he applies hiddenly the strategy $\mathfrak{S}_2$ for a_2 w.r.t. the history of the game $G_k(\mathfrak{A}_2, \mathfrak{B}_2)$ up to now to get an element $b_2 \in B_2$. Finally he answers the move of the *Spoiler* with the move $\langle b_1, b_2\rangle$ for the game $G_k(\mathfrak{A}_1 \times \mathfrak{A}_2, \mathfrak{B}_1 \times \mathfrak{B}_2)$. ■

1.5.2 Decidability and finite model property for first-order logic

Definition 1.5.2.1 [Finite model property (FMP)]**:**

A class of structures $\mathcal{K}$ for a RFOL language $\mathfrak{L}$ has the **finite model property FMP** if for any sentence φ of the language $\mathfrak{L}$:

$$Th(\mathcal{K}) \models \varphi \iff Th(\mathcal{K}^{fin}) \models \varphi,$$

i.e., $Th(\mathcal{K}) = Th(\mathcal{K}^{fin})$.

An equivalent formulation is the following:
A class of structures $\mathcal{K}$ for a RFOL language $\mathfrak{L}$ has **FMP** if for any sentence φ of the language $\mathfrak{L}$:

$$Th(\mathcal{K}) \not\models \varphi \Rightarrow (\exists \mathfrak{B} \in \mathcal{K}^{fin})[\mathfrak{B} \not\models \varphi].$$

Theorem 1.5.2.2:

Let $\mathfrak{L}$ be a finite RFOL language.

If the theory of a class of structures $\mathcal{K}$ has **FMP** and $Th(\mathcal{K})$ is axiomatized by a finite set of sentences Γ, $\Gamma \subseteq \mathcal{S}ent(\mathfrak{L})$, then $Th(\mathcal{K})$ is decidable.

Proof. To check if a sentence $\varphi \in \mathcal{F}orm(\mathfrak{L})$ is valid in all structures of $\mathcal{K}$, we start to enumerate simultaniously two lists, one with all finite structures $\mathfrak{A}$ and the other with all proofs $\Gamma \vdash \psi$. Since we have that $\mathfrak{L}$ has finitely many non-logical symbols (only relation in this case), then we have a finite number (up to isomorphism, what is in the universe of the structure does not really matter; therefore, in any case we can use the initial segment of natural numbers as a universe) structures of cardinality one, finite number of structures of cardinality two and so on. Also since the theory is axiomatized by the finite set Γ it is recursively enumerable (or semidecidable) by theorem 1.2.1.21, so we can list all of the members of the theory.

If $\varphi \notin Th(\mathcal{K})$ then there is a finite model in which φ is not valid and will show up in the first list.

If $\varphi \in Th(\mathcal{K})$ then it will be listed in the second list by Gödel's Completeness theorem.

Thus, we have an effective procedure for deciding if $\varphi \overset{?}{\in} Th(\mathcal{K})$. We can conclude that $Th(\mathcal{K})$ is decidable. ■

Proposition 1.5.2.3:

Let $\mathfrak{L}$ be a finite RFOL language and let T and T' be theories for $\mathfrak{L}$.

If T' is a finite extension of T (only finitely many non-logical axioms are added to T to form T') and T is decidable, then so is T'.

Proof. Let T' be a finite extension of T and T be decidable.

We may assume that T' is the theory of T with added the non-logical axiom φ. Then for all $\psi \in \mathcal{S}ent(\mathfrak{L})$:

$$\psi \in T \iff \varphi \to \psi \in T'$$

by the Deduction theorem.

By assumption, there is an algorithm which can recognize the theorems of T and we can form effectively $\varphi \to \psi$. Therefore, to decide if a sentence $\psi \in \mathcal{S}ent(\mathfrak{L})$ is a theorem of T', apply the algorithm to $\varphi \to \psi$. ■

1.6 Propositional modal logic

1.6.1 Syntax

We are about to introduce what we will mean by a (formal) (propositional) modal logic language (we may skip the mentioning of "formal" and "propositional" at times and substitute "propositional modal logic" with *PML*). We will use the symbols $\mathcal{ML}$ and variations of it with upper or/and lower indices to denote the languages.

Definition 1.6.1.1 [Propositional modal language]**:**

A **propositional modal language (PML)** $\mathcal{ML}$ consists of a countable **alphabet of propositional variables** $\mathcal{P}VAR_{\mathcal{ML}} = \{p, q, r, \dots, p_1, q_1, \dots, p', q', \dots\}$ (mainly we will use the letters p, q, r and variations of them with upper or/and lower indices), a finite **alphabet of propositional/boolean connectives** $\{\vee, \neg\}$, a finite **alphabet of assisting symbols** $\{,, (,)\}$, a finite **alphabet of constants** $\{\bot, \top\}$ and an enumerable **alphabet of possibility modality**
$\mathcal{N}eccesary_{\mathcal{ML}} = \{\Box_1, \Box_2, \dots\}$.

Definition 1.6.1.2 [Cardinality of a PML]**:**

Let $\mathcal{ML}$ be a PML.

Then $\mathbf{card}(\mathcal{ML}) = \mathbf{card}(\mathcal{N}eccesary_{\mathcal{ML}})$.

Definition 1.6.1.3 [k-modal PML]**:**

Let $\mathcal{ML}$ be a PML and $\mathbf{card}(\mathcal{ML}) = k$ such that $k \in \omega^+$.

Then $\mathcal{ML}$ is called a k**-modal PML**.

If $k = 1$, then $\mathcal{ML}$ is called an **unimodal** PML and if $k = 2$, then $\mathcal{ML}$ is called a **bimodal** PML and so on.

Remark 1.6.1.1:

In this work we will only work with finite PML; therefore, from now on we will only talk about properties of finite PML.

Remark 1.6.1.2:

For the sake of simplicity we will define all the notions in these section for an unimodal languages. They are easily generalized for more than one modality.

Definition 1.6.1.4 [Modal formula]**:**

Let $\mathcal{ML}$ be an unimodal PML.

A **modal formula** of $\mathcal{ML}$ is:

- a propositional variable;
- $\bot$ or $\top$;
- if $\mathtt{A}$ is a modal formula, then so is $\neg\mathtt{A}$;
- if $\mathtt{A}$ and $\mathtt{B}$ are modal formulae, then so is $(\mathtt{A} \vee \mathtt{B})$;
- if $\mathtt{A}$ is a modal formula, then so is $\Box\mathtt{A}$;

Every formula can be constructed by a finite amount of application of the previous rules or the base case.

We will use $\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{D}, \dots$ to denote formulae and variations of them with upper or/and lower indices.

We will denote the **set of all modal formulae for** $\mathcal{ML}$ with $\mathcal{MF}orm(\mathcal{ML})$.

If a formula $\mathtt{A}$ if formed using only the constants $\bot, \top$ and the propositional connectives, that is, it does not have any variables in it, we will call it a **variable free modal formula**.

Remark 1.6.1.3:
We define the other propositional connectives $\{\wedge, \rightarrow, \leftrightarrow\}$ as usual. The modal formula $\Diamond \mathtt{A}$ is obtained as the well-known abbreviation: $\Diamond \mathtt{A} \leftrightharpoons \neg\Box\neg \mathtt{A}$.

The **set of variables occurring** in $\mathtt{A}$ we will denote with $Var[\mathtt{A}]$.

If $\mathtt{A}$ is a formula and $p_1, p_2, \dots, p_n \in \mathcal{P}VAR_{\mathcal{ML}}$ are distinct variables, we use the notation $\mathtt{A}(p_1, p_2, \dots, p_n)$, a **(focused) formula**, to show that we are interested in all the occurring variables p_i in $\mathtt{A}$.

If $\mathtt{A}(p_1, p_2, \dots, p_n)$ is a focused formula and $q_1, q_2, \dots, q_n \in \mathcal{P}VAR_{\mathcal{ML}}$, then $\mathtt{A}(q_1, q_2, \dots, q_n)$ denotes the formula $\mathtt{A}$ where all free occurrences of p_i are replaced by q_i.

We adopt the standard rules for omission of the parentheses.

Definition 1.6.1.5 [Normal modal logic]**:**
A set of $\mathcal{ML}$-formulas which contains:

- all tautologies of the classical propositional calculus;

(K): $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$;

- and closed under the following rules of inference:

Modus Ponens (MP): from $\mathtt{A}$ and $\mathtt{A} \rightarrow \mathtt{B}$ infer $\mathtt{B}$;

Substitution (Subst): given a formula $\mathtt{A}(p_1, \dots, p_n)$, derive the formula $\mathtt{A}[p_1/\mathtt{B}_1, \dots, p_n/\mathtt{B}_n]$ which is obtained by uniformly substituting formulas $\mathtt{B}_1, \dots, \mathtt{B}_n$ instead of the variables $p_1, \dots, p_n$ in $\mathtt{A}$, respectively.

Necessitation (N): from $\mathtt{A}$ infer $\Box\mathtt{A}$;

is called a **normal modal logic**.

Remark 1.6.1.4:
As in the section about first-order logic we will omit the formulations of a standard framework of propositional modal calculus where we can precisely formulate the concepts of **proof, deduction, theorem**. We fix one of these PML proof systems and provability will from now on be stated in terms of it.

1.6.2 Semantics

Now we will discuss briefly the most commonly used semantics of interpreting the modal language in some universe of all possible worlds, that is **Kripke semantics**.

Let us fix an unimodal PML $\mathcal{ML}$.

Definition 1.6.2.1 [Kripke frame]**:**
A **(Kripke) structure or frame** (for $\mathcal{ML}$) will be an ordered pair $\mathfrak{F} = \langle W, R \rangle$ such that:

- W is a non-empty set called a **universe** or **domain of the frame**;
- $R \subseteq W \times W$ a binary relation on W.

We will use the letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}$ to denote frames and variations of them with upper or/and lower indices.

With A, B, C, F we will denote the universes of the frames and variations of them with upper or/and lower indices.

A frame is **finite** if its universe is finite, otherwise it is called **infinite**.

Remark 1.6.2.1:
Let $\mathcal{ML}$ be a finite PML and **card**$(\mathcal{ML}) = k$ such that $k \in \omega^+$ and $\mathfrak{F}$ is a structure for $\mathcal{ML}$.

If $k = 1$, then $\mathfrak{F}$ is called an **unimodal** frame and if $k = 2$, then $\mathfrak{F}$ is called a **bimodal** frame and so on.

Definition 1.6.2.2 [Kripke subframe]**:**
Let $\mathfrak{F} = \langle W, R\rangle$ be a Kripke frame.

$\mathfrak{F}' = \langle W', R'\rangle$ is called a **substructure or subframe of** $\mathfrak{F}$, denoted $\mathfrak{F}' \sqsubseteq_{\mathcal{M}} \mathfrak{F}$ if $W' \subseteq W$ and $R' \leftrightharpoons R \cap (W' \times W')$.

Remark 1.6.2.2:
Sometimes for short we may write that a world $a \in \mathfrak{F}$, and we understand that a is an element of the universe of $\mathfrak{F}$.

Definition 1.6.2.3 [Kripke model]**:**
Let $\mathfrak{F} = \langle W, R\rangle$ be a Kripke frame.

A **(Kripke) model** based on a frame $\mathfrak{F} = \langle W, R\rangle$ is a triple $\mathfrak{M} = \langle W, R, V\rangle$, where V is a function assigning to each propositional variable p a subset of W , i.e., $V : \mathcal{P}VAR_{\mathcal{ML}} \rightarrow \mathcal{P}(W)$. V is called an **assignment** and the idea is that $V(p)$ is the set of all worlds in which p is true.

We will use the letters $\mathfrak{M}, \mathfrak{N}$ to denote frames and variations of them with upper or/and lower indices.

Remark 1.6.2.3:
Sometimes for short we may write that a world $a \in \mathfrak{M}$, and we understand that a is an element of the universe of the frame on which $\mathfrak{M}$ is based upon.

Definition 1.6.2.4 [Truth]**:**
Let $\mathfrak{M} = \langle W, R, V\rangle$ be a Kripke model.

The **satisfiability of a modal formula** A **at a world** $a \in \mathfrak{M}$, denoted $\mathfrak{M}, a \models \mathtt{A}$, is inductively defined as follows:

- If $\mathtt{A} \rightleftharpoons p$ for $p \in \mathcal{P}VAR_{\mathcal{ML}}$, then $\mathfrak{M}, a \models p \Longleftrightarrow a \in V(p)$;
- If $\mathtt{A} \rightleftharpoons \bot$, then $\mathfrak{M}, a \not\models \bot$;
- If $\mathtt{A} \rightleftharpoons \top$, then $\mathfrak{M}, a \models \top$;
- If $\mathtt{A} \rightleftharpoons \neg\mathtt{B}$, then $\mathfrak{M}, a \models \neg\mathtt{B} \Longleftrightarrow \mathfrak{M}, a \not\models \mathtt{B}$;
- If $\mathtt{A} \rightleftharpoons (\mathtt{B}_1 \vee \mathtt{B}_2)$, then $\mathfrak{M}, a \models (\mathtt{B}_1 \vee \mathtt{B}_2) \Longleftrightarrow [\mathfrak{M}, a \models \mathtt{B}_1 \,\mathbb{V}\, \mathfrak{M}, a \models \mathtt{B}_2]$;
- If $\mathtt{A} \rightleftharpoons \Box\mathtt{B}$, then $\mathfrak{M}, a \models \Box\mathtt{B} \Longleftrightarrow (\forall b \in W)[\langle a, b\rangle \in R \Rightarrow \mathfrak{M}, b \models \mathtt{B}]$.

As a result, $\mathfrak{M}, a \models \Diamond\mathtt{B} \Longleftrightarrow (\exists b \in W)[\langle a, b\rangle \in R \,\&\, \mathfrak{M}, b \models \mathtt{B}]$.

Let A be a modal formula and V, V' be assignments in a frame $\mathfrak{F} = \langle W, R\rangle$, such that $(\forall p \in Var[\mathtt{A}])[V(p) = V'(p)]$. Then for all $a \in W$:

$$\langle W, R, V\rangle, a \models \mathtt{A} \Longleftrightarrow \langle W, R, V'\rangle, a \models \mathtt{A}.$$

I.e., the truth value of A depends only on the variables occurring in A.

We shall say that a **modal formula** A **is true in a model** $\mathfrak{M}$, denoted $\mathfrak{M} \models \mathtt{A}$, if A is satisfied at all worlds in $\mathfrak{M}$.

A **modal formula** A **is said to be true in a frame** $\mathfrak{F}$ (or **valid** in a frame $\mathfrak{F}$) and a world a, denoted $\mathfrak{F}, a \models \mathtt{A}$, if A is true in all models based on $\mathfrak{F}$.

We shall say that a **modal formula** A **is valid in a class** $\mathcal{K}$ **of frames**, denoted $\mathcal{K} \models \mathtt{A}$, if A is valid in all frames in $\mathcal{K}$.

A **frame** $\mathfrak{F}$ **is said to be weaker than a frame** $\mathfrak{F}'$, denoted $\mathfrak{F} \leq \mathfrak{F}'$, if for all modal formulas A, if $\mathfrak{F} \models \mathtt{A}$ then $\mathfrak{F}' \models \mathtt{A}$.

A **modal formula** $\mathtt{A}$ **is said to be satisfiable** if there is a frame $\mathfrak{F}$, a model $\mathfrak{M}$ based on $\mathfrak{F}$ and a world $x \in \mathfrak{F}$ such that $\mathfrak{M}, x \models \mathtt{A}$.

A **modal formula** $\mathtt{A}$ **is said to be (generally) valid** if it is valid in all Kripke frames.

Now we can give a semantical characterization of (at least some) modal logics by establishing a connection between logics and frames.

Let $\mathcal{K}$ be an arbitrary class of frames. Then

$$Log(\mathcal{K}) \risingdotseq \{\mathtt{A} \in \mathcal{MF}orm(\mathcal{ML}) \mid (\forall \mathfrak{F} \in \mathcal{K})[\mathfrak{F} \models \mathtt{A}]\}$$

is a modal logic called **the logic of** $\mathcal{K}$.

A modal logic L is said to be **sound w.r.t.** $\mathcal{K}$ (or $\mathcal{K}$-sound) if

$$(\forall \mathtt{A} \in L)(\forall \mathfrak{F} \in \mathcal{K})[\mathfrak{F} \models \mathtt{A}],$$

i.e., $L \subseteq Log(\mathcal{K})$.

L is **complete w.r.t.** $\mathcal{K}$ (or $\mathcal{K}$-complete) if

$$(\forall \mathtt{A} \in \mathcal{MF}orm(\mathcal{ML}))[(\forall \mathfrak{F} \in \mathcal{K})[\mathfrak{F} \models \mathtt{A}] \Rightarrow \mathtt{A} \in L],$$

i.e., $Log(\mathcal{K}) \subseteq L$.

We say that L is **determined** (or **characterized**) by $\mathcal{K}$ if L is both $\mathcal{K}$-sound and $\mathcal{K}$-complete, that is, $Log(\mathcal{K}) = L$. If L is determined by some class of frames, we call L **Kripke complete**. A Kripke complete logic L can be characterized by different classes of frames. If L is Kripke complete then it is clearly determined by the class $Fr(L)$ of all frames for L , i.e., $L = Log(Fr(L))$.

Remark 1.6.2.4:

There are many other important notions and properties which are not noted here and one may consult (Chagrov and Zakharyaschev, 1997) and (Kurucz, Wolter, Zakharyaschev, and Gabbay, 2003).

Definition 1.6.2.5 [Modal product of two unimodal frames]**:**

Let $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle U, S\rangle$ be two unimodal Kripke frames.

Then $\mathfrak{F} \underset{mod}{\times} \mathfrak{G} = \langle F \times G, \mathbb{H}, \mathbb{V}\rangle$ is called **the modal product of** $\mathfrak{F}$ **and** $\mathfrak{G}$ is a bimodal Kripke frame and is defined as follows:

- The universe is $F \times G$;
- $\langle\langle a_1, b_1\rangle, \langle a_2, b_2\rangle\rangle \in \mathbb{H} \iff [\langle a_1, a_2\rangle \in R \,\&\, b_1 = b_2]$;
- $\langle\langle a_1, b_1\rangle, \langle a_2, b_2\rangle\rangle \in \mathbb{V} \iff [a_1 = a_2 \,\&\, \langle b_1, b_2\rangle \in S]$;

We will use ⬓ for the modality which uses the $\mathbb{H}$ for horizontal, and ◧ for the modality which uses the $\mathbb{V}$ for vertical relation.

Their meaning is defined as usual:

$$\mathfrak{M}, w \models ⬓A \iff (\forall w' \in \mathfrak{F})[\langle w, w'\rangle \in \mathbb{H} \Rightarrow \mathfrak{M}, w' \models A].$$

$$\mathfrak{M}, w \models ◧A \iff (\forall w' \in \mathfrak{G})[\langle w, w'\rangle \in \mathbb{V} \Rightarrow \mathfrak{M}, w' \models A].$$

Definition 1.6.2.6 [Product of Kripke complete unimodal logics]**:**

Let L_1 and L_2 be two Kripke complete unimodal logics.

Then their **product** is defined as following:

$$L_1 \times L_2 = Log(\{\mathfrak{F}_1 \underset{mod}{\times} \mathfrak{F}_2 \mid \mathfrak{F}_1 \in Fr(L_1) \,\&\, \mathfrak{F}_2 \in Fr(L_2)\}).$$

We will note a structural operation on frames which leave modal satisfaction unaffected.

Definition 1.6.2.7 [Bounded morphism]**:**

Let $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{F}' = \langle W', R'\rangle$ be frames.

A function $f : W \to W'$ assigning to each world in $\mathfrak{F}$ a world in $\mathfrak{F}'$ is called a **bounded morphism** from $\mathfrak{F}$ to $\mathfrak{F}'$ if the following conditions are satisfied:

1. $(\forall a \in W)(\forall b \in W)[\langle a, b\rangle \in R \Rightarrow \langle f(a), f(b)\rangle \in R']$;
2. $(\forall a \in W)(\forall b' \in W')[\langle f(a), b'\rangle \in R' \Rightarrow (\exists b \in W)[\langle a, b\rangle \in R \,\&\, f(b) = b']]$.

$\mathfrak{F}'$ is said to be a **bounded morphic image** of $\mathfrak{F}$ if there exists a surjective bounded morphism from $\mathfrak{F}$ to $\mathfrak{F}'$.

Bounded morphic images give rise to the following lemma:

Lemma 1.6.2.8 [Bounded morphism lemma]**:**

Let $\mathfrak{F}$ and $\mathfrak{F}'$ be frames.

If $\mathfrak{F}'$ is a bounded morphic image of $\mathfrak{F}$ then $\mathfrak{F} \preceq \mathfrak{F}'$.

Proof. See (Chagrov and Zakharyaschev, 1997), Theorem 2.15. ∎

1.7 Correspondence theory

Let $\mathfrak{F} = \langle W, R\rangle$ be an unimodal Kripke structure for an unimodal PML language $\mathcal{ML}$. On the other hand on may think of this structure as a FOL structure for the RFOL $\mathfrak{L}(R, \doteq)$ which is the FOL language with one relation symbol R and $\doteq$. Depending on the context we will determine whether we are talking about a structure from the viewpoint of modal or first-order logic.

Let us fix an unimodal PML $\mathcal{ML}$ language.

Definition 1.7.0.1:

Let $\mathtt{A} \in \mathcal{MF}orm(\mathcal{ML})$ and $\varphi \in \mathcal{S}ent(\mathfrak{L}(R, \doteq))$.

