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Dimiter Todorov Georgiev

Algorithmic Methods for Non-Classical Logics

DISSERTATION

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Table of Contents

Abstract

The area of the dissertation is the vast family of algorithms surrounding modal logic. The author has chosen to focus on computability and algorithms for modal correspondence theory, or the questions of modal definability of first-order formulas and first-order definability of modal formulas.

The scope of the dissertation is to define a deterministic version of the SQEMA algorithm, to show that it still succeeds on the two well-known classes of elementary modal formulas – the Sahlqvist and Inductive classes, and to show some modal and first-order definability results for two classes of Kripke frames, which are interesting in the area of artificial intelligence, namely, the KD45-frames and the Euclidean frames.

The results in the dissertation can be grouped in the following groups.

1. **Results about the algorithm Deterministic SQEMA, Sahlqvist and Inductive formulas.**

These include:

- Defining a new deterministic version of the SQEMA algorithm with additional simplification rules for the universal modality.
- A proof of termination of Deterministic SQEMA.
- A new invariant for Deterministic SQEMA executions on Sahlqvist formulas.
- A proof that Deterministic SQEMA succeeds on all Sahlqvist formulas.
- A new invariant for Deterministic SQEMA executions on Inductive formulas.
- A proof that Deterministic SQEMA succeeds on all Inductive formulas.

2. **Results about applying Deterministic SQEMA to formulas of the Pre-Contact Logic language, and results about Sahlqvist PCL formulas**

- Defining a modified translation of PCL formulas into $ML(\Box, [U])$.
- Proving that the above translation converts Sahlqvist PCL formulas into Sahlqvist $ML(\Box, [U])$ formulas.
- Modifying the existing Deterministic SQEMA implementation at `http://www.fmi.uni-sofia.bg/fmi/logic/sqema` to accept PCL formulas and succeed on all Sahlqvist PCL formulas by using the modified translation.

3. **Computability and complexity results about the correspondence problems in the class of all** KD45 **Kripke frames**

The results of this group are:

- A proof that all modal formulas of the basic modal language are first-order definable in the class of all KD45 frames.
- A proof that the problem of deciding whether first-order formulas are modally definable in the basic modal language in the class of KD45 frames is PSPACE-complete.

- A proof that all modal formulas of the basic modal language extended with the universal modality are first-order definable in the class of all KD45 frames.

- A proof that the problem of deciding modal definability in the basic modal language extended with the universal modality of first-order formulas in the class of KD45 frames is PSPACE-complete.

4. **Computability and complexity results about the correspondence problems in the class of all Euclidean Kripke frames** - this group of results was examined in collaboration with Tinko Tinchev and Philippe Balbiani.

- A proof that all modal formulas of the basic modal language have a first-order definition in the class of all Euclidean Kripke frames.

- A proof that the problem of deciding whether a first-order formula is valid in the class of all Euclidean Kripke frames is undecidable.

- A proof that the problem of deciding whether a first-order formula is modally definable in the class of all Euclidean Kripke frames is undecidable.

Structure

The dissertation consists of 7 parts.

1. **"Introduction"**. First, there is a brief introduction, which does an overview of the subject matter and explains briefly the referred articles with the main results of the dissertation.

2. **"Preliminaries"**. Then, there is a section on preliminaries. It explains the required background knowledge to understand the rest of the dissertation.

3. **"Deterministic SQEMA"**. This section describes the algorithm Deterministic SQEMA and the main results - the invariants for Sahlqvist and Inductive formulas and the proofs that the algorithm succeeds on both classes of formulas. There is also a modified translation of Pre-Contact Logic formulas into formulas that Deterministic SQEMA can work with. It is shown that thus Deterministic SQEMA succeeds on all Sahlqvist PCL formulas.

4. "ML($\Box$) **and** $\mathcal{C}_{\mathrm{KD45}}$". Here the definability problems in the class of all KD45-frames are shown to be decidable. The modal definability problem is shown to be PSPACE-complete.

5. "ML($\Box, [U]$) **and** $\mathcal{C}_{\mathrm{KD45}}$". As in the previous section, this section discusses definability problems in the class of all KD45-frames. This time, however, the modal language is ML($\Box, [U]$) instead of ML($\Box$).

6. "ML($\Box$) **and** $\mathcal{C}_{\mathrm{K5}}$". Here the author in co-authorship with Tinko Tinchev and Philippe Balbiani shows that every modal formula has a first-order definition over the class of all Euclidean frames. Then the authors show that the problems of deciding first-order validity and modal definability of first-order formulas in

the class of all Euclidean frames are both undecidable.

7. **“Conclusion”** A summary of the results is given here.

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The author's findings are briefly presented in the introduction below. Then they are discussed in detail in the main body of this work. Finally, the results are summarized in the conclusion. Authenticity claims and a list of accompanying publications can also be found in the conclusion.

1 Introduction

The problems of modal and first-order definability are important topics in Modal Logic. The first major result in the area was Sahlqvist's class of first-order definable formulas in [48]. This lead to van Benthem's question of whether the problems are decidable in [56][57] as well as the introduction of van Benthem's algorithm for finding first-order correspondents of Sahlqvist formulas. Chagrova, at first in [12] and [13], later with Chagrov in [9], [10], and [11], showed that the problem of first-order definability of modal formulas and the problem of modal definability of first-order formulas are both undecidable over the class of all Kripke frames.

Despite these findings, the two definability problems do have algorithmic solutions for some classes of Kripke frames, such as some of those used in the study of artificial intelligence. For example, Balbiani and Tinchev in [2] and [1] show decidability of first-order and modal definability problems over the class of all equivalence relations with respect to the two modal languages $ML(\Box)$ and $ML(\Box, [U])$. However, in [3], Balbiani and Tinchev develop a technique for reducing the problem of deciding the validity of first-order sentences over some classes of frames to the problem of modal definability of first-order sentences over the same classes of frames, thus proving the undecidability of the modal definability problem over a large number of classes of frames.

It is also possible to find classes of modal formulas, other than the Sahlqvist class, which have algorithmically computable first-order correspondents over the class of all frames. For example, the class of inductive formulas, introduced in [31][54][16][32].

There are also other algorithms for finding first-order correspondents, for example in [23] Gabbay and Ohlbach introduced the SCAN algorithm, and in [52], Szalas introduced DLS. SCAN is based on a resolution procedure applied on a Skolemized translation of the modal formula into the second-order logic, while DLS works on the same translation, but is based on a transformation procedure using a lemma by Ackermann. Both algorithms use a procedure of unskolemization, which is not always successful.

A new non-deterministic algorithm, SQEMA (Second-order Quantifier Elimination using a Modal Ackermann lemma), for computing first-order correspondents in modal logic is introduced in [17][18][19][21][20][15]. It is based on a modal version of the well-known Ackermann Lemma. SQEMA works directly on the modal formulas without translating them into the second-order logic and without using Skolemization. Thus SQEMA succeeds not only on all Sahlqvist formulas, but also on the class of inductive formulas mentioned above. There are examples of modal formulas on which SQEMA succeeds, while both SCAN and DLS fail, e.g.: $(\Box(\Box p \leftrightarrow q) \to p)$. As proved in [17][18][19] [15] SQEMA only succeeds on d-persistent (for languages without nominals) or di-persistent (for reversive languages with nominals) — and hence, like in [8][16][30][32][15], canonical formulas, i.e., whenever successful, it not only computes a local first-order correspondent of the input modal formula, but also proves its canonicity. This extends to any set of modal formulas on which SQEMA succeeds. Thus, SQEMA can also be used as an automated prover of canonical model completeness of modal logics.

An implementation of SQEMA in Java was given in [25]. Some additional simplifications were added to the implementation thanks to a suggestion by Renate Schmidt, which helps the implementation succeed on formulas such as $((\Box\Diamond p \to \Diamond\Box p) \vee (\Box p \to \Diamond p))$, by simplifying $(\Box p \wedge \Box\neg p)$ to $\Box\bot$.

The universal modality and nominals were introduced in [45].

In [26], the author showed a version of SQEMA that was augmented to $\mathrm{ML}(\Box, [U])$, the basic modal language extended by adding the universal modality.

In [18], SQEMA for a reversive language with nominals is discussed, promising an extension with $[U]$. In [21], SQEMA with downwards monotonicity for Ackermann's rule is presented. In [55][20], an extension of SQEMA for a reversive language with $[U]$ and nominals is introduced, with the output being in the first-order μ-calculus.

In [27], the author has given a definition a deterministic and terminating strategy for using the SQEMA rules for the language with universal modality, at most countably infinitely many couples of converse modalities, and nominals, $\mathrm{ML}(T, U)$. The author has shown that Deterministic SQEMA always succeeds on all Sahlqvist and inductive formulas. The existing proof for the original SQEMA in [17] does not hold for this version of the algorithm, for example for the input formula $(\neg\Box\Box p \vee \Diamond p)$. When the input contains no nominals, the author has shown, similarly to [17], that Deterministic SQEMA succeeds on d-persistent formulas. In the case when the input is from a hybrid temporal language with the universal modality, like in [18], Deterministic SQEMA succeeds on di-persistent formulas. The the axiomatic system for $\mathrm{ML}(T, U)$ and its completeness was shown in [27], following closely [45][46][24][8]. In a similar way to [8][30][16][53][32][18], canonicity of di-persistent formulas is shown. Also, in this dissertation, the axiomatic system for hon-hybrid languages is shown, following the same sources, and canonicity of d-persistent formulas is shown. Thus Deterministic SQEMA can be used to prove canonicity of a formula.

The article [27] also shows how to extend Deterministic SQEMA to the language of pre-contact logics. There, a modified form of the translation from [5] is used. With it, we obtain Sahlqvist fromulas from Sahlqvist formulas of the pre-contact language, as defined in [4]. This shows that Deterministic SQEMA succeeds on them. Completeness of all pre-contact formulas is shown in [5].

In [28][29], definability questions over the class of all KD45 Kripke frames are explored. This class is axiomaized by the canonical and decidable normal modal logic KD45. The interest in this class has arisen from the importance of this logic for the study of artificial intelligence. That is why KD45 has been well-analyzed in the literature over the years. At first, it was studied as the logic system DE4 in [37] and [49]. It was later considered as a normal extension of K5 in [42] and [43]. The completeness and complexity of KD45 was examined in [34] and [35], as well as systems with mixed S5 and KD45 modalities, which can also be seen in [58], [41] and [40]. A tableau system is presented in [33]. A good overview of the subject area can be seen in [7].

In [28][29], the author has shown that all $ML(\Box)$ and $ML(\Box, [U])$ formulas are first-order definable over the class of all KD45-frames, and that the modal definability of FOL formulas in each of the languages $ML(\Box)$ and $ML(\Box, [U])$ over the class of all KD45-frames is in PSPACE. Ideas from [2] and [1] have been used for the core of the proofs, and a technique from [3] has been employed to show PSPACE-hardness in one case.

The modal logic K5 and the properties of its frames are examined by Nagle in [42] and later with Thomason in [43]. These topics are also examined by Halpern and Rêgo in [36].

In [6], Balbiani, the author and Tinchev have shown that the all $ML(\Box)$ formulas have a first-order definition over the class of all K5-frames (or, Euclidean frames). It was also demonstrated using the main technique of [3] that the problem of modal definability of first-order formulas over this class of frames is undecidable.

This work summarizes the results for SQEMA and Deterministic SQEMA in [26] and [27], and also the results of [28][29] and [6]. In Section 3 it can be seen that Deterministic SQEMA can be extended to languages without nominals, using the topological proof technique of [17] to show that it only succeeds on d-persistent formulas of non-hybrid languages, thus showing that it can be used to prove axiomatic completeness (canonicity). Thus the results of [26] have been covered, including the rules for the universal modality, introduced there. In the same section, the results of [27] are shown, including the extension of Deterministic SQEMA to the language of pre-contact logics. In Section 4 and Section 5, the results of [28][29] are shown, namely decidability of the two definability problems over the class of all KD45 frames with respect to each of the languages $ML(\Box)$ and $ML(\Box, [U])$. In Section 6, the results of [6] are discussed in detail, showing that all modal formulas of $ML(\Box)$ have a first-order definition over the class of all Euclidean frames, and that the problem of modal definability of first-order formulas over this class of frames is undecidable.

Background material for first-order logic can be found in [50]; for modal logic in [39] and [8]; for model theory in [14] and [22]; for computational complexity in [44]. This paper uses Stockmeyer's theorem for the complexity of the decidability of theorems in the first-order theory of equality in [51].

2 Preliminaries

Our topics of discussion will focus on several different modal and first-order languages.

2.1 Modal Languages

First, let us define the modal languages that we are discussing.

When speaking of words in a formal language, it helps to have a symbol to specify that a word or a symbol occurs in another word. Here we use the symbol $\nearrow$ in the following way. We denote by $a \nearrow b$ iff the word or symbol a occurs in the word b. The negation of occurrence is denoted by $\not\nearrow$.

Let us first define the symbols that we use as our modalities. Let $Box =_{def} \{\Box_0, \Box_1, \Box_2, \dots\}$ be a countably infinite set of symbols which we call *boxes*. Let $RevBox =_{def} \{\Box_0^{-1}, \Box_1^{-1}, \Box_2^{-1}, \dots\}$ be a countably infinite set of *reversed boxes*. Similarly to that, we define $Diamond =_{def} \{\Diamond_0, \Diamond_1, \Diamond_2, \dots\}$ as the set of *diamonds*, and $RevDiamond =_{def} \{\Diamond_0^{-1}, \Diamond_1^{-1}, \Diamond_2^{-1}, \dots\}$ as the set of *reversed diamonds*. We treat the zeroth index of boxes and diamonds in a special way. We denote $\Box_0$ by $[U]$ and we call it *the universal box*, and we denote $\Diamond_0$ by $\langle U \rangle$, which we call *the universal diamond*.

Let $PROP =_{def} \{p_1, p_2, \dots\}$ be a countably infinite set of symbols called *propositional variables*. We denote propositional variables by the letters p and q. Let $NOM =_{def} \{c_1, c_2, \dots\}$ be a countably infinite set of *nominals*.

We assume that the sets $PROP$, NOM, Box, and $RevBox$ are pairwise disjoint.

We use the capital Latin letters A and B for modal formulas. The general syntax of our modal languages is as follows:

$$A ::= \bot \mid \top \mid p_i \mid c_j \mid \neg A \mid (A \vee A) \mid (A \wedge A) \mid \Diamond_s A \mid \Diamond_d^{-1} A \mid \Box_s A \mid \Box_d^{-1} A$$

where $i \in \mathbb{N}, j \in J, s \in S, d \in D$, $D \subseteq S \subseteq \mathbb{N}$, $S \neq \emptyset$, and also J is either $\emptyset$ or $\mathbb{N}$.

We may also use $\rightarrow$ and $\leftrightarrow$ as defined symbols, where $(A \rightarrow B)$ stands for $(\neg A \vee B)$, and $(A \leftrightarrow B)$ is the formula $((A \rightarrow B) \wedge (B \rightarrow A))$.

We may omit braces, using the standard precedence rules.

We refer to propositional variables and nominals as *atomic formulas*, and for simplicity, we generally do not consider $\bot$ and $\top$ as atomic formulas.

A reader who is well-versed in modal logic will notice that we are using both boxes and diamonds as our modal symbols, as well as both conjunctions and disjunctions as our logical connectives. This is due to the fact that in our topics of discussion below, we sometimes require the use of formulas in *negation normal form*, with negations occurring only before atomic formulas, $\bot$, or $\top$.

As we see below in 2.2, it is justified to use the notations $\bigwedge\{A_1, \ldots, A_n\}$ and $\bigvee\{A_1, \ldots, A_n\}$ with their usual meaning. However, when describing an algorithm such as Deterministic SQEMA, it helps to specify the order of the conjunctions and disjunctions, so we denote by $\bigwedge(A_1, \ldots, A_n)$ for $n \geq 0$ and different A_i the formula $(A_1 \wedge \ldots (A_{n-1} \wedge A_n) \ldots)$ if $n > 0$, $\top$ otherwise; we denote by $\bigvee(A_1, \ldots, A_n)$ for $n \geq 0$ and different A_i the formula $(A_1 \vee \ldots (A_{n-1} \vee A_n) \ldots)$ if $n > 0$, $\bot$ otherwise.

We say that a modal operator (a box, a reversed box, a diamond, or a reversed diamond) *occurs positively in A* iff it occurs within the scope of an even number of negations in A. We say that a modal operator *occurs negatively in A* iff it occurs within the scope of an odd number of negations in A.

We assume that the reader knows the definition of whether a formula A is *positive in p*, *negative in p*. We also assume that the reader recognizes positive and negative occurrences of propositional variables and nominals.

For this work, we assume that a formula A is *positive* iff all occurrences of propositional variables in it are positive. Here, we disregard the occurrences of nominals. Again, we assume that a formula A is *negative* iff all occurrences of propositional variables in it are negative, again disregarding any occurrences of nominals.

Let L be one of the modal languages defined above. If $J = \mathbb{N}$, we say that L is a *hybrid modal language*, or just a *hybrid language*, otherwise if $J = \emptyset$, we say that L is a *non-hybrid modal language*, or just a *non-hybrid language*. If $D = S$, we say that L is a *reversive language*, or a *temporal language*. If $0 \in S$, we say that *L contains the universal modality*.

For simplicity, if L is a modal language and $\Box$ is one of its boxes or reversed boxes, we denote by $\Box^{-1}$ the corresponding reversed box or box (if it is in L). The same denotation applies to diamonds. Sometimes we treat $[U]$ and $\Box_0^{-1}$ and also $\langle U \rangle$ and $\Diamond_0^{-1}$ the same way, because, as we see below, their semantics are the same.

The *basic modal language* ML($\Box$) is a modal language with $J = \emptyset$, $S = \{1\}$, and $D = \emptyset$. The *basic modal language extended by adding the universal modality*, ML($\Box, [U]$), is a modal language with $J = \emptyset$, $S = \{0, 1\}$ and $D = \emptyset$. We use just $\Box$ and $\Diamond$ for the non-universal boxes and diamonds when discussing these two languages.

If L is a modal language and $A \in L$, we say that A is a *modal formula*.

If L is a modal language, then $NOM(L)$ is the set of nominals, which occur in formulas of L.

If A is a modal formula, then $PROP(A)$ is the set of propositional variables occurring in A, and $NOM(A)$ is the set of nominals occurring in A.

Generally, if speaking of a *language*, we mean a modal language.

2.2 Kripke Semantics of Modal Languages

We use the popular Kripke semantics for modal languages. A *Kripke frame* or a *Kripke structure* is a tuple consisting of a non-empty set of *possible worlds*, or *states*, and one or more binary *accessibility relations*. Often we say simply *frame* instead of a Kripke frame. When discussing the languages ML($\Box$) and ML($\Box, [U]$), we may use Kripke frames with just one accessibility relation, but generally if we have more than one non-universal modality in a given language, we need Kripke frames that have as many accessibility relations as are the non-universal modalities in the language.

Let L be a modal language with S being the set of indices of its boxes. We say that the tuple $\mathrm{F} = \langle W, \mathcal{R} \rangle$ is *a Kripke frame for* L iff $W \neq \emptyset$ is *the universe of* F, and for all $i \in S$, $\mathcal{R}(i) \subseteq W \times W$ are the accessibility relations, where if $0 \in S$, $\mathcal{R}(0) = W \times W$. If a state $w \in W$, we say that w is *in* F. If S is finite, we may omit $\mathcal{R}(0)$ if $\Box_0$ is present in the language, and we may represent F by the tuple $\langle W, R_{i_1}, \dots, R_{i_n} \rangle$, where $n > 0$, $W \neq \emptyset$, $S = \{i_1, \dots, i_n\}$ and $R_{i_1}, \dots, R_{i_n}$ are binary relations over W. Thus $\langle W, R \rangle$ is a Kripke frame for ML($\Box$) and ML($\Box, [U]$). Also if we require a Kripke frame where only two of the relations matter, we may use the notation $\langle W, R_1, R_2 \rangle$, etc.

Let $\mathrm{F} = \langle W, \mathcal{R} \rangle$ be a Kripke frame for the modal language L. A *Kripke model for* L, or just a *model for* L, is a tuple $\mathrm{M} = \langle \mathrm{F}, V, H \rangle$, where $V : PROP \mapsto \mathbb{P}(W)$ is a *valuation*, and $H : NOM(L) \mapsto W$ is an *assignment*. If the language L does not contain any nominals, we may simply use the tuple $\langle \mathrm{F}, V \rangle$ because H would be the empty function. We say that a model $\mathrm{M} = \langle \mathrm{F}, V, H \rangle$ is *based on* or is *over* F. If $w \in W$, we say that w is *in* M. A model for a hybrid language $\mathrm{M} = \langle \mathrm{F}, V, H \rangle$ is *named* iff H is surjective. Instead of F, we may use the sequence $W, \mathcal{R}$ in the tuple. Thus $\mathrm{M} = \langle W, \mathcal{R}, V, H \rangle$ is a Kripke model. For non-hybrid languages, we may omit the valuation H from the tuple because there are no nominals to evaluate.

Now we inductively define the valuation, or *the extension* of a formula A in a model $\mathrm{M} = \langle W, \mathcal{R}, V, H \rangle$, denoted by $[\![A]\!]_{\mathrm{M}}$. It is assumed that M is a model for a modal language which contains A. $[\![\bot]\!]_{\mathrm{M}} = \emptyset$, $[\![\top]\!]_{\mathrm{M}} = W$, $[\![p]\!]_{\mathrm{M}} = V(p)$, $[\![c]\!]_{\mathrm{M}} = \{H(c)\}$, $[\![\neg A]\!]_{\mathrm{M}} = W \setminus [\![A]\!]_{\mathrm{M}}$, $[\![(A_1 \vee A_2)]\!]_{\mathrm{M}} = [\![A_1]\!]_{\mathrm{M}} \cup [\![A_2]\!]_{\mathrm{M}}$, $[\![(A_1 \wedge A_2)]\!]_{\mathrm{M}} = [\![A_1]\!]_{\mathrm{M}} \cap [\![A_2]\!]_{\mathrm{M}}$, $[\![\Diamond_i A]\!]_{\mathrm{M}} = \{w \in W \mid \exists v \in [\![A]\!]_{\mathrm{M}} : \langle w, v \rangle \in \mathcal{R}(i)\}$, $[\![\Diamond_i^{-1} A]\!]_{\mathrm{M}} = \{w \in W \mid \exists v \in [\![A]\!]_{\mathrm{M}} : \langle v, w \rangle \in \mathcal{R}(i)\}$, $[\![\Box_i A]\!]_{\mathrm{M}} = [\![\neg \Diamond_i \neg A]\!]_{\mathrm{M}}$, $[\![\Box_i^{-1} A]\!]_{\mathrm{M}} = [\![\neg \Diamond_i^{-1} \neg A]\!]_{\mathrm{M}}$. As promised above, we can clearly see that the semantics of the modalities $[U]$ and $\Box_0^{-1}$ are identical.

Now, some commonly used notations in modal logic. Let M be a Kripke model for a language L and let $A \in L$. Let w be in M. We say that A *is true in* M *at* w, denoted by $\mathrm{M}, w \Vdash A$, iff $w \in [\![A]\!]_{\mathrm{M}}$. We say that A *is true in* M, denoted by $\mathrm{M} \Vdash A$, iff for all w in M, it is the case that $\mathrm{M}, w \Vdash A$. Let F be a Kripke frame for L. A *is valid in* F *at a state* w *in* F, denoted by $\mathrm{F}, w \Vdash A$, iff for all models M over F, we have that $\mathrm{M}, w \Vdash A$. A *is valid in* F, denoted by $\mathrm{F} \Vdash A$, iff for all models M over F: $\mathrm{M} \Vdash A$ iff for all states w in F: $\mathrm{F}, w \Vdash A$. We say that A is *valid* iff it is valid in the class of all frames for L.

2.3 Modal Formulas as Operators

Let L be a modal language and let $A \in L$. Let all propositional variables occurring in A be among $p_1 \dots p_n$ and all nominals occurring in A be among $c_1, \dots, c_m$. Let $\mathrm{F} = \langle W, \mathcal{R} \rangle$ be a Kripke frame for L. We define the value of the *operator* $[\![A]\!]$, $[\![A]\!](s_1, \dots, s_n, w_1, \dots, w_m) \subseteq W$, for all $s_\ell \subseteq W$ and $w_k \in W$ in the following way:

$[\![c_k]\!](w_k) = \{w_k\}$, $[\![p_\ell]\!](s_\ell) = s_\ell$, $[\![\neg A]\!](\bar{s}, \bar{w}) = W \setminus [\![A]\!](\bar{s}, \bar{w})$, $[\![(A \vee B)]\!](\bar{s}, \bar{w}) = [\![A]\!](\bar{s}, \bar{w}) \cup [\![B]\!](\bar{s}, \bar{w})$, $[\![(A \wedge B)]\!](\bar{s}, \bar{w}) = [\![A]\!](\bar{s}, \bar{w}) \cap [\![B]\!](\bar{s}, \bar{w})$, $[\![\Diamond_i A]\!](\bar{s}, \bar{w}) = \{w \in W \mid \exists v \in [\![A]\!](\bar{s}, \bar{w}) : \langle w, v \rangle \in \mathcal{R}(i)\}$, $[\![\Diamond_i^{-1} A]\!](\bar{s}, \bar{w}) = \{w \in W \mid \exists v \in [\![A]\!](\bar{s}, \bar{w}) : \langle v, w \rangle \in \mathcal{R}(i)\}$, $[\![\Box_i A]\!](\bar{s}, \bar{w}) = [\![\neg \Diamond_i \neg A]\!](\bar{s}, \bar{w})$, $[\![\Box_i^{-1} A]\!](\bar{s}, \bar{w}) = [\![\neg \Diamond_i^{-1} \neg A]\!](\bar{s}, \bar{w})$.

By the definition of semantics of modal formulas above, it is not hard to check that the extension of A in some model M depends on A, the valuations of $p_1, \dots, p_n$ and the assignments of $c_1, \dots, c_m$ in M.

Thus, it is a simple check to see that if $\mathrm{F} = \langle W, \mathcal{R} \rangle$ is a Kripke frame for L and $A \in L$, then for all $s_1, \dots, s_n \subseteq W$ and all $w_1, \dots, w_m \in W$, $[\![A]\!](\bar{s}, \bar{w}) = [\![A]\!]_{\mathrm{M}}$, where $\mathrm{M} = \langle \mathrm{F}, V, H \rangle$ is any model over F with $V(p_1) = s_1, \dots, V(p_n) = s_n, H(c_1) = w_1, \dots, H(c_m) = w_m$.

2.4 First-Order Languages

We use the standard definition of a *first-order language*, and a *first-order theory*, as defined in [50]. In this thesis, we only work with first-order languages with countably infinitely many individual variables $VAR = \{x_1, x_2, \dots\}$, with equality, without constants and without functional symbols, and only unary ($\{P_1, P_2, \dots\}$) and infix binary ($\{r_1, r_2, \dots\}$) predicate symbols.