We say that φ **defines** $\mathtt{A}$ or alternatively $\mathtt{A}$ **defines** φ if for every structure $\mathfrak{F}$:

$$\mathfrak{F} \models \mathtt{A} \Longleftrightarrow \mathfrak{F} \models \varphi.$$

Definition 1.7.0.2:

- A modal formula $\mathtt{A}$ is called **FOL definable** if there exists a FOL sentence $\varphi \in \mathcal{S}ent(\mathfrak{L}(R, \doteq))$ which defines her.

- A FOL sentence φ is called **modally definable** if there exists a modal formula $\mathtt{A} \in \mathcal{MF}orm(\mathcal{ML})$ which defines her.

- Let $\mathtt{A} \in \mathcal{MF}orm(\mathcal{ML})$ and $\varphi \in \mathcal{S}ent(\mathfrak{L}(R, \doteq))$. They are called **equivalent** if φ defines $\mathtt{A}$ or $\mathtt{A}$ defines φ.

In the end of the 60-ties and the beginning of the 70-ties, Henrik Sahlqvist managed to separate a syntactical class of modal formulae with the splendid property for each modal formula from the class there exists a FOL formula having the same models and many other good properties. Johan van Benthem demonstrates an algorithm which can syntactically transform every formula from the Sahlqvist class into a FOL equivalent. Benthem continues to pose questions about formulae other than the one in Sahlqvist's class and in time three problems are formulated:

FO-def Is there an algorithm which given a modal formula can determine whether it is FO definable?

MD-def Is there an algorithm which given a FOL sentence can determine whether it is modally definable?

Corr Is there an algorithm which given a modal formula and a FOL sentence can determine whether they are equivalent?

Lilia Chagrova proved in her dissertation that all three problems are undecidable over the class of all Kripke frames $\mathcal{K}_{Kripke}$. So why not restrict the problems to some smaller classes of structures and see what happens?

Remark 1.7.0.1:

In this case when we restrict the problems to some smaller classes of structures, the previous definitions will stay the same, but "relativized" w.r.t. a class of structures $\mathcal{K}$. For example "φ **defines** $\mathtt{A}$" will become "φ **defines** $\mathtt{A}$ w.r.t. the class of structures $\mathcal{K}$".

In their paper (Balbiani and Tinchev, 2005) Balbiani and Tinchev proved that all problems over the class of all partitions $\mathcal{K}_{equiv}$ are decidable and are in fact **PSPACE-complete**. After this result they formulated a more general method to obtain lower bounds for the complexity of the problem of modal definability over specific classes of frames called **stable classes**. They relate the problem of deciding the modal definability of sentences w.r.t. a stable class of frames $\mathcal{K}$ to the problem of deciding the validity of sentences in $\mathcal{K}$. In this respect, a special role plays the notion of FOL relativization, so we can understand their method.

Remark 1.7.0.2:
All notions can be extended for PML languages and FOL languages having the same number of modalities to relation symbols and the FOL language having formal equality.

1.7.1 Relativization in FOL

Let us fix a $\mathfrak{L}$ RFOL language until the end of this subsection.

Definition 1.7.1.1 [Relativization of formulae]**:**
Let $\chi, \varphi \in \mathcal{Form}(\mathfrak{L})$ and $x \in \mathcal{Var}_{\mathfrak{L}}$. Let $Var^{free}[\chi] = \{y_1, \dots, y_m\}$.

The relativization of χ **w.r.t.** φ **and an individual variable** x, denoted $(\chi)^{\varphi}_{x}$, is inductively defined as following:

- If $\chi \coloneqq (y_i \doteq y_j)$ for some $1 \le i, j \le m$, then:

$$(\chi)^{\varphi}_{x} \leftrightharpoons (y_i \doteq y_j);$$

- If $\chi \coloneqq p(y_{i_1}, \dots, y_{i_k})$ for some indices $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ and k-ary predicate symbol p, then:

$$(\chi)^{\varphi}_{x} \leftrightharpoons p(y_{i_1}, \dots, y_{i_k});$$

- If $\chi \coloneqq (\chi_1 \vee \chi_2)$, then:

$$(\chi)^{\varphi}_{x} \leftrightharpoons (\chi_1)^{\varphi}_{x} \vee (\chi_2)^{\varphi}_{x};$$

- If $\chi \coloneqq \neg\chi_1$, then:

$$(\chi)^{\varphi}_{x} \leftrightharpoons \neg(\chi_1)^{\varphi}_{x}.$$

- If $\chi \coloneqq \exists z \chi_1$, then:

$$(\chi)^{\varphi}_{x} \leftrightharpoons \exists z(\varphi[x/z] \wedge (\chi_1)^{\varphi}_{x}),$$

where $\varphi[x/z]$ denotes the simultaneous substitution of all free occurrences of the individual variable x in φ by the individual variable z.

When we write $(\chi)^{\varphi}_{x}$, we will always assume that $Var[\chi] \cap Var[\varphi] = \emptyset$.

Proposition 1.7.1.2:
Let $\chi, \varphi \in \mathcal{Form}(\mathfrak{L})$ and $x \in \mathcal{Var}_{\mathfrak{L}}$.
Then $Var^{free}[(\chi)^{\varphi}_{x}] \subseteq (Var^{free}[\varphi] \setminus \{x\}) \cup Var^{free}[\chi]$.

Proof. It can be proven with induction on the formula χ. ■

Corollary 1.7.1.2.1:
Let $\chi \in \mathcal{Sent}(\mathfrak{L})$, $\varphi \in \mathcal{Form}(\mathfrak{L})$ and $x \in \mathcal{Var}_{\mathfrak{L}}$.
Then $Var^{free}[(\chi)^{\varphi}_{x}] \subseteq Var^{free}[\varphi] \setminus \{x\}$.

Definition 1.7.1.3 [Relativized substructure]**:**
Let $\mathfrak{A}$ and $\mathfrak{A}_0$ are structures for $\mathfrak{L}$.

$\mathfrak{A}_0$ is called a **relativized substructure or relativized reduct** of $\mathfrak{A}$ if there exist a FOL formula $\varphi(x, x_1, \dots, x_n) \in \mathcal{Form}(\mathfrak{L})$ and there exists a list of individuals $\bar{a}$ in A such that $\mathfrak{A}_0$ is the substructure of $\mathfrak{A}$ with universe $\{b \mid b \in A \mathbin{\&} \mathfrak{A} \models \varphi[\![b, \bar{a}]\!]\}$. In this case we say that $\mathfrak{A}_0$ is called a **relativized substructure** of $\mathfrak{A}$ w.r.t. $\varphi(x, x_1, \dots, x_n)$ and $\bar{a}$.

Remark 1.7.1.1:

$\mathfrak{A}$ possesses a relativized reduct w.r.t. $\varphi(x, x_1, \dots, x_n)$ and $\bar{a}$ if and only if $\mathfrak{A} \models \exists x \varphi[\![\bar{a}]\!]$.

Theorem 1.7.1.4 [Relativization theorem]**:**

Let $\mathfrak{A}$ and $\mathfrak{A}_0$ are structures for $\mathfrak{L}$, $\varphi(x, x_1, \dots, x_n) \in \mathcal{F}orm(\mathfrak{L})$ and $\bar{a}$ be a list of individuals in A.

If $\mathfrak{A}_0$ is a **relativized substructure** of $\mathfrak{A}$ w.r.t. $\varphi(x, x_1, \dots, x_n)$ and $\bar{a}$, then for all FOL formula $\chi(y_1, \dots, y_m)$ and all list of individuals $\bar{c}$ in A_0:

$$\mathfrak{A} \models (\chi)_x^{\varphi}[\![\bar{a}; \bar{c}]\!] \iff \mathfrak{A}_0 \models \chi[\![\bar{c}]\!].$$

Proof. One may consult (Hodges, 2008), Theorem 5.1.1. ■

1.7.2 Stable classes of frames and modal definability

Definition 1.7.2.1 [Stable class of frames]**:**

Let $\mathcal{K}$ be a class of frames.

$\mathcal{K}$ is called a **stable class of frames** if there exists a first-order formula $\varphi(x, x_1, \dots, x_n)$ and there exists a sentence ψ such that:

(1) for all frames $\mathfrak{F}$ in $\mathcal{K}$, for all lists $\bar{a}$ of individuals in $\mathfrak{F}$ and for all frames $\mathfrak{F}'$, if $\mathfrak{F}'$ is the relativized reduct of $\mathfrak{F}$ w.r.t. $\varphi(x, x_1, \dots, x_n)$ and $\bar{a}$ then $\mathfrak{F}'$ is in $\mathcal{K}$;

(2) for all frames $\mathfrak{F}_0$ in $\mathcal{K}$, there exists frames $\mathfrak{F}$, $\mathfrak{F}'$ in $\mathcal{K}$ and there exists a list $\bar{a}$ of individuals in $\mathfrak{F}$ such that:

(a) $\mathfrak{F}_0$ is the relativized reduct of $\mathfrak{F}$ w.r.t. $\varphi(x, x_1, \dots, x_n)$ and $\bar{a}$;
(b) $\mathfrak{F} \models \psi$ and $\mathfrak{F}' \not\models \psi$;
(c) $\mathfrak{F} \preceq \mathfrak{F}'$.

In this case, $\langle \varphi(x, x_1, \dots, x_n), \psi \rangle$ is called a **witness of the stability of** $\mathcal{K}$.

Theorem 1.7.2.2:

If $\mathcal{K}$ is stable then the problem of deciding the validity of sentences in $\mathcal{K}$ is reducible to the problem of deciding the modal definability of sentences w.r.t. $\mathcal{K}$.

Proof. See in (Balbiani and Tinchev, 2017), Theorem 1. ■

This tight relationship between the problem of deciding the modal definability of sentences w.r.t. $\mathcal{K}$ and the problem of deciding the validity of sentences in $\mathcal{K}$ constitutes the main result of the method of Balbiani and Tinchev.

1.8 Some history on related theories

Let us have a RFOL language $\mathfrak{L}(R, \dot{=})$ with formal equality $\dot{=}$ having only one binary relation symbol R and a RFOL language $\mathfrak{L}(R_1, R_2, \dot{=})$ with formal equality $\dot{=}$ having only two binary relation symbols R_1 and R_2.

Let $\mathcal{K}_{equiv}$ be the class of all structures for $\mathfrak{L}(R, \dot{=})$ such that the predicate symbol is interpreted as an equivalence relation on the universe of the structure.

In (Janiczak, 1953) we have a proof of the decidability of the theory of the class $\mathcal{K}_{equiv}$. It is folklore that $\mathcal{K}_{equiv}$ has **FMP** and the validity of sentences restricted to the class $\mathcal{K}_{equiv}$ is **PSPACE-complete**. Nevertheless one can consult (Balbiani and Tinchev, 2006) for a proof. In (Boerger, Grädel, and Gurevich, 1997) we have a proof that (finite) satisfiability problem restricted to the class $\mathcal{K}_{equiv}$ is **PSPACE-complete**.

Let $\mathcal{K}_{2S5}$ the class of all structures for $\mathfrak{L}(R_1, R_2, \dot{=})$ such that the relation symbols are interpreted as two equivalence relations on the universe of the structure.

Rogers in (H. Rogers, 1956) and Janiczak in (Janiczak, 1953) independently of each other proved that $Th(\mathcal{K}_{2S5})$ is undecidable through different methods. The theory is finitely axiomatizable so by corollary 1.4.0.5.1 it is hereditarily undecidable.

The **monadic second-order (MSO)** extension of the first-order logic is obtained by adding new unary predicate variables and quantifiers over them. Usually in this way the expressive power of FOL is increased.

In (Ershov, Lavrov, Taimanov, and Taitslin, 1965) Ershov proves that MSO logic is decidable over the class of structures with one equivalence relation. But taking into account Janiczak's result (Janiczak, 1953) that $Th(\mathcal{K}_{2S5})$ is undecidable the direct generalization of Ershov's result for MSO logic with more than one equivalence relation is impossible.
In their work (Georgiev and Tinchev, 2008) they restrict the equivalence relations and study the MSO logic over structures with finite number of unary predicates and equivalence relations **in local agreement**, the latter meaning that the equivalence classes of every element of the universe, modulo the respective equivalence relations, are linearly ordered (form a chain) w.r.t. set-theoretic inclusion. Using Ehrenfeucht–Fraïssé games they show that the MSO logic is decidable over the class of all structures with unary predicates and equivalence relations in local agreement. Moreover, they show that over these structures every MSO formula has a translation in the first-order language which has exactly the same models. The translated FOL formula is very complex, compared to the original MSO formula.

Chapter 2

A tale of three theories

2.1 Formulation of the problem

Let us have a RFOL language $\mathfrak{L}(R_1, R_2, \doteq)$ with formal equality $\doteq$ having only two binary relation symbols R_1 and R_2.

Since we have Downward Löwenheim–Skolem theorem, from the last condition we get that $Th(\mathfrak{A}) = Th(\mathfrak{B})$, because being an elementary substructure yields elementarily equivalence between the structures in question.

Then applying it to our case with the language $\mathfrak{L}(R_1, R_2, \doteq)$ that has cardinality $\mathbf{card}(\mathfrak{L}(R_1, R_2, \doteq)) < \aleph_0 = \mathbf{card}(\omega)$ and the semantic definition of a theory of a class of structures $\mathcal{K}$ to be $Th(\mathcal{K}) = \bigcap_{\mathfrak{A} \in \mathcal{K}} Th(\mathfrak{A})$, we can limit ourselves to only consider structures with an *enumerable* (at most countable) universe. Therefore, from here until the end of this chapter we will only work with enumerable structures and classes of enumerable structures (even if not said explicitly).

The subject of our studies will be a particular type of structures for this language $\mathfrak{L}(R_1, R_2, \doteq)$: all structures $\mathfrak{A} = \langle A, R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}} \rangle$, where the interpretations of $R_1^{\mathfrak{A}}$ and $R_2^{\mathfrak{A}}$ are such that $R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}}, R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}} \in \mathcal{E}quiv(A)$, i.e., they are all equivalence relations on A.

Let us define three classes of structures of this type such that each consecutive class is a refinement of the previous:

$$\mathcal{K}_{commute} \leftrightharpoons \{\langle A, R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}} \rangle \mid R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}}, R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}} \in \mathcal{E}quiv(A)\}$$
$$\mathcal{K}_{rectangle} \leftrightharpoons \{\mathfrak{A}_1 \underset{mod}{\times} \mathfrak{A}_2 \mid \mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{K}_{equiv}\}$$
$$\mathcal{K}_{square} \leftrightharpoons \{\mathfrak{A} \underset{mod}{\times} \mathfrak{A} \mid \mathfrak{A} \in \mathcal{K}_{equiv}\}$$

Remark 2.1.0.1:

$\mathcal{K}_{square} \subseteq \mathcal{K}_{rectangle} \subseteq \mathcal{K}_{commute}$

In this chapter we are going to ask ourselves the following questions concerning the theories of these three classes $Th(\mathcal{K}_{commute})$, $Th(\mathcal{K}_{rectangle})$, $Th(\mathcal{K}_{square})$ respectively:

1. How can the structures of the classes be represented in a strict mathematical manner with a strong intuitive meaning?
2. $Th(\mathcal{K}_{commute}) \overset{?}{\subsetneq} Th(\mathcal{K}_{rectangle}) \overset{?}{\subsetneq} Th(\mathcal{K}_{square})$?
3. Are there classes axiomatizable?
4. Are the theories decidable?
5. Do the classes have the finite model property?

2.2 How can we describe the structures?

We will concern ourselves with the class $\mathcal{K}_{commute}$.

Let $\mathfrak{A} \in \mathcal{K}_{commute}$ be such that $\mathfrak{A} = \langle A, R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}} \rangle$. Then from lemma 1.3.1.2 $R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}} \in \mathcal{E}quiv(A)$, and, thus, A is a set of blocks w.r.t. the equivalence relation $R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}}$.

We will prove the following simple proposition:

Proposition 2.2.0.1:

Let $c \in A$ and $a, b \in [c]_{R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}}}$.

Then $[a]_{R_1^{\mathfrak{A}}} \subseteq [c]_{R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}}}$ and $[b]_{R_2^{\mathfrak{A}}} \subseteq [c]_{R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}}}$ and $[a]_{R_1^{\mathfrak{A}}} \cap [b]_{R_2^{\mathfrak{A}}} \neq \emptyset$.

Proof. Since $a, b \in [c]_{R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}}}$, then we have $\langle c, a \rangle \in R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}}$ and $\langle c, b \rangle \in R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}}$. But $R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}} \in \mathcal{E}quiv(A)$, so then $\langle b, a \rangle \in R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}}$. By the definition of composition of relations, then $(\exists d \in A)[\langle b, d \rangle \in R_2^{\mathfrak{A}} \,\&\, \langle d, a \rangle \in R_1^{\mathfrak{A}}]$. Let $d_0 \in A$ be a witness. Then $d_0 \in [a]_{R_1^{\mathfrak{A}}}$ and $d_0 \in [b]_{R_2^{\mathfrak{A}}}$; therefore, $[a]_{R_1^{\mathfrak{A}}} \cap [b]_{R_2^{\mathfrak{A}}} \neq \emptyset$ is true.

Now let $e \in [a]_{R_1^{\mathfrak{A}}}$. Then $\langle a, e \rangle \in R_1^{\mathfrak{A}}$. Also, by assumption, we have that $\langle c, a \rangle \in R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}}$, so by definition of composition of relations $(\exists d \in A)[\langle c, d \rangle \in R_2^{\mathfrak{A}} \,\&\, \langle d, a \rangle \in R_1^{\mathfrak{A}}]$. Let $d_0 \in A$ be a witness. $R_1^{\mathfrak{A}} \in \mathcal{E}quiv(A)$, so then $\langle d_0, e \rangle \in R_1^{\mathfrak{A}}$. As a result we obtain $\langle c, e \rangle \in R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}}$; thus, $[a]_{R_1^{\mathfrak{A}}} \subseteq [c]_{R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}}}$.

The reasoning for $[b]_{R_2^{\mathfrak{A}}} \subseteq [c]_{R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}}}$ is similar. ■

Now let $c \in A$ and let $p \leftrightharpoons [c]_{R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}}}$.

Let us enumerate all the blocks of $R_1^{\mathfrak{A}}$ w.r.t. p: $\{a_\alpha\}_{\alpha<\lambda}$ and enumerate all the blocks of $R_2^{\mathfrak{A}}$ w.r.t. p: $\{b_\beta\}_{\beta<\mu}$, where $\mathbf{card}(p/R_1^{\mathfrak{A}}) = \lambda$ and $\mathbf{card}(p/R_2^{\mathfrak{A}}) = \mu$ ($p/R_1^{\mathfrak{A}}$ is the quotient set p w.r.t. $R_1^{\mathfrak{A}}$).

Let us denote $c_{\alpha,\beta} \leftrightharpoons a_\alpha \cap b_\beta$. We have that:

- $c_{\alpha,\beta} \neq \emptyset$ (by proposition 2.2.0.1);
- $c_{\alpha,\beta} \cap c_{\alpha',\beta'} = \emptyset$ for $\langle \alpha, \beta \rangle \neq \langle \alpha', \beta' \rangle$;
- $\bigcup\limits_{\substack{\alpha<\lambda \\ \beta<\mu}} c_{\alpha,\beta} = p$.

I.e., the family $\{c_{\alpha,\beta}\}_{\substack{\alpha<\lambda \\ \beta<\mu}}$ is a partition of p. So we can think of p as a "**matrix of the type** $\lambda \times \mu$" of non-empty, mutually disjoint sets. We will call an element of the family $\{a_\alpha\}_{\alpha<\lambda}$ a "**row**" and we will call an element of the family $\{b_\beta\}_{\beta<\mu}$ a "**column**". A set $c_{\alpha,\beta}$ we will call a "**cell**".

$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
$\dots$	..	..	**67**	..	..	**1**	$\dots$
$\dots$	..	..	..	..	..	..	$\dots$
$\dots$	**2**	$\aleph_0$	**21**	..	..	**2**	$\dots$
$\dots$	..	..	..	..	**500**	..	$\dots$
$\dots$	$\aleph_1$	..	**2**	..	..	2^{179}	$\dots$
$\iddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

FIGURE 2.1: An example "**matrix**" with "**rows**" ($R_1^{\mathfrak{A}}$ classes) and "**columns**" ($R_2^{\mathfrak{A}}$ classes). In the intersections is written or omitted with ".." the cardinality of the respective "**cells**"

In the other direction, if we have such a family $\{c_{\alpha,\beta}\}_{\substack{\alpha<\lambda\\ \beta<\mu}}$ a partition of p, how can we define the interpretation of the relation symbols R_1 and R_2 on p so to generate a structure in $\mathcal{K}_{commute}$?

Let:

$$\langle a, b\rangle \in R_1^{\mathfrak{B}} \iff (\exists\alpha < \lambda)[a, b \in \bigcup_{\beta<\mu} c_{\alpha,\beta}]$$
$$\langle a, b\rangle \in R_2^{\mathfrak{B}} \iff (\exists\beta < \mu)[a, b \in \bigcup_{\alpha<\lambda} c_{\alpha,\beta}].$$

Then $\mathfrak{B} = \langle p, R_1^{\mathfrak{B}}, R_2^{\mathfrak{B}}\rangle$ is a structure with two equivalence relations, which commute and $\#_{R_1^{\mathfrak{B}}\circ R_2^{\mathfrak{B}}} = 1$. Thus,$\mathfrak{B} \in \mathcal{K}_{commute}$.

So all the structures $\mathfrak{A} = \langle A, R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}}\rangle \in \mathcal{K}_{commute}$ are a collection of matrices $\{M(\gamma)\}_{\gamma<\xi}$ of the type $\lambda_\gamma \times \mu_\gamma$, $\gamma < \xi$, for $\#_{R_1^{\mathfrak{A}}\circ R_2^{\mathfrak{A}}} = \xi$.

Remark 2.2.0.1:

By using Dubreil-Jacotin theorem we get a similar characterization of the relationships between the $R_1^{\mathfrak{A}}$ and $R_2^{\mathfrak{A}}$ blocks w.r.t. a block $[c]_{R_1^{\mathfrak{A}}\circ R_2^{\mathfrak{A}}}$.

One of the benefits of this representation is that the construction of such interesting structures can be done easier as demonstrateed in a proof of in section 2.5.

Remark 2.2.0.2:

Let $\mathfrak{A} \in \mathcal{K}_{rectangle}$. If:

1. $\mathfrak{A}$ is finite; thus, there is a natural number $n \in \omega$ such that $\mathbf{card}(A) = n$ **and**;
2. $A = A_1 \times A_2$, $\mathbf{card}(A_1) = k_1$, $\mathbf{card}(A_2) = k_2$, such that $k_1, k_2 \in \omega$, $n = k_1.k_2$ **and**;
3. $\#_{R_1^{\mathfrak{A}}} = m_1$, $\#_{R_2^{\mathfrak{A}}} = m_2$,

then $k_2|m_1$, $k_1|m_2$, $\#_{R_1^{\mathfrak{A}}\circ R_2^{\mathfrak{A}}} = m_1.m_2$. In particular if $\mathfrak{A} \in \mathcal{K}_{square}$ and is finite, then $\mathbf{card}(\mathfrak{A})$ and $\#_{R_1^{\mathfrak{A}}\circ R_2^{\mathfrak{A}}}$ are always square natural numbers.

2.3 Do they differ?

2.3.1 $Th(\mathcal{K}_{commute})$ is a proper subtheory of $Th(\mathcal{K}_{rectangle})$

Let us define the formula $\varphi_=(x,y)$ of $\mathfrak{L}(R_1, R_2, \dot{=})$ in the following manner:

$$\varphi_=(x,y) \leftrightharpoons (R_1(x,y) \wedge R_2(x,y)).$$

Let $\psi_=$ be the following sentence:

$$\psi_= \leftrightharpoons \forall x \forall y(\varphi_=(x,y) \leftrightarrow x \dot{=} y).$$

Then $\psi_=$ is true for all structures of $\mathcal{K}_{rectangle}$. Let $\mathfrak{A} \in \mathcal{K}_{rectangle}$. Let $a, b \in A$, then:

$$\mathfrak{A} \models \varphi_=(x,y)[\![a,b]\!] \iff$$
$$[\langle pr_1(a), pr_1(b)\rangle \in R_1^{\mathfrak{A}} \mathbin{\&} pr_2(a) = pr_2(b) \mathbin{\&} pr_1(a) = pr_1(b) \mathbin{\&} \langle pr_2(a), pr_2(b)\rangle \in R_2^{\mathfrak{A}}] \iff$$
$$[pr_1(a) = pr_1(b) \mathbin{\&} pr_2(a) = pr_2(b)] \iff$$
$$\mathfrak{A} \models (x \dot{=} y)[\![a,b]\!].$$

Therefore, we can conclude that $\mathfrak{A} \models \psi_=$.

Let $\mathfrak{A}_0$ be defined as: $\mathfrak{A}_0 = \langle \{0,1\}, R_1^{\mathfrak{A}_0}, R_2^{\mathfrak{A}_0} \rangle$, where $R_1^{\mathfrak{A}_0} = R_2^{\mathfrak{A}_0} = A_0 \times A_0$. Then $\mathfrak{A}_0 \in \mathcal{K}_{commute}$ and $\mathfrak{A}_0 \not\models \psi_=$.

2.3.2 $Th(\mathcal{K}_{rectangle})$ is a proper subtheory of $Th(\mathcal{K}_{square})$

Let us define a number of formulae this time:

$$\varphi_{R_1 \circ R_2}(x,y) \leftrightharpoons \exists z(R_1(x,z) \wedge R_2(z,y)).$$
$$\varphi_{oneBlockR_1 \circ R_2} \leftrightharpoons \exists x \forall y(\varphi_{R_1 \circ R_2}(x,y)).$$
$$\varphi_{twoOrLessIndividuals} \leftrightharpoons \exists x \exists y \forall z(x \dot{=} z \vee y \dot{=} z).$$
$$\varphi_{oneIndividual} \leftrightharpoons \exists x \forall y(x \dot{=} y).$$

The intended semantics of the formulae is explained in the name of the formula. Let ψ_{dot} be the following sentence:

$$\psi_{dot} \leftrightharpoons \varphi_{oneBlockR_1 \circ R_2} \wedge \varphi_{twoOrLessIndividuals} \to \varphi_{oneIndividual}.$$

Then ψ_{dot} is true for all structures of $\mathcal{K}_{square}$. Let $\mathfrak{A} \in \mathcal{K}_{square}$. Let $a, b \in A$. If $\mathfrak{A}$ has more than one equivalence class in $R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}}$ and if $\mathfrak{A}$ has more than two individuals in its universe, then ψ_{dot} is trivially true.

Let $\mathfrak{A} \models \varphi_{oneBlockR_1 \circ R_2} \wedge \varphi_{twoOrLessIndividuals}$. Then:

$$\mathfrak{A} \models \varphi_{oneBlockR_1 \circ R_2} \wedge \varphi_{twoOrLessIndividuals} \iff$$
$$\mathfrak{A} \text{ has exactly one block w.r.t. } R_1^{\mathfrak{A}} \circ R_2^{\mathfrak{A}} \text{ of cardinality one.}$$

By remark 2.2.0.2 the cardinality of the universe of a finite structure from the class $\mathcal{K}_{square}$ is a square number; therefore, we have $\mathfrak{A} \models \varphi_{oneIndividual}$, and; therefore, we have equivalence between the last the expressions.

Let $\mathfrak{A}_0$ be defined as: $\mathfrak{A}_0 = \langle \{0,1\} \times \{2\}, R_1^{\mathfrak{A}_0}, R_2^{\mathfrak{A}_0} \rangle$, where $R_1^{\mathfrak{A}_0} = \{0,1\}^2$ and $R_2^{\mathfrak{A}_0} = \{2\}^2$. Then $\mathfrak{A}_0 \in \mathcal{K}_{rectangle}$ and $\mathfrak{A}_0 \not\models \psi_{dot}$.

2.4 Are the classes axiomatizable?

Proposition 2.4.0.1:
$\mathcal{K}_{commute}$ is finitely axiomatizable.

Proof. Let Γ be the set of consisting of the sentences:

$$\varphi_1 \leftrightharpoons \forall x R_1(x,x).$$
$$\varphi_2 \leftrightharpoons \forall x \forall y (R_1(x,y) \to R_1(y,x)).$$
$$\varphi_3 \leftrightharpoons \forall x \forall y \forall z (R_1(x,y) \land R_1(y,z) \to R_1(x,z)).$$
$$\varphi_4 \leftrightharpoons \forall x R_2(x,x).$$
$$\varphi_5 \leftrightharpoons \forall x \forall y (R_2(x,y) \to R_2(y,x)).$$
$$\varphi_6 \leftrightharpoons \forall x \forall y \forall z (R_2(x,y) \land R_2(y,z) \to R_2(x,z)).$$
$$\varphi_7 \leftrightharpoons \forall x \forall y (\exists z (R_1(x,z) \land R_2(z,y)) \leftrightarrow \exists z (R_2(x,z) \land R_1(z,y))).$$

Let $\varphi_{\mathcal{K}_{commute}} \leftrightharpoons \bigwedge \Gamma$. Then for any structure $\mathfrak{A}$ for $\mathfrak{L}(R_1, R_2, \dot{=})$:

$$\mathfrak{A} \in \mathcal{K}_{commute} \iff \mathfrak{A} \models \varphi_{\mathcal{K}_{commute}}.$$

■

Corollary 2.4.0.1.1:
$\mathcal{K}^{fin}_{commute}$ is finitely axiomatizable.

Proof. By remembering definition 1.2.2.5 and using the previous proposition 2.4.0.1, we have that the same $\varphi_{\mathcal{K}_{commute}}$ finitely axiomatizes $\mathcal{K}^{fin}_{commute}$. ■

Even though this class of structures is finitely axiomatizable, its theory is undecidable as shown in section 2.5. Moreover by being finitely axiomatizable and applying corollary 1.4.0.5.1 it is hereditarily undecidable. Then Janiczak's theorem about the undecidability of $Th(\mathcal{K}_{2S5})$ is an immediate corollary (only remove axiom φ_7).

Remark 2.4.0.1:
There is use to try and prove that in a finite RFOL language $\mathcal{K}^{fin}$ is not axiomatizable, because it is not true.

Let $\mathfrak{L}$ be a finite RFOL language and let $\mathcal{K}$ be some class of finite structures for $\mathfrak{L}$. Then for every $n \in \omega^+$, there are a finite number of structures in $\mathcal{K}$ of cardinality n up to isomorphism. That is because for a structure $\mathfrak{A} \in \mathcal{K}$, $\mathbf{card}(A) = n$, we can write a sentence $\varphi_{\mathfrak{A}}$ such that for all structures $\mathfrak{B}$ for $\mathfrak{L}$ $[\mathfrak{B} \models \varphi_{\mathfrak{A}} \iff \mathfrak{A} \cong \mathfrak{B}]$. Therefore, if ψ_n is the sentence saying that in the universe there are exactly n elements, the set $\{\psi_n \to (\varphi_{\mathfrak{A}_1} \lor \ldots \lor \varphi_{\mathfrak{A}_{k_n}}) \mid n \in \omega^+\}$ axiomatizes $\mathcal{K}$.

Remark 2.4.0.2:
All $\mathcal{K}_{rectangle}$, $\mathcal{K}_{square}$, $\mathcal{K}^{fin}_{rectangle}$ and $\mathcal{K}^{fin}_{square}$ are not closed w.r.t. isomorphisms. That is because if we take a structure $\mathfrak{A} \in \mathcal{K}_{rectangle}$, then it is of the type $\langle A_1 \times A_2, R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}} \rangle$. Let the set B be such that $\mathbf{card}(A) = \mathbf{card}(B)$ and the elements of B are **not tuples**. Then $\mathfrak{A} \cong \langle B, R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}} \rangle$, but $\langle B, R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}} \rangle \notin \mathcal{K}_{rectangle}$ (the same reasoning can be applied for the other classes).

Therefore, $\mathcal{K}_{rectangle}$, $\mathcal{K}_{square}$, $\mathcal{K}^{fin}_{rectangle}$ and $\mathcal{K}^{fin}_{square}$ are not axiomatizable.

The question is if we close the classes w.r.t. isomorphisms, can we (finitely) axiomatize the new classes?

Let us denote with $I(\mathcal{K}) \leftrightharpoons \{\mathfrak{A} \mid (\exists \mathfrak{B} \in \mathcal{K})[\mathfrak{A} \cong \mathfrak{B}]\}$ the **closure of the class $\mathcal{K}$ w.r.t. isomorphisms**.

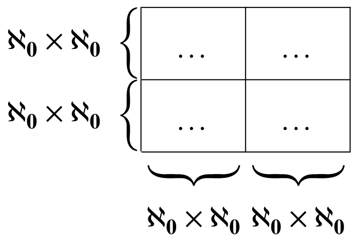
Proposition 2.4.0.2:
$I(\mathcal{K}_{rectangle})$ and $I(\mathcal{K}_{square})$ are not axiomatizable.

Proof. We will do the proof for $I(\mathcal{K}_{rectangle})$. The same proof can be used for $I(\mathcal{K}_{square})$.

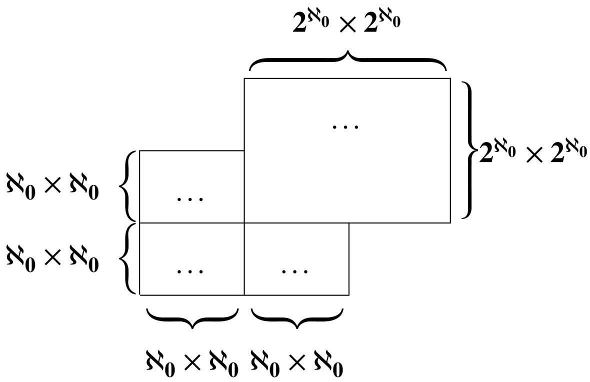
Suppose it is axiomatizable. Then there exist a set of sentences of $\mathfrak{L}(R_1, R_2, \dot{=})$ Σ such that for all structures $\mathfrak{A}$ for $\mathfrak{L}(R_1, R_2, \dot{=})$:

$$[\mathfrak{A} \models \Sigma \iff (\exists \mathfrak{B} \in I(\mathcal{K}_{rectangle}))[\mathfrak{B} \cong \mathfrak{A}]].$$

Let $\mathfrak{A} \in \mathcal{K}_{rectangle}$ (also $\mathfrak{A} \in \mathcal{K}_{square}$) be such a structure that it has four matrices and each of the matrices is of the type $\aleph_0 \times \aleph_0$:



Let $\mathfrak{B}$ be such a structure that it has four matrices and three of the matrices is of the type $\aleph_0 \times \aleph_0$ and one is of the type $2^{\aleph_0} \times 2^{\aleph_0}$ (any cardinal numbers α, β such that $\alpha \geq \aleph_0$, $\beta \geq \aleph_0$ and at least one of them $> \aleph_0$ will be sufficient for forming the matrix):



Then $\mathfrak{A} \equiv \mathfrak{B}$ (for every $n \in \omega$ we can prove that the *Duplicator* has a winning strategy for the n-round Ehrenfeucht–Fraïssé games, and; thus,, $\mathfrak{A} \equiv_n \mathfrak{B}$).

An alternative proof of $\mathfrak{A} \equiv \mathfrak{B}$ is using Downward Löwenheim–Skolem theorem. By applying it we get a countable elementary substructure $\mathfrak{C} \preccurlyeq \mathfrak{B}$. Then $\mathfrak{C} \equiv \mathfrak{B}$. We can say with a formula that there are exactly four matrices in the universe of $\mathfrak{B}$, so then $\mathfrak{C}$ has exactly four matrices. Can we say that some matrix is finite in $\mathfrak{C}$? If we could, then we can describe it with a first-order formula, but then it must be true in $\mathfrak{B}$, which is not the case. Thus, all the matrices of $\mathfrak{C}$ are infinite, but $\mathfrak{C}$ is countable, so the matrices must be also countable. As a result $\mathfrak{C} \cong \mathfrak{A}$ which implies $\mathfrak{C} \equiv \mathfrak{A}$; therefore, $\mathfrak{A} \equiv \mathfrak{B}$.

As a result $\mathfrak{B} \models \Sigma$. But $\mathfrak{B} \notin I(\mathcal{K}_{rectangle})$. We obtained a contradiction.

■

Proposition 2.4.0.3:
$I(\mathcal{K}^{fin}_{rectangle})$ and $I(\mathcal{K}^{fin}_{square})$ are not finitely axiomatizable.

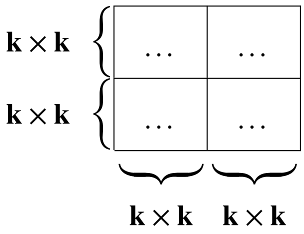
Proof. We will do the proof for $I(\mathcal{K}^{fin}_{rectangle})$. The same proof can be used for $I(\mathcal{K}_{square})$.

Suppose it is finitely axiomatizable. Then there exist a sentence φ of $\mathfrak{L}(R_1, R_2, \dot{=})$ such that for all structures $\mathfrak{A}$ for $\mathfrak{L}(R_1, R_2, \dot{=})$:

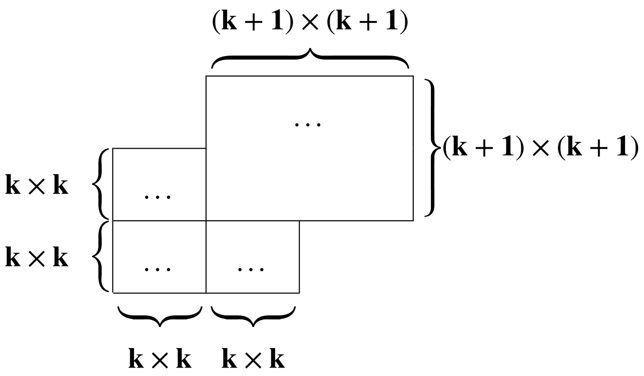
$$[\mathfrak{A} \text{ is finite } \Rightarrow [\mathfrak{A} \models \varphi \Longleftrightarrow \mathfrak{A} \in \mathcal{K}^{fin}_{rectangle}]].$$

Let $qr(\varphi) = k$.

Let $\mathfrak{A} \in \mathcal{K}^{fin}_{rectangle}$ (also $\mathfrak{A} \in \mathcal{K}^{fin}_{square}$) be such a structure that it has four matrices and each of the matrices is of the type $k \times k$:



Let $\mathfrak{B}$ be such a structure that it has four matrices and three of the matrices is of the type $k \times k$ and one is of the type $(k+1) \times (k+1)$ (any cardinal numbers α, β such that $\alpha \geq k$, $\beta \geq k$ and at least one of them $> k$ will be sufficient for forming the matrix):



Then $\mathfrak{A} \equiv_k \mathfrak{B}$ (we can prove that the *Duplicator* has a winning strategy for the k-round Ehrenfeucht–Fraïssé games).

As a result $\mathfrak{B} \models \varphi$. But $\mathfrak{B} \notin I(\mathcal{K}^{fin}_{rectangle})$. We obtained a contradiction. ■

2.5 Undecidability of $Th(\mathcal{K}_{commute})$

We are going to define a class of structures which will be of interest to us:

$$\mathcal{K}_{irref,\ sym} \leftrightharpoons \{\langle A, R^{\mathfrak{A}}\rangle \mid R^{\mathfrak{A}} \text{ is symmetric and irreflexive in } A\}$$
$$\text{for the language } \mathfrak{L}(R, \dot{=}).$$

Remark 2.5.0.1:

In (H. Rogers, 1956) it is demonstrated that the theory $Th(\mathcal{K}_{irref,\ sym})$ with added non-logical axiom $\forall x \forall y \forall z (R(x,y) \wedge R(y,z) \rightarrow \neg R(x,z))$ is undecidable. This theory is finitely axiomatizable so by corollary 1.4.0.5.1 it is hereditarily undecidable; therefore, we have that $Th(\mathcal{K}_{irref,\ sym})$ is hereditarily undecidable as well.

In (Lavrov, 1963) there is a proof that the sets $Th(\mathcal{K}_{irref,\ sym})$ and $\mathcal{S}ent(\mathfrak{L}(R_1, R_2, \dot{=})) \setminus Th(\mathcal{K}^{fin}_{irref,\ sym})$ are recursively inseparable from which follows the undecidability of $Th(\mathcal{K}_{irref,\ sym})$ and $Th(\mathcal{K}^{fin}_{irref,\ sym})$. Moreover, in (Ershov, 1980) there is a proposition stating that $\mathcal{K}^{fin}_{irref,\ sym}$ has a hereditarily undecidable theory.

Now we will use the method of Relative elementary definability to demonstrate that $\mathcal{K}_{commute}$ has an undecidable theory.

Let $\mathcal{K}^{uni}_{commute}$ be all structures from $\mathcal{K}_{commute}$ which have exactly one matrix.

Let $k, m \in \omega^+$ be positive natural numbers such that $k \neq m$. For simplifying the following steps let us fix $k = 1$ and $m = 2$.

Theorem 2.5.0.1:

The class $\mathcal{K}_{irref,\ sym}$ is relatively elementary definable in the class $\mathcal{K}^{uni}_{commute}$.

Proof. Let $\mathfrak{A} \in \mathcal{K}_{irref,\ sym}$, $\mathfrak{A} = \langle A, R^{\mathfrak{A}}\rangle$.

Let us define the following formulae for the language $\mathfrak{L}(R_1, R_2, \dot{=})$:

$$\varphi_{onePointCell}(x) \leftrightharpoons \forall y (R_1(x,y) \wedge R_2(x,y) \rightarrow y \dot{=} x).$$
$$\varphi_{twoPointCell}(x,y) \leftrightharpoons R_1(x,y) \wedge R_2(x,y) \wedge x \dot{\neq} y \wedge$$
$$\forall z (R_1(x,z) \wedge R_2(x,z) \rightarrow z \dot{=} x \vee z \dot{=} y).$$
$$\mathcal{P}oint(x) \leftrightharpoons \varphi_{onePointCell}(x) \wedge \forall y (\neg(x \dot{=} y) \wedge (R_1(x,y) \vee R_2(x,y)) \rightarrow \neg \varphi_{onePointCell}(y)).$$
$$\mathcal{E}dge(x,y) \leftrightharpoons \neg(x \dot{=} y) \wedge \mathcal{P}oint(x) \wedge \mathcal{P}oint(y) \wedge \exists x_1 \exists x_2 \exists y_1 \exists y_2 (\varphi_{twoPointCell}(x_1, x_2) \wedge$$
$$\varphi_{twoPointCell}(y_1, y_2) \wedge R_1(x, y_1) \wedge R_2(y, y_1) \wedge R_2(x, x_1) \wedge R_1(y, x_1)).$$
$$\mathcal{E}quality(x,y) \leftrightharpoons x \dot{=} y.$$

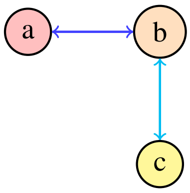
$\mathcal{E}dge(x,y)$ is a *possible definition* for the binary relation symbol R of $\mathfrak{L}(R, \dot{=})$. We will call elements satisfying $\mathcal{P}oint(x)$ *1*-points and elements satisfying $\mathcal{E}dge(x,y)$ *2*-edges (by the choices for k and m).