Clearly for such languages if they do not contain unary predicate symbols, Kripke frames can be used as structures, interpreting each of the symbols r_i with $\mathcal{R}(i)$.

We are mainly interested in two kinds of first-order languages, although Section 6 deals with another kind of first-order language.

First, we have languages with all of the above kinds of symbols, which we call *standard translation languages*.

Then we have languages which have no unary predicate symbols. These language we call the *correspondence languages*. Often, it is known in the context of discussion which binary predicate symbols are in the language, and thus we use simply the name FOL. In most of the paper, FOL is simply the language where all possible binary predicate symbols occur. In sections 4, 5, and 6, FOL refers to the first-order language with a single binary predicate symbol r.

We use the standard definitions of *assignment*, *a sentence*, *validity*, and *satisfiability*. For example, if F is a Kripke frame for the basic modal language, $w_1, \ldots, w_n$ are states in F and $\psi(x_1, \ldots, x_n)$ is a FOL formula with free variables among $x_1, \ldots, x_n$, we denote by $\mathrm{F} \vDash \psi[w_1, \ldots, w_n]$ iff ψ is satisfied on F with an assignment of the individual variables such that x_i is assigned to w_i for every i such that $1 \leq i \leq n$. In the same case, if we need to use a *modified assignment*, we denote by $\mathrm{F} \vDash \psi[w_1, \ldots, w_n, x_i \mapsto v]$ where $i \leq n$ and we have denoted the fact that $\mathrm{F} \vDash \psi[w_1, \ldots, w_{i-1}, v, w_{i+1}, \ldots, w_n]$.

2.5 Decidability and Complexity

We use the standard definitions of decidability (as seen in [50]) and complexity (see [44]).

2.6 The Correspondence Problems

The following two definitions of definability are adapted from [8], see Definition 3.2 on page 126.

Definition 1 (First-order Definability) Let $\mathcal{C}$ be a class of Kripke frames. We say that the class $\mathcal{C}' \subseteq \mathcal{C}$ is *first-order definable over* $\mathcal{C}$ iff there is a FOL formula ψ, such that for all $\mathrm{F} \in \mathcal{C}$, it is the case that $\mathrm{F} \in \mathcal{C}'$ iff $\mathrm{F} \vDash \psi$. In this case, we say that ψ *defines* $\mathcal{C}'$ *over* $\mathcal{C}$.

Definition 2 (Modal Definability) Let $\mathcal{C}$ be a class of Kripke frames for some modal language L. We say that the class $\mathcal{C}' \subseteq \mathcal{C}$ is *modally definable over* $\mathcal{C}$ *in* L iff there is a modal formula $A \in L$, such that for all $\mathrm{F} \in \mathcal{C}$, it is the case that $\mathrm{F} \in \mathcal{C}'$ iff $\mathrm{F} \Vdash A$. In this case, we say that A *defines* $\mathcal{C}'$ *over* $\mathcal{C}$ *in* L.

There are two major correspondence problems, as outlined in the introductions of [1], [2], and[10]. Here, we give them names for simplicity:

Definition 3 (First-order Definability of Modal Formulas) Let L be a modal language and $\mathcal{C}$ be a class of frames for L. The first-order definability problem for L over $\mathcal{C}$ is the following: Given a modal formula $A \in L$, is there a FOL formula ψ, such that for all $\mathrm{F} \in \mathcal{C}$, it is the case that $\mathrm{F} \Vdash A$ iff $\mathrm{F} \vDash \psi$? If there is such a formula ψ, we say that ψ is *a first-order definition of A over $\mathcal{C}$*. We also say that A and ψ are globally correspondent over $\mathcal{C}$.

Definition 4 (Modal Definability of First-order Formulas) Let L be a modal language and $\mathcal{C}$ be a class of frames for L. The modal definability problem for L over $\mathcal{C}$ is the following: Given a first-order formula $\psi \in \mathrm{FOL}$, is there a modal formula $A \in L$, such that for all $\mathrm{F} \in \mathcal{C}$, it is the case that $\mathrm{F} \Vdash A$ iff $\mathrm{F} \vDash \psi$? If there is such a formula A, we say that A is *a modal definition of ψ over $\mathcal{C}$*. We also say that A and ψ are globally correspondent over $\mathcal{C}$.

Given a modal language L, we are interested in whether or not the above two problems are decidable over certain classes of frames for L.

If the above problems are undecidable over the class of all frames for L, are there interesting algorithms for deciding parts of some of the problems, which succeed on as many formulas as possible?

Note that the above definitions are for global definability and global correspondence. Sometimes it is easier to solve a definability problem about the local version of correspondence. Local correspondence is defined later, in Definition 36, and it can be seen that it implies global correspondence.

2.7 Standard Translation

We make use of the Standard Translation in Section 5 and Section 6 as defined in [8], chapter 2.4, see Definition 2.45. Clearly, if the modal language contains nominals, we require that the first-order language contains matching first-order constant symbols. However, we do not use the regular Standard Translation for hybrid languages, as we see below in Section 3. There we make use of a modified standard translation for formulas without propositional variables, see Definition 41.

2.8 P-Morphisms, Disjoint Unions, and Generated Subframes

In sections 4, 5 and 6, we make use of the following well-known model-theoretical constructions for modal logic, see [8], pages 139–143.

Definition 5 (P-Morphic Image) If $\mathrm{F}_1 = \langle W_1, \mathcal{R}_1 \rangle$ and $\mathrm{F}_2 = \langle W_2, \mathcal{R}_2 \rangle$ are frames for some non-hybrid modal language L, with a set of indices of its boxes S and a set of indices of its reversed boxes D, we say that a function $f : W_1 \mapsto W_2$ is a *p-morphism* (or a *bounded morphism*) of F_1 to F_2 iff the following conditions hold:

1. (The Forward Condition)
If $i \in S$ and $\langle x, y \rangle \in \mathcal{R}_1(i)$ then $\langle f(x), f(y) \rangle \in \mathcal{R}_2(i)$.

2. (The Back Condition for Diamonds)

If $i \in S, x \in W_1, y' \in W_2$ and $\langle f(x), y' \rangle \in \mathcal{R}_2(i)$ then there is $y \in W_1$ such that $\langle x, y \rangle \in \mathcal{R}_1(i)$ and $f(y) = y'$.

3. (The Back Condition for Reversed Diamonds)

If $i \in D, x \in W_1, y' \in W_2$ and $\langle f(x), y' \rangle \in \mathcal{R}_2(i)^{-1}$ then there is $y \in W_1$ such that $\langle x, y \rangle \in \mathcal{R}_1(i)^{-1}$ and $f(y) = y'$.

If f is surjective, we say that F_2 is a *p-morphic image of* F_1 (or a *bounded morphic image of* F_1). In this case, it can be shown by induction on $A \in L$ that if $\mathrm{F}_1 \Vdash A$ then $\mathrm{F}_2 \Vdash A$, see [8], pages 139–143.

Definition 6 (Disjoint Union) Let L be a non-hybrid language without the universal modality with a set of indices of boxes S. Let I be a non-empty index set and let $\{\mathrm{F}_i \mid i \in I\}$ be a pairwise disjoint family of frames for L, where for every $i \in I$, $\mathrm{F}_i = \langle W_i, \mathcal{R}_i \rangle$. The *disjoint union* $\biguplus_{i \in I}\{\mathrm{F}_i\} = \langle W, \mathcal{R} \rangle$ is a frame for L defined by $W =_{def} \bigcup_{i \in I}\{W_i\}$ and for all $s \in S$, $\mathcal{R}(s) =_{def} \bigcup_{i \in I}\{\mathcal{R}_i(s)\}$.

Definition 7 (Generated Subframe) Let L be a non-hybrid language without the universal modality, with a set of indices of boxes S and a set of indices of reversed boxes D. Let $\mathrm{F} = \langle W, \mathcal{R} \rangle$ be a frame for a non-hybrid language and let w be a state in F. We say that $\mathrm{F}' = \langle W', \mathcal{R}' \rangle$, which is a frame for L, is *the point-generated subframe, or simply the generated subframe, of* F *at* w iff F' is such that W' is the smallest set such that the following conditions hold:

1. $w \in W' \subseteq W$,
2. For all $i \in S$, $\mathcal{R}'(i) = \mathcal{R}(i) \cap (W' \times W')$
3. For all $i \in S$, if $x \in W'$ and $\langle x, y \rangle \in \mathcal{R}(i)$, then $y \in W'$.
4. For all $i \in D$, if $x \in W'$ and $\langle x, y \rangle \in \mathcal{R}(i)^{-1}$, then $y \in W'$.

It can be shown by induction on $A \in L$ that $\mathrm{F}, w \nVdash A$ iff $\mathrm{F}', w \nVdash A$, see [8], pages 139–143.

2.9 General Frames

In Section 3, we make extensive use of general frames to show axiomatic completeness of certain formulas. Background information can be seen in [8], pages 27–29.

Definition 8 (General Frame) This is adapted from Definition 1.32 of [8], page 29. Let $\mathrm{F} = \langle W, \mathcal{R} \rangle$ be a frame for a modal language L. We say that $\langle \mathrm{F}, \mathbb{W} \rangle$ is a *general frame for* L, iff $\mathbb{W} \subseteq \mathbb{P}(W)$ is non-empty and the following conditions hold:

- $\mathbb{W}$ is closed under $[\![\neg p_1]\!]$ (relative complement)
- $\mathbb{W}$ is closed under $[\![(p_1 \vee p_2)]\!]$ (union)
- $\mathbb{W}$ is closed under $[\![\Diamond p_1]\!]$ for all diamonds and reversed diamonds $\Diamond$ of L.

$\mathbb{W}$ is the set of *admissible valuations*. It can be checked that $W, \emptyset \in \mathbb{W}$. A frame F with universe W can also be considered to be the *full general frame* $\langle \mathrm{F}, \mathbb{P}(W)\rangle$. We use $\mathfrak{g}$ for general frames, $\mathbb{W}$ for sets of admissible valuations. If $\mathfrak{g} = \langle \mathrm{F}, \mathbb{W}\rangle$, then we denote by $\mathfrak{g}_\#$ the *underlying frame* of $\mathfrak{g}$, F. We say that a state w is *in* $\mathfrak{g}$ iff it is in $\mathfrak{g}_\#$. A Kripke model $\mathrm{M} = \langle \mathfrak{g}_\#, V, H\rangle$ is a *model over* $\mathfrak{g}$ iff for each propositional variable p, $V(p) \in \mathbb{W}$. Note that we place no restrictions on the assignments of nominals. We say that a modal formula $A \in L$ is *valid in* $\mathfrak{g}$, denoted by $\mathfrak{g} \Vdash A$, iff it is true in all models over $\mathfrak{g}$. We say that $A \in L$ is *valid in* $\mathfrak{g}$ *and a state* w *in* $\mathfrak{g}$, denoted by $\mathfrak{g}, w \Vdash A$, iff for all models M over $\mathfrak{g}$, we have that $\mathrm{M}, w \Vdash A$.

Definition 9 (Finite Intersection Property) Let W be a non-empty set and $\mathbb{W}_0 \subseteq \mathbb{P}(W)$ be a non-empty family of subsets of W. We say that $\mathbb{W}_0$ has *the finite intersection property (abbreviated as the fip)* iff for every non-empty finite subset $\mathbb{W}_1 \subseteq \mathbb{W}_0$, we have that $\bigcap \mathbb{W}_1 \neq \emptyset$.

Definition 10 (Types of General Frames) This is an adaptation of Definition 5.65 of [8], page 308. Let L be a modal language with S being the set of indices of its diamonds and D being the set of indices of its reversed diamonds. Let $\mathfrak{g} = \langle \mathrm{F}, \mathbb{W}\rangle$ be a general frame for L, where $\mathrm{F} = \langle W, \mathcal{R}\rangle$. Then $\mathfrak{g}$ is called:
1. *differentiated* iff for all $v, w \in W$:
$v = w$ iff for all $s \in \mathbb{W}$: $v \in s \Leftrightarrow w \in s$
2. *tight* iff for all $v, w \in W$, all $i \in S$ and all $j \in D$:
$\langle v, w\rangle \in \mathcal{R}(i)$ iff for all $s \in \mathbb{W}$: $w \in s \Rightarrow v \in [\![\Diamond_i p]\!](s)$
$\langle w, v\rangle \in \mathcal{R}(j)$ iff for all $s \in \mathbb{W}$: $w \in s \Rightarrow v \in [\![\Diamond_j^{-1} p]\!](s)$
Note that the two conditions above are trivially true for $\langle U\rangle$ and $\Diamond_0^{-1}$. Thus we can ignore these conditions for the universal modality.
3. *compact* iff $\bigcap \mathbb{W}_0 \neq \emptyset$ for every subset $\mathbb{W}_0$ of $\mathbb{W}$ which has the fip (and hence, is non-empty)
4. *descriptive* iff $\mathfrak{g}$ is differentiated, tight and compact
5. *discrete* iff every singleton of an element of W is admissible in $\mathfrak{g}$.

If $\mathfrak{g}$ is discrete, then an induction on all $A \in L$ shows that if M is a model over $\mathfrak{g}$, then $[\![A]\!]_{\mathrm{M}} \in \mathbb{W}$ for any formula. If L is a non-hybrid modal language and $\mathfrak{g}$ is any general frame for L, again an induction on all $A \in L$ shows that all extensions of A in models over $\mathfrak{g}$ are admissible sets.

Sometimes we may say just a *descriptive frame* instead of a descriptive general frame, and just a *discrete frame* instead of a discrete general frame.

It can be shown that any discrete frame is differentiated and tight.

2.10 D-Persistence and Di-Persistence

Definition 11 Let $\mathcal{C}$ be a class of general frames for some language L and let $A \in L$ be a modal formula. We say that A is *persistent with respect to the class* *of frames $\mathcal{C}$ and the language L* iff for all $\mathfrak{g} \in \mathcal{C}$, it is the case that $\mathfrak{g}_{\#} \Vdash A$ iff $\mathfrak{g} \Vdash A$. In this case, if $\mathcal{C}$ is the class of all descriptive frames for L, then A is called *d-persistent with respect to* L, and if $\mathcal{C}$ is the class of all discrete frames for L, then A is called *di-persistent with respect to* L.

2.11 Normal Modal Logics, Completeness and Canonicity

Here, we follow the axiomatic system for nominals and universal modality, described in [45][46][24], with some differences in the proofs.

Let L be a modal language with J being the set of indices of its nominals, S being the set of indices of its boxes, and D being the set of indices of its reversed boxes. In this section, we only allow L to contain nominals if L also contains the universal modality. Formally, $J = \mathbb{N} \Rightarrow 0 \in S$.

We show an axiomatic system for the language L. For simplicity of the axiomatic system, we add implications and remove diamonds, reversed diamonds, $\wedge$, $\vee$, $\neg$ and $\top$, using them only as *defined symbols*, see [50]. Therefore, our language L for this section becomes:

$$A ::= \bot \mid p_i \mid c_j \mid (A \to A) \mid \Box_s A \mid \Box_d^{-1} A$$

for all $i \in \mathbb{N}$, $j \in J$, $s \in S$ and $d \in D$.

Definition 12 (Admissible Form) Let $\#$ be a symbol, which is not in the alphabet of L. $\#$ is an *admissible form*. If $AF(\#)$ is an admissible form, then so are $\Box AF(\#)$ and $(A \to AF(\#))$ for any box or reversed box $\Box$ of L and any formula $A \in L$. The formula, obtained by replacing all occurrences of $\#$ with $A \in L$ in $AF(\#)$ is denoted by $AF(A)$.

Definition 13 (Uniform Substitution) Let A_1 and A' be modal formulas in L. We denote by $A_1[p/A']$ the word obtained from A_1, where each occurrence of p (if any) has been replaced with A'. According to the definition of modal formulas, the word thus constructed is also a formula in L, A_2. We call the rule for obtaining A_2 from A_1 *uniform substitution* of p by A' in A_1.

We use the same notation for *nominal substitution*, replacing a nominal with another nominal, as the notation for uniform substitution.

Axioms: (We assume that the axioms with $[U]$ or $\langle U\rangle$ are only used if $0 \in S$)
The axioms of propositional calculus.
(K) $(\Box_s(p \to q) \to (\Box_s p \to \Box_s q))$ for every $s \in S$
(T for U) $([U]p \to p)$
(B for U) $(p \to [U]\langle U\rangle p)$
(4 for U) $([U]p \to [U][U]p)$

(U) $([U]p \to \Box_s p)$ for every $s \in S$
(GP) $(p \to \Box_i \Diamond_i^{-1} p)$ for every $i \in D$
(HF) $(p \to \Box_i^{-1} \Diamond_i p)$ for every $i \in D$
(Nom1) $\langle U \rangle c$, only if $J = \mathbb{N}$, which also implies that $0 \in S$
(Nom2) $(\langle U \rangle (c \wedge p) \to [U](c \to p))$, only if $J = \mathbb{N}$, which implies also $0 \in S$

Rules:

$$\text{Modus Ponens (MP): } \frac{A, (A \to B)}{B},$$

$$\text{Gen: } \frac{A}{\Box A} \text{for every box or reversed box } \Box,$$

$$\text{Uniform Substitution: } \frac{A}{A[p/A']}, \text{ Nominal Substitution: } \frac{A}{A[c'/c'']},$$

$$\text{Cov*: } \frac{AF(\neg c) \text{ for some } c \not\nearrow \ AF(\#)}{AF(\bot)}.$$

See 7.3 of [8] for an alternative axiomatic system without the Cov* rule.

Clearly, we may ignore the rules Nominal Substitution and Cov* if the language is not hybrid.

A *normal modal logic for* L, or just *logic for* L, is a set of formulas $\Lambda \subseteq L$ such that Λ contains all axioms and is closed under applications of the five rules.

We denote by K_L the smallest logic for L.

Let $A \in L$ be a formula. We denote the smallest logic for L which contains A by $K_L + A$, and A is called *the axiom of* $K_L + A$. If $\Gamma \subseteq L$ is a set of formulas, we denote by $K_L + \Gamma$ the smallest logic for L which contains Γ, and we say that Γ is *the set of axioms of* $K_L + \Gamma$.

Let Λ be a logic for L. We denote by $\vdash_\Lambda A$ iff $A \in \Lambda$. We use the capital greek letters Γ, Δ, Σ for sets of formulas of L. A Λ-*theory* Γ is a set of formulas $\Gamma \subseteq L$ such that $\Lambda \subseteq \Gamma$ and Γ is closed under applications of MP and, only if L is a hybrid language, also the infinitary rule Cov:

$$\text{Cov: } \frac{AF(\neg c) \text{ for all } c}{AF(\bot)}.$$

The Λ-*theory of a set of formulas* $\Gamma \subseteq L$, $Th_\Lambda(\Gamma)$, is the smallest Λ-theory such that $\Gamma \subseteq Th_\Lambda(\Gamma)$. Despite the infinitary rule, the deduction lemma holds:

Lemma 14 (Deduction Lemma) $(A \to B) \in Th_\Lambda(\Gamma)$ iff $B \in Th_\Lambda(\Gamma \cup \{A\})$.

Proof The left to right direction is obvious. Let $B \in Th_\Lambda(\Gamma \cup \{A\})$ and let $\Gamma' =_{def} \{A' \mid (A \to A') \in Th_\Lambda(\Gamma)\}$. Easily, $A \in \Gamma'$ and $\Lambda \subseteq Th_\Lambda(\Gamma) \subseteq \Gamma'$. Also, Γ' is closed under applications of MP. If L is a hybrid language, to see that Γ' is closed under applications of Cov, let $AF(\#)$ be an admissible form, and suppose that for each nominal c, $AF(\neg c) \in \Gamma'$. Then, by propositional reasoning, for each nominal c: $(A \to AF(\neg c)) \in Th_\Lambda(\Gamma)$. Applying Cov to $(A \to AF(\#))$, we get that $(A \to AF(\bot)) \in Th_\Lambda(\Gamma)$, therefore $AF(\bot) \in \Gamma'$, so Γ' is closed under applications of Cov. Therefore, $Th_\Lambda(\Gamma \cup \{A\}) \subseteq \Gamma'$, so by the definition of Γ', $(A \to B) \in Th_\Lambda(\Gamma)$. $\Box$

A set of formulas $\Gamma \subseteq L$ is Λ-*consistent* iff $\bot \notin Th_\Lambda(\Gamma)$, and is Λ-*inconsistent*, otherwise. Γ is a *complete* Λ-*theory*, iff Γ is a Λ-consistent Λ-theory, and for every formula A, it is the case that either $A \in \Gamma$ or $\neg A \in \Gamma$. Γ is a *maximal* Λ-*theory*, iff Γ is a Λ-consistent Λ-theory, and for any set of formulas $\Sigma \subseteq L$ such that $\Gamma \subsetneq \Sigma$, Σ is Λ-inconsistent.

Corollary 15 A theory is maximal iff it is complete.

Proof First, let Γ be a complete Λ-theory and let for some set $\Sigma \subseteq L$ such that $\Gamma \subseteq \Sigma$, $A \in \Sigma \setminus \Gamma$. Then $\neg A \in \Gamma$, so by propositional reasoning $\bot \in Th_\Lambda(\Sigma)$. Now, let Γ be a maximal Λ-theory, let $A \in L$ and let $A \notin \Gamma$. Then, $\bot \in Th_\Lambda(\Gamma \cup \{A\})$, so by the deduction lemma, $(A \to \bot) \in \Gamma$, so $\neg A \in \Gamma$. $\Box$

Note that the classical Lindenbaum lemma here has the following form:

Lemma 16 (Lindenbaum Lemma) Let Γ be Λ-consistent. Then Γ can be extended to a complete Λ-theory.

Proof Let $A_1, A_2, \ldots$ be an enumeration of all formulas of the countable language L, for example, the lexicographical order. We construct by induction an infinite chain of Λ-consistent Λ-theories $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$ with the property that for every $i \geq 1$, either $A_i \in \Gamma_i$ or $\neg A_i \in \Gamma_i$ in the following way. Let Γ_0 be $Th_\Lambda(\Gamma)$. Thus Γ_0 is a Λ-consistent Λ-theory. Suppose that Γ_i is defined for some $i \geq 0$.

1. If $\Gamma_i \cup \{A_{i+1}\}$ is Λ-consistent, let $\Gamma_{i+1} =_{def} Th_\Lambda(\Gamma_i \cup \{A_{i+1}\})$.

2. If $\Gamma_i \cup \{A_{i+1}\}$ is Λ-inconsistent, then $\neg A_{i+1} \in \Gamma_i$. There are two cases.

2.1. If L has no nominals or if A_{i+1} is not in the form $AF(\bot)$, then let $\Gamma_{i+1} =_{def} \Gamma_i$.

2.2. If L is a hybrid language and A_{i+1} is $AF(\bot)$ for some admissible form $AF(\#)$, then we show that there is some nominal c such that $\Gamma_i \cup \{\neg AF(\neg c)\}$ is Λ-consistent. Suppose for the sake of contradiction that for all nominals c: $\Gamma_i \cup \{\neg AF(\neg c)\}$ is Λ-inconsistent. Then by the deduction lemma, for all c: $(\neg AF(\neg c) \to \bot) \in \Gamma_i$, hence for all c: $AF(\neg c) \in \Gamma_i$. Because Γ_i is a Λ-theory, by Cov, $AF(\bot) \in \Gamma_i$, so $A_{i+1} \in \Gamma_i$. Thus Γ_i is Λ-inconsistent, which contradicts the Λ-consistency of Γ_i. We conclude that there is a nominal c such that $\Gamma_i \cup \{\neg AF(\neg c)\}$ is Λ-consistent. Let $\Gamma_{i+1} =_{def} Th_\Lambda(\Gamma_i \cup \{\neg AF(\neg c)\})$.

According to the construction, Γ_{i+1} is a Λ-consistent Λ-theory, which is an extension of Γ_i.

Let $\Gamma^+ =_{def} \bigcup_{i=0}^{\infty} \Gamma_i$.

First note that $\perp \notin \Gamma^+$ because for all $i \geq 0$, $\perp \notin \Gamma_i$.

Now we show that Γ^+ is closed under applications of Modus Ponens. Let $A, (A \to B) \in \Gamma^+$. Then there is a step $i \geq 1$ such that $A, (A \to B) \in \Gamma_i$. But Γ_i is closed under applications of MP, so $B \in \Gamma_i \subseteq \Gamma^+$.

If L contains nominals, we show that Γ^+ is closed under applications of Cov. Let there be some $AF(\#)$ such that for all c: $AF(\neg c) \in \Gamma^+$ and suppose for the sake of contradiction that $AF(\perp) \notin \Gamma^+$. There is an index $i \geq 1$ such that $AF(\perp)$ is A_i, and by case 2.2 of the construction, there is a nominal c' such that $\neg AF(\neg c') \in \Gamma_i \subseteq \Gamma^+$. By propositional reasoning, $\perp \in \Gamma^+$, contradiction. Therefore, Γ^+ is closed under applications of Cov.

Because every formula of L is A_i for some $i \geq 1$, by the construction either $A_i \in \Gamma_i$ or $\neg A_i \in \Gamma_i$. Thus Γ^+ is a complete Λ-theory. $\square$

We denote by $\Gamma \vdash_\Lambda A$ iff $A \in Th_\Lambda(\Gamma)$. Thus $\emptyset \vdash_\Lambda A$ iff $\vdash_\Lambda A$. We denote by $\mathrm{M}, w \Vdash \Gamma$ iff for all $A \in \Gamma$, $\mathrm{M}, w \Vdash A$. We say that A is a *local semantic consequence* of Γ over the class $\mathcal{C}$ of frames, denoted by $\Gamma \Vdash_{\mathcal{C}} A$, or, if $\Gamma = \emptyset$, as $\mathcal{C} \Vdash A$, iff for every frame $\mathrm{F} \in \mathcal{C}$, every model M over F and every state w from F, it is the case that if $\mathrm{M}, w \Vdash \Gamma$, then $\mathrm{M}, w \Vdash A$. The *class of frames of* Λ, $Fr(\Lambda)$, is the class $\mathcal{C}$ of all frames F for L such that $\mathrm{F} \Vdash \Lambda$. Γ is *satisfiable on* $\mathcal{C}$ iff there is an $\mathrm{F} \in \mathcal{C}$, an M over F and a w in F such that $\mathrm{M}, w \Vdash \Gamma$.

Our goal is to examine the relationship between $\vdash$ and $\Vdash$.

Definition 17 (Weak Soundness With Respect To L) We say that a normal modal logic $\Lambda \subseteq L$ is *weakly sound with respect to* L iff $\vdash_\Lambda A$ implies $Fr(\Lambda) \Vdash A$.