Let $\{a_\alpha\}_{\alpha<\lambda}$ is an enumeration of the elements of A, where $\mathbf{card}(A) = \lambda$.

We can construct such a structure $\mathfrak{B} \in \mathcal{K}^{uni}_{commute}$ using $\mathfrak{A}$ having a matrix (only one block in the composition of the relations) such that:

$$\mathbf{card}(c_{\alpha,\beta}) = \begin{cases} 1, & \text{if } \alpha = \beta \\ 2, & \text{if } \langle a_\alpha, a_\beta\rangle \in R^{\mathfrak{A}} \\ 3, & \text{otherwise} \end{cases}$$

3 was chosen as an arbitrary number different from k and m.



1	**3**	**2**
3	**1**	**2**
2	**2**	**1**

FIGURE 2.2: Example for a three node graph

Here is such a construction.

Let f, g be functions such that:

- $Dom(f) = Dom(g) = A \times A$;
- for $a, b \in A$, $f(a, b)$ is a **two element set** and if $\langle a, b\rangle \neq \langle a', b'\rangle$ for $a', b' \in A$, then $f(a, b) \cap f(a', b') = \emptyset$;
- for $a, b \in A$, $g(a, b)$ is a **three element set** and if $\langle a, b\rangle \neq \langle a', b'\rangle$ for $a', b' \in A$, then $g(a, b) \cap g(a', b') = \emptyset$;
- $\bigcup Range(f) \cap \bigcup Range(g) = \emptyset$ and $(\bigcup Range(f) \cup \bigcup Range(g)) \cap A = \emptyset$.

For example such functions satisfying these conditions are:

$$f(a, b) \leftrightharpoons \{\langle 0, \langle a, b\rangle\rangle, \langle 1, \langle a, b\rangle\rangle\};$$
$$g(a, b) \leftrightharpoons \{\langle 2, \langle a, b\rangle\rangle, \langle 3, \langle a, b\rangle\rangle, \langle 4, \langle a, b\rangle\rangle\},$$

for $a, b \in A$. WLOG we can assume that the elements of A are not ordered pairs with first coordinate an integer between 0 and 4.

Then let:

- $B \leftrightharpoons A \cup \bigcup\{f(a_\alpha, a_\beta) \mid \langle a_\alpha, a_\beta\rangle \in R^{\mathfrak{A}}\} \cup \bigcup\{g(a_\alpha, a_\beta) \mid a_\alpha \neq a_\beta \,\&\, \langle a_\alpha, a_\beta\rangle \notin R^{\mathfrak{A}}\}$;
- $\langle a, b\rangle \in R_1^{\mathfrak{B}} \iff (\exists\alpha < \lambda)[a, b \in \{a_\alpha\} \cup \bigcup\{f(a_\alpha, b_\beta) \mid \alpha \neq \beta \,\&\, \beta < \lambda\} \cup \bigcup\{g(a_\alpha, b_\beta) \mid \alpha \neq \beta \,\&\, \beta < \lambda\}]$, for $a, b \in B$;
- $\langle a, b\rangle \in R_2^{\mathfrak{B}} \iff (\exists\beta < \lambda)[a, b \in \{a_\beta\} \cup \bigcup\{f(a_\alpha, b_\beta) \mid \alpha \neq \beta \,\&\, \alpha < \lambda\} \cup \bigcup\{g(a_\alpha, b_\beta) \mid \alpha \neq \beta \,\&\, \alpha < \lambda\}]$, for $a, b \in B$.

Let $C \leftrightharpoons \{a \in B \mid \mathfrak{B} \models \mathcal{P}oint[\![a]\!]\}$. Then we have that the structure $\mathfrak{B}$ satisfies the following conditions:

- $C \neq \emptyset$;
- there exists a bijection $h : A \rightarrowtail\!\!\!\!\twoheadrightarrow C$, such that whenever $a, b \in A$ it is true that $\langle a, b\rangle \in R^{\mathfrak{A}} \iff \mathfrak{B} \models \mathcal{E}dge[\![h(a), h(b)]\!]$.

Now if we take the quotient of C w.r.t. the congruence $\mathcal{E}quality(x, y)$, because of the simplicity for $\mathcal{E}quality(x, y)$ and the choice for $k = 1$, the elements of the quotient set will be singletons. I.e., we trivially fulfill one of Ershov's conditions for the application of theorem 1.4.1.1 (in the its full form we may need the points of A to be represented in B by some configurations and then we need to do factorization w.r.t. $\mathcal{E}quality$). Let $R^{\mathfrak{C}} \leftrightharpoons \{\langle a, b\rangle \mid a, b \in C \,\&\, \mathfrak{B} \models \mathcal{E}dge[\![a, b]\!]\}$. The structure $\mathfrak{C} = \langle C, R^{\mathfrak{C}}\rangle$ is already isomorphic to $\mathfrak{A}$ and we have not yet applied factorization w.r.t. $\mathcal{E}quality(x, y)$. We will not need to care for the congruence. If $k \neq 1$, that will not be the case.

We can prove by induction on the formula $\psi(x_1, x_2, \ldots, x_n) \in \mathcal{F}orm(\mathfrak{L}(R, \dot{=}))$ that for all $c_1, c_2, \ldots, c_n \in C$:

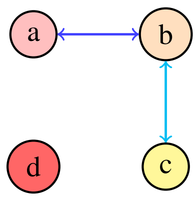
$$\mathfrak{B} \models \psi^*[\![c_1, c_2, \ldots, c_n]\!] \iff \mathfrak{C} \models \psi[\![c_1, c_2, \ldots, c_n]\!],$$

where $\psi^* \leftrightharpoons \bar{\psi}$ as in the proof of theorem 1.4.1.1.

It is immediate now that $\mathfrak{A} \cong \mathfrak{C}$.

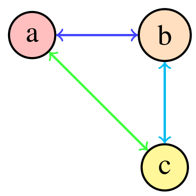
■

The following figures are some examples how we can represent a finite graph in a structure of $\mathcal{K}^{uni}_{commute}$:



1	3	2	3
3	1	2	3
2	2	1	3
3	3	3	1

1	2
2	1



1	2	2
2	1	2
2	2	1

Theorem 2.5.0.2:
$Th(\mathcal{K}^{uni}_{commute})$ is hereditarily undecidable, and, therefore, undecidable.

Proof. By theorem 2.5.0.1 we get that $\mathcal{K}_{irref,\ sym}$ is relatively elementary definable in the class $\mathcal{K}^{uni}_{commute}$ and since $\mathcal{K}_{irref,\ sym}$ is hereditarily undecidable we enter the conditions of theorem 1.4.1.1 making $\mathcal{K}^{uni}_{commute}$ hereditarily undecidable. ■

Theorem 2.5.0.3:
$Th(\mathcal{K}_{commute})$ is hereditarily undecidable, and, therefore, undecidable.

Proof. From theorem 2.5.0.2 we have that $Th(\mathcal{K}^{uni}_{commute})$ is hereditarily undecidable, but $Th(\mathcal{K}_{commute})$ is a subtheory of $Th(\mathcal{K}^{uni}_{commute})$ making it also undecidable. By remark 1.4.0.1 $Th(\mathcal{K}_{commute})$ is also hereditarily undecidable. ■

Corollary 2.5.0.3.1 [Janiczak, Rogers]**:**
$Th(\mathcal{K}_{2S5})$ is undecidable.

Corollary 2.5.0.3.2:
$Th(\mathcal{K}_{2S5})$ is hereditarily undecidable.

Theorem 2.5.0.4:
$Th(\mathcal{K}^{fin}_{commute})$ is hereditarily undecidable.

Proof. From remark 2.5.0.1 we have that $Th(\mathcal{K}^{fin}_{irref,\ sym})$ is hereditarily undecidable.

Remark that if the structure $\mathfrak{A}$ is finite, then the construction in the proof of theorem 2.5.0.1 shows that $\mathfrak{B}$ is also finite. Therefore, $Th(\mathcal{K}^{fin}_{irref,\ sym})$ is relatively elementary definable in $Th((\mathcal{K}^{fin}_{commute})^{uni})$ and by theorem 1.4.1.1 $Th((\mathcal{K}^{fin}_{commute})^{uni})$ is hereditarily undecidable, rending $Th(\mathcal{K}^{fin}_{commute})$ also hereditarily undecidable. ■

Corollary 2.5.0.4.1:

$Th(\mathcal{K}^{fin}_{2S5})$ is hereditarily undecidable.

Remark 2.5.0.2:

Since $Th(\mathcal{K}_{commute}) \subseteq Th(\mathcal{K}^{fin}_{commute})$, then theorem 2.5.0.3 is a corollary of theorem 2.5.0.4.

Lemma 2.5.0.5:

There exists a theory for $\mathfrak{L}(R_1, R_2, \dot{=})$ having $Th(\mathcal{K}_{commute})$ as a subtheory which is finitely axiomatizable and decidable.

Proof. Let $\psi \leftrightharpoons \varphi_{\mathcal{K}_{commute}} \wedge \forall x \forall y (R_1(x, y) \leftrightarrow R_2(x, y))$. Then this extension of the theory is **the theory of $\mathcal{K}_{equiv}$**, which is decidable ($\varphi_{\mathcal{K}_{commute}}$ is used in proposition 2.4.0.1).

There are a lot of syntactical complete (i.e., for every formula φ of the language of the theory either φ or $\neg\varphi$ is a theorem of theory) extensions of the theory $Th(\mathcal{K}_{commute})$. If we take a finite structure $\mathfrak{A} \in \mathcal{K}^{fin}_{commute}$, then the formula $\varphi_{\mathfrak{A}}$ characterizing the structure up to isomorphism can be added to the theory to obtain, yet again another finitely axiomatizable and decidable extension.
For example let the structure $\mathfrak{A}$ for $\mathfrak{L}(R_1, R_2, \dot{=})$ be defined as in this figure:

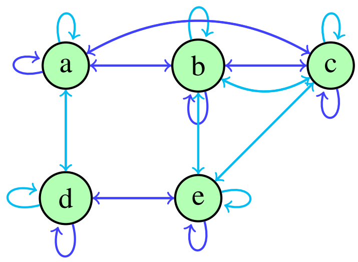


FIGURE 2.6: $R_1^{\mathfrak{A}}$ is in dark blue and $R_2^{\mathfrak{A}}$ is in cyan

It is immediate that $\mathfrak{A} \in \mathcal{K}_{commute}$. Thus, $Th(\mathcal{K}_{commute}) \subseteq Th(\mathfrak{A})$.

The $Th(\mathfrak{A})$ is finitely axiomatizable and the problem of validity of a sentence in it is decidable. ■

Corollary 2.5.0.5.1:

$Th(\mathcal{K}_{commute})$ is not essentially undecidable.

Lemma 2.5.0.6:

$\mathcal{K}_{commute}$ does not have **FMP**.

Proof. If $\mathcal{K}_{commute}$ had **FMP**, then by being finitely axiomatizable by proposition 2.4.0.1, then by theorem 1.5.2.2 it will have a decidable theory; hence, a contradiction. ■

In the next section we will see what happens with some of its subclasses $\mathcal{K}_{rectangle}$ and $\mathcal{K}_{square}$.

2.6 Decidability of $Th(\mathcal{K}_{rectangle})$ and $Th(\mathcal{K}_{square})$

Remark 2.6.0.1:

Since in $\mathcal{K}_{rectangle}$ and in $\mathcal{K}_{square}$ both model $\forall x \forall y (x \doteq y \leftrightarrow R_1(x, y) \wedge R_2(x, y))$; therefore, automatically all cells have cardinality one.

Now we will take an alternative approach to see the decidability of $Th(\mathcal{K}_{rectangle})$.

Let us have a structure $\mathfrak{A} = \langle A, R^{\mathfrak{A}} \rangle$ for the language $\mathfrak{L}(R, \doteq)$. We will effectively generate two new in a way expansions of $\mathfrak{A}$ for the language $\mathfrak{L}(R_1, R_2, \doteq)$ in the following manner:

Let $\mathfrak{A}^{=2} = \langle A, R_1^{\mathfrak{A}^{=2}}, R_2^{\mathfrak{A}^{=2}} \rangle$ be such that the interpretation of $R_1^{\mathfrak{A}^{=2}}$ is the same as that of $R^{\mathfrak{A}}$ and the interpretation of $R_2^{\mathfrak{A}^{=2}}$ will be that of equality of individuals of A (formal equality in the structure $\mathfrak{A}$). Similarly, we generate an expansion $\mathfrak{A}^{=1} = \langle A, R_1^{\mathfrak{A}^{=1}}, R_2^{\mathfrak{A}^{=1}} \rangle$ such that the interpretation of $R_1^{\mathfrak{A}^{=1}}$ will be that of equality of individuals of A and the interpretation of $R_2^{\mathfrak{A}^{=1}}$ is the same as that of $R^{\mathfrak{A}}$.

We can also obtain a structure $\mathfrak{A} = \langle A, R^{\mathfrak{A}} \rangle$ for the language $\mathfrak{L}(R, \doteq)$ when having $\mathfrak{A}^{=i} = \langle A, R_1^{\mathfrak{A}^{=i}}, R_2^{\mathfrak{A}^{=i}} \rangle$ for the language $\mathfrak{L}(R, \doteq)$ by having the universes to be the same and having the interpretation of R be the same as $R_i^{\mathfrak{A}^{=i}}$.

Let us take a formula φ from $\mathfrak{L}(R, \doteq)$. We can effectively generate two formulae $\varphi^{=2}$ and $\varphi^{=1}$ like this:

- For $\varphi^{=2}$ we substitute all occurrences of the predicate symbol R for R_1, and we substitute all occurrences of the formal equality $\doteq$ for R_2.
- For $\varphi^{=1}$ we substitute all occurrences of the formal equality $\doteq$ for R_1, and we substitute all occurrences of the predicate symbol R for R_2.

In turn by taking a formula φ from $\mathfrak{L}(R_1, R_2, \doteq)$ we can obtain a formula from the language $\mathfrak{L}(R, \doteq)$ by substituting all occurrences of the symbol R_1 with R and substituting all occurrences of the symbol R_2 with $\doteq$. We will denote it as $\varphi[R_1/R, R_2/\doteq]$ or $tr_2(\varphi)$. We can do also this translation $\varphi[R_1/\doteq, R_2/R]$ or $tr_1(\varphi)$.

We will show an example:

Let $\varphi \leftrightharpoons \forall x \forall y \forall z((R(x, y) \wedge x \doteq y) \vee x \doteq y)$ be a formula from $\mathfrak{L}(R, \doteq)$. Then $\varphi^{=1} \eqcirc \forall x \forall y \forall z((R_2(x, y) \wedge R_1(x, z)) \vee R_1(x, y))$ and $tr_1(\varphi^{=1}) \eqcirc \forall x \forall y \forall z((R(x, y) \wedge x \doteq z) \vee x \doteq y)$. If we want to return to the language $\mathfrak{L}(R_1, R_2, \doteq)$ we do not know which $\doteq$ comes from a substitution of the symbol R_i with $\doteq$ or was originally $\doteq$; thus, we do not have injectivity of the translation, but at least every formula from $\mathfrak{L}(R_1, R_2, \doteq)$ has a translation (totality).

We can prove:

Lemma 2.6.0.1:

For any formula $\varphi(x_1, \ldots, x_n)$ from the language $\mathfrak{L}(R, \doteq)$, for any structure $\mathfrak{A}$ for $\mathfrak{L}(R, \doteq)$ and for any individuals $a_1, \ldots, a_n \in A$ we have:

$$\mathfrak{A} \models \varphi[\![a_1, \ldots, a_n]\!] \iff \mathfrak{A}^{=2} \models \varphi^{=2}[\![a_1, \ldots, a_n]\!] \iff \mathfrak{A}^{=1} \models \varphi^{=1}[\![a_1, \ldots, a_n]\!].$$

Proof. Induction on the construction of the formula $\varphi(x_1, \ldots, x_n)$. ■

Corollary 2.6.0.1.1:

If the structure $\mathfrak{A}$ for $\mathfrak{L}(R, \doteq)$ has a decidable theory, then so do the structures $\mathfrak{A}^{=1}$ and $\mathfrak{A}^{=2}$.

Remark 2.6.0.2:

Let $\mathcal{K}^{=i} \leftrightharpoons \{\mathfrak{A}^{=i} \mid \mathfrak{A} \in \mathcal{K}\}$ for $i = 1, 2$.

Corollary 2.6.0.1.2:
If the class of structures $\mathcal{K}$ for $\mathfrak{L}(R, \dot{=})$ has a decidable theory, then so do the classes $\mathcal{K}^{=1}$ and $\mathcal{K}^{=2}$.

Proof. Let $\mathcal{K}$ have a decidable theory. Therefore, for any sentence φ in the language $\mathfrak{L}(R, \dot{=})$:

$$\begin{gathered}\varphi \notin Th(\mathcal{K}) \iff \\ (\exists \mathfrak{A} \in \mathcal{K})[\mathfrak{A} \not\models \varphi] \iff \\ (\exists \mathfrak{A} \in \mathcal{K})[\mathfrak{A} \models \neg\varphi] \overset{lemma\ 2.6.0.1}{\iff} \\ (\exists \mathfrak{A} \in \mathcal{K})[\mathfrak{A}^{=1} \models \neg\varphi^{=1}] \iff \\ (\exists \mathfrak{A} \in \mathcal{K})[\mathfrak{A}^{=1} \not\models \varphi^{=1}] \iff \\ (\exists \mathfrak{A} \in \mathcal{K}^{=1})[\mathfrak{A} \not\models \varphi^{=1}] \iff \\ \varphi^{=1} \notin Th(\mathcal{K}^{=1}).\end{gathered}$$

Therefore, $Th(\mathcal{K}^{=1})$ is decidable. The same goes for $Th(\mathcal{K}^{=2})$. ■

Lemma 2.6.0.2:
For any formula $\varphi(x_1, \dots, x_n)$ from the language $\mathfrak{L}(R_1, R_2, \dot{=})$, for any structure $\mathfrak{A}$ for $\mathfrak{L}(R, \dot{=})$ and for any individuals $a_1, \dots, a_n \in A$ we have:

$$\mathfrak{A}^{=i} \models \varphi[\![a_1, \dots, a_n]\!] \iff \mathfrak{A} \models tr_i(\varphi)[\![a_1, \dots, a_n]\!]$$

for $i = 1, 2$.

Proof. Induction on the construction of the formula $\varphi(x_1, \dots, x_n)$. ■

We remind that $\mathcal{K}_{equiv}$ is the class of all structures for $\mathfrak{L}(R, \dot{=})$ such that the predicate symbol is interpreted as an equivalence relation on the universe of the structure.

Why was all this introduced and why is it useful? The reason is that it gives us a deconstruction of the models of $\mathcal{K}_{rectangle}$.

If we have $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{K}_{equiv}$, then $\mathfrak{A}_1^{=2} \times \mathfrak{A}_2^{=1}$ will be such a structure that:

- the universe is $A_1 \times A_2$;
- the interpretation of $R_1^{\mathfrak{A}_1^{=2} \times \mathfrak{A}_2^{=1}}$ is such that for any $\langle a, b\rangle, \langle c, d\rangle \in A_1 \times A_2$:

$$\begin{gathered}\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in R_1^{\mathfrak{A}_1^{=2} \times \mathfrak{A}_2^{=1}} \iff \langle a, c\rangle \in R_1^{\mathfrak{A}_1^{=2}} \,\&\, \langle b, d\rangle \in R_1^{\mathfrak{A}_2^{=1}} \iff \\ \langle a, c\rangle \in R_1^{\mathfrak{A}_1^{=2}} \,\&\, b = d.\end{gathered}$$

- the interpretation of $R_2^{\mathfrak{A}_1^{=2} \times \mathfrak{A}_2^{=1}}$ is such that for any $\langle a, b\rangle, \langle c, d\rangle \in A_1 \times A_2$:

$$\begin{gathered}\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in R_2^{\mathfrak{A}_1^{=2} \times \mathfrak{A}_2^{=1}} \iff \langle a, c\rangle \in R_2^{\mathfrak{A}_1^{=2}} \,\&\, \langle b, d\rangle \in R_2^{\mathfrak{A}_2^{=1}} \iff \\ a = c \,\&\, \langle b, d\rangle \in R_2^{\mathfrak{A}_2^{=1}}.\end{gathered}$$

Proposition 2.6.0.3:

(1) $\mathfrak{A}_1 \underset{mod}{\times} \mathfrak{A}_2 = \mathfrak{A}_1^{=2} \times \mathfrak{A}_2^{=1}$, i.e., the direct product and the modal product coincides for these specific structures.

(2) $\mathcal{K}_{rectangle} \overset{def.}{=} \{\mathfrak{A}_1 \underset{mod}{\times} \mathfrak{A}_2 \mid \mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{K}_{equiv}\} = \mathcal{K}_{equiv} \underset{mod}{\times} \mathcal{K}_{equiv} \overset{2.6.0.3.(1)}{=}$ $\{\mathfrak{A}_1^{=2} \times \mathfrak{A}_2^{=1} \mid \mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{K}_{equiv}\} = \mathcal{K}_{equiv}^{=2} \times \mathcal{K}_{equiv}^{=1}$.

2.6.1 Decidability of $Th(\mathcal{K}_{rectangle})$

Now we will talk about the decidability of the theory of $\mathcal{K}_{rectangle}$. By using old results on decidability of generalized products and powers from the 50-ties started by Mostowski and continued by Feferman and Vaught, we will prove a corner case corollary which will yield one means with which we will show the decidability of $Th(\mathcal{K}_{rectangle})$. The original papers are (Mostowski, 1952) and (Feferman and Vaught, 1959).

Before we prove the decidability of the theories, we will make some preparations.

Let $\mathfrak{L}$ be a finite RFOL language with or without formal equality $\doteq$. In order for the proof of the proposition and theorem to go smoothly, we will think that the first-order predicate formulae for $\mathfrak{L}$ have some additional properties.

- First, we wish φ to not contain the connectives and quantifier $\{\leftrightarrow, \rightarrow, \forall\}$ (usage of equivalent transformations);
- Second, when we write $\varphi(x_1, \dots, x_n)$ we will mean that $Var^{free}[\varphi] \cup Var^{bound}[\varphi] \subseteq \{x_1, \dots, x_n\}$;
- Third, we wish that for φ, $Var^{free}[\varphi] \cap Var^{bound}[\varphi] = \emptyset$ (usage of the 1.2.1.14);
- Forth, we wish that if we have a formula $\exists x\varphi$, then the variable x does not occur as a bounded variable in φ (usage of the 1.2.1.14);

Until the end of the proof of proposition 2.6.1.3 we will think of the formulae of $\mathfrak{L}$ to have these properties.

We will need to evaluate $\varphi(x_1, \dots, x_n)$ in $\mathfrak{A} \times \mathfrak{B}$ for $\mathfrak{A}$ and $\mathfrak{B}$ some structures over $\mathfrak{L}$. Then the $x_1, \dots, x_n$ are ordered tuples with their first coordinate from A and second coordinate from B. Let us take fresh distinct variables $y_1, \dots, y_n, z_1, \dots, z_n$ and then associate with each variable x_i the variables y_i, z_i.

We will now define a very specific finite set of ordered pairs of formulae for each formula $\varphi(x_1, \dots, x_n)$ of the language $\mathfrak{L}$ which will be evaluated in a product of two structures.

Definition 2.6.1.1:

Let $\varphi(x_1, \dots, x_n)$ be a formula from $\mathfrak{L}$.

Then we define $\langle\!\langle \varphi \rangle\!\rangle \leftrightharpoons \{\langle \psi_i^1(y_1, \dots, y_n), \psi_i^2(z_1, \dots, z_n)\rangle \mid i \in I\}$, where I is a non-empty finite set of indices, using induction on the construction of the formula.

- If $\varphi(x_1, \dots, x_n) \eqcirc (x_i \doteq x_j)$ for some $1 \le i, j \le n$, then:

$$\langle\!\langle \varphi \rangle\!\rangle \leftrightharpoons \{\langle (y_i \doteq y_j), (z_i \doteq z_j)\rangle\};$$

- If $\varphi(x_1, \dots, x_n) \eqcirc p(x_{i_1}, \dots, x_{i_k})$ for some indices $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ and k-ary predicate symbol p, then:

$$\langle\!\langle \varphi \rangle\!\rangle \leftrightharpoons \{\langle p(y_{i_1}, \dots, y_{i_k}), p(z_{i_1}, \dots, z_{i_k})\rangle\};$$

- If $\varphi(x_1, \dots, x_n) \eqcirc (\varphi_1 \vee \varphi_2)$ and we have $\langle\!\langle \varphi_1 \rangle\!\rangle, \langle\!\langle \varphi_2 \rangle\!\rangle$ by the induction hypothesis, then:

$$\langle\!\langle \varphi \rangle\!\rangle \leftrightharpoons \langle\!\langle \varphi_1 \rangle\!\rangle \cup \langle\!\langle \varphi_2 \rangle\!\rangle;$$

- If $\varphi(x_1, \dots, x_n) \eqcirc (\varphi_1 \wedge \varphi_2)$ and we have $\langle\!\langle \varphi_1 \rangle\!\rangle, \langle\!\langle \varphi_2 \rangle\!\rangle$ by the induction hypothesis, then:

$$\langle\!\langle \varphi \rangle\!\rangle \leftrightharpoons \{\langle (\psi_1^1 \wedge \psi_2^1), (\psi_1^2 \wedge \psi_2^2)\rangle \mid \langle \psi_1^1, \psi_1^2\rangle \in \langle\!\langle \varphi_1 \rangle\!\rangle \;\&\; \langle \psi_2^1, \psi_2^2\rangle \in \langle\!\langle \varphi_2 \rangle\!\rangle\};$$

- If $\varphi(x_1, \dots, x_n) \coloneqq \neg\psi$ and we have $\langle\!\langle \psi \rangle\!\rangle = \{\langle \chi_i^1, \chi_i^2 \rangle \mid i \in I\}$ by the induction hypothesis, then:

$$\langle\!\langle \varphi \rangle\!\rangle \leftrightharpoons \{\langle \bigwedge_{i \in J} \neg\chi_i^1, \bigwedge_{j \in I \setminus J} \neg\chi_j^2 \rangle \mid J \in \mathcal{P}(I)\}$$

(the size of the new set stays finite, but jumps exponentially);

- If $\varphi(x_1, \dots, x_n) \coloneqq \exists x_i \psi$ and we have $\langle\!\langle \psi \rangle\!\rangle = \{\langle \chi_i^1(y_1, \dots, y_n), \chi_i^2(z_1, \dots, z_n) \rangle \mid i \in I\}$ by the induction hypothesis, then:

$$\langle\!\langle \varphi \rangle\!\rangle \leftrightharpoons \{\langle \exists y_i \chi^1 \exists z_i \chi^2 \rangle \mid \langle \chi^1, \chi^2 \rangle \in \langle\!\langle \psi \rangle\!\rangle\}.$$

Then we can prove this proposition for this effective mapping with induction on the construction of a formula from the language $\mathfrak{L}$:

Proposition 2.6.1.2:

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be structures for $\mathfrak{L}$.

For all formulae $\varphi(x_1, \dots, x_n)$ for $\mathfrak{L}$ and any n individuals $\langle a_1, b_1 \rangle, \dots, \langle a_n, b_n \rangle$ from $A_1 \times A_2$ we have that:

$$\mathfrak{A}_1 \times \mathfrak{A}_2 \models \varphi[\![\langle a_1, b_1 \rangle, \dots, \langle a_n, b_n \rangle]\!] \iff$$
$$(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle \varphi \rangle\!\rangle)[\mathfrak{A}_1 \models \psi^1[\![a_1, \dots, a_n]\!] \;\&\; \mathfrak{A}_2 \models \psi^2[\![b_1, \dots, b_n]\!]].$$

Proof. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be structures for $\mathfrak{L}$.

We will prove the proposition using induction on the construction of φ in $\mathfrak{L}$.

- If $\varphi \coloneqq (x_i \doteq x_j)$ for some $1 \le i, j \le n$. Let $\langle a_1, b_1 \rangle, \dots, \langle a_n, b_n \rangle$ be individuals from $A_1 \times A_2$. Then:

$$\mathfrak{A}_1 \times \mathfrak{A}_2 \models (x_i \doteq x_j)[\![\langle a_1, b_1 \rangle, \dots, \langle a_n, b_n \rangle]\!] \iff$$
$$\langle a_i, b_i \rangle = \langle a_j, b_j \rangle \iff$$
$$a_i = a_j \;\&\; b_i = b_j \iff$$
$$\mathfrak{A}_1 \models (y_i \doteq y_j)[\![a_1, \dots, a_n]\!] \;\&\; \mathfrak{A}_2 \models (z_i \doteq z_j)[\![b_1, \dots, b_n]\!] \iff$$
$$(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle (x_i \doteq x_j) \rangle\!\rangle)[\mathfrak{A}_1 \models \psi^1[\![a_1, \dots, a_n]\!] \;\&\; \mathfrak{A}_2 \models \psi^2[\![b_1, \dots, b_n]\!]].$$

- If $\varphi \coloneqq p(x_{i_1}, \dots, x_{i_k})$ for some indices $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ and k-ary predicate symbol p. Let $\langle a_1, b_1 \rangle, \dots, \langle a_n, b_n \rangle$ be individuals from $A_1 \times A_2$. Then:

$$\mathfrak{A}_1 \times \mathfrak{A}_2 \models p(x_{i_1}, \dots, x_{i_k})[\![\langle a_1, b_1 \rangle, \dots, \langle a_n, b_n \rangle]\!] \iff$$
$$\langle \langle a_1, b_1 \rangle, \dots, \langle a_n, b_n \rangle \rangle \in p^{\mathfrak{A}_1 \times \mathfrak{A}_2} \iff$$
$$\langle a_1, \dots, a_n \rangle \in p^{\mathfrak{A}_1} \;\&\; \langle b_1, \dots, b_n \rangle \in p^{\mathfrak{A}_2} \iff$$
$$\mathfrak{A}_1 \models p(y_{i_1}, \dots, y_{i_k})[\![a_1, \dots, a_n]\!] \;\&\; \mathfrak{A}_2 \models p(z_{i_1}, \dots, z_{i_k})[\![b_1, \dots, b_n]\!] \iff$$
$$(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle p(x_{i_1}, \dots, x_{i_k}) \rangle\!\rangle)[\mathfrak{A}_1 \models \psi^1[\![a_1, \dots, a_n]\!] \;\&\; \mathfrak{A}_2 \models \psi^2[\![b_1, \dots, b_n]\!]].$$

- If $\varphi \doteqdot (\varphi_1 \vee \varphi_2)$ and we have induction hypothesis for φ_1 and φ_2. Let $\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle$ be individuals from $A_1 \times A_2$. Then:

$$\begin{gathered}
\mathfrak{A}_1 \times \mathfrak{A}_2 \models (\varphi_1 \vee \varphi_2)[\![\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle]\!] \iff \\
\mathfrak{A}_1 \times \mathfrak{A}_2 \models \varphi_1[\![\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle]\!] \mathbin{\mathbb{V}} \mathfrak{A}_1 \times \mathfrak{A}_2 \models \varphi_2[\![\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle]\!] \overset{(i.h)}{\iff} \\
(\exists\langle \psi_1^1, \psi_1^2\rangle \in \langle\!\langle \varphi_1\rangle\!\rangle)[\mathfrak{A}_1 \models \psi_1^1[\![a_1, \ldots, a_n]\!] \mathbin{\&} \mathfrak{A}_2 \models \psi_1^2[\![b_1, \ldots, b_n]\!]] \mathbin{\mathbb{V}} \\
(\exists\langle \psi_2^1, \psi_2^2\rangle \in \langle\!\langle \varphi_2\rangle\!\rangle)[\mathfrak{A}_1 \models \psi_2^1[\![a_1, \ldots, a_n]\!] \mathbin{\&} \mathfrak{A}_2 \models \psi_2^2[\![b_1, \ldots, b_n]\!]] \iff \\
(\exists\langle \psi^1, \psi^2\rangle \in \langle\!\langle \varphi_1\rangle\!\rangle \cup \langle\!\langle \varphi_2\rangle\!\rangle)[\mathfrak{A}_1 \models \psi^1[\![a_1, \ldots, a_n]\!] \mathbin{\&} \mathfrak{A}_2 \models \psi^2[\![b_1, \ldots, b_n]\!]] \iff \\
(\exists\langle \psi^1, \psi^2\rangle \in \langle\!\langle (\varphi_1 \vee \varphi_2)\rangle\!\rangle)[\mathfrak{A}_1 \models \psi^1[\![a_1, \ldots, a_n]\!] \mathbin{\&} \mathfrak{A}_2 \models \psi^2[\![b_1, \ldots, b_n]\!]].
\end{gathered}$$

- If $\varphi \doteqdot (\varphi_1 \wedge \varphi_2)$ and we have induction hypothesis for φ_1 and φ_2. Let $\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle$ be individuals from $A_1 \times A_2$. Then:

$$\begin{gathered}
\mathfrak{A}_1 \times \mathfrak{A}_2 \models (\varphi_1 \wedge \varphi_2)[\![\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle]\!] \iff \\
\mathfrak{A}_1 \times \mathfrak{A}_2 \models \varphi_1[\![\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle]\!] \mathbin{\&} \mathfrak{A}_1 \times \mathfrak{A}_2 \models \varphi_2[\![\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle]\!] \overset{(i.h)}{\iff} \\
(\exists\langle \psi_1^1, \psi_1^2\rangle \in \langle\!\langle \varphi_1\rangle\!\rangle)[\mathfrak{A}_1 \models \psi_1^1[\![a_1, \ldots, a_n]\!] \mathbin{\&} \mathfrak{A}_2 \models \psi_1^2[\![b_1, \ldots, b_n]\!]] \mathbin{\&} \\
(\exists\langle \psi_2^1, \psi_2^2\rangle \in \langle\!\langle \varphi_2\rangle\!\rangle)[\mathfrak{A}_1 \models \psi_2^1[\![a_1, \ldots, a_n]\!] \mathbin{\&} \mathfrak{A}_2 \models \psi_2^2[\![b_1, \ldots, b_n]\!]] \iff \\
(\exists\langle \psi_1^1, \psi_1^2\rangle \in \langle\!\langle \varphi_1\rangle\!\rangle)(\exists\langle \psi_2^1, \psi_2^2\rangle \in \langle\!\langle \varphi_2\rangle\!\rangle) \\
[\mathfrak{A}_1 \models (\psi_1^1 \wedge \psi_2^1)[\![a_1, \ldots, a_n]\!] \mathbin{\&} \mathfrak{A}_2 \models (\psi_1^2 \wedge \psi_2^2)[\![b_1, \ldots, b_n]\!]] \iff \\
(\exists\langle (\psi_1^1 \wedge \psi_1^2), (\psi_2^1 \wedge \psi_2^2)\rangle \in \langle\!\langle (\varphi_1 \wedge \varphi_2)\rangle\!\rangle) \\
[\mathfrak{A}_1 \models (\psi_1^1 \wedge \psi_2^1)[\![a_1, \ldots, a_n]\!] \mathbin{\&} \mathfrak{A}_2 \models (\psi_1^2 \wedge \psi_2^2)[\![b_1, \ldots, b_n]\!]] \iff \\
(\exists\langle \psi^1, \psi^2\rangle \in \langle\!\langle (\varphi_1 \wedge \varphi_2)\rangle\!\rangle)[\mathfrak{A}_1 \models \psi^1[\![a_1, \ldots, a_n]\!] \mathbin{\&} \mathfrak{A}_2 \models \psi^2[\![b_1, \ldots, b_n]\!]].
\end{gathered}$$

- If $\varphi \doteqdot \neg\varphi_1$ and we have induction hypothesis for φ_1. Let $\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle$ be individuals from $A_1 \times A_2$. Assume that $\langle\!\langle \varphi_1\rangle\!\rangle = \{\langle \psi_i^1, \psi_i^2\rangle \mid i \in I\}$. Then:

$$\begin{gathered}
\mathfrak{A}_1 \times \mathfrak{A}_2 \models \neg\varphi_1[\![\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle]\!] \iff \\
\mathfrak{A}_1 \times \mathfrak{A}_2 \not\models \varphi_1[\![\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle]\!] \overset{(i.h)}{\iff} \\
(\forall\langle \psi^1, \psi^2\rangle \in \langle\!\langle \varphi_1\rangle\!\rangle)[\mathfrak{A}_1 \not\models \psi^1[\![a_1, \ldots, a_n]\!] \mathbin{\mathbb{V}} \mathfrak{A}_2 \not\models \psi^2[\![b_1, \ldots, b_n]\!]] \overset{def. \langle\!\langle \varphi_1\rangle\!\rangle}{\iff} \\
(\forall i \in I)[\mathfrak{A}_1 \not\models \psi_i^1[\![a_1, \ldots, a_n]\!] \mathbin{\mathbb{V}} \mathfrak{A}_2 \not\models \psi_i^2[\![b_1, \ldots, b_n]\!]] \iff \\
(\forall i \in I)[\mathfrak{A}_1 \models \neg\psi_i^1[\![a_1, \ldots, a_n]\!] \mathbin{\mathbb{V}} \mathfrak{A}_2 \models \neg\psi_i^2[\![b_1, \ldots, b_n]\!]]. \qquad \text{(i)}
\end{gathered}$$

Let the last equivalent reformulation be denoted as (i).

First ($\Rightarrow$).

Let $\mathfrak{A}_1 \times \mathfrak{A}_2 \models \neg\varphi_1[\![\langle a_1, b_1\rangle, \ldots, \langle a_n, b_n\rangle]\!]$. Then we have (i). Let:

$$J_1 \leftrightharpoons \{i \mid i \in I \mathbin{\&} \mathfrak{A}_1 \models \neg\psi_i^1[\![a_1, \ldots, a_n]\!]\}$$

and

$$J_2 \leftrightharpoons \{i \mid i \in I \mathbin{\&} \mathfrak{A}_2 \models \neg\psi_i^2[\![b_1, \ldots, b_n]\!]\}.$$

By (i) it holds $J_1 \cup J_2 = I$. Therefore, we have:

$$\mathfrak{A}_1 \models \bigwedge_{i \in J_1} \neg\psi_i^1 [\![a_1, \dots, a_n]\!] \;\&\; \mathfrak{A}_2 \models \bigwedge_{i \in J_2} \neg\psi_i^2 [\![b_1, \dots, b_n]\!].$$

Thus, from $I \setminus J_1 \subseteq J_2$ follows:

$$\mathfrak{A}_1 \models \bigwedge_{i \in J_1} \neg\psi_i^1 [\![a_1, \dots, a_n]\!] \;\&\; \mathfrak{A}_2 \models \bigwedge_{i \in I \setminus J_1} \neg\psi_i^2 [\![b_1, \dots, b_n]\!].$$

But $\langle \bigwedge_{i \in J_1} \neg\psi_i^1, \bigwedge_{i \in I \setminus J_1} \neg\psi_i^2 \rangle \in \langle\!\langle \neg\varphi_1 \rangle\!\rangle$; therefore, we get:

$$(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi_1 \rangle\!\rangle)[\mathfrak{A}_1 \models \psi^1 [\![a_1, \dots, a_n]\!] \;\&\; \mathfrak{A}_2 \models \psi^2 [\![b_1, \dots, b_n]\!]].$$

Remark 2.6.1.1:
$\bigwedge_{i \in \emptyset} \neg\chi_i = \forall x (x \doteq x)$, i.e., it is the trivial truth, for whatever formulae χ_i.

Now ($\Leftarrow$).
Let $(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi_1 \rangle\!\rangle)[\mathfrak{A}_1 \models \psi^1 [\![a_1, \dots, a_n]\!] \;\&\; \mathfrak{A}_2 \models \psi^2 [\![b_1, \dots, b_n]\!]]$.
Let $\langle \psi^1, \psi^2 \rangle$ be witnesses. Then by the definition of $\langle\!\langle \neg\varphi_1 \rangle\!\rangle$ there exists a $J \subseteq I$, such that $\psi^1 \equiv \bigwedge_{i \in J} \neg\psi_i^1$ and $\psi^2 \equiv \bigwedge_{i \in I \setminus J} \neg\psi_i^2$.
Then:

$$[\mathfrak{A}_1 \models \bigwedge_{i \in J} \neg\psi_i^1 [\![a_1, \dots, a_n]\!]] \;\&\; [\mathfrak{A}_2 \models \bigwedge_{i \in I \setminus J} \neg\psi_i^2 [\![b_1, \dots, b_n]\!]] \iff$$
$$(\forall i \in J)[\mathfrak{A}_1 \models \neg\psi_i^1 [\![a_1, \dots, a_n]\!]] \;\&\; (\forall i \in I \setminus J)[\mathfrak{A}_2 \models \neg\psi_i^2 [\![b_1, \dots, b_n]\!]] \Rightarrow$$
$$(\forall i \in I)[\mathfrak{A}_1 \models \neg\psi_i^1 [\![a_1, \dots, a_n]\!] \vee \mathfrak{A}_2 \models \neg\psi_i^2 [\![b_1, \dots, b_n]\!]].$$

But the last expression is (i) which is equivalent with:

$$\mathfrak{A}_1 \times \mathfrak{A}_2 \models \neg\varphi_1 [\![\langle a_1, b_1 \rangle, \dots, \langle a_n, b_n \rangle]\!].$$

- If $\varphi \rightleftharpoons \exists x_1 \varphi_1$ and we have induction hypothesis for φ_1 (WLOG let x_i be x_1). Let $\langle a_1, b_1\rangle, \dots, \langle a_n, b_n\rangle$ be individuals from $A_1 \times A_2$. Then:

$$
\begin{gathered}
\mathfrak{A}_1 \times \mathfrak{A}_2 \models \exists x_1 \varphi_1[\![\langle a_1, b_1\rangle, \langle a_2, b_2\rangle, \dots, \langle a_n, b_n\rangle]\!] \iff \\
(\exists u \in A_1 \times A_2)[\mathfrak{A}_1 \times \mathfrak{A}_2 \models \varphi_1[\![u, \langle a_2, b_2\rangle, \dots, \langle a_n, b_n\rangle]\!]] \iff \\
(\exists a'_1 \in A_1)(\exists b'_1 \in A_2)(\exists u \in A_1 \times A_2) \\
[u = \langle a'_1, b'_1\rangle \mathbin{\&} \mathfrak{A}_1 \times \mathfrak{A}_2 \models \varphi_1[\![u, \langle a_2, b_2\rangle, \dots, \langle a_n, b_n\rangle]\!]] \iff \\
(\exists a'_1 \in A_1)(\exists b'_1 \in A_2)(\exists \langle \psi^1, \psi^2\rangle \in \langle\!\langle \varphi_1 \rangle\!\rangle) \\
[\mathfrak{A}_1 \models \psi^1[\![a'_1, a_2, \dots, a_n]\!] \mathbin{\&} \mathfrak{A}_2 \models \psi^2[\![b'_1, b_2, \dots, b_n]\!]] \iff \\
(\exists \langle \psi^1, \psi^2\rangle \in \langle\!\langle \varphi_1 \rangle\!\rangle)(\exists a'_1 \in A_1)(\exists b'_1 \in A_2) \\
[\mathfrak{A}_1 \models \psi^1[\![a'_1, a_2, \dots, a_n]\!] \mathbin{\&} \mathfrak{A}_2 \models \psi^2[\![b'_1, b_2, \dots, b_n]\!]] \iff \\
(\exists \langle \psi^1, \psi^2\rangle \in \langle\!\langle \varphi_1 \rangle\!\rangle) \\
(\exists a'_1 \in A_1)[\mathfrak{A}_1 \models \psi^1[\![a'_1, a_2, \dots, a_n]\!]] \mathbin{\&} (\exists b'_1 \in A_2)[\mathfrak{A}_2 \models \psi^2[\![b'_1, b_2, \dots, b_n]\!]] \iff \\
(\exists \langle \psi^1, \psi^2\rangle \in \langle\!\langle \varphi_1 \rangle\!\rangle) \\
[\mathfrak{A}_1 \models \exists y_1 \psi^1[\![a_1, a_2, \dots, a_n]\!] \mathbin{\&} \mathfrak{A}_2 \models \exists z_1 \psi^2[\![b_1, b_2, \dots, b_n]\!]] \iff \\
(\exists \langle \psi^1, \psi^2\rangle \in \langle\!\langle \exists x_1 \varphi_1 \rangle\!\rangle)[\mathfrak{A}_1 \models \psi^1[\![a_1, a_2, \dots, a_n]\!] \mathbin{\&} \mathfrak{A}_2 \models \psi^2[\![b_1, b_2, \dots, b_n]\!]].
\end{gathered}
$$

∎

This corollary has been formulated and given a sketchy proof in the book of Ershov, but here we will prove it in full:

Proposition 2.6.1.3:

Let $\mathfrak{L}$ be a finite RFOL language

(1) If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are structures for the language $\mathfrak{L}$ such that $Th(\mathfrak{A}_1)$ and $Th(\mathfrak{A}_2)$ are decidable, then $Th(\mathfrak{A}_1 \times \mathfrak{A}_2)$ is also decidable.