Proposition 18 Let F be a frame for L which validates all premises of one of the five finitary rules - MP, Gen, Cov*, Uniform Substitution, and Nominal Substitution. Then F validates the conclusion of the rule.

Proof MP preserves truth in a point and a model. Gen preserves global truth in a model. Nominal substitution trivially preserves validity in a frame. Uniform substitution preserves validity in a frame. It remains to show that Cov* preserves validity in a frame.

Let $\mathrm{F} \nVdash AF(\perp)$, let $c \not\nearrow AF(\#)$. We show that $\mathrm{F} \nVdash AF(\neg c)$. We do this by showing that for all models $\mathrm{M} = \langle \mathrm{F}, V, H \rangle$ and states w of M, if $\mathrm{M}, w \nVdash AF(\perp)$, then there is a model $\mathrm{M}' = \langle \mathrm{F}, V, H' \rangle$ where $H'(c) = w'$ for some state w' of F and $H'(c') = H(c')$ for any other nominal c', denoted as $\mathrm{M}' = \mathrm{M}[c \mapsto w']$, such that $\mathrm{M}', w \nVdash AF(\neg c)$. The proof is by induction on $AF(\#)$.

Let $AF(\#)$ be $\#$. Then M' is $\mathrm{M}[c \mapsto w]$.

Let $AF(\#)$ be $(\gamma \to AF'(\#))$. Let $\mathrm{M}, w \Vdash \gamma$ and $\mathrm{M}, w \nVdash AF'(\bot)$. By the induction hypothesis, there is a model $\mathrm{M}' = \mathrm{M}[c \mapsto w']$ for some state w' of M, such that $\mathrm{M}', w \nVdash AF'(\neg c)$, so $\mathrm{M}', w \nVdash AF(\neg c)$.

Let $AF(\#)$ be $\Box AF'(\#)$ for some box or reversed box $\Box$ of L. Let $\mathrm{M}, w \nVdash \Box AF'(\bot)$, so there is some w_1 from M, such that $\langle w, w_1\rangle \in R_\Box$ and $\mathrm{M}, w_1 \nVdash AF'(\bot)$. By the i.h., there is a model $\mathrm{M}' = \mathrm{M}[c \mapsto w']$ for some state w' of M, such that $\mathrm{M}', w_1 \nVdash AF'(\neg c)$. Therefore, $\mathrm{M}', w \nVdash AF(\neg c)$. □

Corollary 19 Every normal modal logic for the modal language L is weakly sound with respect to L. □

Definition 20 (Strong Soundness With Respect To L) We say that a normal modal logic $\Lambda \subseteq L$ is *strongly sound with respect to* L iff for any set $\Gamma \subseteq L$, $\Gamma \vdash_\Lambda A$ implies $\Gamma \Vdash_{Fr(\Lambda)} A$.

Any normal modal logic for a language without nominals is strongly sound, because Modus Ponens preserves local truth in a model at a point.

Definition 21 (Weak Completeness With Respect To L, First Form) We say that a weakly sound with respect to L normal modal logic $\Lambda \subseteq L$ is *weakly complete with respect to* L iff $Fr(\Lambda) \Vdash A$ implies $\vdash_\Lambda A$.

Definition 22 (Weak Completeness With Respect To L, Second Form) We say that a weakly sound with respect to L normal modal logic $\Lambda \subseteq L$ is *weakly complete with respect to* L iff $\{A\}$ being Λ-consistent implies that $\{A\}$ is satisfiable on $Fr(\Lambda)$.

Definition 23 (Strong Completeness With Respect To L, First Form) We say that a strongly sound with respect to L normal modal logic $\Lambda \subseteq L$ is *strongly complete with respect to* L iff for any set $\Gamma \subseteq L$, $\Gamma \Vdash_{Fr(\Lambda)} A$ implies $\Gamma \vdash_\Lambda A$.

Definition 24 (Strong Completeness With Respect To L, Second Form) We say that a strongly sound with respect to L normal modal logic $\Lambda \subseteq L$ is *strongly complete with respect to* L iff for any set $\Gamma \subseteq L$, Γ being Λ-consistent implies that Γ is satisfiable on $Fr(\Lambda)$.

Proposition 25 The two forms of weak completeness are equivalent. The two forms of strong completeness are equivalent.

Proof We prove both results simultaneously. Following [8]:

Let $\mathcal{C} =_{def} Fr(\Lambda)$. Suppose the second form for Λ and suppose that the first form does not hold for Λ. Then, there is a set of formulas $\Gamma \cup \{A\} \subseteq L$, such that

$\Gamma \Vdash_{\mathcal{C}} A$, but $\Gamma \nvdash_\Lambda A$, and also if we are proving weak completeness, $\Gamma = \emptyset$. Suppose that $\Gamma \cup \{(A \to \bot)\}$ is Λ-inconsistent. Then $\bot \in \mathit{Th}_\Lambda(\Gamma \cup \{(A \to \bot)\})$, and by propositional reasoning, $A \in \mathit{Th}_\Lambda(\Gamma \cup \{(A \to \bot)\})$, so, by the deduction lemma, $((A \to \bot) \to A) \in \mathit{Th}_\Lambda(\Gamma)$, and by propositional reasoning, $A \in \mathit{Th}_\Lambda(\Gamma)$, contradiction. Therefore, $\Gamma \cup \{(A \to \bot)\}$ is Λ-consistent. But also it is not satisfiable on (any frame of) $\mathcal{C}$, contradicts the second form.

For the other direction, let the first form hold for Λ. Let $\Gamma \subseteq L$ be a Λ-consistent set, and also if we are proving weak completeness, let Γ be a singleton. Suppose that for all $\mathrm{F} \in \mathcal{C}$, for all M over F and all worlds w in F, there exists a formula $A \in \Gamma$, such that $\mathrm{M}, w \nVdash A$, and so $\mathrm{M}, w \nVdash \Gamma$. Then $\Gamma \Vdash_{\mathcal{C}} \bot$, and by the first form, $\Gamma \vdash_\Lambda \bot$, contradiction. $\square$

Clearly, strong soundness implies weak soundness and strong completeness implies weak completeness. Therefore, sometimes we speak of *sound and complete* logics and we mean weakly sound and complete logics.

Here, similarly to [8][30][16][53][32][18][27], we show that:

1. For a hybrid language L and di-persistent formula $A \in L$, the logics K_L and $K_L + A$ are weakly sound and complete with respect to L.

2. For a non-hybrid language L and d-persistent formula $A \in L$, the logics K_L and $K_L + A$ are strongly sound and complete with respect to L.

In both cases, we show the second form of completeness, as it is the most convenient.

Definition 26 Let $\Box$ be a box or a reversed box of L. Then $\Box\Gamma$ is the set $\{A \mid \Box A \in \Gamma\}$.

Lemma 27 Let Γ, Σ and Δ be Λ-consistent Λ-theories. Let $\Box$ be any box or reversed box of L. Then

1. The set $\Gamma' =_{def} \Box\Gamma$ is a Λ-theory and if for some formula A, $\Box A \notin \Gamma$, then Γ' is Λ-consistent.
2. If $0 \in S$, then $[U]\Gamma$ is Λ-consistent, $[U]\Gamma \subseteq \Gamma$, and $[U]\Gamma \subseteq \Box\Gamma$.
3. If Γ is complete, then $\Box A \notin \Gamma$ iff there is a complete Λ-theory Σ such that $\Box\Gamma \subseteq \Sigma$ and $A \notin \Sigma$.
4. For all $i \in D$, if Γ and Σ are complete, then $\Box_i\Gamma \subseteq \Sigma$ iff $\Box_i^{-1}\Sigma \subseteq \Gamma$.
5. If $0 \in S$, Γ and Σ are complete, then $[U]\Gamma \subseteq \Sigma$ iff $[U]\Sigma \subseteq \Gamma$.
6. If $0 \in S$, Γ, Σ and Δ are complete, $[U]\Gamma \subseteq \Sigma$, and $[U]\Sigma \subseteq \Delta$, then $[U]\Delta \subseteq \Gamma$.
7. If $0 \in S$, Γ and Σ are complete, and $[U]\Gamma \subseteq \Sigma$, then $[U]\Gamma = [U]\Sigma$.

Proof We only show 1. The proofs for the rest are standard, and follow easily by the axioms, 1., the deduction lemma and the Lindenbaum lemma.

Let $(A \to B), A \in \Gamma'$, therefore $\Box(A \to B), \Box A \in \Gamma$. Because of (K), $\vdash_\Lambda (\Box(A \to B) \to (\Box A \to \Box B))$, therefore, by MP, $\Box B \in \Gamma$, so $B \in \Gamma'$.

Now, let L be a hybrid language and let for all c, $AF(\neg c) \in \Gamma'$. Then for all c, $\Box AF(\neg c) \in \Gamma$, so by Cov, $\Box AF(\bot) \in \Gamma$ and hence $AF(\bot) \in \Gamma'$.

Finally, if $A \in L$ and $\Box A \notin \Gamma$, then $A \notin \Gamma'$ and hence Γ' is Λ-consistent. □

Let Λ be a logic for L. Let W be either the set of all complete Λ-theories Σ, such that $[U]\Gamma \subseteq \Sigma$ for a given complete Λ-theory Γ, if L contains the universal modality, or the set of all complete Λ-theories, otherwise. Let $\mathcal{R} : S \mapsto \mathbb{P}(W \times W)$ be such that if $0 \in S$, then $\mathcal{R}(0) = W \times W$, and for all $i > 0$, such that $i \in S$, $\langle \Sigma_1, \Sigma_2 \rangle \in \mathcal{R}(i)$ iff $\Box_i \Sigma_1 \subseteq \Sigma_2$. Clearly F is a frame for L. Thus F is called either *the* Λ*-canonical frame for* Γ, if $0 \in S$, or just *the canonical frame for* Λ, or *the* Λ*-canonical frame*, otherwise.

Proposition 28 Let L contain nominals (and hence, L also contains the universal modality). Let Λ be a logic for L, and let Γ be a complete Λ-theory. If $\mathrm{F} = \langle W, \mathcal{R} \rangle$ is the Λ-canonical frame for Γ, then

1. for every $\Sigma \in W$ at least one $c \in \Sigma$.
2. for every c there is exactly one $\Sigma \in W$ such that $c \in \Sigma$.

Proof 1. Let Σ be a complete Λ-theory. Suppose that for all c, $c \notin \Sigma$. Then, by the completeness of Σ, for all c, $\neg c \in \Sigma$. Therefore, by Cov, $\bot \in \Sigma$, contradiction.

2. First, we show that for every c, there is a $\Sigma \in W$ such that $c \in \Sigma$. Suppose this is not the case, so there is a c such that for all $\Sigma \in W$, $c \notin \Sigma$. Then $\neg c \in \Sigma$, therefore by Lemma 27, $[U]\neg c \in \Gamma$, which contradicts axiom (Nom1). Second, let for some c there be $\Sigma_1, \Sigma_2 \in W$, such that $c \in \Sigma_1 \cap \Sigma_2$. Let $A \in \Sigma_1$. Then, $(c \wedge A) \in \Sigma_1$. Suppose $A \notin \Sigma_2$, then $(c \to \neg A) \in \Sigma_2$. Now, there are two cases. First, if $[U](c \to \neg A) \in \Sigma_2$, then because of the definition of W, $(c \to \neg A) \in \Sigma_1$, contradiction. Second, if $[U](c \to \neg A) \notin \Sigma_2$, then $\neg[U](c \to \neg A) \in \Sigma_2$, so $\langle U \rangle(c \wedge A) \in \Sigma_2$, then because of (Nom2), $[U](c \to A) \in \Sigma_2$, but $[U]\Sigma_2 \subseteq \Sigma_2$, so $A \in \Sigma_2$, contradiction. So, we have that $\Sigma_1 \subseteq \Sigma_2$. The converse inclusion is proven similarly, therefore $\Sigma_1 = \Sigma_2$. □

It easily follows that all axioms of K_L are valid in any Λ-canonical frame, because they are valid on all frames for L.

We are now ready to define the Λ-*canonical model* for a given complete Λ-theory Γ if L contains the universal modality, or just the Λ-*canonical model*, otherwise. Let $\mathrm{F} = \langle W, \mathcal{R} \rangle$ be the Λ-canonical frame (for Γ), then we define $\mathrm{M} =_{def} \langle \mathrm{F}, V, H \rangle$, where $V(p) =_{def} \{\Sigma \in W \mid p \in \Sigma\}$; if L contains nominals, $H(c) =_{def} \Sigma$, where Σ is the only element of W, such that $c \in \Sigma$, or otherwise H is the empty function. The definition of H is correct by Proposition 28. It follows that if L contains nominals, then M is a *named model*.

Lemma 29 (Truth Lemma) Let $\mathrm{M} = \langle\langle W, \mathcal{R}\rangle, V, A\rangle$ be the Λ-canonical model for some complete Λ-theory Γ, if L contains the universal modality, or just the Λ-canonical model otherwise. Then for any formula $A \in L$ and any world Σ in M, $A \in \Sigma$ iff $\mathrm{M}, \Sigma \Vdash A$.

Proof Induction on $A \in L$. For atomic A and for $\bot$, the result follows by the definition of the canonical model. For $(A \to B)$, the result follows by the induction hypothesis and propositional reasoning.

For $\Box_i^{-1}A$: first, let $\Box_i^{-1}A \in \Sigma$. Let $W' =_{def} \{\Sigma' \in W \mid \Box_i\Sigma' \subseteq \Sigma\}$. We show that W' contains all Λ-complete Λ-theories Σ', such that $\Box_i\Sigma' \subseteq \Sigma$. This trivially holds if L does not contain the universal modality, so let $0 \in S$. Because by Lemma 27, for any Λ-consistent Λ-theory Σ', $[U]\Sigma' \subseteq \Box_i\Sigma'$, we have that for all Λ-complete Σ', such that $[U]\Sigma' \subseteq \Box_i\Sigma' \subseteq \Sigma$, it is the case that $[U]\Sigma' = [U]\Sigma = [U]\Gamma$, therefore $\Sigma' \in W$. By Lemma 27, for all $\Sigma' \in W'$, $\Box_i^{-1}\Sigma \subseteq \Sigma'$, so $A \in \Sigma'$. By the induction hypothesis, for all $\Sigma' \in W'$: $\mathrm{M}, \Sigma' \Vdash A$, so, by the definition of $\mathcal{R}(i)$ and the definition of Kripke semantics, $\mathrm{M}, \Sigma \Vdash \Box_i^{-1}A$. Now, let $\mathrm{M}, \Sigma \Vdash \Box_i^{-1}A$. Then, using the same definition of W', we have that for all $\Sigma' \in W'$, we can use the induction hypothesis and find that $A \in \Sigma'$. Because W' contains exactly all Λ-complete Λ-theories Σ', such that $\Box_i^{-1}\Sigma \subseteq \Sigma'$, then it follows that $\Box_i^{-1}A \in \Sigma$. For $\Box_i A$, the result follows by Lemma 27. $\Box$

Definition 30 Let Λ be a logic for L and let Λ be valid in any Λ-canonical frame. Then the logic Λ is *canonical with respect to* L. Let $A \in L$ and let A be valid in any Λ-canonical frame for $K_L + A$. Then we say that the formula A is *canonical with respect to* L.

Theorem 31 *If a logic Λ for L is canonical with respect to L, then it is also weakly consistent and weakly complete with respect to L. If L is a non-hybrid language, then Λ is also strongly consistent and strongly complete with respect to L.*

Proof We use the second form of completeness. Like [8][30][16][53][32][18][27]:

We have already shown the soundness results above.

Now, let us show completeness.

Let Γ be a Λ-consistent set. In the case of a hybrid language, let Γ also be a singleton. By the Lindenbaum lemma, there is a complete Λ-theory Γ^+ extending Γ. Let M be the Λ-canonical model M for Γ^+ if L contains the universal modality, or just the Λ-canonical model otherwise. Let the universe of M be W. By the Truth Lemma 29, Γ^+ is satisfiable in M at Γ^+, therefore Γ also is. The frame of M, F, also validates Λ by the fact that Λ is canonical. If L is hybrid, by the second form of weak completeness, Λ is weakly complete with respect to L. If L is non-hybrid, by the second form of strong completeness, Λ is strongly complete with respect to L. $\square$

The following theorem lists three well-known facts in literature, which we need for our further discussions.

Theorem 32 *(1) K_L is a canonical logic with respect to L. (2) If L is a hybrid language containing the universal modality, then for any di-persistent with respect to L formula $A \in L$, it is the case that $K_L + A$ is canonical with respect to L. (3) If L does not contain nominals, then any logic $K_L + A$ is canonical with respect to L for any d-persistent with respect to L formula $A \in L$.*

Proof For (1), we have already shown that all axioms of K_L are valid on any K_L-canonical frame because they are valid on all frames for L.

For (2) and (3), like in [8][30][16][53][32][18][27]:

Let $\Lambda =_{def} K_L + A$. We assume that $\perp \notin \Lambda$ because otherwise the result trivially holds.

Let Γ be a complete Λ-theory, let $\mathrm{F} = \langle W, \mathcal{R} \rangle$ be the Λ-canonical frame for Γ (or just the Λ-canonical frame if L does not contain the universal modality). Let $\mathrm{M} = \langle \mathrm{F}, V, H \rangle$ be the Λ-canonical model for Γ (or just the Λ-canonical model if L does not contain the universal modality). Now, we construct a general frame over M in the following way. For all modal formulas $B \in L$, denote $\widehat{B} =_{def} \{\Sigma \in W \mid B \in \Sigma\}$. Let $\mathbb{W} =_{def} \{\widehat{B} \mid B \in L\}$ and let $\mathfrak{g} =_{def} \langle \mathrm{F}, \mathbb{W} \rangle$.

We show that $\mathfrak{g}$ is a general frame and $\mathfrak{g} \Vdash A$.

The first follows directly from the Truth Lemma 29, which allows us to check the closure conditions for $\mathbb{W}$, namely, relative complement ($W \backslash \widehat{B} = \widehat{\neg B}$), union ($\widehat{B_1} \cup \widehat{B_2} = \widehat{(B_1 \vee B_2)}$), and operator $[\![\Diamond p_1]\!]$ for any diamond and reversed diamond $\Diamond$ of L ($[\![\Diamond p_1]\!](\widehat{B}) = \widehat{\Diamond B}$).

For the second, first note that for any complete Λ-theory Σ, $A \in \Lambda \subseteq \Sigma$, so by the Truth Lemma 29, $\mathrm{M} \Vdash A$, so $[\![A]\!]_{\mathrm{M}} = W$. Clearly M is a model over $\mathfrak{g}$. If $PROP(A) \cup NOM(A) = \emptyset$, then we are done. Otherwise, let all propositional variables occurring in A be among $p_1, \ldots, p_n$, and let all nominals occurring in A be among $c_1, \ldots, c_m$, where $m = 0$ if L does not contain nominals. Then clearly for any model M' over $\mathfrak{g}$, $[\![A]\!]_{\mathrm{M}'} = [\![A]\!](s_1, \ldots, s_n, w_1, \ldots, w_m)$ for some $s_1, \ldots, s_n \in \mathbb{W}$ and $w_1, \ldots, w_m \in W$, by the definition of $\mathbb{W}$ as the extensions in M of all possible formulas, and the fact that if L has nominals, then every w_i contains a nominal, is equal to the following set:
$[\![A[p_1/A_1, \ldots, p_n/A_n, c_1/c'_1, \ldots, c_m/c'_m]]\!]_{\mathrm{M}}$ for some formulas $A_1, \ldots, A_n$ and some nominals $c'_1, \ldots, c'_m$. However, $\Lambda \subseteq \Sigma$ for any complete Λ-theory Σ, and Λ is closed under applications of uniform substitution and nominal substitution. Therefore, for all $\Sigma \in W$: $A[p_1/A_1, \ldots, p_n/A_n, c_1/c'_1, \ldots, c_m/c'_m] \in \Sigma$. So, by the truth lemma, $\mathfrak{g} \Vdash A$.

To prove (2), let us show that if L is a hybrid language with the universal modality, then $\mathfrak{g}$ is discrete. But this follows by the fact that every nominal is a modal formula. Thus there is a discrete general frame over F which validates A, and A is di-persistent, therefore $\mathrm{F} \Vdash A$ and this makes Λ canonical.

To show (3), let L be a non-hybrid modal language. We must show that $\mathfrak{g}$ is a descriptive frame. Then the result follows by the facts that $\mathfrak{g} \Vdash A$ and A is d-persistent.

To show that $\mathfrak{g}$ is differentiated, let Σ and Θ be distinct complete Λ-theories from W. It is not hard to show that there is a formula $B \in \Sigma$ such that $B \notin \Theta$. Equivalently $\Sigma \in \widehat{B}$ and $\Theta \notin \widehat{B}$. Thus, we established the contrapositive for the differentiation condition equivalence.

For tightness, let Σ and Θ be two complete Λ-theories from W, and let for some $i \in S$, $\langle \Sigma, \Theta \rangle \notin \mathcal{R}(i)$. Equivalently, $\Diamond B \notin \Sigma$, but $B \in \Theta$ for some $B \in L$, so equivalently, $\Theta \in \widehat{B}$ but $\Sigma \notin \widehat{\Diamond B}$, and we've established the contrapositive of the first equivalence for tightness. The second tightness condition is proved similarly for every $i \in D$.

For compactness, first we need a lemma on Λ-consistent Λ-theories. Let Σ be a Λ-consistent set of L-formulas and let $\bot \in Th_\Lambda(\Sigma)$. Then there are formulas $B_1, \dots, B_n \in \Sigma$, such that $\vdash_\Lambda ((B_1 \wedge \cdots \wedge B_n) \to \bot) \in \Lambda$. This can be proven by first showing that for any formula $B \in Th_\Lambda(\Sigma)$, there is a finite sequence of formulas $B_1, \dots, B_n$, such that B_n is B and each element of the sequence is either Σ, from Λ, or is derived by an application of Modus Ponens by preceding formulas. This follows by the fact that Modus Ponens is a finitary rule and theories in non-hybrid languages do not have to be closed under applications of the infinitary rule Cov. Then we can get the desired conclusion by propositional reasoning and n applications of the deduction lemma.

Let W_0 be any non-empty family of admissible (over $\mathfrak{g}$) sets with the fip. We must show that it has a non-empty intersection. But $W_0 =_{def} \{\widehat{B} \mid B \in \Sigma\}$, where Σ is a non-empty set of L-formulas. It follows that Σ is Λ-consistent, because suppose for the sake of contradiction that it's not. Then, there are $B_1, \dots, B_n \in \Sigma$: $\vdash_\Lambda (B_1 \wedge \cdots \wedge B_n) \to \bot$. This implies that there is no complete Λ-theory Θ such that $\psi_1, \dots, \psi_n \in \Theta$. But then $\widehat{\psi_1} \cap \cdots \cap \widehat{\psi_n} = \emptyset$, contradicts our assumption on W_0. Now, let $\Delta = [U]\Gamma$ if Λ has the universal modality, or let $\Delta = \emptyset$, otherwise. We show that Δ is Λ-consistent. Because $\bot \notin \Lambda$, if $\Delta = \emptyset$, it is trivially Λ-consistent, so suppose that $\Delta \neq \emptyset$. By the definition of $\mathcal{R}(0)$ and the fact that $\mathcal{R}(0)$ is reflexive, it follows that $\Delta \subseteq \Gamma$, and therefore Δ is Λ-consistent. Now that we know Δ is Λ-consistent, we see that the set $W_1 =_{def} \{\widehat{B} \mid B \in \Delta\}$ has the fip, because otherwise there are $B_1, \dots, B_m \in \Delta$ such that $\vdash_\Lambda (B_1 \wedge \cdots \wedge B_m) \to \bot$, contradiction. Now, we prove that $\Sigma \cup \Delta$ is Λ-consistent. Suppose for the sake of contradiction that it is not, so there are $A_1, \dots, A_n \in \Sigma$ and $B_1, \dots, B_m \in \Delta$ such that $\vdash_\Lambda A_1 \wedge \cdots \wedge A_n \wedge B_1 \wedge \cdots \wedge B_m \to$

$\bot$. Because both Σ and Δ are Λ-consistent, we have that $n \geq 1$ and $m \geq 1$. But then $\widehat{A_1} \cap \cdots \cap \widehat{A_n} \cap \widehat{B_1} \cap \cdots \cap \widehat{B_m} = \emptyset$. Let $W' = \widehat{A_1} \cap \cdots \cap \widehat{A_n}$, and let $W'' = \widehat{B_1} \cap \cdots \cap \widehat{B_m}$. Because both W_0 and W_1 have the fip, both W' and W'' are non-empty. Let $\Sigma_0 \in W'$, it follows that $\Sigma_0 \notin W''$. Because of the definitions of W' and W'', there are i: $1 \leq i \leq n$ and j: $1 \leq j \leq m$ such that $\Sigma_0 \in \widehat{A_i}$ and $\Sigma_0 \notin \widehat{B_j}$. First, $\Sigma_0 \in W$, so by the definition of Δ, $\Delta \subseteq \Sigma_0$, and therefore $B_j \in \Sigma_0$. Second, $\Sigma_0 \notin \widehat{B_j}$, so therefore $B_j \notin \Sigma_0$, contradiction with $B_j \in \Sigma_0$. We conclude that $\Sigma \cup \Delta$ is Λ-consistent. Now, $\Sigma \cup \Delta$ can be extended, by Lindenbaum's lemma, to a complete Λ-theory Σ^+. Because of the definition of Δ, and the fact that $\Delta \subseteq \Sigma \subseteq \Sigma^+$, $\Sigma^+ \in W$, and we almost immediately obtain that $\Sigma^+ \in \bigcap\{\widehat{B} \mid B \in \Sigma\}$, so W_0 has a non-empty intersection. $\Box$

2.12 Finite Model Property

Definition 33 Let L be a modal language and let $\mathcal{C}$ be a class of frames for L. We say that $\mathcal{C}$ has *the finite model property* iff for every formula $A \in L$ such that $\mathcal{C} \not\Vdash A$, there exists a finite model M over some finite frame $\mathrm{F} \in \mathcal{C}$ and a state w in M, such that $\mathrm{M}, w \not\Vdash A$. If L is such that if L contains nominals, it also contains the universal modality, and if $\Lambda \subseteq L$ is a normal modal logic, we say that Λ has *the finite model property* iff $\mathit{Fr}(\Lambda)$ has the finite model property.

Usually, the method of *filtration* is used to prove that a logic has the finite model property, see [8]. In this work, we use that the logics S5 and KD45 in the context of ML($\Box$) have the finite model property. These are well-known facts in literature, and we use Ehrenfeucht-Fraïssé games and Ehrenfeucht's theorem to show them, as explained in [22] - see Lemma 97 in Section 4. We show a similar result with respect to ML($\Box, [U]$) in Section 5, using the properties of the Standard Translation.