(2) If $\mathcal{K}_1$ and $\mathcal{K}_2$ are classes of structures for language $\mathfrak{L}$ such that $Th(\mathcal{K}_1)$ and $Th(\mathcal{K}_2)$ are decidable, then $Th(\mathcal{K}_1 \times \mathcal{K}_2)$ is also decidable.

(3) Let $\mathcal{K}$ be a class of structures for language $\mathfrak{L}$ such that $Th(\mathcal{K})$ is decidable and let $\mathcal{K}' \leftrightharpoons \{\mathfrak{A} \times \mathfrak{A} \mid \mathfrak{A} \in \mathcal{K}\}$. Then $Th(\mathcal{K}')$ is also decidable.

Proof. Proof of (1):

Let $\mathfrak{A}_1, \mathfrak{A}_2$ be structures for $\mathfrak{L}$ such that $Th(\mathfrak{A}_1)$ and $Th(\mathfrak{A}_2)$ are decidable.

Let $\varphi \in \mathcal{S}ent(\mathfrak{L})$. Then we construct $\langle\!\langle \varphi \rangle\!\rangle$ following the definition. We have two cases for the validity of φ in $\mathfrak{A}_1 \times \mathfrak{A}_2$.

(i) Let $\mathfrak{A}_1 \times \mathfrak{A}_2 \models \varphi$. Then by proposition 2.6.1.2 we have that $(\exists \langle \psi^1, \psi^2\rangle \in \langle\!\langle \varphi \rangle\!\rangle)[\mathfrak{A}_1 \models \psi^1 \mathbin{\&} \mathfrak{A}_2 \models \psi^2]$.

(ii) Let $\mathfrak{A}_1 \times \mathfrak{A}_2 \not\models \varphi$. Then by proposition 2.6.1.2 we have that $(\forall \langle \psi^1, \psi^2\rangle \in \langle\!\langle \varphi \rangle\!\rangle)[\mathfrak{A}_1 \not\models \psi^1 \vee \mathfrak{A}_2 \not\models \psi^2]$.

Our decision procedure will be the following:

One by one we analyze the tuples $\langle \psi^1, \psi^2\rangle \in \langle\!\langle \varphi \rangle\!\rangle$. For each tuple $\langle \psi^1, \psi^2\rangle \in \langle\!\langle \varphi \rangle\!\rangle$, since $Th(\mathfrak{A}_1)$ and $Th(\mathfrak{A}_2)$ are decidable, we can recognize if $\mathfrak{A}_1 \models \psi^1$ and if $\mathfrak{A}_2 \models \psi^2$.

Let for there exists a tuple $\langle \psi^1, \psi^2\rangle \in \langle\!\langle \varphi \rangle\!\rangle$ such that $\mathfrak{A}_1 \models \psi^1$ and $\mathfrak{A}_2 \models \psi^2$, then we stop the procedure and we state that $\mathfrak{A}_1 \times \mathfrak{A}_2 \models \varphi$.

Let for all tuples $\langle \psi^1, \psi^2\rangle \in \langle\!\langle \varphi \rangle\!\rangle$ is true that $\mathfrak{A}_1 \not\models \psi^1$ or $\mathfrak{A}_2 \not\models \psi^2$, then we stop the procedure and we state that $\mathfrak{A}_1 \times \mathfrak{A}_2 \not\models \varphi$.

Since $\langle\!\langle\varphi\rangle\!\rangle$ is constructed from φ effectively and $\langle\!\langle\varphi\rangle\!\rangle$ is a finite non-empty set of pairs of sentences, then we have a decision procedure for the validity problem for $\mathfrak{A}_1 \times \mathfrak{A}_2$.

Proof of (2): Let $\mathcal{K}_1$ and $\mathcal{K}_2$ are classes of structures for $\mathfrak{L}$ such that $Th(\mathcal{K}_1)$ and $Th(\mathcal{K}_2)$ are decidable. Note that $\mathcal{K}_1 \times \mathcal{K}_2 = \{\mathfrak{A}_1 \times \mathfrak{A}_2 \mid \mathfrak{A}_1 \in \mathcal{K}_1 \;\&\; \mathfrak{A}_2 \in \mathcal{K}_2\}$.

For all $\varphi \in Sent(\mathfrak{L})$ we have that

$$\varphi \notin Th(\mathcal{K}_i) \iff (\exists \mathfrak{A}_i \in \mathcal{K}_i)[\mathfrak{A}_i \not\models \varphi],$$

for $i = 1, 2$. Since $Th(\mathcal{K}_i)$ is decidable, then given a sentence $\varphi \in Sent(\mathfrak{L})$ the problem $\varphi \notin Th(\mathcal{K}_i)$ is also decidable for $i = 1, 2$. Let (*) denote this fact.

Let $\varphi \in Sent(\mathfrak{L})$.

$$\begin{gathered}
\varphi \notin Th(\mathcal{K}_1 \times \mathcal{K}_2) \iff \\
(\exists \mathfrak{A}_1 \in \mathcal{K}_1)(\exists \mathfrak{A}_2 \in \mathcal{K}_2)[\mathfrak{A}_1 \times \mathfrak{A}_2 \not\models \varphi] \iff \\
(\exists \mathfrak{A}_1 \in \mathcal{K}_1)(\exists \mathfrak{A}_2 \in \mathcal{K}_2)[\mathfrak{A}_1 \times \mathfrak{A}_2 \models \neg\varphi] \overset{prop.\ 2.6.1.2}{\iff} \\
(\exists \mathfrak{A}_1 \in \mathcal{K}_1)(\exists \mathfrak{A}_2 \in \mathcal{K}_2)(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle)[\mathfrak{A}_1 \models \psi^1 \;\&\; \mathfrak{A}_2 \models \psi^2] \iff \\
(\exists \mathfrak{A}_1 \in \mathcal{K}_1)(\exists \mathfrak{A}_2 \in \mathcal{K}_2)(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle)[\mathfrak{A}_1 \not\models \neg\psi^1 \;\&\; \mathfrak{A}_2 \not\models \neg\psi^2] \iff \\
(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle)(\exists \mathfrak{A}_1 \in \mathcal{K}_1)(\exists \mathfrak{A}_2 \in \mathcal{K}_2)[\mathfrak{A}_1 \not\models \neg\psi^1 \;\&\; \mathfrak{A}_2 \not\models \neg\psi^2] \iff \\
(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle)[(\exists \mathfrak{A}_1 \in \mathcal{K}_1)[\mathfrak{A}_1 \not\models \neg\psi^1] \;\&\; (\exists \mathfrak{A}_2 \in \mathcal{K}_2)[\mathfrak{A}_2 \not\models \neg\psi^2]].
\end{gathered}$$

Now we will demonstrate a prodecure which given a sentence $\varphi \in Sent(\mathfrak{L})$ can determine if $\varphi \notin Th(\mathcal{K}_1 \times \mathcal{K}_2)$.

We construct $\langle\!\langle \neg\varphi \rangle\!\rangle$ following the definition. One by one we analyze the tuples $\langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle$.

Let for there exists a tuple $\langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle$ such that $(\exists \mathfrak{A}_1 \in \mathcal{K}_1)[\mathfrak{A}_1 \not\models \neg\psi^1]$ and $(\exists \mathfrak{A}_2 \in \mathcal{K}_2)[\mathfrak{A}_2 \not\models \neg\psi^2]$. By (*) we have that the latter are decidable problems. By 2.6.1.3.(1) $\mathfrak{A}_1 \times \mathfrak{A}_2 \models \neg\varphi$. Then we stop the procedure and we state that $\mathfrak{A}_1 \times \mathfrak{A}_2 \not\models \varphi$.

Let for all tuples $\langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle$ is true that $\neg(\exists \mathfrak{A}_1 \in \mathcal{K}_1)[\mathfrak{A}_1 \not\models \neg\psi^1]$ or $\neg(\exists \mathfrak{A}_2 \in \mathcal{K}_2)[\mathfrak{A}_2 \not\models \neg\psi^2]$. By (*) we have that the latter are decidable problems. By 2.6.1.3.(1) $\mathfrak{A}_1 \times \mathfrak{A}_2 \not\models \neg\varphi$. Then we stop the procedure and we state that $\mathfrak{A}_1 \times \mathfrak{A}_2 \models \varphi$.

Since $\langle\!\langle \neg\varphi \rangle\!\rangle$ is constructed from $\neg\varphi$ effectively and $\langle\!\langle \neg\varphi \rangle\!\rangle$ is a finite non-empty set of pairs of sentences, then we have a decision procedure for the validity problem for $\mathcal{K}_1 \times \mathcal{K}_2$.

Proof of (3): Let $\mathcal{K}$ $\mathfrak{L}$ such that $Th(\mathcal{K})$ is decidable.

For all $\varphi \in Sent(\mathfrak{L})$ we have that

$$\varphi \notin Th(\mathcal{K}) \iff (\exists \mathfrak{A}_i \in \mathcal{K})[\mathfrak{A} \not\models \varphi].$$

Since $Th(\mathcal{K})$ is decidable, then given a sentence $\varphi \in Sent(\mathfrak{L})$ the problem $\varphi \notin Th(\mathcal{K})$ is also decidable. Let (**) denote this fact.

Let $\varphi \in Sent(\mathfrak{L})$.

$$\begin{gathered}
\varphi \notin Th(\mathcal{K}') \iff \\
(\exists \mathfrak{A} \in \mathcal{K})([\mathfrak{A} \times \mathfrak{A} \not\models \varphi] \iff \\
(\exists \mathfrak{A} \in \mathcal{K})([\mathfrak{A} \times \mathfrak{A} \models \neg\varphi] \overset{prop.\ 2.6.1.2}{\iff} \\
(\exists \mathfrak{A} \in \mathcal{K})(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle)[\mathfrak{A} \models \psi^1 \;\&\; \mathfrak{A} \models \psi^2] \iff \\
(\exists \mathfrak{A} \in \mathcal{K})(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle)[\mathfrak{A} \models (\psi^1 \wedge \psi^2)] \iff \\
(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle)(\exists \mathfrak{A} \in \mathcal{K})[\mathfrak{A} \not\models \neg(\psi^1 \wedge \psi^2)].
\end{gathered}$$

Now we will demonstrate a prodecure which given a sentence $\varphi \in Sent(\mathfrak{L})$ can determine if $\varphi \notin Th(\mathcal{K}')$.

We construct $\langle\!\langle \neg\varphi \rangle\!\rangle$ following the definition. One by one we analyze the tuples $\langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle$.

Let for there exists a tuple $\langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle$ such that $(\exists \mathfrak{A} \in \mathcal{K})[\mathfrak{A} \not\models \neg(\psi^1 \wedge \psi^2)]$. By (**) we have that the latter is a decidable problem. By 2.6.1.3.(1) $\mathfrak{A} \times \mathfrak{A} \models \neg\varphi$. Then we stop the procedure and we state that $\mathfrak{A} \times \mathfrak{A} \not\models \varphi$.

Let for all tuples $\langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle$ is true that $\neg(\exists \mathfrak{A} \in \mathcal{K})[\mathfrak{A} \not\models \neg(\psi^1 \wedge \psi^2)]$. By (**) we have that the latter are decidable problems. By 2.6.1.3.(1) $\mathfrak{A} \times \mathfrak{A} \not\models \neg\varphi$. Then we stop the procedure and we state that $\mathfrak{A} \times \mathfrak{A} \models \varphi$.

Since $\langle\!\langle \neg\varphi \rangle\!\rangle$ is constructed from $\neg\varphi$ effectively and $\langle\!\langle \neg\varphi \rangle\!\rangle$ is a finite non-empty set of pairs of sentences, then we have a decision procedure for the validity problem for $\mathcal{K}'$.

■

Theorem 2.6.1.4:
The theory of $\mathcal{K}_{rectangle}$ is decidable.

Proof. We know $\mathcal{K}_{equiv}$ has a decidable theory discussed in subsection 1.8. Therefore, by corollary 2.6.0.1.2, so do $\mathcal{K}^{=2}_{equiv}$ and $\mathcal{K}^{=1}_{equiv}$. As a result from applying proposition 2.6.1.3.(2) we have that $\mathcal{K}^{=2}_{equiv} \times \mathcal{K}^{=1}_{equiv}$ has a decidable theory which by proposition 2.6.0.3.(2) means that $\mathcal{K}_{rectangle}$ has a decidable theory. ■

Remark 2.6.1.2:
By proposition 1.5.2.3 $\mathcal{K}^{uni}_{rectangle}$ has a decidable theory since it is a finite extension of $\mathcal{K}_{rectangle}$ with the addional non-logical axiom $\forall x \forall y \varphi_{R_1 \circ R_2}(x, y)$.

2.6.2 Decidability of $Th(\mathcal{K}_{square})$

Remark 2.6.2.1:

$$\mathcal{K}_{square} \overset{def.}{=} \{\mathfrak{A} \underset{mod}{\times} \mathfrak{A} \mid \mathfrak{A} \in \mathcal{K}_{equiv}\} \overset{2.6.0.3.(1)}{=} \{\mathfrak{A}^{=2} \times \mathfrak{A}^{=1} \mid \mathfrak{A} \in \mathcal{K}_{equiv}\}.$$

Proposition 2.6.2.1:
The theory of $\mathcal{K}_{square}$ is decidable.

Proof. Let φ be a sentence from $\mathfrak{L}(R_1, R_2, \doteq)$. Then:

$$\varphi \notin Th(\mathcal{K}_{square}) \iff$$
$$(\exists \mathfrak{A} \in \mathcal{K}_{equiv})[\mathfrak{A}^{=2} \times \mathfrak{A}^{=1} \not\models \varphi] \iff$$
$$(\exists \mathfrak{A} \in \mathcal{K}_{equiv})[\mathfrak{A}^{=2} \times \mathfrak{A}^{=1} \models \neg\varphi] \overset{prop.\ 2.6.1.2}{\iff}$$
$$(\exists \mathfrak{A} \in \mathcal{K}_{equiv})(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle)[\mathfrak{A}^{=2} \models \neg\psi^1 \;\&\; \mathfrak{A}^{=1} \models \neg\psi^2] \overset{lemma\ 2.6.0.1}{\iff}$$
$$(\exists \mathfrak{A} \in \mathcal{K}_{equiv})(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle)[\mathfrak{A} \models \neg tr_2(\psi^1) \;\&\; \mathfrak{A} \models \neg tr_1(\psi^2)] \iff$$
$$(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle)(\exists \mathfrak{A} \in \mathcal{K}_{equiv})[\mathfrak{A} \models (\neg tr_2(\psi^1) \wedge \neg tr_1(\psi^2))] \iff$$
$$(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle)(\exists \mathfrak{A} \in \mathcal{K}_{equiv})[\mathfrak{A} \models \neg(tr_2(\psi^1) \vee tr_1(\psi^2))] \iff$$
$$(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle)(\exists \mathfrak{A} \in \mathcal{K}_{equiv})[\mathfrak{A} \not\models (tr_2(\psi^1) \vee tr_1(\psi^2))] \iff$$
$$(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle)[(tr_2(\psi^1) \vee tr_1(\psi^2)) \notin Th(\mathcal{K}_{equiv})].$$

Where $tr_2(\psi^1)$ and $tr_1(\psi^2)$ are translations of formulae from the language of $\mathfrak{L}(R_1, R_2, \dot{=})$ to formulae of the language of $\mathfrak{L}(R, \dot{=})$. We have that $\langle\langle \neg\varphi \rangle\rangle$ is a non-empty finite set of pairs of sentences and that $\chi \in Th(\mathcal{K}_{equiv})$ is a decidable problem, rending $\chi \notin Th(\mathcal{K}_{equiv})$ also a decidable problem for $\chi \in \mathcal{S}ent(\mathfrak{L}(R_1, R_2, \dot{=}))$. Then we can conclude that $\mathcal{K}_{square}$ also has a decidable theory.

■

Remark 2.6.2.2:

By proposition 1.5.2.3 $\mathcal{K}^{uni}_{square}$ has a decidable theory since it is a finite extension of $\mathcal{K}_{square}$ with the addional non-logical axiom $\forall x \forall y \varphi_{R_1 \circ R_2}(x, y)$.

2.7 $\mathcal{K}_{rectangle}$ and $\mathcal{K}_{square}$ have FMP

2.7.1 $\mathcal{K}_{rectangle}$ has FMP

We will show that $\mathcal{K}_{rectangle}$ has **FMP** and also show decidability of the theory of $\mathcal{K}^{fin}_{rectangle}$.

Lemma 2.7.1.1:

Let $\mathfrak{L}$ be a RFOL language.

(1) If $\mathcal{K}_1$ and $\mathcal{K}_2$ be classes of the structures for $\mathfrak{L}$ have **FMP**, then $\mathcal{K}_1 \times \mathcal{K}_2$ has **FMP**.
(2) Let $\mathcal{K}$ be a class of structures for language $\mathfrak{L}$ such that it has **FMP** and let $\mathcal{K}' \leftleftharpoons \{\mathfrak{A} \times \mathfrak{A} \mid \mathfrak{A} \in \mathcal{K}\}$. Then $\mathcal{K}'$ is also has **FMP**.

Proof. Proof of (1): Let $\varphi \in \mathcal{S}ent(\mathfrak{L})$ and let $\mathfrak{A}$ be an arbitrary structure from $\mathcal{K}_1 \times \mathcal{K}_2$ such that $\mathfrak{A} \not\models \varphi$.

Since $\mathfrak{A} \in \mathcal{K}_1 \times \mathcal{K}_2$, then there exist structures $\mathfrak{A}_1 \in \mathcal{K}_1$ and $\mathfrak{A}_2 \in \mathcal{K}_2$ such that $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2$. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be witnesses.

Then:

$$\mathfrak{A} \not\models \varphi \iff$$
$$\mathfrak{A}_1 \times \mathfrak{A}_2 \not\models \varphi \iff$$
$$\mathfrak{A}_1 \times \mathfrak{A}_2 \models \neg\varphi \overset{prop. 2.6.1.2}{\iff}$$
$$(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle)[\mathfrak{A}_1 \models \psi^1 \,\&\, \mathfrak{A}_2 \models \psi^2] \iff$$
$$(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle)[\mathfrak{A}_1 \not\models \neg\psi^1 \,\&\, \mathfrak{A}_2 \not\models \neg\psi^2] \overset{def.\ 1.5.2.1}{\Rightarrow}$$
$$(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle)(\exists \mathfrak{B}_1 \in \mathcal{K}_1^{fin})(\exists \mathfrak{B}_2 \in \mathcal{K}_2^{fin})[\mathfrak{B}_1 \not\models \neg\psi^1 \,\&\, \mathfrak{B}_2 \not\models \neg\psi^2] \iff$$
$$(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle)(\exists \mathfrak{B}_1 \in \mathcal{K}_1^{fin})(\exists \mathfrak{B}_2 \in \mathcal{K}_2^{fin})[\mathfrak{B}_1 \models \psi^1 \,\&\, \mathfrak{B}_2 \models \psi^2] \iff$$
$$(\exists \mathfrak{B}_1 \in \mathcal{K}_1^{fin})(\exists \mathfrak{B}_2 \in \mathcal{K}_2^{fin})(\exists \langle \psi^1, \psi^2 \rangle \in \langle\!\langle \neg\varphi \rangle\!\rangle)[\mathfrak{B}_1 \models \psi^1 \,\&\, \mathfrak{B}_2 \models \psi^2] \overset{prop. 2.6.1.2}{\iff}$$
$$(\exists \mathfrak{B}_1 \in \mathcal{K}_1^{fin})(\exists \mathfrak{B}_2 \in \mathcal{K}_2^{fin})[\mathfrak{B}_1 \times \mathfrak{B}_2 \models \neg\varphi] \iff$$
$$(\exists \mathfrak{B} \in \mathcal{K}_1^{fin} \times \mathcal{K}_2^{fin})[\mathfrak{B} \models \neg\varphi] \iff$$
$$(\exists \mathfrak{B} \in \mathcal{K}_1^{fin} \times \mathcal{K}_2^{fin})[\mathfrak{B} \not\models \varphi].$$

Thus, $\mathcal{K}_1 \times \mathcal{K}_2$ has **FMP**.

Proof of (2): Let $\varphi \in \mathcal{S}ent(\mathfrak{L})$ and let $\mathfrak{C}$ be an arbitrary structure from $\mathcal{K}'$ such that $\mathfrak{C} \not\models \varphi$.

Since $\mathfrak{C} \in \mathcal{K}'$, then there exists a structure $\mathfrak{A} \in \mathcal{K}$ such that $\mathfrak{C} = \mathfrak{A} \times \mathfrak{A}$. Let $\mathfrak{A}$ be a witness.

Then:

$$\begin{gathered}
\mathfrak{C} \not\models \varphi \iff \\
\mathfrak{A} \times \mathfrak{A} \not\models \varphi \iff \\
\mathfrak{A} \times \mathfrak{A} \models \neg\varphi \overset{prop.2.6.1.2}{\iff} \\
(\exists\langle\psi^1, \psi^2\rangle \in \langle\!\langle\neg\varphi\rangle\!\rangle)[\mathfrak{A} \models \psi^1 \;\&\; \mathfrak{A} \models \psi^2] \iff \\
(\exists\langle\psi^1, \psi^2\rangle \in \langle\!\langle\neg\varphi\rangle\!\rangle)[\mathfrak{A} \models (\psi^1 \wedge \psi^2)] \iff \\
(\exists\langle\psi^1, \psi^2\rangle \in \langle\!\langle\neg\varphi\rangle\!\rangle)[\mathfrak{A} \not\models \neg(\psi^1 \wedge \psi^2)] \overset{def.\ 1.5.2.1}{\Rightarrow} \\
(\exists\langle\psi^1, \psi^2\rangle \in \langle\!\langle\neg\varphi\rangle\!\rangle)(\exists\mathfrak{B} \in \mathcal{K}^{fin})[\mathfrak{B} \not\models \neg(\psi^1 \wedge \psi^2)] \iff \\
(\exists\langle\psi^1, \psi^2\rangle \in \langle\!\langle\neg\varphi\rangle\!\rangle)(\exists\mathfrak{B} \in \mathcal{K}^{fin})[\mathfrak{B} \models (\psi^1 \wedge \psi^2)] \iff \\
(\exists\langle\psi^1, \psi^2\rangle \in \langle\!\langle\neg\varphi\rangle\!\rangle)(\exists\mathfrak{B} \in \mathcal{K}^{fin})[\mathfrak{B} \models \psi^1 \;\&\; \mathfrak{B} \models \psi^2)] \iff \\
(\exists\mathfrak{B} \in \mathcal{K}^{fin})(\exists\langle\psi^1, \psi^2\rangle \in \langle\!\langle\neg\varphi\rangle\!\rangle)[\mathfrak{B} \models \psi^1 \;\&\; \mathfrak{B} \models \psi^2)] \overset{prop.2.6.1.2}{\iff} \\
(\exists\mathfrak{B} \in \mathcal{K}^{fin})[\mathfrak{B} \times \mathfrak{B} \models \neg\varphi] \iff \\
(\exists\mathfrak{B}' \in (\mathcal{K}')^{fin})[\mathfrak{B}' \models \neg\varphi] \iff \\
(\exists\mathfrak{B}' \in (\mathcal{K}')^{fin})[\mathfrak{B}' \not\models \varphi].
\end{gathered}$$

Thus, $\mathcal{K}'$ has **FMP**. ■

Proposition 2.7.1.2:

(1) If we have two classes of finite structures $\mathcal{K}_1$ and $\mathcal{K}_2$ for the same RFOL language $\mathfrak{L}$, then $\mathcal{K}_1 \times \mathcal{K}_2$ is also a class of finite structures.

(2) If we have a class of finite structures $\mathcal{K}$ for $\mathfrak{L}(R, \doteq)$, then $\mathcal{K}^{=2}$ and $\mathcal{K}^{=1}$ are also classes of finite structures.

Proof. Proof of (1):

Each of the universes of structures of $\mathcal{K}_1 \times \mathcal{K}_2$ is a Cartesian product of a finite universe of a structure of $\mathcal{K}_1$ and a finite universe of a structure of $\mathcal{K}_2$ which is again a finite universe.

Proof of (2): To each structure of the class $\mathcal{K}$ we only add an interpretation of a new relation symbol of the language and nothing is added to the universe of the structure making the resulting structure again finite. ■

Remark 2.7.1.1:

For a class of structures $\mathcal{K}$ for the language $\mathfrak{L}(R, \doteq)$, $(\mathcal{K}^{=i})^{fin} = (\mathcal{K}^{fin})^{=i}$ for $i = 1, 2$.

Remark 2.7.1.2:

Let $\mathcal{K}^{fin}_{equiv}$ be the class of finite partitions.

$\mathcal{K}^{fin}_{rectangle} = (\mathcal{K}^{fin}_{equiv})^{=2} \times (\mathcal{K}^{fin}_{equiv})^{=1} \overset{2.7.1.1}{=} (\mathcal{K}^{=2}_{equiv})^{fin} \times (\mathcal{K}^{=1}_{equiv})^{fin}$ is also a class of finite structures.

Proposition 2.7.1.3:

Let $\mathcal{K}$ be a class of structures for $\mathfrak{L}(R, \doteq)$.

If $\mathcal{K}$ has **FMP** then $\mathcal{K}^{=i}$ has **FMP** for $i = 1, 2$.

Proof. Let $\mathcal{K}$ has **FMP**. We will show that $\mathcal{K}^{=1}$ (the proof for $\mathcal{K}^{=2}$ is analogous).

Let $\varphi \in \mathcal{S}ent(\mathfrak{L}(R_1, R_2, \dot{=}))$.

$$\varphi \notin Th(\mathcal{K}^{=1}) \iff$$
$$(\exists \mathfrak{A}^{=1} \in \mathcal{K}^{=1})[\mathfrak{A}^{=1} \not\models \varphi] \iff$$
$$(\exists \mathfrak{A}^{=1} \in \mathcal{K}^{=1})[\mathfrak{A}^{=1} \models \neg\varphi] \overset{lemma\ 2.6.0.1}{\iff}$$
$$(\exists \mathfrak{A} \in \mathcal{K})[\mathfrak{A} \models \neg tr_1(\varphi)] \iff$$
$$(\exists \mathfrak{A} \in \mathcal{K})[\mathfrak{A} \not\models tr_1(\varphi)] \overset{def.\ 1.5.2.1}{\Rightarrow}$$
$$(\exists \mathfrak{B} \in \mathcal{K}^{fin})[\mathfrak{B} \not\models tr_1(\varphi)] \iff$$
$$(\exists \mathfrak{B} \in \mathcal{K}^{fin})[\mathfrak{B} \models \neg tr_1(\varphi)] \overset{lemma\ 2.6.0.1}{\iff}$$
$$(\exists \mathfrak{B}^{=1} \in (\mathcal{K}^{fin})^{=1})[\mathfrak{B}^{=1} \models \neg\varphi] \iff$$
$$(\exists \mathfrak{B}^{=1} \in (\mathcal{K}^{fin})^{=1})[\mathfrak{B}^{=1} \not\models \varphi] \overset{rem.\ 2.7.1.1}{\iff}$$
$$(\exists \mathfrak{B}^{=1} \in (\mathcal{K}^{=1})^{fin})[\mathfrak{B}^{=1} \not\models \varphi].$$

Thus, $\mathcal{K}^{=1}$ has **FMP**. ■

Theorem 2.7.1.4:
$\mathcal{K}_{rectangle}$ has **FMP**.

Proof. From $\mathcal{K}_{equiv}$ has **FMP** we have from 2.7.1.3 and 2.7.1.1.(1) that $\mathcal{K}^{=2}_{equiv} \times \mathcal{K}^{=1}_{equiv}$ has **FMP**, i.e., $\mathcal{K}_{rectangle}$ has **FMP** by remark 2.7.1.2. ■

Corollary 2.7.1.4.1:
$Th(\mathcal{K}^{fin}_{rectangle})$ is decidable.

Proof. $\mathcal{K}_{rectangle}$ has **FMP** by theorem 2.7.1.4; therefore, $Th(\mathcal{K}^{fin}_{rectangle}) = Th(\mathcal{K}_{rectangle})$.
$\mathcal{K}_{rectangle}$ is decidable by theorem 2.6.1.4, so $Th(\mathcal{K}^{fin}_{rectangle})$ is also decidable. ■

2.7.2 $\mathcal{K}_{square}$ has FMP

Remark 2.7.2.1:
Let $\mathcal{K}^{fin}_{equiv}$ be the class of finite partitions.
$\mathcal{K}^{fin}_{square} = \{\mathfrak{A}^{=2} \times \mathfrak{A}^{=1} \mid \mathfrak{A} \in \mathcal{K}^{fin}_{equiv}\}$ is also a class of finite structures.

Theorem 2.7.2.1:
$\mathcal{K}_{square}$ has **FMP**.

Proof. Let $\varphi \in Sent(\mathfrak{L}(R_1, R_2, \doteq))$.

$$\begin{gathered}
\varphi \notin Th(\mathcal{K}_{square}) \iff \\
(\exists \mathfrak{A} \in \mathcal{K}_{equiv})[\mathfrak{A}^{=2} \times \mathfrak{A}^{=1} \not\models \varphi] \iff \\
(\exists \mathfrak{A} \in \mathcal{K}_{equiv})[\mathfrak{A}^{=2} \times \mathfrak{A}^{=1} \models \neg\varphi] \overset{prop.2.6.1.2}{\iff} \\
(\exists \mathfrak{A} \in \mathcal{K}_{equiv})(\exists \langle \psi^1, \psi^2 \rangle \in \langle\langle \neg\varphi \rangle\rangle)[\mathfrak{A}^{=2} \models \psi^1 \,\&\, \mathfrak{A}^{=1} \models \psi^2] \overset{lemma\ 2.6.0.1}{\iff} \\
(\exists \mathfrak{A} \in \mathcal{K}_{equiv})(\exists \langle \psi^1, \psi^2 \rangle \in \langle\langle \neg\varphi \rangle\rangle)[\mathfrak{A} \models tr_2(\psi^1) \,\&\, \mathfrak{A} \models tr_1(\psi^2)] \iff \\
(\exists \mathfrak{A} \in \mathcal{K}_{equiv})(\exists \langle \psi^1, \psi^2 \rangle \in \langle\langle \neg\varphi \rangle\rangle)[\mathfrak{A} \models (tr_2(\psi^1) \wedge tr_1(\psi^2))] \iff \\
(\exists \mathfrak{A} \in \mathcal{K}_{equiv})(\exists \langle \psi^1, \psi^2 \rangle \in \langle\langle \neg\varphi \rangle\rangle)[\mathfrak{A} \not\models \neg(tr_2(\psi^1) \wedge tr_1(\psi^2))] \iff \\
(\exists \langle \psi^1, \psi^2 \rangle \in \langle\langle \neg\varphi \rangle\rangle)(\exists \mathfrak{A} \in \mathcal{K}_{equiv})[\mathfrak{A} \not\models \neg(tr_2(\psi^1) \wedge tr_1(\psi^2))] \overset{def.\ 1.5.2.1}{\Rightarrow} \\
(\exists \langle \psi^1, \psi^2 \rangle \in \langle\langle \neg\varphi \rangle\rangle)(\exists \mathfrak{B} \in \mathcal{K}^{fin}_{equiv})[\mathfrak{B} \not\models \neg(tr_2(\psi^1) \wedge tr_1(\psi^2))] \iff \\
(\exists \langle \psi^1, \psi^2 \rangle \in \langle\langle \neg\varphi \rangle\rangle)(\exists \mathfrak{B} \in \mathcal{K}^{fin}_{equiv})[\mathfrak{B} \models (tr_2(\psi^1) \wedge tr_1(\psi^2))] \iff \\
(\exists \langle \psi^1, \psi^2 \rangle \in \langle\langle \neg\varphi \rangle\rangle)(\exists \mathfrak{B} \in \mathcal{K}^{fin}_{equiv})[\mathfrak{B} \models tr_2(\psi^1) \,\&\, \mathfrak{B} \models tr_1(\psi^2)] \overset{lemma\ 2.6.0.1}{\iff} \\
(\exists \mathfrak{B} \in \mathcal{K}^{fin}_{equiv})(\exists \langle \psi^1, \psi^2 \rangle \in \langle\langle \neg\varphi \rangle\rangle)[\mathfrak{B} \models tr_2(\psi^1) \,\&\, \mathfrak{B} \models tr_1(\psi^2)] \overset{lemma\ 2.6.0.1}{\iff} \\
(\exists \mathfrak{B} \in \mathcal{K}^{fin}_{equiv})(\exists \langle \psi^1, \psi^2 \rangle \in \langle\langle \neg\varphi \rangle\rangle)[\mathfrak{B}^{=2} \models \psi^1 \,\&\, \mathfrak{B}^{=1} \models \psi^2] \overset{prop.2.6.1.2}{\iff} \\
(\exists \mathfrak{B} \in \mathcal{K}^{fin}_{equiv})[\mathfrak{B}^{=2} \times \mathfrak{B}^{=1} \models \neg\varphi] \iff \\
(\exists \mathfrak{B} \in \mathcal{K}^{fin}_{equiv})[\mathfrak{B}^{=2} \times \mathfrak{B}^{=1} \not\models \varphi].
\end{gathered}$$

Therefore, $\mathcal{K}_{square}$ has **FMP**. ■

Corollary 2.7.2.1.1:
$Th(\mathcal{K}^{fin}_{square})$ is decidable.

Proof. $\mathcal{K}_{square}$ has **FMP** by theorem 2.7.2.1; therefore, $Th(\mathcal{K}^{fin}_{square}) = Th(\mathcal{K}^{fin}_{square})$. $\mathcal{K}_{square}$ is decidable by proposition 2.6.2.1, so $Th(\mathcal{K}^{fin}_{square})$ is also decidable. ■

2.7.3 Another way to see that $\mathcal{K}_{rectangle}$ has FMP

We will not be satisfied with having only one method demonstrateing that $\mathcal{K}_{rectangle}$ has **FMP**. We will show another method using Ehrenfeucht–Fraïssé games and the fact that $\mathcal{K}_{equiv}$ has **FMP** and the methods used by Tinchev and Balbiani for obtaining it by reducing the cardinality of structures in $\mathcal{K}_{equiv}$ to finite ones as in (Balbiani and Tinchev, 2006).

This method gives us more information about the exact upper bound of the complexity of the membership problem to $Th(\mathcal{K}_{rectangle})$ than the previous. In the previous the amount of pairs in a set $\langle\langle\varphi\rangle\rangle$ for a sentence φ jumps exponentially on each encounter of the propositional connective $\neg$ as well as the lengths of the formulae jump drastically (syntactically a lot of formulae are generated on each step which are logically equivalent; for each quantifier $\forall$ we get formulae of exponential length); thus, rending the previous solution to have an exponential space complexity.

We will use a property about the Ehrenfeucht–Fraïssé game strategies formulated for the direct product $\times$ of the "playing boards" for which have lemma 1.5.1.6.

Proposition 2.7.3.1:
Let $\mathfrak{A}$ be a structure for the language $\mathfrak{L}(R, \doteq)$ and let $k \in \omega$. Then the *Duplicator* has a winning strategy for $G_k(\mathfrak{A}, \mathfrak{A}^{=2})$ and $G_k(\mathfrak{A}, \mathfrak{A}^{=1})$.

Proof. In the structures $\mathfrak{A}^{=2}$ and $\mathfrak{A}^{=1}$ we have only added a new relation symbol interpreted with the formal equality of the structure $\mathfrak{A}$, so it is trivial for the *Duplicator* to win the k-round games by just copying the moves of the *Spoiler*. ∎

Let us denote by $red_2(k)$ the composition of the two refinements used in (Balbiani and Tinchev, 2006) to reduce structures of the class $\mathcal{K}_{equiv}$ to finite ones. The first refinement cuts down on the size of the blocks of the equivalence relation to blocks having no more than k elements and the second refinement cuts down on the number of blocks with a specific cardinality, that is, for $1 \leq i \leq k$ there are no more than k blocks of cardinality i.

Proposition 2.7.3.2:
$\mathcal{K}^{=2}_{equiv}$ and $\mathcal{K}^{=1}_{equiv}$ have **FMP**.

Proof. First of all we remind that $\mathcal{K}_{equiv}$ has **FMP**.

We will show that $\mathcal{K}^{=2}_{equiv}$ has **FMP**. The reasoning for $\mathcal{K}^{=1}_{equiv}$ is analogous. Let φ be a sentence of $\mathfrak{L}(R_1, R_2, \doteq)$ and let $qr(\varphi)$.

Let (i) be the following corollary of proposition 2.7.3.1 and the Fraïssé–Hintikka theorem:

$$(\forall \mathfrak{A} \in \mathcal{K}_{equiv})[\mathfrak{A} \equiv^{*}_{k} \mathfrak{A}^{=2}],$$

whereby the * we denote that the formulae must be translated between the structures in the manner described in the beginning of the section. Note that $qr(tr_i(\varphi)) = qr(\varphi) = k$ for $i = 1, 2$.

Let (ii) denote the fact that:

$$(\forall \mathfrak{A} \in \mathcal{K}_{equiv})[(\mathfrak{A}^{red_2(k)})^{=2} \cong (\mathfrak{A}^{=2})^{red_2(k)}].$$

Let (iii) denote the facts:

$$(\forall \mathfrak{A} \in \mathcal{K}_{equiv})[\mathfrak{A}^{red_2(k)} \equiv_k \mathfrak{A}] \textit{ and } (\forall \mathfrak{A} \in \mathcal{K}^{=i}_{equiv})[\mathfrak{A}^{red_2(k)} \equiv_k \mathfrak{A}],$$

for $i = 1, 2$.

Then:

$$\begin{gathered}
\varphi \notin Th(\mathcal{K}^{=2}_{equiv}) \iff \\
(\exists \mathfrak{A}^{=2} \in \mathcal{K}^{=2}_{equiv})[\mathfrak{A}^{=2} \not\models \varphi] \iff \\
(\exists \mathfrak{A}^{=2} \in \mathcal{K}^{=2}_{equiv})[\mathfrak{A}^{=2} \models \neg\varphi] \overset{(i)}{\iff} \\
(\exists \mathfrak{A} \in \mathcal{K}_{equiv})[\mathfrak{A} \models \neg tr_2(\varphi)] \iff \\
(\exists \mathfrak{A} \in \mathcal{K}_{equiv})[\mathfrak{A} \not\models tr_2(\varphi)] \overset{(iii)}{\iff} \\
(\exists \mathfrak{A} \in \mathcal{K}_{equiv})[\mathfrak{A}^{red_2(k)} \not\models tr_2(\varphi)] \iff \\
(\exists \mathfrak{A} \in \mathcal{K}_{equiv})[\mathfrak{A}^{red_2(k)} \models \neg tr_2(\varphi)] \overset{(i)}{\iff} \\
(\exists \mathfrak{A} \in \mathcal{K}_{equiv})[(\mathfrak{A}^{red_2(k)})^{=2} \models \neg\varphi] \iff \\
(\exists \mathfrak{A} \in \mathcal{K}_{equiv})[(\mathfrak{A}^{red_2(k)})^{=2} \not\models \varphi] \overset{\textit{rem. } 2.7.1.1 \textit{ and } (ii)}{\implies} \\
(\exists \mathfrak{C} \in (\mathcal{K}^{=2}_{equiv})^{fin})[\mathfrak{C} \not\models \varphi].
\end{gathered}$$

Therefore, $\mathcal{K}^{=2}_{equiv}$ has **FMP**. ■

Theorem 2.7.3.3:
$\mathcal{K}_{rectangle}$ has **FMP**.

Proof. Let φ be a sentence in the language $\mathfrak{L}(R_1, R_2, \dot{=})$ such that $qr(\varphi) = k$. By combing all the previous results plus some previous subsection, we get:

$$\varphi \notin Th(\mathcal{K}_{rectangle}) \iff$$
$$(\exists \mathfrak{A}_1 \in \mathcal{K}^{=2}_{equiv})(\exists \mathfrak{A}_2 \in \mathcal{K}^{=1}_{equiv})[\mathfrak{A}_1 \times \mathfrak{A}_2 \not\models \varphi] \overset{prop.\ 2.7.3.2\ and\ lemma\ 1.5.1.6\ and\ (iii)}{\Longrightarrow}$$
$$(\exists \mathfrak{A}_1 \in \mathcal{K}^{=2}_{equiv})(\exists \mathfrak{A}_2 \in \mathcal{K}^{=1}_{equiv})[\mathfrak{A}_1^{red_2(k)} \times \mathfrak{A}_2^{red_2(k)} \models \neg\varphi] \overset{rem.\ 2.7.1.1\ and\ (ii)}{\Longrightarrow}$$
$$(\exists \mathfrak{B}_1 \in (\mathcal{K}^{=2}_{equiv})^{fin})(\exists \mathfrak{B}_2 \in (\mathcal{K}^{=1}_{equiv})^{fin})[\mathfrak{B}_1 \times \mathfrak{B}_2 \not\models \varphi] \iff$$
$$(\exists \mathfrak{C} \in \mathcal{K}^{fin}_{rectangle})[\mathfrak{C} \not\models \varphi].$$

■

2.7.4 Another way to see that $\mathcal{K}_{square}$ has FMP

Proposition 2.7.4.1:
$\mathcal{K}_{square}$ has **FMP**.