2.13 Decidability of Normal Modal Logics

Definition 34 Let L be a modal language such that if L contains nominals, then it also contains the universal modality, and let $\Lambda \subseteq L$ be a normal modal logic. Clearly, there can be an effective encoding of formulas $A \in L$ as natural numbers, so we may treat L as a recursive set and Λ as a subset of $\mathbb{N}$.

We say that Λ is *decidable* iff Λ is a recursive set, up to an encoding of the formulas of L. We say that Λ is *semi-decidable* iff Λ is a recursively enumerable set (up to an encoding of formulas as numbers).

If $\Lambda = K_L + \Gamma$ for some set $\Gamma \subseteq L$, we say that Λ is *recursively axiomatizable* iff the set of axioms Γ is recursive (up to an encoding).

Proposition 35 Let L be a modal language such that, if L contains nominals, then it also contains the universal modality, and let $\Lambda \subseteq L$ be a recursively axiomatizable normal modal logic with a recursive set of axioms Γ. Then Λ is semi-decidable. Moreover, if Λ is canonical, Λ has the finite model property, and the set of (up to isomorphism) finite models for L, which are based on frames that validate Λ, is recursive up to an encoding of finite Kripke models, then Λ is decidable, and the set $\{A \in L \mid Fr(\Lambda) \Vdash A\}$ is recursive.

Proof Clearly, for every formula $A \in \Lambda$, there exists a finite sequence of formulas $A_1, \ldots, A_n$, which we call a *proof of* A *in* Λ, such that A is A_n and every A_i of this sequence is either one of the axioms of K_L, an element of Γ, or is derived by previous elements of the sequence by applying one of the rules Modus Ponens, Gen, Uniform Substitution, Nominal Substitution, or Cov*. Thus, to check whether $A \in \Lambda$, it is enough to enumerate all possible finite sequences of formulas of L, and check whether we have found a proof of A in Λ, which is a recursive predicate because the derivation rules are finitary, because the axioms of K_L are a recursive set, and so is Γ.

Now, let Λ be canonical, let Λ have the finite model property, and let the set of finite Kripke models for L, which are based on frames that validate Λ, be recursive. Because Λ is canonical, Λ is weakly complete, so $\Lambda = \{A \in L \mid Fr(\Lambda) \Vdash A\}$. Because Λ has the finite model property and because the set of finite models for L, which are based on frames that validate Λ, is recursive, then it is easy to obtain a semi-decidable procedure finding, given a formula $A \in L$, whether $Fr(\Lambda) \nVdash A$, by generating all possible finite models for L, checking whether they are based on frames which validate Λ, and finally checking whether the current model invalidates A. Thus, the set $L \setminus \Lambda$ is recursively enumerable, and so is the set Λ. Clearly, the set $\mathbb{N} \setminus \Lambda$ is also recursively enumerable because L is recursive. Thus Λ is recursive (decidable). $\square$

3 Deterministic SQEMA

3.1 Introduction to Deterministic SQEMA

In our discussions here, we will reduce the modal languages that we are discussing to only a few types.

We say that a language L is *of type 1* iff it is a non-hybrid language (either temporal or not, either containing the universal modality or not). We say that a language L is *of type 2* iff it is a hybrid temporal language containing the universal modality.

In our discussion of the algorithm Deterministic SQEMA, we assume that the language of the input formula is either of type *1* or of type *2*. We make a distinction between type *1* and type *2* languages in that, by Theorem 32, a formula of a type *1* language L_1 which is d-persistent with respect to L_1 is canonical with respect to L_1, whereas a formula of a type *2* language L_2 which is di-persistent with respect to L_2 is canonical with respect to L_2.

As we see below, Deterministic SQEMA only succeeds on d-persistent with respect to their language formulas of type *1* languages and on di-persistent with respect to their language formulas of type *2* languages, thus we make sure that Deterministic SQEMA can be used to prove canonicity of the formula with respect to either a type *1* language or a type *2* language. In order to use Deterministic SQEMA to prove canonicity of a formula with respect to another kind of hybrid modal language, some further restrictions on the used rules are required, as shown in [18], and is outside the scope of this work.

3.2 Strategy of Deterministic SQEMA

Definition 36 (Local Correspondence) Let L be a modal language. We say that a formula $A \in L$ and a FOL formula $\psi(x)$ are *locally correspondent with respect to the class of frames $\mathcal{C}$ for L*, denoted by $A \sim \psi(x)$, iff for all frames $\mathrm{F} \in \mathcal{C}$ and all states w in F, it is the case that $\mathrm{F}, w \Vdash A$ iff $\mathrm{F} \vDash \psi[x \mapsto w]$. Clearly if A and $\psi(x)$ are locally correspondent with respect to $\mathcal{C}$, then they are also globally correspondent with respect to $\mathcal{C}$.

If L is a language, denote by $\mathcal{C}_L$ the class of all frames for L.

Our goal is to find, if possible, a local first-order correspondent of a formula A from a language L (of type *1* or *2*), with respect to $\mathcal{C}_L$, and at the same time to prove that A is canonical with respect to L.

To do that, we first extend L to a language L', in the following way. Let S be the set of indices of the diamonds of L, and let D be the set of indices of the reversed diamonds of L. Let L' be the temporal hybrid language with a set of indices of diamonds and reversed diamonds S. Thus we have made sure all modalities of L have their corresponding reversed ones in L' and we have also possibly added nominals to the language. Thus the algorithm works by taking an input in L and working with formulas in L'. Clearly, $\mathcal{C}_L = \mathcal{C}_{L'}$, but the class of general frames for L is not always the class of general frames for L'. Also, if $\mathfrak{g}$ is a general frame for L and $\mathrm{M} = \langle \mathfrak{g}_{\#}, V, H \rangle$ is a model over $\mathfrak{g}$ for L, then it may not always be a model for L', because the valuation H may be the empty function if L doesn't have nominals. Still, it is possible to define the meaning of $\mathfrak{g}, w \Vdash A'$ for some $A' \in L'$ in a similar way to the regular definition of local validity in a general frame, by constructing models for L' over $\mathfrak{g}$.

For denotation purposes, let $L'+[U]$ be L' with an added universal modality, if L' doesn't already have it. The algorithm does not operate on formulas of $L' + [U]$, but it simplifies our description of the properties of the algorithm to be able to use formulas of this language. Clearly, we may see that $\mathcal{C}_L = \mathcal{C}_{L'} = \mathcal{C}_{L'+[U]}$, and also we may define local validity of formulas of $L' + [U]$ in general frames for L.

Definition 37 (Relative Local Frame-Equivalence) Let L be a modal language, let L' be the temporal hybrid extension of L as defined above, and let $L' + [U]$ be the extension of L' with the universal modality, as shown above. For a given k, two formulas $A \in L' + [U]$ and $B \in L' + [U]$ are *locally frame-equivalent with respect to $\mathcal{C}_L$ and c_k*, denoted by $A \sim_k^L B$, iff for every frame F for L and every state w in F, we have that for every model M for $L' + [U]$ over F, $\mathrm{M}, w \Vdash (c_k \to \langle U \rangle A)$ iff for every model M for $L' + [U]$ over F, $\mathrm{M}, w \Vdash (c_k \to \langle U \rangle B)$. If the language L is known in the context of discussion, we may just write $A \sim_k B$.

Let $\mathcal{C}$ be a class of general frames for L. We say that A and B are *locally equivalent with respect to $\mathcal{C}$ and c_k*, denoted by $A \approx_k^{\mathcal{C}} B$, iff for every general frame $\mathfrak{g} \in \mathcal{C}$ and every state w in $\mathfrak{g}$, we have that for every model M for $L' + [U]$ over $\mathfrak{g}$, $\mathrm{M}, w \Vdash (c_k \to \langle U \rangle A)$ iff for every model M for $L' + [U]$ over $\mathfrak{g}$, $\mathrm{M}, w \Vdash (c_k \to \langle U \rangle B)$. If the language L is known and $\mathcal{C}$ is the class of all descriptive frames for L, then we may just write $A \approx_k^d B$ (A and B are *locally d-equivalent with respect to c_k*), or if $\mathcal{C}$ is the class of all discrete frames for L, we may just write $A \approx_k^{di} B$ (A and B are *locally di-equivalent with respect to c_k*).

Let c_k be a nominal. Let $\psi(x_k)$ be a FOL formula. We say that A and $\psi(x_k)$ are *locally correspondent with respect to L and c_k*, denoted by $A \sim_k^L \psi(x_k)$ iff for all frames F for L and all states w in F, it is the case that for every model M for $L' + [U]$ over F, $\mathrm{M}, w \Vdash (c_k \to \langle U \rangle A)$ iff $\mathrm{F} \vDash \psi[x_k \mapsto w]$. If the language L is known, we may simply write $A \sim_k \psi(x_k)$.

An easy argument shows that if $c_k \not\nearrow A$, and $(c_k \wedge A) \sim_k \psi(x_k)$, then $A \sim \psi(x_k)$. Also, if $A \sim_k B$, and if $B \sim_k \psi(x_k)$, then $A \sim_k \psi(x_k)$.

Definition 38 (Pure Formulas) A modal formula A is called *pure* iff it does not contain any occurrences of propositional variables.

Definition 39 (Deterministic SQEMA Strategy) Let L be a modal language of either type *1* or of type *2*. Let $A \in L$ be the input formula of Deterministic SQEMA. Let all nominals occurring in A be among $c_1, \ldots, c_{k-1}$. We try to find a sequence of formulas $A_1, \ldots, A_n \in L'$, such that A_1 is $(c_k \wedge A)$, A_n is pure, and for all A_i and A_j of the sequence, it is the case that $A_i \sim_k A_j$. In addition to that, if L is a type *1* language, we require that for all A_i and A_j of the sequence, $A_i \approx_k^d A_j$ (that is, with respect to all descriptive general frames for L), and if L is a type *2* language, we require that for all A_i and A_j of the sequence, $A_i \approx_k^{di} A_j$ (with respect to the discrete general frames for L).

Proposition 40 Let $A, A_1, \ldots, A_n$ be as in Definition 39.

(1) If L is a type *1* language then A is d-persistent.
(2) If L is a type *2* language then A is di-persistent.

Proof Let $\mathfrak{g}$ be a general frame for L, which is descriptive if L is a type *1* language, and which is discrete if L is a type *2* language. Let w be a world in $\mathfrak{g}$. Then:

$\mathfrak{g}, w \Vdash A$ iff (because $c_k \not\nearrow A$)

$\mathfrak{g}, w \Vdash (c_k \to \langle U\rangle(c_k \wedge A))$ iff (because $A_1 \approx_k^d A_n$ or $A_1 \approx_k^{di} A_n$)

$\mathfrak{g}, w \Vdash (c_k \to \langle U\rangle A_n)$ iff (because A_n is pure and there are no restrictions on assignments of nominals in a model over $\mathfrak{g}$)

$\mathfrak{g}_{\#}, w \Vdash (c_k \to \langle U\rangle A_n)$ iff (because $A_n \sim_k A_1$)

$\mathfrak{g}_{\#}, w \Vdash (c_k \to \langle U\rangle(c_k \wedge A))$ iff (because $c_k \not\nearrow A$)

$\mathfrak{g}_{\#}, w \Vdash A$. □

It remains to find the local first-order correspondent of A given the pure formula A_n.

Definition 41 (Standard Translation for Pure Formulas) In the function definition below, $st(n, x, A)$ stands for *standard translation for pure formulas.* Denoting $st(n, x_\ell, A)$, we assume that A is a pure formula in a hybrid language, that x_ℓ is such that c_ℓ does not occur in A, and that n is such that $\ell < n$ and all nominals occurring in A are among $\{c_1, \ldots, c_{n-1}\}$.

$st(n, x_i, \bot) =_{def} \bot$

$st(n, x_i, \top) =_{def} \top$

$st(n, x_i, c_j) =_{def} (x_i = x_j)$ for all $j \in \mathbb{N}$ such that $i \neq j$

$st(n, x_i, \neg A) =_{def} \neg st(n, x_i, A)$

$st(n, x_i, (A \vee B)) =_{def} (st(n, x_i, A) \vee st(n', x_i, B))$, where n' is the least number such that $n' \geq n$, $n' > i$ and for all x_j, occurring in $st(n, x_i, A)$, $n' > j$.

$st(n, x_i, (A \wedge B)) =_{def} (st(n, x_i, A) \wedge st(n', x_i, B))$, where n' is the least number such that $n' \geq n$, $n' > i$ and for all x_j, occurring in $st(n, x_i, A)$, $n' > j$.

$st(n, x_i, \langle U\rangle A) =_{def} \exists x_n st(n+1, x_n, A)$

$st(n, x_i, \Diamond_0{}^{-1} A) =_{def} \exists x_n st(n+1, x_n, A)$

$st(n, x_i, [U]A) =_{def} \forall x_n st(n+1, x_n, A)$

$st(n, x_i, \Box_0{}^{-1} A) =_{def} \forall x_n st(n+1, x_n, A)$

$st(n, x_i, \Diamond_m A) =_{def} \exists x_n((x_i\ r_m\ x_n) \wedge st(n+1, x_n, A))$

$st(n, x_i, \Diamond_m{}^{-1} A) =_{def} \exists x_n((x_n\ r_m\ x_i) \wedge st(n+1, x_n, A))$

$st(n, x_i, \Box_m A) =_{def} \forall x_n(\neg(x_i\ r_m\ x_n) \vee st(n+1, x_n, A))$

$st(n, x_i, \Box_m{}^{-1} A) =_{def} \forall x_n(\neg(x_n\ r_m\ x_i) \vee st(n+1, x_n, A))$

It is immediate that *st* defines a unique function if the conditions for it hold. It is also clear that the result of *st* can be effectively obtained.

Clearly, the output of this function is a FOL formula with free variables among $\{x_1, \ldots, x_{n-1}\}$.

An easy, but somewhat tedious, induction on pure formulas A shows that, under the above assumptions for n and x_i, for any model $\mathrm{M} = \langle \mathrm{F}, V, H \rangle$ for a hybrid language which contains A and any world w in M, it is the case that:

$$\mathrm{M}, w \Vdash A \text{ iff } \mathrm{F} \vDash st(n, x_i, A)[H(c_1), \ldots, H(c_{n-1}), x_i \mapsto w]$$

We call this the *main property of st*.

Lemma 42 (Standard Translation Lemma) Let c_k be a nominal, and let A be a pure formula of a hybrid language L. Then for A there can be effectively obtained a first-order formula $\psi(x_k)$, such that $A \sim_k \psi(x_k)$.

Proof Let i be such that all nominals occurring in A be among $\{x_1, \ldots, x_{i-1}\}$. Consider ψ: $\forall x_{j_1} \ldots \forall x_{j_m} \exists x_i st(n, x_i, A)$, where $n = i + 1$, and $[j_1, \ldots, j_m]$ are such that $[c_{j_1}, \ldots, c_{j_m}]$ is the list of elements of $NOM(A) \setminus \{c_k\}$ in left-to-right order of initial occurrence in A. Note that ψ can be denoted by $\psi(x_k)$, because the only free variable, if any, is x_k. We show that $A \sim_k \psi(x_k)$. For convenience, denote $\neg A$ by A'.

For given F for L and w in F, let $\mathrm{M} = \langle \mathrm{F}, V, H \rangle$ be a model over F such that $\mathrm{M}, w \Vdash c_k$ and $\mathrm{M} \Vdash A'$. By the main property of st, for every w_1 in F: $\mathrm{F} \vDash st(n, x_i, A')[H(c_1), \ldots, H(c_{i-1}), w_1]$ iff $\mathrm{M}, w_1 \Vdash A'$. Therefore, $\mathrm{F} \vDash \forall x_i st(n, x_i, A')[H(c_1), \ldots, H(c_{i-1})]$. In this way, because x_k is the only free variable in $\exists x_{j_1} \ldots \exists x_{j_m} \forall x_i st(n, x_i, A')$, if any, it becomes apparent that, $\mathrm{F} \vDash \exists x_{j_1} \ldots \exists x_{j_m} \forall x_i st(n, x_i, A')[x_k \mapsto H(c_k)]$. Because $H(c_k) = w$, we have that $\mathrm{F} \vDash \exists x_{j_1} \ldots \exists x_{j_m} \forall x_i st(n, x_i, A')[x_k \mapsto w]$.

Now, for given F for L and w in F, let:

$\mathrm{F} \vDash \exists x_{j_1} \ldots \exists x_{j_m} \forall x_i st(n, x_i, A')[x_k \mapsto w]$.

Clearly there are states $v_{j_1}, \ldots, v_{j_m}$ in F, for which it holds that:

$\mathrm{F} \vDash \forall x_i st(n, x_i, A')[x_k \mapsto w, x_{j_1} \mapsto v_{j_1}, \ldots, x_{j_m} \mapsto v_{j_m}]$.

Now let us define a model $\mathrm{M} =_{def} \langle \mathrm{F}, V, H \rangle$ where V is any valuation, $H(c_k) =_{def} w$, $H(c_{j_1}) =_{def} v_{j_1}$, $\ldots$, $H(c_{j_m}) =_{def} v_{j_m}$, and the value of H for any other nominal is any state of F. By the main property of st, $\mathrm{M} \Vdash A'$ and clearly $\mathrm{M}, w \Vdash c_k$. $\square$

For reducing the size of the problem, we need a lemma for conjunctions.

Lemma 43 (Conjunction Lemma) Let L be a type *1* or a type *2* language and let $A, B \in L$. Let $\psi_1(x_k)$ and $\psi_2(x_k)$ be FOL formulas.

1. Let $A \sim \psi_1(x_k)$ and $B \sim \psi_2(x_k)$. Then $(A \wedge B) \sim (\psi_1(x_k) \wedge \psi_2(x_k))$.
2. If A and B are d-persistent with respect to L, then so is $(A \wedge B)$.
3. If A and B are di-persistent with respect to L, then so is $(A \wedge B)$.

Proof For 1, let w be a world in F, which is a frame for L. Then, by the hypothesis, $\mathrm{F}, w \Vdash A$ iff $\mathrm{F} \Vdash \psi_1[x_k \mapsto w]$ and $\mathrm{F}, w \vDash B$ iff $\mathrm{F} \vDash \psi_2[x_k \mapsto w]$. Let $\mathrm{F}, w \Vdash (A \wedge B)$. Then, $\mathrm{F}, w \Vdash A$ and $\mathrm{F}, w \Vdash B$. Therefore, $\mathrm{F} \vDash \psi_1[w]$ and $\mathrm{F} \vDash \psi_2[x_k \mapsto w]$, so $\mathrm{F} \vDash (\psi_1(x_k) \wedge \psi_2(x_k))[x_k \mapsto w]$. The converse direction is analogous.

For 2 and 3, the result follows directly from the definition of d-persistence and di-persistence. $\square$

It remains to describe how to apply the strategy given in Definition 39.

3.3 Deterministic SQEMA Overview

We follow [17][21]. First, we give a simplified informal definition of the algorithm. Let L be a type *1* or a type *2* language and let $A \in L$ be the input modal formula. The goal is to obtain a nominal c_k, and a pure formula $A' \in L'$, where L' is the hybrid temporal extension of L, such that $c_k \not\nearrow A$ and $A \sim_k A'$. Then by Lemma 42, we obtain a local FOL correspondent of A.

Definition 44 A modal formula A is in *negation normal form* iff negation only occurs in front of $\top$, $\bot$, propositional variables or nominals. (Here we assume that the defined symbol for implication, $\rightarrow$, is not used.)

It is clear that for any modal formula, we may effectively find a semantically equivalent formula in the same modal language, which is in negation normal form.

First, we negate A and rewrite it in negation normal form, obtaining γ. We start eliminating variables by a process similar to Gaussian elimination. Thus, we solve a *system of equations* (actually a conjunction of disjunctions), starting with a system with the single equation $(\neg c_k \vee \gamma)$, such that c_k is a chosen nominal to represent the current state, and is such that $c_k \not\nearrow A$. We eliminate each variable separately, so let p be the current variable to eliminate. The elimination is carried out by applying the following rules, where $\beta(\neg p)$ is a modal formula which is negative in p:

Ackermann rule:

$$\begin{cases} \bigwedge((\alpha_1 \vee p), \ldots, (\alpha_{n_a} \vee p)) \wedge \\ \bigwedge(\beta_1(\neg p), \ldots, \beta_{n_b}(\neg p)) \wedge \\ \bigwedge(\theta_1, \ldots, \theta_{n_t}) \end{cases} \Rightarrow \begin{cases} \bigwedge(\beta_1, \ldots, \beta_{n_b})[p/\neg \bigwedge(\alpha_1, \ldots, \alpha_{n_a})] \wedge \\ \bigwedge(\theta_1, \ldots, \theta_{n_t}) \end{cases}$$

where $p \not\nearrow \{\alpha_1, \ldots, \alpha_{n_a}, \theta_1, \ldots, \theta_{n_t}\}$ and $\bigwedge(\beta_1, \ldots, \beta_{n_b})$ is negative in p.

$\Box$-rule: $(B_1 \vee \Box B_2) \Rightarrow (\Box^{-1} B_1 \vee B_2)$

$\Diamond$-rule: $(\neg c' \vee \Diamond B) \Rightarrow (\neg c' \vee \Diamond c'') \wedge (\neg c'' \vee B)$, where c'' is a new nominal.

Now we are ready to formalize the algorithm.

Definition 45 (Monotonicity of a Formula Relative to a Variable) Let $A \in L$ be a modal formula, let all variables occurring in A be among $p_1, \ldots, p_n$, let all nominals occurring in A be among $c_1, \ldots, c_m$, let $i \leq n$, and let $p_i \in PROP$. For a given frame for L, $\mathrm{F} = \langle W, \mathcal{R} \rangle$, given sets $s_1, \ldots, s_n \subseteq W$, worlds $w_1, \ldots, w_m \in W$, and set $s \subseteq W$, we denote by $A(\bar{s}, \bar{w}, p_i \to s)$ the set $[\![A]\!](s_1, \ldots, s_{i-1}, s, s_{i+1}, \ldots, s_n, \bar{w})$.

We say that A is *upwards monotone in* p iff for all Kripke frames $\mathrm{F} = \langle W, \mathcal{R} \rangle$, all sets of worlds $\bar{s}$ and all worlds $\bar{w}$, if $t_1 \subseteq t_2 \subseteq W$, then $[\![A]\!](\bar{s}, \bar{w}, p \to t_1) \subseteq [\![A]\!](\bar{s}, \bar{w}, p \to t_2)$.

We say that A is *downwards monotone in* p iff for all Kripke frames $\mathrm{F} = \langle W, \mathcal{R} \rangle$, all sets of worlds $\bar{s}$ and all worlds $\bar{w}$, if $t_2 \subseteq t_1 \subseteq W$, then $[\![A]\!](\bar{s}, \bar{w}, p_i \to t_1) \subseteq [\![A]\!](\bar{s}, \bar{w}, p_i \to t_2)$.

Proposition 46 Let A be a formula for some modal language L and let $p \in PROP$. If A is positive in p, then A is upwards monotone in p. If A is negative in p, then A is downwards monotone in p.

Proof By induction on A. $\square$

Note that the converse is not always true. For example, let $B =_{def} (p \wedge (p \vee \neg p))$. B is neither positive nor negative in p, but $B \equiv p$, and hence, by Proposition 46, is positive in p.

Definition 47 (Syntactically Closed and Open Formulas) Let L be a type *1* language, let S be the set of indices of all diamonds in L, and let D be the set of indices of all reversed diamonds in L. Let L' be the hybrid temporal extension of L. We say that a formula $A \in L'$ is *syntactically closed with respect to* L iff all occurrences in A of nominals and $\Diamond_i^{-1}$, such that $i \in S \setminus D$, are positive, and all occurrences in A of $\Box_i^{-1}$, such that $i \in S \setminus D$, are negative. We say that A is *syntactically open with respect to* L iff all occurrences in A of nominals and $\Diamond_i^{-1}$, such that $i \in S \setminus D$, are negative, and all occurrences in A of $\Box_i^{-1}$, such that $i \in S \setminus D$, are positive.

Clearly, negation maps syntactically open formulas to syntactically closed ones and vice versa.

Also, clearly, all formulas in L (which are also formulas in L') are both syntactically closed and open.

Definition 48 (SQEMA rules) Let L be a type *1* or a type *2* language language and let L' be the hybrid temporal extension of L. Let c_k be a nominal. Below, $\Box$ is any box or reversed box of L', $\Diamond$ is any diamond or reversed diamond of L', and the formulas below are formulas of L'.

1. *Semantic equivalence rule.*
   Let $A, B, \sigma \in L'$, and let $A \equiv B$. If L is a type *1* language, let both A and B be syntactically open formulas, and let σ be syntactically closed. Then, within σ, replace occurrences of A with B.
2. *Polarity reversing rule.*
   Replace $\neg A$ with $\neg A[p/\neg p]$.
3. *Positive elimination rule.*
   Let A be positive in p. Then replace $\neg A$ with $\neg A[p/\top]$.
4. *Negative elimination rule.*
   Let A be negative in p. Then replace $\neg A$ with $\neg A[p/\bot]$.
5. $\Box$-*rule.*
   Replace $\neg(B \wedge (A_1 \vee \Box A_2))$ with $\neg(B \wedge (\Box^{-1} A_1 \vee A_2))$.
6. $\Diamond$-*rule.*
   Let c'' be such that $c'' \not\nearrow \{c_k, c', A, B\}$. Then:
   Replace $\neg(B \wedge (\neg c' \vee \Diamond A))$ with $\neg(B \wedge ((\neg c' \vee \Diamond c'') \wedge (\neg c'' \vee A)))$.
7. *The Ackermann rule.* Let $\alpha_1, \ldots, \alpha_{n_a}, \theta_1, \ldots, \theta_{n_t}$ be formulas which contain no occurrences of p, let $\beta_1, \ldots, \beta_{n_b}$ be formulas which are either *negative* in p, or, only if L is a type *2* language, downwards monotone in p. Then replace the formula:
   $\gamma' =_{def} \neg \bigwedge((\alpha_1 \vee p), \ldots, (\alpha_{n_a} \vee p), \beta_1, \ldots, \beta_{n_b}, \theta_1, \ldots, \theta_{n_t})$
   with:
   $\gamma'' =_{def} \neg \bigwedge(\bigwedge(\beta_1, \ldots, \beta_{n_b})[p/\neg \bigwedge(\alpha_1, \ldots, \alpha_{n_a})], \theta_1, \ldots, \theta_{n_t})$

Definition 49 *Equations* are formulas of the kind $(c' \to \Diamond c'')$ or of the kind $(A \veebar B)$, such that A and B are in negation normal form and $\veebar$ is a symbol used in place of $\vee$ in equations to signify that they are equations. A *system* is a formula of the kind $\neg \bigwedge(\chi_1, \ldots, \chi_n)$ for some $n \geq 0$, where $\chi_1, \ldots, \chi_n$ are equations. We use σ for systems of equations and χ for equations. σ is *solved for p* iff there are no occurrences of p in σ. σ is *solved* iff it is pure. Below, we say that c_m is a *new nominal*, if c_m is such that: if $\gamma_1, \ldots, \gamma_n$ are all formulas that have occurred as input or during the execution of any branch of the algorithm so far, then the nominals occurring in $\{\gamma_1, \ldots, \gamma_n\}$ are among $\{c_1, \ldots, c_{m-1}\}$.

The algorithm first splits the input formula, by the Conjunction Lemma 43, into several systems of equations, trying to solve each of them in sequence, by applying the SQEMA rules from Definition 48. We must show that the rules in Definition 48 would allow us to construct the sequence that we require, as described in Definition 39.

Proposition 50 Let L be a type *1* or a type *2* language. Let L' be the hybrid temporal extension of L. Let $\sigma' \in L'$ be a system obtained by applying one of the rules in Definition 39 to the system $\sigma \in L'$. Then $\sigma \sim^L_k \sigma'$.

(1) If L is a type *1* language and if the input formula to any of the rules is a syntactically closed with respect to L system from L' of syntactically open with respect to L equations, then the result also is.

(2) If L is a type *1* language, then $\sigma \approx_k^d \sigma'$ with respect to L.

(3) If L is a type *2* language, then $\sigma \approx_k^{di} \sigma'$ with respect to L.

Proof (1) Let L be a type *1* language. It can be easily checked that all the rules convert a syntactically closed with respect to L system from L' of syntactically open with respect to L equations into another one of the same kind.

For (2), the proof involves a lengthy discussion of the properties of the modal operators of L' with respect to descriptive general frames for L. A simpler variant of the full proof can be seen in [17].

For (3), the proof is given in [27].

A full proof for both (2) and (3) is given in the following subsection of this dissertation, 3.4. □

3.4 Correctness of the SQEMA Rules

This section proves Proposition 50 by using the proofs given in [17] for type *1* languages, and the proofs given in [27] for type *2* languages.

Proposition 51 Let L be a type *1* or a type *2* language. Let L' be the hybrid temporal extension of L. Let $\sigma' \in L'$ be a system obtained by applying one of the following rules from 39 to the system $\sigma \in L'$: the polarity reversing rule, the positive or negative elimination rule, the $\Box$-rule, or the $\Diamond$-rule. Then $\sigma \sim_k^L \sigma'$, and if L is a type *1* language, then $\sigma \approx_k^d \sigma'$ with respect to L, or if L is a type *2* language, then $\sigma \approx_k^{di} \sigma'$ with respect to L.

Proof Let $\mathcal{C}$ be the class of all general frames for L. It is enough to show that $\sigma \approx_k^{\mathcal{C}} \sigma'$, because all frames can be considered to be full general frames, and $\mathcal{C}$ also includes all descriptive and all discrete frames for L.

For the equivalence rule, the result follows immediately.

The rest of listed rules are in the form "Replace $\neg A'$ with $\neg A''$" for some formulas A' and A'' from L'. Let $\mathfrak{g} \in \mathcal{C}$ be any general frame for L, and let w be a world in $\mathfrak{g}$. To prove that $\neg A' \approx_k^{\mathcal{C}} \neg A''$, it is enough to prove that for every model M for L' over $\mathfrak{g}$, such that $[\![c_k]\!]_{\mathrm{M}} = \{w\}$ and $\mathrm{M} \Vdash A'$, there is a model M$'$ for L' over $\mathfrak{g}$, such that $[\![c_k]\!]_{\mathrm{M}'} = \{w\}$ and $\mathrm{M}' \Vdash A''$, and vice versa.

Polarity reversing rule: Because negations of admissible valuations are admissible, we set M$'$ to be equal to M, except $[\![p]\!]_{\mathrm{M}'}$ is set to be the complement of $[\![p]\!]_{\mathrm{M}}$. The implication follows by the definition of Kripke semantics. The converse follows analogously.

Positive elimination rule: Let W be the universe of M. By induction on A, we get that $[\![A]\!]_{\mathrm{M}} \subseteq [\![A[p/\top]]\!]_{\mathrm{M}}$.

First, let $[\![c_k]\!]_{\mathrm{M}} = \{w\}$ and $\mathrm{M} \Vdash A$. We set M' to be equal to M, except $[\![p]\!]_{\mathrm{M}'}$ is set to be W, which is admissible. We have that $W = [\![A]\!]_{\mathrm{M}} \subseteq [\![A[p/\top]]\!]_{\mathrm{M}} = [\![A[p/\top]]\!]_{\mathrm{M}'}$, by the properties of uniform substitution. Therefore, $[\![A[p/\top]]\!]_{\mathrm{M}'} = W$.

Now, let $[\![c_k]\!]_{\mathrm{M}} = \{w\}$ and $\mathrm{M} \Vdash A[p/\top]$. We construct M' in the same way, and it is straightforward to prove that $[\![A]\!]_{\mathrm{M}'} = W$.

Negative elimination rule: Follows from the polarity reversing rule and the positive elimination rule.

$\Box$-rule: Let $R_\Box$ be the (converse) relation of M, which corresponds to $\Box$.

First, let $\mathrm{M} \Vdash (B \wedge (A_1 \vee \Box A_2))$, suppose that $\mathrm{M} \nVdash (B \wedge (\Box^{-1} A_1 \vee A_2))$. Then, there is a $w_1 \in W$: $\mathrm{M}, w_1 \nVdash (\Box^{-1} A_1 \vee A_2)$. Then, $\mathrm{M}, w_1 \nVdash \Box^{-1} A_1$ and $\mathrm{M}, w_1 \nVdash A_2$. Therefore, there is a $w_2 \in W$: $\mathrm{M}, w_2 \nVdash A_1$ and $\langle w_2, w_1 \rangle \in R_\Box$. However, $\mathrm{M} \Vdash (A_1 \vee \Box A_2)$, therefore $\mathrm{M}, w_2 \Vdash \Box A_2$, so $\mathrm{M}, w_1 \Vdash A_2$, contradiction.

Now, let $\mathrm{M} \Vdash (B \wedge (\Box^{-1} A_1 \vee A_2))$, suppose that $\mathrm{M} \nVdash (B \wedge (A_1 \vee \Box A_2))$. Analogously to the above, we derive a contradiction.

$\Diamond$-rule: Let $\mathrm{M} = \langle \mathfrak{g}_\#, V, H \rangle$. Let $R_\Diamond$ be the relation or converse relation of M, corresponding to $\Diamond$.

First, let $\mathrm{M} \Vdash (B \wedge (\neg c' \vee \Diamond A))$, and let $w_1 =_{def} H(c')$. Then, $\mathrm{M}, w_1 \Vdash \Diamond A$. So, there is a $w_2 \in W$: $\langle w_1, w_2 \rangle \in R_\Diamond$ and $\mathrm{M}, w_2 \Vdash A$. We set $\mathrm{M}' =_{def} \langle \mathfrak{g}_\#, V, H' \rangle$, where $H'(c'') =_{def} w_2$, and $H'(c) =_{def} H(c)$ for any other nominal. By the properties of uniform substitution, and by the hypothesis on c'', the condition holds.

Now, let $\mathrm{M} \Vdash (B \wedge ((\neg c' \vee \Diamond c'') \wedge (\neg c'' \vee A)))$. Then, $\mathrm{M} \Vdash (B \wedge (\neg c' \vee \Diamond A))$. $\Box$

Proposition 52 Let L be a type *2* language. Clearly, the hybrid temporal extension of L is also L. Let $\sigma' \in L$ be a system obtained by applying one of the rules in 39 to the system $\sigma \in L$. Then $\sigma \sim_k^L \sigma'$. Also, $\sigma \approx_k^{di} \sigma'$ with respect to L.

Proof By Proposition 51, it remains to show the result only for the Ackermann rule.

It can be easily checked that the extension of any modal formula $A \in L$ at a Kripke model M over some discrete general frame $\mathfrak{g}$ for L, is also an admissible in $\mathfrak{g}$ set.

The Ackermann rule: If β is negative in p, then it is downwards monotone in p by Proposition 46. Let α be $\bigwedge(\alpha_1, \ldots, \alpha_{n_a})$, β be $\bigwedge(\beta_1, \ldots, \beta_{n_b})$, and β be downwards monotone in p. Let M be a model either over a discrete general frame $\mathfrak{g}$ for L, or over a full general frame $\mathfrak{g}$ for L. First, let $\mathrm{M} \Vdash \neg\gamma'$, so $\mathrm{M} \Vdash$

$(\alpha \vee p)$ and $\mathrm{M} \Vdash \beta$. Then, $[\![\neg\alpha]\!]_{\mathrm{M}} \subseteq [\![p]\!]_{\mathrm{M}}$, therefore $W = [\![\beta]\!]_{\mathrm{M}} \subseteq [\![\beta[p/\neg\alpha]]\!]_{\mathrm{M}}$, so $\mathrm{M} \Vdash \neg\gamma''$. Now, let $\mathrm{M} \Vdash \neg\gamma''$, and let $\mathrm{M} = \langle \mathfrak{g}_{\#}, V, A\rangle$. Let $V'(p) =_{def} [\![\neg\alpha]\!]_{\mathrm{M}}$, and let $V'(p') =_{def} V(p')$ for other variables p'. Let $\mathrm{M}' =_{def} \langle \mathfrak{g}_{\#}, V', A\rangle$. Clearly, M' is a model over $\mathfrak{g}$. Then, $\mathrm{M}' \Vdash \neg\gamma'$. $\square$

For the rest of this section, we prove the first result of Proposition 50 regarding type *1* languages and the Ackermann rule. It will take some time to get to the conclusion. The proofs closely follow [17].

Definition 53 (Closed Sets) Let $\mathfrak{g} = \langle \mathrm{F}, \mathbb{W}\rangle$ be a general frame for some language. A subset s of the universe W is called *closed (with respect to* $\mathfrak{g}$*)* iff $s = \bigcap\{s_0 \in \mathbb{W} \mid s \subseteq s_0\}$. We denote by $\mathcal{C}(\mathbb{W})$ the set of all closed with respect to $\mathfrak{g}$ sets.

Definition 54 (Open Sets) Let $\mathfrak{g} = \langle \mathrm{F}, \mathbb{W}\rangle$ be a general frame for some language. A subset s of the universe $W$is called *open (with respect to* $\mathfrak{g}$*)* iff it is the relative complement (to W) of a closed set.

Definition 55 (Clopen Sets) Let $\mathfrak{g} = \langle \mathrm{F}, \mathbb{W}\rangle$ be a general frame for some language. A subset s of the universe W is called *clopen (with respect to* $\mathfrak{g}$*)* iff it is both closed and open.

Definition 56 (Closed Operators and Closed Formulas) Let L be a type *1* language, and let L' be the hybrid temporal extension of L. Let $A \in L'$, let $\mathit{PROP}(A) \subseteq \{p_1, \dots, p_n\}$, and let $\mathit{NOM}(A) \subseteq \{c_1, \dots, c_m\}$. We say that A is a *closed operator in* L iff for every descriptive frame $\mathfrak{g} = \langle \mathrm{F}, \mathbb{W}\rangle$, if $P_1, \dots, P_n \in \mathcal{C}(\mathbb{W})$ and $w_1, \dots, w_m$ are any elements of W, then $[\![A]\!](P_1, \dots, P_n, w_1, \dots, w_m) \in \mathcal{C}(\mathbb{W})$, i.e. when applied to closed sets and any elements (resp. for $\mathit{PROP}(A)$ and $\mathit{NOM}(A)$) with respect to a descriptive frame for L it produces a closed set. We say that A is a *closed formula in* L iff whenever $[\![A]\!]$ is applied to *admissible* sets and any elements in any descriptive frame for L, it produces a closed set.

Thus, if a formula is a closed operator in L, then it is a closed formula in L. The converse is not always true.

Definition 57 (Open Operators and Open Formulas) Similarly to above, a formula from L', the hybrid temporal extension of a type *1* language L, is an *open operator* in L iff whenever its modal operator is applied to open sets and any singletons in a descriptive frame for L it produces an open set; it is an *open formula* in L if whenever its modal operator is applied to admissible sets and any singletons it produces an open set.

Whenever we mention and index set, we also mean that the index set is non-empty.

Proposition 58 To prove that a set s is closed with respect to a general frame $\mathfrak{g} = \langle \mathrm{F}, \mathbb{W} \rangle$ for any language, it is enough to prove that s is equal to the intersection of some non-empty family of closed sets, and in particular, admissible sets (with respect to $\mathfrak{g}$).

Proof Let $\{P_i\}_{i \in I}$ be a family of closed sets over $\mathfrak{g}$, for some index set I, and let $s = \bigcap_{i \in I}\{P_i\}$. Because for all $i \in I$, P_i is a closed set, then, for all $i \in I$, there is a family of admissible over $\mathfrak{g}$ sets Q_{ij}, such that J is an index set with $j \in J$ and $P_i = \bigcap_{j \in J}\{Q_{ij}\}$. This means that $s = \bigcap_{i \in I}\bigcap_{j \in J}\{Q_{ij}\}$, where, for all $i \in I$ and for all $j \in J$, Q_{ij} is an admissible set. This means that there's a family of admissible sets $\{D_k\}_{k \in K}$ for some index set K, such that $\{D_k \mid D_k \in \mathbb{W} \ \& \ k \in K\} = \{Q_{ij} \mid i \in I \ \& \ j \in J\}$. Then, s is an intersection of admissible sets, $s = \bigcap_{k \in K}\{D_k \mid D_k \in \mathbb{W}\}$ so it follows that $s \subseteq D_k$ for all $k \in K$, so we get $s = \bigcap_{k \in K}\{D_k \mid D_k \in \mathbb{W} \ \& \ \mathrm{s} \subseteq D_k\}$. If we add more admissible sets $\{E_n\}_{n \in N}$ for some index set N such that for each $n \in N$, $s \subseteq E_n$, to the family $\{D_k\}_{k \in K}$, we do not change the intersection. So, we get $s = \bigcap\{X \in \mathbb{W} \mid s \subseteq X\}$, which means that s is a closed set with respect to $\mathfrak{g}$. $\Box$

Proposition 59 To prove that a set s is open with respect to a general frame $\mathfrak{g} = \langle W, \mathcal{R}, \mathbb{W} \rangle$ for any language, it is enough to prove that s is equal to the union of some family of open sets, and in particular, admissible sets (with respect to $\mathfrak{g}$).

Proof Let $\{P_i\}_{i \in I}$ be a family of open sets over $\mathfrak{g}$, for some index set I, and let $s = \bigcup_{i \in I}\{P_i\}$. For s to be open, we need to prove that its relative complement to W is closed, so in other words, we need to prove that $W \setminus \bigcup_{i \in I}\{P_i\}$ is closed. $W \setminus \bigcup_{i \in I}\{P_i\} = \bigcap_{i \in I}\{W \setminus P_i\}$. But this set is an intersection of some family of closed sets, and by proposition 58 above, it is closed. Therefore, s, as its relative complement, is open. $\Box$

Proposition 60 Let $\mathrm{F} = \langle W, \mathcal{R} \rangle$ be a frame for a type *1* language L, and let L' be its hybrid temporal extension. Let $\Box$ be any of box $\Box_k$ or reversed box $\Box_k^{-1}$ of L or L'. As an operator, $[\![\Box p_1]\!]$ distributes over arbitrary intersections of subsets of W.

Proof Let R be either $\mathcal{R}(k)$ if $\Box$ is $\Box_k$, or $\mathcal{R}(k)^{-1}$ if $\Box$ is $\Box_k^{-1}$. Let I be an index set, and let $\{P_i\}_{i \in I}$ be a family of subsets of W. We need to prove that $[\![\Box p_1]\!](\bigcap_{i \in I}\{P_i\}) = \bigcap_{i \in I}\{[\![\Box p_1]\!](P_i)\}$.

First, suppose $x \in [\![\Box p_1]\!](\bigcap_{i \in I}\{P_i\})$. Then, for all $y \in W$: $\langle x, y \rangle \in R \Rightarrow y \in \bigcap_{i \in I}\{P_i\}$, or, for all $i \in I$ and for all $y \in W$: $\langle x, y \rangle \in R \Rightarrow y \in P_i$. Equivalently, for all $i \in I$: $x \in [\![\Box p_1]\!](P_i)$, and from here, $x \in \bigcap_{i \in I}\{[\![\Box p_1]\!](P_i)\}$.

For the converse, suppose that $x \in \bigcap_{i\in I}\{[\![\Box p_1]\!](P_i)\}$. Going backwards on the equivalence chain from the first half of the proof, the statement is equivalent to for all $i \in I$ and for all $y \in W$: $\langle x, y\rangle \in R \Rightarrow y \in P_i$, which also means that for all $y \in W$: $\langle x, y\rangle \in R \Rightarrow y \in \bigcap_{i\in I}\{P_i\}$, and from there, $x \in [\![\Box p_1]\!](\bigcap_{i\in I}\{A_i\})$. □

Proposition 61 Let L be a type *1* language. For every box or reversed box $\Box$ of L, $[\![\Box p_1]\!]$ is a closed operator in L. For every diamond or reversed diamond $\Diamond$ of L, $[\![\Diamond p_1]\!]$ is an open operator in L.

Proof Let $\mathfrak{g} = \langle W, \mathcal{R}, \mathbb{W}\rangle$ be any general frame (not just descriptive) for L.

1. $[\![\Box p_1]\!]$. Let $s \in \mathcal{C}(\mathbb{W})$ be a closed set with respect to $\mathfrak{g}$. So, there's a family $\{P_i\}_{i\in I}$ of admissible sets for some index set I, and $s = \bigcap_{i\in I}\{P_i\}$. By proposition 60, we have that $[\![\Box p_1]\!](\bigcap_{i\in I}\{P_i\}) = \bigcap_{i\in I}\{[\![\Box p_1]\!](P_i)\}$, which, according to proposition 58, and the fact that $\mathbb{W}$ is closed under taking relative complements and $[\![\Diamond p_1]\!]$ (or, just $[\![\Box p_1]\!]$), is enough to prove that $[\![\Box p_1]\!](s)$ is a closed set, and from there we get that $[\![\Box p_1]\!]$ is a closed operator in L.

2. $[\![\Diamond p_1]\!]$. Let s be an open set with respect to $\mathfrak{g}$. It is enough to prove that $W \setminus [\![\Diamond p_1]\!](s)$ is a closed set. But $W \setminus [\![\Diamond p_1]\!](s) = [\![\Box p_1]\!](W \setminus s)$, where $W \setminus s$ is a closed set. Then the proof follows immediately by the proof for $[\![\Box p_1]\!]$. □

Proposition 62 Let L be a type *1* language, and let L' be the hybrid temporal extension of L. Let $\mathrm{F} = \langle W, \mathcal{R}\rangle$ be a frame for L. Let $\Diamond$ be either one of the diamonds $\Diamond_k$ or reversed diamonds $\Diamond_k^{-1}$ of L or L'. Let I be an index set (and hence, non-empty) and let $\{P_i\}_{i\in I}$ be a collection of subsets of W. Then, $[\![\Diamond p_1]\!](\bigcap_{i\in I}\{P_i\}) \subseteq \bigcap_{i\in I}\{[\![\Diamond p_1]\!](P_i)\}$.

Proof Let R be either $\mathcal{R}(k)$ if $\Diamond$ is $\Diamond_k$, or $\mathcal{R}(k)^{-1}$, otherwise. Let $x \in [\![\Diamond p_1]\!](\bigcap_{i\in I}\{P_i\})$. Then, for some $y \in W : \langle x, y\rangle \in R \,\&\, y \in \bigcap_{i\in I}\{P_i\}$. Let y be one such a element of W that $\langle x, y\rangle \in R \,\&\, y \in \bigcap_{i\in I}\{P_i\}$, and let i be any index in I. Then $y \in P_i$. We have $\langle x, y\rangle \in R \,\&\, y \in P_i \Rightarrow x \in [\![\Diamond p_1]\!](P_i)$. Because we picked any index $i \in I$, we have that for all $i \in I$: $x \in [\![\Diamond p_1]\!](P_i)$. But then $x \in \bigcap_{i\in I}\{[\![\Diamond p_1]\!](P_i)\}$, and hence $[\![\Diamond p_1]\!](\bigcap_{i\in I}\{P_i\}) \subseteq \bigcap_{i\in I}\{[\![\Diamond p_1]\!](P_i)\}$. □

Proposition 63 Let L be a type *1* language and let L' be its hybrid temporal extension. Let $\mathfrak{g} = \langle W, \mathcal{R}, \mathbb{W}\rangle$ be a descriptive general frame for L. Then (a) Every singleton of a state in W is closed with respect to $\mathfrak{g}$. (b) Let $\Diamond^{-1}$ be any reversed diamond or diamond of L or L'. If $s \subseteq W$ is a closed set with respect to $\mathfrak{g}$, then so is the set $[\![\Diamond^{-1} p_1]\!](s)$.

Proof This proof is adapted from [8], pages 316–318.

(a) Let $w \in W$ be any state in W. Then, from the fact that $\mathfrak{g}$ is descriptive, hence differentiated, it follows directly that $\{w\} = \cap\{P \in \mathbb{W} \mid w \in P\}$.

(b) Let $\Diamond^{-1}$ be either $\Diamond_k^{-1}$ or $\Diamond_k$ of L'. Let $\Diamond$ be the converse of $\Diamond^{-1}$, and let $\Box$ be either $\Box_k$ if $\Diamond$ is $\Diamond_k$, or the converse k-th box. Let s be a closed set with respect to $\mathfrak{g}$. We show that $[\![\Diamond^{-1}p_1]\!](s) = \bigcap\{P \in \mathbb{W} \mid s \subseteq [\![\Box p_1]\!](P)\}$ (1), and from Proposition 58, it then follows immediately that $[\![\Diamond^{-1}p_1]\!](s)$ is closed. The left to right inclusion in (1) is trivial, so assume for the sake of contradiction that $x \notin [\![\Diamond^{-1}p_1]\!](s)$ and for all $P \in \mathbb{W}$: if $s \subseteq [\![\Box p_1]\!](P)$ then $x \in P$.

Let $\mathbb{W}_0 =_{def} \{Q \in \mathbb{W} \mid s \subseteq Q\} \cup \{[\![\Diamond p_1]\!](P) \mid x \in P \ \& \ P \in \mathbb{W}\}$. First, we'll prove that $\mathbb{W}_0$ has the fip (2). For that, we need to show that $\mathbb{W}_0$ is non-empty. Because $W \in \mathbb{W}$, and because $s \subseteq W$, $W \in \mathbb{W}_0$. Suppose that there is a finite subset of $\mathbb{W}_0$ with an empty intersection. So, there are $Q_1, \ldots, Q_n$ and $P_1, \ldots, P_m$ in $\mathbb{W}_0$, such that $s \subseteq Q_i$ for all i such that $1 \leq i \leq n$, $x \in P_j$ for all j such that $1 \leq j \leq m$, and $Q_1 \cap \cdots \cap Q_n \cap [\![\Diamond p_1]\!](P_1) \cap \cdots \cap [\![\Diamond p_1]\!](P_m) = \emptyset$. Let $Q =_{def} Q_1 \cap \cdots \cap Q_n$ and let $P =_{def} P_1 \cap \cdots \cap P_m$. Clearly $x \in P$. By Proposition 62, we have that $[\![\Diamond p_1]\!](P) \subseteq [\![\Diamond p_1]\!](P_1) \cap \cdots \cap [\![\Diamond p_1]\!](P_m)$, and hence, $Q \cap [\![\Diamond p_1]\!](P) = \emptyset$. But $[\![\Diamond p_1]\!](P) = W \setminus [\![\Box p_1]\!](W \setminus P)$, so we get $Q \subseteq [\![\Box p_1]\!](W \setminus P)$, so $s \subseteq [\![\Box p_1]\!](W \setminus P)$, and, by the assumption, it follows that $x \in W \setminus P$, contradiction with $x \in P$. Thus, we proved (2) that $\mathbb{W}_0$ has the fip.

Now, because $\mathfrak{g}$ is a compact frame, (2) implies that $\mathbb{W}_0$ has a non-empty intersection, so there is a $y \in W$ such that: $y \in \bigcap\{Q \in \mathbb{W} \mid s \subseteq Q\}$ (3) and $y \in \bigcap\{[\![\Diamond p_1]\!](P) \mid x \in P \ \& \ P \in \mathbb{W}\}$ (4). It follows immediately from (3) that $y \in s$. Let R be either $\mathcal{R}(k)$ if $\Diamond^{-1}$ is the reversed diamond $\Diamond_k^{-1}$, or $\mathcal{R}(k)^{-1}$, otherwise. We now also prove that it follows from (4) that $\langle y, x\rangle \in R$, i.e. $x \in [\![\Diamond^{-1}p_1]\!](s)$, contradiction with the assumption that $x \notin [\![\Diamond^{-1}p_1]\!](s)$, thus prove that $\bigcap\{P \in \mathbb{W} \mid s \subseteq [\![\Box p_1]\!](P)\} \subseteq [\![\Diamond^{-1}p_1]\!](s)$.

Now, suppose that $\langle y, x\rangle \notin R$. By the tightness of $\mathfrak{g}$, there would be a $P \in \mathbb{W}$ witnessing $\langle y, x\rangle \notin R$, that is, $x \in P$ while $y \notin [\![\Diamond p_1]\!](P)$. But this contradicts (4), thus $\langle y, x\rangle \in R$. $\Box$

Corollary 64 Let L be a type *1* language and let L' be its hybrid temporal extension. Let $\Diamond$ be any diamond of L or any reversed diamond of L'. Then $\Diamond p_1$ is a closed operator in L.

Proof This is exactly point (b) of Proposition 63 above. $\Box$

Corollary 65 Let L be a type *1* language and let L' be its hybrid temporal extension. Let $\Box$ be any box of L or any reversed box of L'. Then $\Box p_1$ is an open operator in L.

Proof Follows by Corollary 64, see the proof of Proposition 61, point 2. $\Box$

Proposition 66 Let $\mathfrak{g}$ be a descriptive general frame for some language. Then every non-empty set of closed sets that has the fip, has a non-empty intersection.

Proof This proof is adapted from [8], pages 316–318.

Follows directly from the fact that $\mathfrak{g}$ is compact, and by examining the definition of a closed set. $\square$

Definition 67 (Downward directed family) . Let $\{P_i\}_{i\in I}$ be a family of subsets of the set W, for some index set I. $\{P_i\}_{i\in I}$ is *a downward directed family of sets* iff for all $x \in \{P_i\}_{i\in I}$ and for all $y \in \{P_i\}_{i\in I}$, there is a $z \in \{P_i\}_{i\in I}$ such that $z \subseteq x$ & $z \subseteq y$.