Proof. Let φ be a sentence in the language $\mathfrak{L}(R_1, R_2, \dot{=})$ such that $qr(\varphi) = k$. By combing all the previous results plus some previous subsections, we get:

$$\varphi \notin Th(\mathcal{K}_{square}) \iff$$
$$(\exists \mathfrak{A} \in \mathcal{K}_{equiv})[\mathfrak{A}^{=2} \times \mathfrak{A}^{=1} \not\models \varphi] \overset{lemma\ 1.5.1.6\ and\ (iii)}{\Longrightarrow}$$
$$(\exists \mathfrak{A} \in \mathcal{K}_{equiv})[(\mathfrak{A}^{=2})^{red_2(k)} \times (\mathfrak{A}^{=1})^{red_2(k)} \not\models \varphi] \overset{(ii)}{\iff}$$
$$(\exists \mathfrak{A} \in \mathcal{K}_{equiv})[(\mathfrak{A}^{red_2(k)})^{=2} \times (\mathfrak{A}^{red_2(k)})^{=1} \not\models \varphi] \Rightarrow$$
$$(\exists \mathfrak{B} \in \mathcal{K}^{fin}_{equiv})[\mathfrak{B}^{=2} \times \mathfrak{B}^{=1} \not\models \varphi] \iff$$
$$(\exists \mathfrak{C} \in \mathcal{K}^{fin}_{square})[\mathfrak{C} \not\models \varphi].$$

Thus,$\mathcal{K}_{square}$ has **FMP**. ■

Chapter 3

Modal definability problem in $\mathcal{K}_{commute}$

In section 1.7 we discussed a method developed by Balbiani and Tinchev for reducing the problem of deciding the validity of sentences over some class of structures to the problem of modal definability over the same class of structures.

We are interested in their results regarding the class of all bi-partitioned frames $\mathcal{K}_{2S5}$, i.e., coincidesthe two relations are interpreted as equivalence relations w.r.t. the universe.

The notion of Stable class of frames is too restricting by fixing the formula ψ so early on. If the conditions in the definition are relaxed it can be proven that the problem of deciding the validity of sentences in $\mathcal{K}_{2S5}$ is reducible to the problem of modal definability problem w.r.t. $\mathcal{K}_{2S5}$.

Theorem 3.0.0.1:

The problem of deciding the validity of sentences in $\mathcal{K}_{2S5}$ is reducible to **MD-def** w.r.t. $\mathcal{K}_{2S5}$.

Proof. See in (Balbiani and Tinchev, 2017), Theorem 10. ■

Corollary 3.0.0.1.1:

MD-def w.r.t. $\mathcal{K}_{2S5}$ is undecidable.

Proof. See in (Balbiani and Tinchev, 2017), Corollary 9. ■

Theorem 3.0.0.2:

The problem of deciding the validity of sentences in $\mathcal{K}_{2S5}^{fin}$ is reducible to **MD-def** w.r.t. $\mathcal{K}_{2S5}^{fin}$.

Proof. See in (Balbiani and Tinchev, 2017), Theorem 11. ■

We will consern ourselves with the class $\mathcal{K}_{commute} \subseteq \mathcal{K}_{2S5}$ and show that the problem of deciding the validity of sentences in $\mathcal{K}_{commute}$ is reducible to **MD-def** w.r.t. $\mathcal{K}_{commute}$ by following the proof of that of $\mathcal{K}_{2S5}$. We will prove it in full reproducing the proof of 3.0.0.1 in the following theorem:

Theorem 3.0.0.3:

The problem of deciding the validity of sentences in $\mathcal{K}_{commute}$ is reducible to **MD-def** w.r.t. $\mathcal{K}_{commute}$.

Proof. Let $\varphi \in \mathcal{F}orm(\mathfrak{L}(R_1, R_2, \dot{=}))$ be defined $\varphi(x, x_1) \leftrightharpoons \neg\exists z(R_1(x_1, z) \wedge R_2(z, x))$. Remark that x_1 and x will differ, because in the structures of $\mathcal{K}_{commute}$ the composition of the two relations is an equivalence relations.

Let $\chi \in Sent(\mathfrak{L}(R_1, R_2, \dot{=}))$ be such that $qr(\chi) = k$ for some $k \in \omega$ and define the sentence ψ to depend on $qr(\chi)$ in the following manner:

$$\psi \leftrightharpoons \exists y_1 \dots \exists y_{k+1}(\bigwedge_{1 \le i < j \le k+1} R_1(y_i, y_j) \wedge \bigwedge_{1 \le i < j \le k+1} \neg R_2(y_i, y_j) \wedge$$
$$\forall z \forall t(R_1(y_1, z) \wedge R_2(z, t) \to R_1(y_1, t)) \wedge$$
$$\forall z(R_1(y_1, z) \to \exists t_1 \dots \exists t_k(\bigwedge_{1 \le i < j \le k} \neg(t_i \dot{=} t_j) \wedge \bigwedge_{1 \le i \le k} R_2(z, t_i)))).$$

The sentence says that there exists an R_2-closed equivalence class modulo R_1 containing at least $k+1$ mutually disjoint equivalence classes modulo R_2 and every R_2-equivalence class has at least k elements. In our terminology it says that there exists a matrix w.r.t. $R_1 \circ R_2$ such that it has only one row with at least $k+1$ cells in it, each of which has at least k elements in it. It looks like this:

$\ge$ **k**	$\ge$ **k**	$\ge$ **k**	...	$\ge$ **k**

$\ge \mathbf{k+1}$

Let $\theta \leftrightharpoons \exists x_1(\exists x \varphi(x, x_1) \wedge \neg(\chi)_x^{\varphi(x,x_1)}) \wedge \psi$ be a sentence of $\mathfrak{L}(R_1, R_2, \dot{=})$. We will prove that $\mathcal{K}_{commute} \models \chi \iff \theta$ is modally definable w.r.t. $\mathcal{K}_{commute}$.

($\Rightarrow$): Let $\mathcal{K}_{commute} \models \chi$. We will show that $\bot$ is a modal definition of θ w.r.t. $\mathcal{K}_{commute}$. FTSOC suppose $(\exists \mathfrak{F} \in \mathcal{K}_{commute})[\mathfrak{F} \models \theta]$ and let $\mathfrak{F}_0$ be a witness.
Then $\mathfrak{F}_0 \models \exists x_1(\exists x \varphi(x, x_1) \wedge \neg(\chi)_x^{\varphi(x,x_1)})$ and let $a_1 \in \mathfrak{F}_0$ be a witness. Therefore, $\mathfrak{F}_0 \models \neg(\chi)_x^{\varphi(x,x_1)}[\![a_1]\!]$, i.e., $\mathfrak{F}_0 \not\models (\chi)_x^{\varphi(x,x_1)}[\![a_1]\!]$. Let $\mathfrak{F}'$ be the relativized reduct of $\mathfrak{F}_0$ w.r.t. $\varphi(x, x_1)$ and a_1 and it exists by remark 1.7.1.1. This means that $\mathfrak{F}'$ is the set of all matrices from $\mathfrak{F}_0$ that do not contain a_1; therefore, $\mathfrak{F}' \in \mathcal{K}_{commute}$. Since $\mathfrak{F}'$ is the relativized reduct of $\mathfrak{F}_0$ w.r.t. $\varphi(x, x_1)$ and a_1, by Relativization theorem we have:

$$\mathfrak{F}_0 \models (\chi)_x^{\varphi(x,x_1)}[\![a_1]\!] \iff \mathfrak{F}' \models \chi. \qquad \text{(i)}$$

Since $\mathfrak{F}_0 \not\models (\chi)_x^{\varphi(x,x_1)}[\![a_1]\!]$, by (i) $\mathfrak{F}' \not\models \chi$. But $\mathfrak{F}' \in \mathcal{K}_{commute}$; therefore, $\mathfrak{F}' \models \chi$. We obtained a contradiction.

($\Leftarrow$): Now let θ be modally definable w.r.t. $\mathcal{K}_{commute}$ and let $\mathtt{A}$ be a modal definition of θ w.r.t. $\mathcal{K}_{commute}$. FTSOC suppose $\mathcal{K}_{commute} \not\models \chi$. Let $\mathfrak{F}_0 = \langle W_0, R_{01}, R_{02} \rangle \in \mathcal{K}_{commute}$ such that $\mathfrak{F}_0 \not\models \chi$. Let $\mathfrak{F}_1 = \langle W_1, R_{11}, R_{12} \rangle \in \mathcal{K}_{commute}$ be the same structure as $\mathfrak{F}$ with the exception that every matrix which has only one row with at least $k+1$ cells in it, each of which has at least k elements in it, is replaced by a matrix such that it has only one row with exactly k cells in it, each of which has exactly k elements in it.

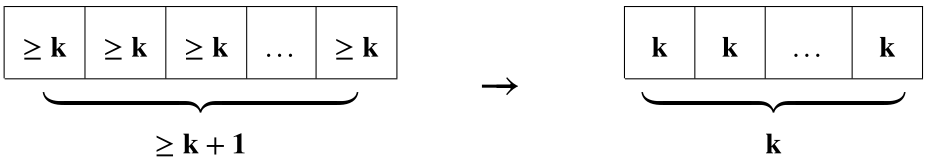


FIGURE 3.1: How the specific matrices in $\mathfrak{F}_0$ are replaced with “smaller” matrices in $\mathfrak{F}_1$

It is immediate that $\mathfrak{F}_1 \not\models \psi$. We can easily prove that the *Duplicator* has a winning strategy for $G_k(\mathfrak{F}_0, \mathfrak{F}_1)$; hence, by Fraïssé–Hintikka theorem we have that $\mathfrak{F}_0 \equiv_k \mathfrak{F}_1$. Therefore, since $\mathfrak{F}_0 \not\models \chi$ and $qr(\chi) = k$, then $\mathfrak{F}_1 \not\models \chi$.

Let W^i be $k+1$ sets such that:

- $\mathbf{card}(W^i) = k$, for $1 \leq i \leq k+1$;
- for $1 \leq i < j \leq k+1$ $W^i \cap W^j = \emptyset$;
- Let $W_u \leftrightharpoons \bigcup_{1 \leq i \leq k+1} W^i$. Then $W_u \cap W_1 = \emptyset$.

Let $a_1 \in W^1$ be a witness of the non-emptiness of W^1.

Let us define the frame $\mathfrak{F} = \langle W, R_1, R_2 \rangle$:

- $W \leftrightharpoons W_1 \cup W_u$;
- $R_1 \leftrightharpoons R_{11} \cup W_u \times W_u$;
- $R_2 \leftrightharpoons R_{12} \cup \bigcup_{1 \leq i \leq k+1} (W^i \times W^i)$.

I.e. we add to $\mathfrak{F}_1$ this matrix:

$$\underbrace{\boxed{W^1}\boxed{W^2}\boxed{W^3}\boxed{\dots}\boxed{W^{k+1}}}_{\mathbf{k+1}}$$

The added equivalence classes form a new equivalence class in the composition of the relations, because the union of the added classes form an equivalence relation and by 1.3.1.4 we have that they commute. Since $\mathfrak{F}_1 \in \mathcal{K}_{commute}$ and the newly added class commutes, then $\mathfrak{F} \in \mathcal{K}_{commute}$.

Let us define the frame $\mathfrak{F}' = \langle W', R'_1, R'_2 \rangle$:

- $W' \leftrightharpoons W_1 \cup \{a_1\}$;
- $R'_1 \leftrightharpoons R_{11} \cup \{\langle a_1, a_1 \rangle\}$;
- $R'_2 \leftrightharpoons R_{12} \cup \{\langle a_1, a_1 \rangle\}$.

Again the added equivalence classes form a new equivalence class in the composition of the relations, because the union of the added classes form an equivalence relation and by 1.3.1.4 we have that they commute. Since $\mathfrak{F}_1 \in \mathcal{K}_{commute}$ and the newly added class commutes, then $\mathfrak{F}' \in \mathcal{K}_{commute}$.

I.e. we add to $\mathfrak{F}_1$ this matrix:

$$\underbrace{\boxed{a_1}}_{\mathbf{1}}$$

Then $\mathfrak{F}_1$ is a relativized reduct of $\mathfrak{F}$ w.r.t. $\varphi(x, x_1)$ and a_1, $\mathfrak{F} \models \psi$, $\mathfrak{F}' \not\models \psi$ and $\mathfrak{F}'$ is a bounded morphic image of $\mathfrak{F}$, for example a bounded morphism is $f : W \twoheadrightarrow W'$:

$$f(x) \leftrightharpoons \begin{cases} a_1, & \textit{if } x \in W_u, \\ x, & \textit{if } x \in W_1. \end{cases}$$

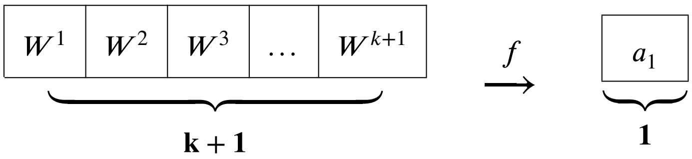


FIGURE 3.2: How a witness bounded morphism f "compresses" the matrix added to $\mathfrak{F}_1$ to form $\mathfrak{F}$

By Bounded morphism lemma we have that $\mathfrak{F} \preceq \mathfrak{F}'$.

Since $\mathfrak{F}' \not\models \psi$, then $\mathfrak{F}' \not\models \theta$. But A is a modal definition of θ w.r.t. $\mathcal{K}_{commute}$, $\mathfrak{F}, \mathfrak{F}' \in \mathcal{K}_{commute}$ and $\mathfrak{F} \preceq \mathfrak{F}'$; therefore, $\mathfrak{F} \not\models \theta$. Since, $\mathfrak{F} \models \psi$, then $\mathfrak{F} \not\models \exists x_1(\exists x \varphi(x, x_1) \wedge \neg(\chi)_x^{\varphi(x,x_1)})$.

$\mathfrak{F}_1$ is a relativized reduct of $\mathfrak{F}$ w.r.t. $\varphi(x, x_1)$ and a_1, so by Relativization theorem we have: i

$$\mathfrak{F} \models (\chi)_x^{\varphi(x,x_1)}[\![a_1]\!] \iff \mathfrak{F}_1 \models \chi. \tag{ii}$$

Moreover $\mathfrak{F} \models \exists x \varphi[\![a_1]\!]$.

But $\mathfrak{F} \not\models \exists x_1(\exists x \varphi(x, x_1) \wedge \neg(\chi)_x^{\varphi(x,x_1)})$; hence, $\mathfrak{F} \models (\chi)_x^{\varphi(x,x_1)}[\![a_1]\!]$. Now using (ii) we get that $\mathfrak{F}_1 \models \chi$, which is a contradiction.

From ($\Rightarrow$) and ($\Leftarrow$) we conclude that $\mathcal{K}_{commute} \models \chi \iff \theta$ is modally definable w.r.t. $\mathcal{K}_{commute}$.

■

Corollary 3.0.0.3.1:

MD-def w.r.t. $\mathcal{K}_{commute}$ is undecidable.

Proof. By theorem 2.5.0.3 the problem of deciding the validity of sentences in $\mathcal{K}_{commute}$ is undecidable; therefore, by theorem 3.0.0.3 we have our statement. ■

Theorem 3.0.0.4:

The problem of deciding the validity of sentences in $\mathcal{K}_{commute}^{fin}$ is reducible to **MD-def** w.r.t. $\mathcal{K}_{commute}^{fin}$.

Proof. Remark that if the frame $\mathfrak{F}_0$ is finite then the construction in the proof of theorem 3.0.0.3 shows that $\mathfrak{F}$ and $\mathfrak{F}'$ are also finite. Therefore, the problem of deciding validity of sentences in $\mathcal{K}_{commute}^{fin}$ is reducible to **MD-def** w.r.t. $\mathcal{K}_{commute}^{fin}$. ■

Corollary 3.0.0.4.1:

MD-def w.r.t. w.r.t. $\mathcal{K}_{commute}^{fin}$ is undecidable.

Chapter 4

Summary and further work

The main results of this work can be summarized in the following table:

Classes and status of validity in them		
Classes of structures	Arbitrary cardinality	Finite cardinality
$\mathcal{K}^{uni}_{commute}$	undecidable	undecidable
$\mathcal{K}_{commute}$	undecidable	undecidable
$\mathcal{K}^{uni}_{rectangle}$	decidable	decidable
$\mathcal{K}_{rectangle}$	decidable	decidable
$\mathcal{K}^{uni}_{square}$	decidable	decidable
$\mathcal{K}_{square}$	decidable	decidable

Also, **MD-def** w.r.t. $\mathcal{K}_{commute}$ is undecidable. In future works we will be interested in the status of **MD-def** and **FO-def** in all classes mentioned. We conjecture that for some of the above mentioned classes **MD-def** is a decidable problem w.r.t. the particular class in question.

An object of interest in our study of **MD-def** will be also some subclasses of structures of $\mathcal{K}_{commute}$ with various "constraints" like the following:

- Let for each $n \in \omega^+$ $\mathcal{K}^{R_1 \leq n}_{commute}$ be the class of all structures from $\mathcal{K}_{commute}$ such that for each matrix in the structure the rows have $\leq n$ number of cells.
- Let for each $n \in \omega^+$ $\mathcal{K}^{R_1 \leq n, R_2 < w}_{commute}$ be the class of all structures from $\mathcal{K}^{R_1 \leq n}_{commute}$ such that for each matrix in the structure the columns have a finite number of cells.

We conjecture that they have **FMP**. The classes $\mathcal{K}^{R_1 \leq n}_{commute}$ for $n \in \omega^+$ are finitely axiomatizable. All the classes of these types can be proven to have decidable theories. The even tighter classes $\mathcal{K}^{R_1 \leq n, R_2 \leq m}_{commute}$ for $n, m \in \omega^+$ are finitely axiomatizable and decidable.

We will also be interested in syntactically complete extensions of $Th(\mathcal{K}_{commute})$. For example let $\mathcal{K}^{1,\infty,\infty,\infty}_{commute}$ be a subclass of $\mathcal{K}_{commute}$ such that each structure is a collection of are infinitely many matrices and each matrix has infinitely many columns, infinitely many rows and all cells are of cardinality 1. We conjecture that the theory of $\mathcal{K}^{1,\infty,\infty,\infty}_{commute}$ is decidable and syntactically complete. Is it possible to describe all syntactically complete extensions of $\mathcal{K}_{commute}$? What about **MD-def** w.r.t. these classes.

Bibliography

Mostowski, A. (1952). “On Direct Products of Theories”. In: *The Journal of Symbolic Logic* 17.1, pp. 1–31.

Janiczak, A. (1953). “Undecidability of some simple formalized theories”. In: *Fundamenta Mathematicae* 40.2, pp. 131–139.

H. Rogers, Jr (1956). “Certain Logical Reduction and Decision Problems”. In: *Annals of Mathematics, Second Series* 64.2, pp. 264–284.

Feferman, S. and R. Vaught (1959). “The first order properties of products of algebraic systems”. In: *Fundamenta mathematical* 47.1, pp. 57–103.

Lavrov, I. A. (1963). “The effective non-separability of the set of identically true formulae and the set of finitely refutable formulae for certain elementary theories”. In: *Algebra i Logika. Sem.* 2.1, pp. 5–18.

Ershov, Yu. L. et al. (1965). “Elementary theories”. In: *Uspekhi Mat. Nauk* 20.4.

Shoenfield, J. R. (1967). *Mathematical logic)*. Addison-Wesley Series in Logic. Addison-Wesley Publishing Company.

Ershov, Yu. L. (1980). *Problems of decidability and constructive models*. Nauka, Moscow, pp. 265–275.

Ebbinghaus, H. and J. Flum (1995). *Finite Model Theory*. Springer-Verlag Berlin Heidelberg, pp. 13–26.

Finberg, D., M. Mainetti, and G.-C. Rota (1996). “The Logic of Commuting Equivalence Relations”. In: *Lecture Notes in Pure and Applied Mathematics* 180, pp. 69–97.

Boerger, E., E. Grädel, and Y. Gurevich (1997). *The classical decision problem*. Springer-Verlag Berlin Heidelberg.

Chagrov, A. and M. Zakharyaschev (1997). *Modal Logic*. Oxford Logic Guides. Clarendon Press.

Kurucz, Agi et al. (2003). *Many-dimensional modal logics: Theory and applications*. Elsevier.

Decidability and Complexity of Definability within the Class of All Partitions (2005), pp. 26–33.

Balbiani, Ph. and T. Tinchev (2006). “Definability over the class of all partitions”. In: *Journal of Logic and Computation* 16.5, pp. 541–557.

Georgiev, G. and T. Tinchev (2008). “Second-order logic on equivalence relations”. In: *Journal of Applied Non-Classical Logics* 18.2-3, pp. 229–246.

Hodges, W. (2008). *Model Theory*. Encyclopedia of Mathematics and its Applications. Cambridge University Press.

Balbiani, Ph. and T. Tinchev (2017). “Undecidable problems for modal definability”. In: *Journal of Logic and Computation* 27.3, 901—920.