Proposition 68 (Esakia's Lemma for $\Diamond$) . Let L be a type *1* language. Let $\Diamond$ be any diamond of $\Diamond_k$ of L. Let $\mathfrak{g} = \langle W, \mathcal{R}, \mathbb{W}\rangle$ be a descriptive frame for L, and let I be some index, hence, non-empty, set. Then for every downward directed family of nonempty closed for $\mathfrak{g}$ sets $\{P_i\}_{i\in I}$, it is the case that $[\![\Diamond p_1]\!](\bigcap_{i\in I}\{P_i\}) = \bigcap_{i\in I}\{[\![\Diamond p_1]\!](P_i)\}$.

Proof The left to right inclusion is a special case of Proposition 62 and follows directly from there. Thus we get $[\![\Diamond p_1]\!](\bigcap_{i\in I}\{P_i\}) \subseteq \bigcap_{i\in I}\{[\![\Diamond p_1]\!](P_i)\}$.

Let R be $\mathcal{R}(k)$. For any $x \in W$, we denote by $R(x)$ the set $\{y \in W \mid \langle x, y\rangle \in R\}$. Now, let $x \in \bigcap_{i\in I}\{[\![\Diamond p_1]\!](P_i)\}$. Suppose for the sake of contradiction that $x \notin [\![\Diamond p_1]\!](\bigcap_{i\in I}\{P_i\})$. Then, for all $y \in \bigcap_{i\in I}\{P_i\}$, $\langle x, y\rangle \notin R$. Then, $R(x) \cap (\bigcap_{i\in I}\{P_i\}) = \emptyset$ (1).

Clearly $x \in \bigcap_{i\in I}\{x_0 \in W \mid \exists y \in P_i : \langle x_0, y\rangle \in R\}$. Then for all $i \in I$, there is a $y \in P_i$ such that $\langle x, y\rangle \in R$. Thus for all $i \in I$, it is the case that $(R(x) \cap P_i) \neq \emptyset$ (2).

Now, because $\{P_i\}_{i\in I}$ is a downward directed family of non-empty of closed (with respect $\mathfrak{g}$) sets, we have that for all $i \in I$, for all $j \in I$, there is a $k \in I$, such that $P_k \subseteq P_i \cap P_j$, and also we have that for all $k \in I$, $P_k \neq \emptyset$. Thus, for all $i \in I$, for all $j \in I$: $P_i \cap P_j \neq \emptyset$. But we also have (2), and that each C_i is non-empty, thus we get that $\{R(x)\} \cup \{P_i\}_{i\in I}$ has the fip. Because of Proposition 63, we have that $R(x)$ is closed, and because of proposition 66, we have that $\bigcap(\{R(x)\} \cup \{P_i\}_{i\in I}) \neq \emptyset$ because $\{R(x)\} \cup \{P_i\}_{i\in I}$ is a set of closed sets with the fip. But $\bigcap(\{R(x)\} \cup \{P_i\}_{i\in I}) \neq \emptyset$ contradicts (1), and hence, $x \in [\![\Diamond p_1]\!](\bigcap_{i\in I}\{P_i\})$.

Finally, we conclude that $[\![\Diamond p_1]\!](\bigcap_{i\in I}\{P_i\}) = \bigcap_{i\in I}\{[\![\Diamond p_1]\!](P_i)\}$. $\square$

Proposition 69 (Esakia's Lemma for $\Diamond^{-1}$ in L') Let L be a type *1* language and let L' be its hybrid temporal temporal extension. Let $\mathfrak{g} = \langle W, \mathcal{R}, \mathbb{W}\rangle$ be a descriptive frame for L, and let I be some index set. Then for every downward directed family of nonempty closed with respect to $\mathfrak{g}$ sets $\{P_i\}_{i\in I}$, and any reversed diamond $\Diamond^{-1}$ of L', it is the case that $[\![\Diamond^{-1}p_1]\!](\bigcap_{i\in I}\{P_i\}) = \bigcap_{i\in I}\{[\![\Diamond^{-1}p_1]\!](P_i)\}$.

Proof The inclusion $[\![\Diamond^{-1}p_1]\!](\bigcap_{i\in I}\{P_i\}) \subseteq \bigcap_{i\in I}\{[\![\Diamond^{-1}p_1]\!](P_i)\}$ is a special case of proposition 62 and follows directly from there.

Let $\Diamond$ be the diamond of L with the same index as $\Diamond^{-1}$.

Let $x \notin [\![\Diamond^{-1}p_1]\!](\bigcap_{i\in I}\{P_i\})$, i.e. $[\![\Diamond p_1]\!](\{x\}) \cap (\bigcap_{i\in I}\{P_i\}) = \emptyset$. Because $\mathfrak{g}$ is descriptive, we have that $\{x\}$ is closed by Proposition 63, and, by Proposition 68, $[\![\Diamond p_1]\!](\{x\})$ is also closed. Hence, $[\![\Diamond p_1]\!](\{x\}) \cup \{P_i\}_{i\in I}$ is a family of closed with respect to $\mathfrak{g}$ sets with the empty intersection, which, by Proposition 66, cannot have the fip. Thus, there is a finite subfamily $\{P_1, \ldots, P_n\} \subseteq \{P_i\}_{i\in I}$ such that $[\![\Diamond p_1]\!](\{x\}) \cap P_1 \cap \cdots \cap P_n = \emptyset$. Because $\{P_i\}_{i\in I}$ is downward directed, there exists a $P \in \{P_i\}_{i\in I}$ such that $P \subseteq \bigcap\{P_1, \ldots, P_n\}$ and $[\![\Diamond p_1]\!](\{x\}) \cap P = \emptyset$. But then $x \notin [\![\Diamond^{-1}p_1]\!](P)$, and hence, $x \notin \bigcap_{i\in I}\{[\![\Diamond^{-1}p_1]\!](P_i)\}$. $\square$

Proposition 70 Let L be a type *1* language and let L' be its hybrid temporal extension. Every syntactically closed with respect to L formula of L' is a closed formula in L, and every syntactically open with respect to L formula of L' is an open formula in L (with respect to the descriptive frames for L).

Proof Because we have descriptive frames for L, we can use the facts that $\Diamond p_1$ for any diamond or reversed diamond $\Diamond$ of L, and $\Box p_1$ for any box or reversed box $\Box$ of L are both closed and open operators in L; $\Diamond^{-1}p_1$ for any reversed diamond $\Diamond^{-1}$ which is not in L is a closed operator L; and $\Box^{-1}p_1$ for any reversed box $\Box^{-1}$ of L' which is not in L is an open operator in L. The proof is by induction, using Definition 56, the above-mentioned facts, and that singletons are closed in descriptive frames (by Proposition 63). $\square$

Proposition 71 Let L be a type *1* language and let L' be its hybrid temporal extension. Let $A \in L'$ be a syntactically closed with respect to L formula, with $\mathit{PROP}(A) \subseteq \{q_1, \ldots, q_n, p\}$, and $\mathit{NOM}(A) \subseteq \{c_1, \ldots, c_m\}$, and let A be positive in p. Let $\mathfrak{g} = \langle W, \mathcal{R}, \mathbb{W}\rangle$ be a descriptive frame for L. Then, for all $Q_1, \ldots, Q_n \in \mathbb{W}$, $x_1, \ldots, x_m \in W$, and $P \in \mathcal{C}(\mathbb{W})$, it is the case that:
$[\![A]\!](Q_1, \ldots, Q_n, P, x_1, \ldots, x_m)$ is closed with respect to $\mathfrak{g}$.

Proof If A is not in negation normal form, we choose A to be a semantically equivalent formula in negation normal form, and clearly we can choose it to also be syntactically closed with respect to L. Because of the semantic equivalence of the two formulas, their operators for $\mathfrak{g}$ will be equal. For the rest of the proof, we will consider A to be in negation normal form.

Because A is in negation normal form, we may assume that reversed boxes of L', which are not in L, do not occur in A, as any such occurrence would have to be negative, and rewriting the formula into negation normal form would change it into a reversed diamond of L'.

We proceed by induction on A.

If A is $\top$, $\bot$, or one of the atoms $q_1, \ldots, q_n, p, c_1, \ldots, c_m$, then it is clear that $[\![A]\!](Q_1, \ldots, Q_n, P, x_1, \ldots, x_m)$ is a closed set, as we use the facts that $\emptyset \in \mathbb{W}$, $W \in \mathbb{W}$, and that every singleton is closed in descriptive frames. Such is also the case if A is the negation of a propositional variable from among $q_1, \ldots, q_n$, because $Q_1, \ldots, Q_n$ are clopen as admissible sets in $\mathfrak{g}$.

The cases where A is the negation of p or one of $c_1, \ldots, c_m$ do not happen, by the definition of A and because A is in negation normal form.

The cases for $\wedge$ and $\vee$ follow since the finite unions and intersections of closed sets are closed. For finite unions, the result is an intersection of finite unions of admissible sets (by applying De Morgan's laws), and for intersections, the result is immediate by the definition of a closed set.

The cases for diamonds and reversed diamonds of L' follow by Corollary 64.

The cases for the boxes and reversed boxes of L follow by Proposition 61. $\square$

Proposition 72 (Esakia's Lemma for Syntactically Closed Formulas) Let L be a type *1* language and let L' be its hybrid temporal extension. Let $A \in L'$ be a syntactically closed in L formula with $PROP(A) \subseteq \{q_1, \ldots, q_n, p\}$ and $NOM(A) \subseteq \{c_1, \ldots, c_m\}$, which is positive in p. Let $\mathfrak{g} = \langle W, \mathcal{R}, \mathbb{W} \rangle$ be a descriptive frame for L. Then, for all $Q_1, \ldots, Q_n \in \mathbb{W}$, $x_1, \ldots, x_m \in W$, and downwards directed family of closed sets $\{P_i\}_{i \in I}$ for some index set I, it is the case that:
$[\![A]\!](Q_1, \ldots, Q_n, \bigcap_{i \in I}\{P_i\}, x_1, \ldots, x_m) = \bigcap_{i \in I}[\![A]\!](Q_1, \ldots, Q_n, P_i, x_1, \ldots, x_m)$.

Proof The proof is by induction on A. For brevity, we'll omit the parameters $Q_1, \ldots, Q_n, x_1, \ldots, x_m$ when writing (sub)formulas. As before, we assume that the formulas are written in negation normal form, i.e. we may also assume that reversed boxes of L' which are not in L do not occur, as all such occurrences have to be negative, and rewriting in negation normal form changes these into reversed diamonds.

The cases when A is $\top$, $\bot$, or among the atoms in A, one of $q_1, \ldots, q_n, p, c_1, \ldots, c_m$ are trivial, as is the case when A is the negation of a propositional variable among $q_1, \ldots, q_n$. As before, the cases where A is the negation of p or the negation of one of $c_1, \ldots, c_m$ do not happen, by the definition of A and because A is in negation normal form.

The inductive step in the case when A is in the form $(B_1 \wedge B_2)$ is also trivial.

Suppose A is of the form $(B_1 \vee B_2)$. We have to show that $[\![B_1]\!](\bigcap_{i \in I}\{P_i\}) \cup [\![B_2]\!](\bigcap_{i \in I}\{P_i\}) = \bigcap_{i \in I}([\![B_1]\!](P_i) \cup [\![B_2]\!](P_i))$.

For the left to right inclusion, let $x \in [\![B_1]\!](\bigcap_{i\in I}\{P_i\}) \cup [\![B_2]\!](\bigcap_{i\in I}\{P_i\})$. By the inductive hypothesis, $x \in \bigcap_{i\in I}[\![B_1]\!](P_i) \cup \bigcap_{i\in I}[\![B_2]\!](P_i)$. Without loss of generality, let $x \in \bigcap_{i\in I}[\![B_1]\!](P_i)$, i.e. for all $i \in I$: $x \in [\![B_1]\!](P_i)$. So, for all $i \in I$: $x \in [\![B_1]\!](P_i) \cap [\![B_2]\!](P_i)$, so $x \in \bigcap_{i\in I}([\![B_1]\!](P_i) \cup [\![B_2]\!](P_i))$.

For the right to left inclusion, let $x \notin [\![B_1]\!](\bigcap_{i\in I}\{P_i\}) \cup [\![B_2]\!](\bigcap_{i\in I}\{P_i\})$. By the inductive hypothesis, $x \notin \bigcap_{i\in I}[\![B_1]\!](P_i) \cup \bigcap_{i\in I}[\![B_2]\!](P_i)$. Thus, there are $P_1, P_2 \in \{P_i\}_{i\in I}$ such that $x \notin [\![B_1]\!](P_1)$ and $x \notin [\![B_2]\!](P_2)$. Because $\{P_i\}_{i\in I}$ is downward directed, there is a $P \in \{P_i\}_{i\in I}$, such that $P \subseteq P_1 \cap P_2$. Because B_1 and B_2 are positive in p and hence (by Proposition 46) upwards monotone in p, it follows that $x \notin [\![B_1]\!](P)$ and $x \notin [\![B_2]\!](P)$, and from here we get that $x \notin \bigcap_{i\in I}([\![B_1]\!](P_i) \cup [\![B_2]\!](P_i))$.

Suppose A is of the form $\Diamond B$ for some diamond or reversed diamond $\Diamond$ of L. We have to show that $[\![\Diamond B]\!](\bigcap_{i\in I}\{P_i\}) = \bigcap_{i\in I}[\![\Diamond B]\!](P_i)$. By the inductive hypothesis, we have $[\![\Diamond B]\!](\bigcap_{i\in I}\{P_i\}) = [\![\Diamond p_1]\!]\bigcap_{i\in I}[\![B]\!](P_i)$. If $[\![B]\!](P_i) = \emptyset$ for some $i \in I$, then $[\![\Diamond p_1]\!]\bigcap_{i\in I}[\![B]\!](P_i) = \emptyset = \bigcap_{i\in I}[\![\Diamond B]\!](P_i)$, so we may assume that $[\![B]\!](P_i) \neq \emptyset$ for all $i \in I$. Then by Proposition 71, $\{[\![B]\!](P_i) \mid i \in I\}$ is a family of non-empty closed sets. Moreover, we prove that $\{[\![B]\!](P_i) \mid i \in I\}$ is downward directed. Because, consider any finite number of elements of $\{[\![B]\!](P_i) \mid i \in I\}$, $[\![B]\!](P_1), \ldots, [\![B]\!](P_k)$. Then there is some $P \in \{P_i\}_{i\in I}$ such that $P \subseteq \bigcap_{j=1}^{k}\{P_j\}$. But then, $[\![B]\!](P) \in \{[\![B]\!](P_i) \mid i \in I\}$ and $[\![B]\!](P) \subseteq \bigcap_{j=1}^{k}[\![B]\!](P_j)$ by the upward monotonicity of B in p. Now, we may apply Proposition 68 (Esakia's Lemma for $\Diamond$) and conclude that $[\![\Diamond B]\!](\bigcap_{i\in I}\{P_i\}) = \bigcap_{i\in I}[\![\Diamond B]\!](P_i)$.

The case when A is of the form $\Diamond^{-1}B$ for some reversed diamond $\Diamond^{-1}$ of L' which is not in L is verbatim the same, except that we appeal to Proposition 69 (Esakia's Lemma for $\Diamond^{-1}$) instead of Proposition 68.

Lastly, consider the case when A is of the form $\Box B$ for some box $\Box$ of L. The result follows by the inductive hypothesis and the fact that $[\![\Box p_1]\!]$ distributes over arbitrary intersections of subsets of W, as seen in Proposition 60. $\Box$

Proposition 73 Let L be a type *1* language. Let $\mathfrak{g} = \langle W, \mathcal{R}, \mathbb{W}\rangle$ be a descriptive frame for L. Let s be a closed set with respect to $\mathfrak{g}$, $s = \bigcap\{P \in \mathbb{W} \mid s \subseteq P\}$. Then, $\{P \in \mathbb{W} \mid s \subseteq P\}$ is a downward directed family of closed with respect to $\mathfrak{g}$ sets.

Proof First, every admissible set is closed, so $\{P \in \mathbb{W} \mid s \subseteq P\}$ is a family of closed with respect to $\mathfrak{g}$ sets. To see that $\{P \in \mathbb{W} \mid s \subseteq P\}$ is downwards directed, let $P_1, P_2 \in \{P \in \mathbb{W} \mid s \subseteq P\}$. Then $s \subseteq P_1 \cap P_2$, and also $P_1 \cap P_2$ is an admissible set, therefore $P_1 \cap P_2 \in \{P \in \mathbb{W} \mid s \subseteq P\}$. $\Box$

To finish the proof of the first result of Proposition 50 regarding type *1* languages and the Ackermann rule, we only require one more result. The proof then follows by the second result of Proposition 50 and by the following Proposition 74:

Proposition 74 (Ackermann Lemma for Descriptive Frames) Let L be a type *1* language and let L' be its hybrid temporal extension. Let:
1. $\mathfrak{g} = \langle W, \mathcal{R}, \mathbb{W} \rangle$ be a descriptive frame for L, or a Kripke frame for L (in this case, we take $\mathbb{W} = \mathbb{P}(W)$).
2. $\{q_1, \ldots, q_n, p\}$ be different propositional variables from $PROP$.
3. A be a syntactically closed with respect to L formula in L' with $PROP(A) \subseteq \{q_1, \ldots, q_n\}$ and $NOM(A) \subseteq \{c_1, \ldots, c_m\}$
4. B be a syntactically open with respect to L formula in L' with $PROP(B) \subseteq \{q_1, \ldots, q_n, p\}$ and $NOM(A) \subseteq \{c_1, \ldots, c_m\}$, which is negative in p
Then for all $Q_1, \ldots, Q_n \in \mathbb{W}$ and all $x_1, \ldots, x_m \in W$, it is the case that:
$[\![B]\!](Q_1, \ldots, Q_n, [\![A]\!](Q_1, \ldots, Q_n, x_1, \ldots, x_n), x_1, \ldots, x_n) = W$
if and only if there is a $P \in \mathbb{W}$ such that:
$[\![A]\!](Q_1, \ldots, Q_n, x_1, \ldots, x_n) \subseteq P$ and
$[\![B]\!](Q_1, \ldots, Q_n, P, x_1, \ldots, x_n) = W$.

Proof The implication from right to left follows by Proposition 46, by the downwards monotonicity of B in p.

For the converse, let $Q_1, \ldots, Q_n \in \mathbb{W}$ and let $x_1, \ldots, x_m \in W$. We denote:
$A_0 =_{def} [\![A]\!](Q_1, \ldots, Q_n, x_1, \ldots, x_n)$, where $A_0 \subseteq W$;
$B_0(P) =_{def} [\![B]\!](Q_1, \ldots, Q_n, P, x_1, \ldots, x_n)$,
where $P \subseteq W$ and $B_0(P) : \mathbb{P}(W) \mapsto \mathbb{P}(W)$ is a function over subsets of W.

If $\mathfrak{g}$ is a Kripke frame, i.e. $\mathbb{W} = \mathbb{P}(W)$, we simply take $P = A_0$, because all subsets of W are admissible.

Now, assume that $\mathfrak{g}$ is a descriptive frame and $B_0(A_0) = W$. Let B' be the negation of B written in negation normal form. Then B' is a syntactically closed formula, B' is positive in p, and we again denote $B_0'(P)$ to be the corresponding function over subsets of W with $B_0'(P) =_{def} [\![B']\!](Q_1, \ldots, Q_n, P, x_1, \ldots, x_n)$. We have that and $B_0'(A_0) = \emptyset$. We need to find an admissible set $P \in \mathbb{W}$ such that $A_0 \subseteq P$ and $B_0'(P) = \emptyset$. Since A is a syntactically closed formula, it follows by Proposition 70 that A_0 is a closed with respect to $\mathfrak{g}$ subset of W and hence that $A_0 = \cap\{S \in \mathbb{W} \mid A_0 \subseteq S\}$. By Proposition 73, $\{S \in \mathbb{W} \mid A_0 \subseteq S\}$ is a downward directed family of closed sets with respect to $\mathfrak{g}$. Hence $\emptyset = B_0'(A_0) = B_0'(\bigcap\{S \in \mathbb{W} \mid A_0 \subseteq S\}) = \bigcap\{B_0'(S) \mid S \in \mathbb{W} \ \& \ A_0 \subseteq S\}$, by Proposition 72 for B', which requires B' to be positive in p. Again by proposition 70, $\{B_0'(S) \mid S \in \mathbb{W} \ \& \ A_0 \subseteq S\}$ is a family of closed with respect to $\mathfrak{g}$ sets with an empty intersection. Hence, by the compactness of $\mathfrak{g}$ (Proposition 66), $\{B_0'(S) \mid S \in \mathbb{W} \ \& \ A_0 \subseteq S\}$ doesn't have the fip, and we can also see that it's non-empty (take for example $S = W$). Hence there is a finite subfamily, $S_1, \ldots, S_k \in \{S \in \mathbb{W} \mid A_0 \subseteq S\}$, such that $\bigcap_{j=1}^{k}\{B_0'(S_j)\} = \emptyset$. But then $S_0 =_{def} \bigcap_{j=1}^{k}\{S_j\}$ is an admissible set, $A_0 \subseteq S_0$, and $B_0'(S_0) = \emptyset$, i.e. $B_0(S) = W$. Hence, we can choose $P = S_0$. $\square$

3.5 The Algorithm Deterministic SQEMA

We now describe a deterministic version of the SQEMA algorithm from [17].

If σ is $\neg\bigwedge(\chi_1, \ldots, \chi_m)$, we denote by $\sigma[\chi_j//\chi_1', \ldots, \chi_m']$: $\neg\bigwedge(\chi_1, \ldots, \chi_{j-1}, \chi_1', \ldots, \chi_m', \chi_{j+1}, \ldots, \chi_n)$.

We denote by $\sigma[p//\neg p]$ the system of equations, produced from σ, where, simultaneously, every positive occurrence of p has been replaced with $\neg p$ and every occurrence of $\neg p$ has been replaced with p.

Below, by boxes we mean any box or reversed box, and by diamond we mean any diamond or reversed diamond.

Definition 75 (The algorithm Deterministic SQEMA)

INPUT: $A \in L$, where L is a type *1* or a type *2* language and L' is its hybrid temporal extension.
OUTPUT: $\langle success, fol(A)\rangle$ or $\langle failure\rangle$

STEP 1: Rewrite A in negation normal form. Then, distribute all boxes, which are not in the scope of a diamond, and all disjunctions, over conjunctions as much as possible, using the semantic equivalences:
Rule 1.1: $\Box(A_1 \wedge A_2) \equiv (\Box A_1 \wedge \Box A_2)$
Rule 1.2: $((A_1 \wedge A_2) \vee A_3) \equiv ((A_1 \vee A_3) \wedge (A_2 \vee A_3))$
Rule 1.3: $(A_1 \vee (A_2 \wedge A_3)) \equiv ((A_1 \vee A_2) \wedge (A_1 \vee A_3))$

Thus, obtain $A \equiv \bigwedge(A_1, \ldots, A_n)$ where no further applications of rules 1.1, 1.2 or 1.3 are possible on any A_i. Now reserve the nominal c_k, such that all nominals occurring in A are among $c_1, \ldots, c_{k-1}$, and use it throughout the steps. Proceed with STEP 2, applied separately on each of the subformulas A_i, and if it succeeds for all A_i, proceed to STEP 5. Otherwise, if anyone of the branches for a single i fails, then return $\langle failure\rangle$ as output and stop.

STEP 2: Let A_i be one of the conjuncts from STEP 1. Let A' be the *normalized* form, of $\neg A_i$, which we define below, but for now it suffices to know that it means that A' is in negation normal form, and any variable, that occurs only positively or negatively in $\neg A_i$ has been replaced, by the positive or negative elimination rules, with $\top$, or $\bot$, respectively. Now, construct the equation $(\neg c_k \veebar A')$, where c_k is the nominal from STEP 1. By the equivalence rule, try solving σ: $\neg\bigwedge((\neg c_k \veebar A'))$ by proceeding to STEP 3, and then return the result to STEP 1.

STEP 3: Let the current system be σ. For every permutation of $PROP(\sigma)$, try it as the *variable elimination order*, trying to eliminate each variable in that order with a new, empty *backtracking stack* to be used with the current permutation, by proceeding to STEP 4. If a permutation succeeds, and thus, all propositional variables have been eliminated from the current system, proceed to STEP 5. If all elimination orders fail, report failure for the current system and go back to executing STEP 2.

STEP 4: Take the propositional variable p that has to be eliminated and the system σ_0 as input. Save a backtracking context $\langle p, \sigma_0 \rangle$, to the stack for the application of the polarity reversing rule, but only if the input hasn't come out of the stack. Deterministically apply the SQEMA rules in order to try eliminating all occurrences of p, converting σ_0 to σ_1. Use the deterministic strategy for SQEMA rules application which is shown below. If p has been eliminated, report success and return the normalized form of σ_1 (defined below) to STEP 3 to try eliminating the remaining variables. If this fails, check if the backtracking stack is empty. If it is empty, report failure to eliminate p and resume executing STEP 3 to try other permutations. Otherwise, backtrack to the context $\langle p', \sigma'_0 \rangle$ from the top of the stack, which may apply to a previous variable, then execute STEP 4 with p' and $\sigma'_0[p' // \neg p']$, skipping the saving of backtracking context.

STEP 5: If this step is reached by all branches of the execution, then all propositional variables have been eliminated from all systems resulting from the input formula. Let all pure systems be $\sigma_1, \ldots, \sigma_n$. For each pure system σ_i, let $NOM(\sigma_i) \setminus \{c_k\} = \{c_{j^i_1}, \ldots, c_{j^i_{l_i}}\}$, and let c_{m_i} be such that all nominals occurring in $\{c_k, \sigma_i\}$ are among $c_1, \ldots, c_{m_i - 1}$. Using the Standard Translation Lemma 42, let $fol_i(A)$ be: $\forall x_{j^i_1} \ldots \forall x_{j^i_{l_i}} \exists x_{m_i} st(m_i + 1, x_{m_i}, \sigma_i)$. Let $fol(A)$ be $\bigwedge(fol_1(A), \ldots, fol_n(A))$, by the Conjunction Lemma 43. Return the result $\langle success,\ fol(A) \rangle$. Stop.

Now, we define: a) the normalization of a formula used in STEP 2 with diamond extraction, b) the normalization of a system of equations used in STEP 4, and c) the deterministic SQEMA rules application strategy.

Definition 76 (Main normalization rules) We say that A is *normalized* iff the following rules cannot be applied any more to any subformula of A:

1. $\neg\neg A_1 \Rightarrow A_1$.
2. $\neg(A_1 \vee A_2) \Rightarrow (\neg A_1 \wedge \neg A_2)$.
3. $\neg(A_1 \wedge A_2) \Rightarrow (\neg A_1 \vee \neg A_2)$.
4. $((A_1 \wedge A_2) \vee A_3) \Rightarrow ((A_1 \vee A_3) \wedge (A_2 \vee A_3))$ (this is the same as SQEMA's Rule 1.2 in STEP 1).
5. $(A_1 \vee (A_2 \wedge A_3)) \Rightarrow ((A_1 \vee A_2) \wedge (A_1 \vee A_3))$ (this is the same as SQEMA's Rule 1.3 in STEP 1).

We say that these three rules are the *main normalization rules*, or just the *main rules*.

Clearly, a normalized formula is also in negation normal form.

Definition 77 (Normalized equation) We say that an equation $(\gamma_1 \veebar \gamma_2)$ is *normalized* iff both γ_1 and γ_2 are normalized formulas. Note that a formula, which is an equation, may be a normalized equation but not a normalized formula. For example, consider $(\neg c \veebar (p \wedge \neg p))$, which is clearly a normalized equation, but it is not a normalized formula, because the main rule 5. is applicable.

Definition 78 (Normalized system) We say that a system σ is normalized iff all equations in σ are normalized. Note that as a formula, no system is normalized because it is clearly not in negation normal form.

In the rules given below, we are influenced by the work of Hughes and Cresswell in [39]. In Chapter Three, they discuss how an S5 formula is reducible to a formula of modal depth one, using rules similar to the rules below for the universal modality. This is no coincidence, because the universal modality is a kind of an S5 modality. Also, the author's conjecture is that because of the way Deterministic SQEMA uses normalization, the algorithm succeeds on formulas of modal depth one; it is a fact that formulas of modal depth 1 have first-order correspondents, noticed by van Benthem in [56][57].

We use the table below, where $\Box$ is any box or reversed box, and $\Diamond$ is any diamond or reversed diamond:

For $j \in \{1, 2\}$, we use U_j for either $[U]$ or $\langle U \rangle$, we use $\overset{\wedge}{\vee}$ for either $\vee$ or $\wedge$.

Replace	with	Replace	with
$(c_1 \to \langle U \rangle c_2)$	$\top$	$(\langle U \rangle \gamma_1 \vee \gamma_2)$, for $\gamma_2 \equiv \neg\gamma_1$	$\top$
$U_1 U_2 \gamma$	$U_2 \gamma$	$(\langle U \rangle \gamma \vee \Diamond\gamma)$	$\langle U \rangle \gamma$
$\Box U_1 \gamma$	$(U_1 \gamma \vee \Box\bot)$	$(\langle U \rangle \gamma \vee \gamma)$	$\langle U \rangle \gamma$
$[U](U_1\gamma_1 \overset{\wedge}{\vee} U_2\gamma_2)$	$(U_1\gamma_1 \overset{\wedge}{\vee} U_2\gamma_2)$	$(\langle U \rangle \gamma \wedge \Diamond\gamma)$	$\Diamond\gamma$
$[U](U_1\gamma_1 \overset{\wedge}{\vee} \gamma_2)$	$(U_1\gamma_1 \overset{\wedge}{\vee} [U]\gamma_2)$	$(\langle U \rangle \gamma \wedge \gamma)$	γ
$[U]\neg c$	$\bot$	$([U]\gamma_1 \wedge \gamma_2)$, for $\gamma_2 \equiv \neg\gamma_1$	$\bot$
$\Diamond U_1 \gamma$	$(U_1\gamma \wedge \Diamond\top)$	$([U]\gamma \wedge \Box\gamma)$	$[U]\gamma$
$\langle U \rangle(U_1\gamma_1 \overset{\wedge}{\vee} U_2\gamma_2)$	$(U_1\gamma_1 \overset{\wedge}{\vee} U_2\gamma_2)$	$([U]\gamma \wedge \gamma)$	$[U]\gamma$
$\langle U \rangle(U_1\gamma_1 \overset{\wedge}{\vee} \gamma_2)$	$(U_1\gamma_1 \overset{\wedge}{\vee} \langle U \rangle\gamma_2)$	$([U]\gamma \vee \Box\gamma)$	$\Box\gamma$
$\langle U \rangle c$	$\top$	$([U]\gamma \vee \gamma)$	γ

Replace	with	Replace	with
$\neg\top$	$\bot$	$\neg\bot$	$\top$
$(\gamma \vee \top)$	$\top$	$(\gamma \wedge \top)$	γ
$(\gamma \vee \bot)$	γ	$(\gamma \wedge \bot)$	$\bot$
$(\gamma \vee \gamma)$	γ	$(\gamma \wedge \gamma)$	γ
$(\gamma_1 \vee \gamma_2)$, for $\gamma_1 \equiv \neg\gamma_2$	$\top$	$(\gamma_1 \wedge \gamma_2)$, for $\gamma_1 \equiv \neg\gamma_2$	$\bot$
$(\gamma_1 \vee \gamma_2)$ (in CNF)	$(\gamma_2 \vee \gamma_1)$	$(\gamma_1 \wedge \gamma_2)$ (in CNF)	$(\gamma_2 \wedge \gamma_1)$
$(\gamma_1 \vee (\gamma_2 \vee \gamma_3))$ (in CNF)	$((\gamma_1 \vee \gamma_2) \vee \gamma_3)$	$(\gamma_1 \wedge (\gamma_2 \wedge \gamma_3))$ (in CNF)	$((\gamma_1 \wedge \gamma_2) \wedge \gamma_3)$
$\Box\top$	$\top$	$\Diamond\bot$	$\bot$
$(\Diamond\gamma_1 \vee \Diamond\gamma_2)$	$\Diamond(\gamma_1 \vee \gamma_2)$	$(\Box\gamma_1 \wedge \Box\gamma_2)$	$\Box(\gamma_1 \wedge \gamma_2)$

Definition 79 (Eliminating Normalization Procedure) We say that a procedure is an *eliminating normalization procedure* iff its input is a modal formula, its output is a modal formula, it is terminating, deterministic, and it converts its input formula A into negation normal form by applying the main rules 1., 2., and 3. to subformulas of A as much as possible, finally obtaining A', and then the procedure uses subformula rewriting by applying all the main rules to subformulas of A' until none of them may be applied further, while also possibly applying the rules in the above tables, using the rules marked with (in CNF) only for re-arranging a conjunctive normal form's subformulas into some kind of ordering (without looping forever), replaces all variables, which either occur only positively or occur only negatively in the formula with $\top$ or $\bot$ respectively, does no other kinds of modifications to the formula, and its output does not contain any subformulas of the kind $\neg\top$, $\neg\bot$, $(\bot \overset{\wedge}{\vee} \gamma)$, or $(\gamma \overset{\wedge}{\vee} \bot)$, and contains no variables which occur only positively or only negatively.

a) It is clear how we can obtain a formula in negation normal form for a given γ, such that $\Box_0^{-1}$ and $\Diamond_0^{-1}$ do not occur, because these are semantically equivalent to $[U]$ and $\langle U\rangle$. We use this procedure to reduce the number of subformulas in the output, by applying the above equivalence rules for some obvious boolean and modal laws, as well as the above rules for the universal modality. Then, we define a procedure for constructing a *conjunctive normal form*, using the standard definition of this notion. It is clear how this normal form can be constructed. During this construction, also perform *diamond extraction*, applying the rule $(\Diamond A' \vee \Diamond A'') \equiv \Diamond(A' \vee A'')$. Attempt to eliminate semantically equivalent or opposite members of any disjunction, by comparing their normal forms. The output must not have subformulas of the kind $\neg\top$, $\neg\bot$, $(\bot \overset{\wedge}{\vee} \gamma)$, or $(\gamma \overset{\wedge}{\vee} \bot)$.

Two improvements can be made: during the elimination, a tableaux method for the input language could be used to prove an equivalence, instead of comparing normal forms. Also, in the conjunction construction phase, modal resolution can be performed, as in example 6.14 of [18].

This is the normalization procedure for γ, which produces the *normal form of* γ: First, convert γ to negation normal form, then convert the result to conjunctive normal form simultaneously performing diamond extraction, by the equivalence rule, then perform box extraction using the semantic equivalence $(\Box A_1 \wedge \Box A_2) \equiv \Box(A_1 \wedge A_2)$, and finally replace any variables that occur either only positively or only negatively in γ with $\top$, or $\bot$, respectively. Repeat the whole process until no further changes to the formula can be made.

Clearly, the normalization procedure for single formulas that Deterministic SQEMA uses is an eliminating normalization procedure by Definition 79.

Definition 80 (System Normalization Procedure) We say that a procedure to convert a system $\neg \bigwedge(\chi_1, \ldots, \chi_n)$ into another system σ' is a *system normalization procedure* iff it produces A_0 from $\bigwedge(\chi_1, \ldots, \chi_n)$ by applying the rules $\neg\top \Rightarrow \bot$, $\neg\bot \Rightarrow \top$, $(\gamma \vee \bot) \Rightarrow \gamma$, and $(\bot \vee \gamma) \Rightarrow \gamma$ to its subformulas as much as possible, then produces A_1 from A_0 via an eliminating normalization procedure by Definition 79, and finally forms its result σ' as either $\neg \bigwedge(A_1)$ if A_1 is of the kind $(\neg c \vee A_2)$, or otherwise $\neg \bigwedge((\bot \veebar A_1))$.

b) Now, we normalize a system of equations σ. Let σ be $\neg \bigwedge(\chi_1, \ldots, \chi_n)$. Let A' be the normal form of $\bigwedge(\chi_1, \ldots, \chi_n)$. If A' is of the kind $(\neg c \vee A'')$, then the *normal form of* σ is $\neg \bigwedge((\neg c \veebar A''))$; otherwise, it is $\neg \bigwedge((\bot \veebar A'))$.

Clearly, the normalization procedure of Deterministic SQEMA for systems is equivalent to a system normalization procedure according to Definition 80.

c) The deterministic strategy for applying the SQEMA rules for a given variable p is to use the *step* function (given below) repeatedly until either a formula without occurrences of p is reached, or *failure* is obtained.

Definition 81 (Deterministic SQEMA Step) We describe a single step of the strategy, which is uniquely defined for σ and p.

(1) If $p \not\nearrow \sigma$, then the result is σ.
(2) Else, if σ is $\neg \bigwedge\{(\alpha_1 \veebar p), \ldots, (\alpha_{n_a} \veebar p), \beta_1, \ldots, \beta_{n_b}, \theta_1, \ldots, \theta_{n_t}\}$, where $n_a \geq 0$, $n_b \geq 0$, $n_t \geq 0$, $p \not\nearrow \{\alpha_1, \ldots, \alpha_{n_a}, \theta_1, \ldots, \theta_{n_t}\}$, and $\beta_1, \ldots, \beta_{n_b}$ are formulas which are *negative* in p, then we can apply the Ackermann rule for p and σ. Let for $1 \leq l \leq n_b$, β'_l be obtained from β_l by replacing all occurrences of $\neg p$ with $\bigwedge(\alpha_1, \ldots, \alpha_{n_a})$. Then the result for σ is $\neg \bigwedge(\beta'_1, \ldots, \beta'_{n_b}, \theta_1, \ldots, \theta_{n_t})$.
(3) If we are not in any of the above two cases, then there is at least one positive occurrence of p in σ, which is not in an equation of the kind $(\alpha \veebar p)$, such that $p \not\nearrow \alpha$. For convenience, let σ be $\neg \bigwedge(\chi_1, \ldots, \chi_m)$,

let j be the least number, such that p occurs positively in χ_j, χ_j is not as described, and let χ_j be $(A' \veebar A_1)$.

(3.1) If A_1 is $(A_2 \wedge A_3)$, then, by the equivalence rule, the result for σ is $\sigma[\chi_j//(A' \veebar A_2), (A' \veebar A_3)]$.

(3.2) If A_1 is $(A_2 \vee A_3)$, then there are three cases. If $p \not\nearrow A_2$, then by the equivalence rule the result for σ is $\sigma[\chi_j//((A' \vee A_2) \veebar A_3)]$. Otherwise, if $p \not\nearrow A_3$, then, by the equivalence rule, the result for σ is $\sigma[\chi_j//((A' \vee A_3) \veebar A_2)]$. Otherwise, the result for σ is *failure*.

(3.3) If A_1 is $\Box A_2$, by the box rule, the result for σ is $\sigma[\chi_j//(\Box^{-1}A' \veebar A_1)]$.

(3.4) If A_1 is $\Diamond A_2$ and A' is either $\neg c'$ or $(\bot \vee \neg c')$, then, by the diamond rule, let c'' be a new nominal, then the result for σ' is $\sigma[\chi_j//(c' \to \Diamond c''), (\neg c'' \veebar A_1)]$.

(3.5) If we are not in any of the above four cases, the result for σ is *failure*.

It is immediate that the above describes a uniquely defined effective function over the systems of equations and propositional variables. We denote the function by *step*.

Immediately by the definition of *step*, we have that $\sigma \approx_k^d step(\sigma, p)$ for type *1* input languages, and $\sigma \approx_k^{di} step(\sigma, p)$ for type *2* input languages, by the correctness of the SQEMA rules, Proposition 50.

We prove that the application of *step* can be composed only finitely many times for a starting σ and a given p, before reaching either a σ', such that $p \not\nearrow \sigma'$, or *failure*.

Indeed, if the result is ever obtained by (1), (2), (3.5), or the failing condition of (3.2), it is clear that this is the final application of *step*. Therefore, suppose there is an infinite sequence of results, obtained by (3.1), (3.3), (3.4), or the non-failing conditions of (3.2). Then, there is an infinite sequence $\sigma_0, \sigma_1, \ldots$, and let $S_0, S_1, \ldots$ be the sum of lengths of right-hand sides of equations in the corresponding σ-s. It is clear that $S_0 > 0$ and for $i < j$, $S_i > S_j$, which is impossible. Therefore, we can only apply *step* a finite number of times. $\Box$

Lemma 82 (Main Deterministic SQEMA Lemma) Let σ be a system of equations that Deterministic SQEMA works on. Then the following hold:

i. σ is a normalized system, except in STEP 4 between an application of the Ackermann Rule (which is case 2. of the function *step*) and the subsequent system normalization.

ii. In the left-hand side of each equation, there are no conjunctions outside the scope of a box or a diamond.

iii. If we are in STEP 4 of Deterministic SQEMA with a variable p to eliminate, then the left-hand side of any equation in σ does not contain occurrences of p.

iv. If L, the modal language of the input formula, is a type *1* language, and if L' is its hybrid temporal extension, then σ is a syntactically closed with respect to L formula of L' with its equations being syntactically open with respect to L formulas of L'.

Proof Condition iv. above can be checked immediately by checking all possible rules that Deterministic SQEMA, including *step*, applies to its systems.

In STEP 2, we take the conjunct A_i and use an eliminating normalization procedure to convert $\neg A_i$ into A', then we form a system with the only equation $(\neg c \veebar A')$. Clearly, the only equation in this system is normalized, i.e. the main rules cannot be applied further on any subformula of A' or on any subformula of $\neg c$. Also, there are no conjunctions or variables in the left-hand side of the equation.

In STEP 4, we run the Deterministic SQEMA strategy, which is to run the *step* function, converting the current system into another one, occasionally backtracking and replacing the current system with a system, where a variable's polarity has been switched.

In STEP 4, after eliminating a variable from a system, we use an eliminating normalization procedure to convert $\neg\sigma$ to a normalized formula A_0, then we construct a new system σ', which is either $\neg \bigwedge((\neg c \vee A_1))$ if A_0 is $(\neg c \vee A_1)$ or otherwise $\neg \bigwedge((\bot \veebar A_0))$. Because A_0 is normalized, then so is the system σ'. Also, there are no conjunctions or variables in the left-hand side of the single equation of σ'.

Let p be the current variable to eliminate in STEP 4. It remains only to see that if σ has the desired properties, then $step(\sigma, p)$ is either *failure* or a system with the desired properties.

Let p be the current variable to eliminate in STEP 4. Let σ be the input system, which is normalized, there are no conjunctions in the left-hand side of any equation of σ outside the scope of a box or a diamond, and p does not occur in any left-hand side of any equation of σ. Let $\sigma' =_{def} step(\sigma, p)$ be the output system, in the cases when the output is not *failure*.

1. If the result is obtained by 1., the conditions hold.

2. If the result is obtained by 2., the Ackermann Rule, we only need to prove conditions ii. and iii. We have that σ is $\neg \bigwedge((\alpha_1 \vee p), \ldots, (\alpha_{n_a} \vee p), \beta_1, \ldots, \beta_{n_b}, \theta_1, \ldots, \theta_{n_t})$. Because $p \not\nearrow \sigma'$, clearly iii. holds for σ'. Because iii. holds for σ, then the left-hand side of any β_j hasn't changed. This fact and the fact that ii. holds for σ shows that ii. holds for σ'.

3. If the result is obtained by 3.1, then we have split an equation on a conjunction into two equations. Let χ be the equation that has been split. Let χ be $(A_1 \veebar (A_2 \wedge A_3))$. The equation χ has been split into two equations of σ', $\chi_1 =_{def} (A_1 \veebar A_2)$ and $\chi_2 =_{def} (A_1 \veebar A_3)$. Because the system σ is normalized, we have that the formula $(A_2 \wedge A_3)$ is normalized, so there are no

possible applications of the main rules on any subformula of $(A_2 \wedge A_3)$. It is enough to show that there are no possible applications of the main rules on any subformula of any of the two formulas A_2 and A_3. But this follows by the fact that A_2 and A_3 are subformulas of $(A_2 \wedge A_3)$.

4. If the result is obtained by 3.2, then it is either *failure*, in which case the condition holds, or σ'. Let $\chi =_{def} (A_1 \veebar (A_2 \vee A_3))$ be the changed equation of σ, and let w.l.o.g. the resulting equation of σ' be $\chi' =_{def} ((A_1 \vee A_2) \veebar A_3)$.

Consider $(A_1 \vee A_2)$. Suppose for the sake of contradiction that A_2 contains a conjunction which is outside the scope of a box or a diamond. Let A_2' be the longest subformula of A_2, which is a conjunction not in the scope of a box or a diamond. Then either A_2' is A_2, or A_2' w.l.o.g. occurs in a subformula $(A_2' \vee A_2'')$ of A_2. In both cases, either the main rule 4. or the main rule 5. is applicable to $(A_2 \vee A_3)$, contradiction. We conclude that A_2 does not have any conjunctions which are not in the scope of a box or a diamond. Then clearly $(A_1 \vee A_2)$ also does not have any conjunctions outside the scope of a box or a diamond.

All of A_1, A_2, A_3 are already normalized formulas so it remains to see that no main rule is applicable to $(A_1 \vee A_2)$. Suppose one of the main rules 1., 2., or 3. is applicable to a subformula of $(A_1 \vee A_2)$, then it is also applicable to a subformula of either A_1 or A_2, contradiction. Now suppose that main rule 4. or main rule 5. is applicable to a subformula of $(A_1 \vee A_2)$. Because it is not applicable to subformulas of either A_1 or A_2, it must be applicable directly to $(A_1 \vee A_2)$. But $(A_1 \vee A_2)$ contains no conjunctions outside the scope of a box or a diamond, contradiction. Therefore χ' is a normalized equation.

5. If the result is obtained by 3.3, then the Box-Rule of SQEMA was applied. Let the changed equation in σ be $\chi =_{def} (A_1 \veebar \Box A_2)$, and the resulting equation in σ' be $\chi' =_{def} (\Box^{-1} A_1 \veebar A_2)$. Clearly, $\Box^{-1} A_1$ has no conjunctions outside the scope of a box or a diamond. Clearly, A_2 is a normalized formula because it is a subformula of the normalized formula $\Box A_2$. Now suppose for the sake of contradiction that $\Box^{-1} A_1$ is not a normalized formula. Then some main rule 1., 2., 3., 4., or 5. can be applied to a subformula of $\Box^{-1} A_1$. Then the same main rule can be applied to a subformula of A_1, contradiction.

6. If the result is obtained by 3.4, then the Diamond-Rule of SQEMA was applied. Let the changed equation in σ be $\chi =_{def} (A_1 \veebar \Diamond A_2)$, with A_1 being either $\neg c$ or $(\perp \vee \neg c)$, and the resulting equations in σ' be $\chi_1' =_{def} (c \to \Diamond c')$ and $\chi_2' =_{def} (\neg c' \veebar A_2)$. Clearly both equations are normalized and both equations do not have in their left-hand side any conjunctions or variables.

7. If the result is obtained by 3.5, then the result is *failure* and the conditions hold. □

Corollary 83 Let σ be a system that Deterministic SQEMA works on in STEP 4, let p be a current variable to eliminate, and let $step(\sigma, p) = failure$. Then there is an equation χ in the system σ, such that p occurs positively in

χ and χ is either $(A_1 \veebar (A_2 \vee A_3))$ such that $p \nearrow A_2$ and $p \nearrow A_3$, or χ is $(A_1 \veebar \Diamond A_2)$ such that A_1 is not a negated nominal and is not $(\bot \vee \neg c)$.

Proof Clearly *failure* may only be obtained either from 3.2, in which case there is nothing to prove, or from 3.5. So let it be obtained from 3.5. Because the result is not obtained from either 1 or 2, then let the first equation of σ, where p occurs positively and it is not in the form $(\alpha \veebar p)$ such that $p \not\nearrow \alpha$, be χ. Suppose for the sake of contradiction that χ is not of the kind $(A_1 \veebar \Diamond A_2)$ such that A_1 is not a negated nominal and is not $(\bot \vee \neg c)$. Then there are the following cases:

1. χ is of the kind $(c' \to \Diamond c'')$. Contradicts the fact that $p \nearrow \chi$.

2. χ is of the kind $(A_1 \veebar p')$. If p' is p, then because χ is not in the form $(\alpha \veebar p)$ such that $p \not\nearrow \alpha$, then $p \nearrow A_1$. But this contradicts Lemma 82. So p' is not p. But then by Lemma 82, $p \not\nearrow \chi$, contradiction.

3. χ is either of the kind $(A_1 \veebar \neg p')$, of the kind $(A_1 \veebar c)$, or of the kind $(A_1 \veebar \neg c)$. Then because p occurs positively in χ, then we get $p \nearrow A_1$, contradicts Lemma 82.

4. χ is either of the kind $(A_1 \veebar (A_2 \overset{\wedge}{\vee} A_3))$ or of the kind $(A_1 \veebar \Box A_2)$. Contradicts the fact that the *failure* was obtained from 3.5.

5. χ is of the kind $(A_1 \veebar \Diamond A_2)$. Clearly in this case, if A_1 is either a negated nominal or $(\bot \vee \neg c)$, then the result is obtained from 3.4, contradiction. So the only possibility here is that A_1 is not a negated nominal and is not $(\bot \vee \neg c)$, but this contradicts the supposed property of χ.

In all cases, we have reached a contradiction. Thus we conclude that χ is of the kind $(A_1 \veebar \Diamond A_2)$ such that A_1 is not a negated nominal and is not $(\bot \vee \neg c)$. $\Box$

Also by Lemma 82, the SQEMA rules have been applied with accordance to Definition 48, so by Theorem 32, we conclude that the formulas that Deterministic SQEMA succeeds on are canonical with respect to L.

This concludes our definition of Deterministic SQEMA and the proof for its soundness and termination.

We now use the above Main Deterministic SQEMA Lemma 82 and its Corollary 83 to prove that Deterministic SQEMA succeeds on two popular elementary classes of canonical formulas, *the Sahlqvist class* and *class of Inductive formulas.*

3.6 Examples

Here $\Diamond$ means $\Diamond_1$ as in ML($\Box$) and $\Box$ means $\Box_1$.

3.6.1 $(c_1 \rightarrow \neg\Diamond c_1)$

Consider the formula $(c_1 \rightarrow \neg\Diamond c_1)$.

In STEP 1, we rewrite it in negation normal form, obtaining:
$(\neg c_1 \vee \Box\neg c_1)$.

It is impossible to apply rules 1.1, 1.2 or 1.3 anymore. This we have a single conjunct, $A_1 = (\neg c_1 \vee \Box\neg c_1)$.

In STEP 2, we need to normalize $\neg A_1 = \neg(\neg c_1 \vee \Box\neg c_1)$.

By Definition 79, first we obtain the negation normal form of $\neg A_1$, which is $(c_1 \wedge \Diamond c_1)$.

None of the main rules are applicable to $(c_1 \wedge \Diamond c_1)$.

Now, we reserve a nominal, c_2, which does not occur in $(c_1 \wedge \Diamond c_1)$. The initial equation is $(\neg c_2 \veebar (c_1 \wedge \Diamond c_1))$. The initial system is:
$\sigma_1 = \neg \bigwedge((\neg c_2 \veebar (c_1 \wedge \Diamond c_1)))$.

In STEP 3, we pick the empty elimination order because there are no propositional variables in σ_1. Clearly we have succeeded in eliminating all propositional variables in the empty elimination order, so we proceed with STEP 5.

In STEP 5, we have the pure system $\sigma_1 = \neg \bigwedge((\neg c_2 \veebar (c_1 \wedge \Diamond c_1)))$. We take c_3 as the next available nominal, and we set $\mathit{fol_1} = \forall x_1 \exists x_3 st(4, x_3, \sigma_1)$. The result is $\bigwedge(\forall x_1 \exists x_3 st(4, x_3, \sigma_1))$.

Let us see how st works here.
$st(4, x_3, \neg \bigwedge((\neg c_2 \vee (c_1 \wedge \Diamond c_1)))) \equiv \neg st(4, x_3, (\neg c_2 \vee (c_1 \wedge \Diamond c_1))) \equiv$
$\equiv \neg(st(4, x_3, \neg c_2) \vee st(4, x_3, (c_1 \wedge \Diamond c_1))) \equiv$
$\equiv \neg(\neg st(4, x_3, c_2) \vee (st(4, x_3, c_1) \wedge st(4, x_3, \Diamond c_1))) \equiv$
$\equiv \neg(\neg(x_3 = x_2) \vee ((x_3 = x_1) \wedge \exists x_4((x_3\ r_1\ x_4) \wedge st(5, x_4, c_1)))) \equiv$
$\equiv \neg(\neg(x_3 = x_2) \vee ((x_3 = x_1) \wedge \exists x_4((x_3\ r_1\ x_4) \wedge (x_4 = x_1))))$

So, the final result is:
$\forall x_1 \exists x_3 \neg(\neg(x_3 = x_2) \vee ((x_3 = x_1) \wedge \exists x_4((x_3\ r_1\ x_4) \wedge (x_4 = x_1))))$

After some simplification, we obtain the result:
$\psi(x_2) \equiv \forall x_1((x_2 = x_1) \rightarrow \neg(x_2\ r_1\ x_1))$.
which can be further simplified to:
$\psi(x_2) \equiv \neg(x_2\ r_1\ x_2)$, the local irreflexivity condition.

3.6.2 $((\Box\Diamond p \rightarrow \Diamond\Box p) \vee (\Box p \rightarrow \Diamond p))$

Consider the formula $((\Box\Diamond p \rightarrow \Diamond\Box p) \vee (\Box p \rightarrow \Diamond p))$. This is a formula on which the classical SQEMA fails.

In STEP 1, we rewrite it in negation normal form, obtaining:
$(\Diamond\Box p \vee \Diamond\Box\neg p \vee \Diamond p \vee \Diamond\neg p)$.

It is impossible to apply rules 1.1, 1.2 or 1.3 anymore. This we have a single conjunct, $A_1 = (\Diamond\Box p \vee \Diamond\Box\neg p \vee \Diamond p \vee \Diamond\neg p)$.

In STEP 2, we need to normalize $\neg A_1 = \neg(\Diamond\Box p \vee \Diamond\Box\neg p \vee \Diamond p \vee \Diamond\neg p)$.

By Definition 79, first we obtain the negation normal form of $\neg A_1$, which is $(\Box\Diamond\neg p \wedge (\Box\Diamond p \wedge (\Box\neg p \wedge \Box p)))$.

Now, none of the main rules are applicable to the above, but the box extraction rule is applicable. Thus we obtain $(\Box\Diamond\neg p \wedge (\Box\Diamond p \wedge \Box(\neg p \wedge p)))$.

The next rule that is applied is replacing $(\gamma_1 \wedge \gamma_2)$ with $\perp$ when the normalized forms of γ_1 and $\neg\gamma_2$ are the same formula. Here, both of these are $\neg p$ and thus we obtain the formula $(\Box\Diamond\neg p \wedge (\Box\Diamond p \wedge \Box\perp))$.

We apply box extraction again, obtaining $(\Box\Diamond\neg p \wedge \Box(\Diamond p \wedge \perp))$.

Now we replace $(\gamma \wedge \perp)$ with $\perp$, obtaining $(\Box\Diamond\neg p \wedge \Box\perp)$.

Box extraction is applied again, obtaining $\Box(\Diamond\neg p \wedge \perp)$.

Using the same rule as before, we obtain $\Box\perp$. No more rules can be applied.

Now, we reserve a nominal, c_1, which does not occur in $\Box\perp$. The initial equation is $(\neg c_1 \veebar \Box\perp)$. The initial system is $\sigma_1 = \neg \bigwedge((\neg c_1 \veebar \Box\perp))$.

In STEP 3, we pick the empty elimination order because there are no propositional variables in σ_1. Clearly we have succeeded eliminating all variables in the empty elimination order, so we proceed with STEP 5.

In STEP 5, we have the pure system $\sigma_1 = \neg \bigwedge((\neg c_1 \veebar \Box\perp))$. We take c_2 as the next available nominal, and we set $\mathit{fol_1} = \exists x_2 st(3, x_2, \sigma_1)$. The result is $\bigwedge(\exists x_2 st(3, x_2, \sigma_1))$.

After some simplification and renaming of individual variables, we obtain the result $\exists z_1(x\ r_1\ z_1)$.

Here we can see the value of the box extracting rule, allowing Deterministic SQEMA to succeed where the classic SQEMA fails.

3.6.3 $(\langle U\rangle p \to \langle U\rangle\Diamond p)$

Let us consider the formula $(\langle U\rangle p \to \langle U\rangle\Diamond p)$.

In STEP 1, we rewrite it to negation normal form, obtaining
$A_1 = ([U]\neg p \vee \langle U\rangle\Diamond p)$, where it is impossible to apply rules 1.1, 1.2, or 1.3.

In STEP 2, we normalize $\neg A_1 = \neg([U]\neg p \vee \langle U\rangle\Diamond p)$.

We obtain the negation normal form of $\neg A_1$, which is $(\langle U\rangle p \wedge [U]\Box\neg p)$. No more rules for normalization can be applied, except for the ones marked with CNF, which do not apply here.

Now we reserve the nominal c_1.

The initial system is $\sigma_1 = \neg \bigwedge((\neg c_1 \veebar (\langle U\rangle p \wedge [U]\Box\neg p)))$.

In STEP 3, we chose $\langle p\rangle$ as the variable eliminating order with an empty backtracking stack and proceed to STEP 4.

In STEP 4, we save a backtracking context $\langle p, \sigma_1\rangle$, which will not be necessary, as we see below.

We now apply the Deterministic SQEMA strategy, which is to call *step* as many times as possible, until obtaining either *failure* or a pure system:

First, *step* splits on a conjunction: $\sigma_2 = \neg \bigwedge((\neg c_1 \veebar \langle U\rangle p), (\neg c_1 \veebar [U]\Box\neg p)$

Then, *step* applies the $\Diamond$-rule:
$\sigma_3 = \neg \bigwedge((c_1 \to \langle U\rangle c_2), (\neg c_2 \veebar p), (\neg c_1 \veebar [U]\Box\neg p))$

Now, *step* applies the Ackermann rule:
$\sigma_4 = \neg \bigwedge((c_1 \to \langle U\rangle c_2), (\neg c_1 \veebar [U]\Box\neg c_2))$

Because σ_4 is a pure system, we return to STEP 3, going to STEP 5:

We choose the new nominal c_3 and set $\mathit{fol}_1 = \forall x_2 \exists x_3 st(4, x_3, \sigma_4)$. Let $\mathit{fol} = \bigwedge(\mathit{fol}_1)$, which is our result. After simplification and renaming of the individual variables, we obtain: $\forall y_1 \exists z_1(z_1\ r_1\ y_1)$.

3.6.4 $((\Diamond\Box(p \to q) \wedge \Diamond\Box(q \to p)) \to \Diamond\Box(p \leftrightarrow q))$

Let us consider the formula $((\Diamond\Box(p \to q) \wedge \Diamond\Box(q \to p)) \to \Diamond\Box(p \leftrightarrow q))$.

As we see below, Deterministic SQEMA fails on this formula, however, in [19], the original authors of SQEMA present an extension of SQEMA, SQEMA$^{\mathrm{sub}}$, which is capable of handling this formula by converting it using a *reversible substitution* into an inductive formula, on which SQEMA can succeed. The substitution would be the following:

1. $T(q_1) = \neg p \vee q$.
2. $T(q_2) = p \vee \neg q$.
3. $T(q_3) = p \vee q$.

The reversed substitution is:

1. $S(p) = q_3 \wedge q_2$
2. $S(q) = q_3 \wedge q_1$.

Thus the formula becomes: $((\Diamond\Box q_1 \wedge \Diamond\Box q_2) \to \Diamond\Box(q_1 \wedge q_2))$.

A good future work item for Deterministic SQEMA and its implementation is to include this extension of SQEMA.

Now, let us go back to the original formula and see how Deterministic SQEMA fails on it.

In STEP 1, we first obtain a negation normal form of the input formula, which is:
$A_1 = \Box\Diamond(p \wedge \neg q) \vee \Box\Diamond(q \wedge \neg p) \vee \Diamond\Box((\neg p \vee q) \wedge (\neg q \vee p))$.
There is a single conjunct in the above, so now we proceed to STEP 2.
In STEP 2, we have to normalize $\neg A_1$. First, we obtain a negation normal form:
$\Diamond\Box(\neg p \vee q) \wedge \Diamond\Box(\neg q \vee p) \wedge \Box\Diamond((p \wedge \neg q) \vee (q \wedge \neg p))$.
Now, we distribute $\vee$ over $\wedge$ and obtain:
$\Diamond\Box(\neg p \vee q) \wedge \Diamond\Box(\neg q \vee p) \wedge \Box\Diamond((p \vee q) \wedge (p \vee \neg p) \wedge (q \vee \neg q) \wedge (\neg p \vee \neg q))$.
Now, we may apply the rule of opposites:
$\Diamond\Box(\neg p \vee q) \wedge \Diamond\Box(\neg q \vee p) \wedge \Box\Diamond((p \vee q) \wedge \top \wedge (q \vee \neg q) \wedge (\neg p \vee \neg q))$.
And again:
$\Diamond\Box(\neg p \vee q) \wedge \Diamond\Box(\neg q \vee p) \wedge \Box\Diamond((p \vee q) \wedge \top \wedge \top \wedge (\neg p \vee \neg q))$.
Then we eliminate $\top$ from the conjunction:
$\Diamond\Box(\neg p \vee q) \wedge \Diamond\Box(\neg q \vee p) \wedge \Box\Diamond((p \vee q) \wedge (\neg p \vee \neg q))$.

No more rules can be applied, except for the rules marked with CNF, so we allocate a nominal, c_1. The initial system is:
$\sigma_1 = \neg \bigwedge((\neg c_1 \veebar (\Diamond\Box(\neg p \vee q) \wedge \Diamond\Box(\neg q \vee p) \wedge \Box\Diamond((p \vee q) \wedge (\neg p \vee \neg q)))))$.

In STEP 3, we choose the elimination order $\langle p, q \rangle$ and an empty backtracking stack, then we proceed to STEP 4 with p and σ_1.

In STEP 4, we save the backtracking context $\langle p, \sigma_1 \rangle$ and start applying the *step* function.

First, $step(\sigma_1, p)$ splits on conjunction:
$\sigma_2 = \neg \bigwedge($
$\quad (\neg c_1 \veebar \Diamond\Box(\neg p \vee q)),$
$\quad (\neg c_1 \veebar (\Diamond\Box(\neg q \vee p) \wedge \Box\Diamond((p \vee q) \wedge (\neg p \vee \neg q)))))$.
And again, $step(\sigma_2, p)$ splits on conjunction:
$\sigma_3 = \neg \bigwedge($

$(\neg c_1 \veebar \Diamond\Box(\neg p \vee q))$,
$(\neg c_1 \veebar \Diamond\Box(\neg q \vee p))$,
$(\neg c_1 \veebar \Box\Diamond((p \vee q) \wedge (\neg p \vee \neg q))))$.

Now, the $\Diamond$-rule is applied by $\mathit{step}(\sigma_3, p)$:

$\sigma_4 = \neg \bigwedge($
$(\neg c_1 \veebar \Diamond\Box(\neg p \vee q))$,
$(c_1 \rightarrow \Diamond c_2)$,
$(\neg c_2 \veebar \Box(\neg q \vee p))$,
$(\neg c_1 \veebar \Box\Diamond((p \vee q) \wedge (\neg p \vee \neg q))))$.

Next, $\mathit{step}(\sigma_4, p)$ applies the $\Box$-rule:

$\sigma_5 = \neg \bigwedge($
$(\neg c_1 \veebar \Diamond\Box(\neg p \vee q))$,
$(c_1 \rightarrow \Diamond c_2)$,
$(\Box^{-1}\neg c_2 \veebar (\neg q \vee p))$,
$(\neg c_1 \veebar \Box\Diamond((p \vee q) \wedge (\neg p \vee \neg q))))$.

And now $\mathit{step}(\sigma_5, p)$ applies case 3.2:

$\sigma_6 = \neg \bigwedge($
$(\neg c_1 \veebar \Diamond\Box(\neg p \vee q))$,
$(c_1 \rightarrow \Diamond c_2)$,
$((\Box^{-1}\neg c_2 \vee \neg q) \veebar p)$,
$(\neg c_1 \veebar \Box\Diamond((p \vee q) \wedge (\neg p \vee \neg q))))$.

$\mathit{step}(\sigma_6, p)$ applies the $\Box$-rule:

$\sigma_7 = \neg \bigwedge($
$(\neg c_1 \veebar \Diamond\Box(\neg p \vee q))$,
$(c_1 \rightarrow \Diamond c_2)$,
$((\Box^{-1}\neg c_2 \vee \neg q) \veebar p)$,
$(\Box^{-1}\neg c_1 \veebar \Diamond((p \vee q) \wedge (\neg p \vee \neg q))))$.

And $\mathit{step}(\sigma_7, p)$ fails.

Now, the configuration $\langle p, \sigma_1 \rangle$ is popped from the backtracking stack and the polarity of p is reversed, obtaining:

$\sigma_8 = \neg \bigwedge((\neg c_1 \veebar (\Diamond\Box(p \vee q) \wedge \Diamond\Box(\neg q \vee \neg p) \wedge \Box\Diamond((\neg p \vee q) \wedge (p \vee \neg q)))))$.

First, $\mathit{step}(\sigma_8, p)$ splits on conjunction:

$\sigma_9 = \neg \bigwedge($
$(\neg c_1 \veebar \Diamond\Box(p \vee q))$,
$(\neg c_1 \veebar (\Diamond\Box(\neg q \vee \neg p) \wedge \Box\Diamond((\neg p \vee q) \wedge (p \vee \neg q)))))$.

Now, $\mathit{step}(\sigma_9, p)$ applies the $\Diamond$-rule:

$\sigma_{10} = \neg \bigwedge($
$(\neg c_1 \veebar \Diamond c_3)$,
$(\neg c_3 \veebar \Box(p \vee q))$,
$(\neg c_1 \veebar (\Diamond\Box(\neg q \vee \neg p) \wedge \Box\Diamond((\neg p \vee q) \wedge (p \vee \neg q)))))$.

$\mathit{step}(\sigma_{10}, p)$ applies the $\Box$-rule:

$\sigma_{11} = \neg \bigwedge($
$(\neg c_1 \veebar \Diamond c_3)$,
$(\Box^{-1}\neg c_3 \veebar (p \vee q))$,
$(\neg c_1 \veebar (\Diamond\Box(\neg q \vee \neg p) \wedge \Box\Diamond((\neg p \vee q) \wedge (p \vee \neg q)))))$.

$\mathit{step}(\sigma_{11}, p)$ applies case 3.2:

$\sigma_{12} = \neg \bigwedge($

$(\neg c_1 \veebar \Diamond c_3)$,
$((\Box^{-1}\neg c_3 \vee q) \veebar p)$,
$(\neg c_1 \veebar (\Diamond\Box(\neg q \vee \neg p) \wedge \Box\Diamond((\neg p \vee q) \wedge (p \vee \neg q)))))$.

$step(\sigma_{12}, p)$ splits on a conjunction:

$\sigma_{13} = \neg \bigwedge($
$(\neg c_1 \veebar \Diamond c_3)$,
$((\Box^{-1}\neg c_3 \vee q) \veebar p)$,
$(\neg c_1 \veebar \Diamond\Box(\neg q \vee \neg p))$,
$(\neg c_1 \veebar \Box\Diamond((\neg p \vee q) \wedge (p \vee \neg q))))$.

$step(\sigma_{13}, p)$ applies the $\Box$-rule:

$\sigma_{14} = \neg \bigwedge($
$(\neg c_1 \veebar \Diamond c_3)$,
$((\Box^{-1}\neg c_3 \vee q) \veebar p)$,
$(\neg c_1 \veebar \Diamond\Box(\neg q \vee \neg p))$,
$(\Box^{-1}\neg c_1 \veebar \Diamond((\neg p \vee q) \wedge (p \vee \neg q))))$.

$step(\sigma_{14}, p)$ fails.

The backtracking context is empty, so we return to STEP 3 with failure.

In STEP 3, we try the second permutation of the variables, $\langle q, p\rangle$, create a new empty backtracking context, and proceed to STEP 4.

In STEP 4, we save the backtracking context $\langle q, \sigma_1\rangle$.

The current system is:

$\sigma_1 = \neg \bigwedge((\neg c_1 \veebar (\Diamond\Box(\neg p \vee q) \wedge \Diamond\Box(\neg q \vee p) \wedge \Box\Diamond((p \vee q) \wedge (\neg p \vee \neg q)))))$.

First, $step(\sigma_1, q)$ splits on conjunction:

$\sigma_{15} = \neg \bigwedge($
$(\neg c_1 \veebar \Diamond\Box(\neg p \vee q))$,
$(\neg c_1 \veebar (\Diamond\Box(\neg q \vee p) \wedge \Box\Diamond((p \vee q) \wedge (\neg p \vee \neg q)))))$.

$step(\sigma_{15}, q)$ applies the $\Diamond$-rule:

$\sigma_{16} = \neg \bigwedge($
$(\neg c_1 \veebar \Diamond c_4)$,
$(\neg c_4 \veebar \Box(\neg p \vee q))$,
$(\neg c_1 \veebar (\Diamond\Box(\neg q \vee p) \wedge \Box\Diamond((p \vee q) \wedge (\neg p \vee \neg q)))))$.

$step(\sigma_{16}, q)$ applies the $\Box$-rule:

$\sigma_{17} = \neg \bigwedge($
$(\neg c_1 \veebar \Diamond c_4)$,
$(\Box^{-1}\neg c_4 \veebar (\neg p \vee q))$,
$(\neg c_1 \veebar (\Diamond\Box(\neg q \vee p) \wedge \Box\Diamond((p \vee q) \wedge (\neg p \vee \neg q)))))$.

$step(\sigma_{17}, q)$ applies case 3.2:

$\sigma_{18} = \neg \bigwedge($
$(\neg c_1 \veebar \Diamond c_4)$,
$((\Box^{-1}\neg c_4 \vee \neg p) \veebar q)$,
$(\neg c_1 \veebar (\Diamond\Box(\neg q \vee p) \wedge \Box\Diamond((p \vee q) \wedge (\neg p \vee \neg q)))))$.

$step(\sigma_{18}, q)$ splits on conjunction:

$\sigma_{19} = \neg \bigwedge($
$(\neg c_1 \veebar \Diamond c_4)$,
$((\Box^{-1}\neg c_4 \vee \neg p) \veebar q)$,
$(\neg c_1 \veebar \Diamond\Box(\neg q \vee p))$,
$(\neg c_1 \veebar \Box\Diamond((p \vee q) \wedge (\neg p \vee \neg q))))$.

$step(\sigma_{19}, q)$ applies the $\Box$-rule:
$\sigma_{20} = \neg \bigwedge($
$(\neg c_1 \veebar \Diamond c_4)$,
$((\Box^{-1}\neg c_4 \vee \neg p) \veebar q)$,
$(\neg c_1 \veebar \Diamond\Box(\neg q \vee p))$,
$(\Box^{-1}\neg c_1 \veebar \Diamond((p \vee q) \wedge (\neg p \vee \neg q))))$.
$step(\sigma_{20}, q)$ fails.
Backtracking occurs to context $\langle q, \sigma_1 \rangle$. The polarity of q is reversed, obtaining:
$\sigma_{21} = \neg \bigwedge((\neg c_1 \veebar (\Diamond\Box(\neg p \vee \neg q) \wedge \Diamond\Box(q \vee p) \wedge \Box\Diamond((p \vee \neg q) \wedge (\neg p \vee q)))))$.
$step(\sigma_{21}, q)$ splits on conjunction:
$\sigma_{22} = \neg \bigwedge($
$(\neg c_1 \veebar \Diamond\Box(\neg p \vee \neg q))$,
$(\neg c_1 \veebar (\Diamond\Box(q \vee p) \wedge \Box\Diamond((p \vee \neg q) \wedge (\neg p \vee q)))))$.
$step(\sigma_{22}, q)$ splits on conjunction:
$\sigma_{23} = \neg \bigwedge($
$(\neg c_1 \veebar \Diamond\Box(\neg p \vee \neg q))$,
$(\neg c_1 \veebar \Diamond\Box(q \vee p))$,
$(\neg c_1 \veebar \Box\Diamond((p \vee \neg q) \wedge (\neg p \vee q))))$.
$step(\sigma_{23}, q)$ applies the $\Diamond$-rule:
$\sigma_{24} = \neg \bigwedge($
$(\neg c_1 \veebar \Diamond\Box(\neg p \vee \neg q))$,
$(\neg c_1 \veebar \Diamond c_5)$,
$(\neg c_5 \veebar \Box(q \vee p))$,
$(\neg c_1 \veebar \Box\Diamond((p \vee \neg q) \wedge (\neg p \vee q))))$.
$step(\sigma_{24}, q)$ applies the $\Box$-rule:
$\sigma_{25} = \neg \bigwedge($
$(\neg c_1 \veebar \Diamond\Box(\neg p \vee \neg q))$,
$(\neg c_1 \veebar \Diamond c_5)$,
$(\Box^{-1}\neg c_5 \veebar (q \vee p))$,
$(\neg c_1 \veebar \Box\Diamond((p \vee \neg q) \wedge (\neg p \vee q))))$.
$step(\sigma_{25}, q)$ applies case 3.2:
$\sigma_{26} = \neg \bigwedge($
$(\neg c_1 \veebar \Diamond\Box(\neg p \vee \neg q))$,
$(\neg c_1 \veebar \Diamond c_5)$,
$((\Box^{-1}\neg c_5 \vee p) \veebar q)$,
$(\neg c_1 \veebar \Box\Diamond((p \vee \neg q) \wedge (\neg p \vee q))))$.
$step(\sigma_{26}, q)$ applies the $\Box$-rule:
$\sigma_{27} = \neg \bigwedge($
$(\neg c_1 \veebar \Diamond\Box(\neg p \vee \neg q))$,
$(\neg c_1 \veebar \Diamond c_5)$,
$((\Box^{-1}\neg c_5 \vee p) \veebar q)$,
$(\Box^{-1}\neg c_1 \veebar \Diamond((p \vee \neg q) \wedge (\neg p \vee q))))$.
$step(\sigma_{27}, q)$ fails.
Because there is nothing to backtrack to, STEP 4 reports failure and returns to STEP 3.

In STEP 3, there are no more permutations of the propositional variables to try. STEP 3 reports failure and returns to STEP 2.

In STEP 2, we are unable to solve the system σ_1 and report failure to STEP 1. In STEP 1, the result of Deterministic SQEMA is *failure*.

3.6.5 More Examples

Here, we skip some of the steps to illustrate some more examples:

Take the formula $(\Box\Box p \to \Diamond p)$.

The initial equation is $(\neg c_1 \veebar \Box(\Box p \wedge \neg p))$.
The elimination order is $\langle p \rangle$.
The initial system is
σ_0: $\neg \bigwedge((\neg c_1 \veebar \Box(\Box p \wedge \neg p)))$.
step(σ_0, p) applies the $\Box$-rule, and the result is
σ_1: $\neg \bigwedge((\Box^{-1}\neg c_1 \veebar (\Box p \wedge \neg p)))$.
Then *step*(σ_1, p) is, by the semantic equivalence rule,
σ_2: $\neg \bigwedge((\Box^{-1}\neg c_1 \veebar \Box p), (\Box^{-1}\neg c_1 \veebar \neg p))$
Then the $\Box$-rule is applied:
σ_3: $\neg \bigwedge((\Box^{-1}\Box^{-1}\neg c_1 \veebar p), (\Box^{-1}\neg c_1 \veebar \neg p))$
Finally, the Ackermann rule is applied:
σ_4: $\neg \bigwedge((\Box^{-1}\neg c_1 \vee \bigwedge(\Box^{-1}\Box^{-1}\neg c_1)))$
After applying *st* and simplification, the following formula is the result:
$\exists z_1((x\ r_1\ z_1) \wedge \exists z_2((x\ r_1\ z_2) \wedge (z_2 r_1 z_1)))$

Now, let us consider $(\Box\Diamond(p \wedge q) \vee (\Box\Diamond((p \vee q) \wedge (\neg p \vee \neg q)) \vee \Box\Diamond(\neg p \wedge \neg q)))$

A single system is produced, starting with elimination order $\langle p, q \rangle$:
σ_1: $\neg \bigwedge((\neg c_1 \veebar (\Diamond\Box(p \vee q) \wedge (\Diamond\Box(\neg p \vee \neg q) \wedge \Diamond\Box((p \vee \neg q) \wedge (q \vee \neg p))))))$
Then, the equation is split on a conjunction:
σ_2: $\neg \bigwedge((\neg c_1 \veebar \Diamond\Box(p \vee q)), (\neg c_1 \veebar (\Diamond\Box(\neg p \vee \neg q) \wedge \Diamond\Box((p \vee \neg q) \wedge (q \vee \neg p)))))$
The $\Diamond$-rule is applied:
σ_3: $\neg \bigwedge((c_1 \to \Diamond c_2), (\neg c_2 \veebar \Box(p \vee q)), (\neg c_1 \veebar (\Diamond\Box(\neg p \vee \neg q) \wedge \Diamond\Box((p \vee \neg q) \wedge (q \vee \neg p)))))$
The $\Box$-rule is applied:
σ_4: $\neg \bigwedge((c_1 \to \Diamond c_2), (\Box^{-1}\neg c_2 \veebar (p \vee q)), (\neg c_1 \veebar (\Diamond\Box(\neg p \vee \neg q) \wedge \Diamond\Box((p \vee \neg q) \wedge (q \vee \neg p)))))$
Now, the equivalence rule is applied on a disjunction (3.2 of *step*):
σ_5: $\neg \bigwedge((c_1 \to \Diamond c_2), ((\Box^{-1}\neg c_2 \vee q) \veebar p), (\neg c_1 \veebar (\Diamond\Box(\neg p \vee \neg q) \wedge \Diamond\Box((p \vee \neg q) \wedge (q \vee \neg p)))))$
Then, an equation is split on a conjunction:
σ_6: $\neg \bigwedge((c_1 \to \Diamond c_2), ((\Box^{-1}\neg c_2 \vee q) \veebar p), (\neg c_1 \veebar \Diamond\Box(\neg p \vee \neg q)), (\neg c_1 \veebar \Diamond\Box((p \vee \neg q) \wedge (q \vee \neg p))))$
Now, the $\Diamond$-rule is applied:
σ_7: $\neg \bigwedge((c_1 \to \Diamond c_2), ((\Box^{-1}\neg c_2 \vee q) \veebar p), (\neg c_1 \veebar \Diamond\Box(\neg p \vee \neg q)), (c_1 \to \Diamond c_3), (\neg c_3 \veebar \Box((p \vee \neg q) \wedge (q \vee \neg p))))$
The $\Box$-rule:
σ_8: $\neg \bigwedge((c_1 \to \Diamond c_2), ((\Box^{-1}\neg c_2 \vee q) \veebar p), (\neg c_1 \veebar \Diamond\Box(\neg p \vee \neg q)), (c_1 \to \Diamond c_3), (\Box^{-1}\neg c_3 \veebar ((p \vee \neg q) \wedge (q \vee \neg p))))$
Now, split on conjunction:
σ_9: $\neg \bigwedge((c_1 \to \Diamond c_2), ((\Box^{-1}\neg c_2 \vee q) \veebar p), (\neg c_1 \veebar \Diamond\Box(\neg p \vee \neg q)), (c_1 \to \Diamond c_3), (\Box^{-1}\neg c_3 \veebar (p \vee \neg q)), (\Box^{-1}\neg c_3 \veebar (q \vee \neg p)))$

step, point 3.2:

σ_{10}: $\neg \bigwedge((c_1 \to \Diamond c_2), ((\Box^{-1}\neg c_2 \vee q) \veebar p), (\neg c_1 \veebar \Diamond\Box(\neg p \vee \neg q)), (c_1 \to \Diamond c_3), ((\Box^{-1}\neg c_3 \vee \neg q) \veebar p), (\Box^{-1}\neg c_3 \veebar (q \vee \neg p)))$

Applying the Ackermann rule followed by applying the system normalization procedure gives us the following system:

σ_{11}: $\neg \bigwedge((\bot \veebar ((\neg c_1 \vee \Diamond\Box(\Box^{-1}\neg c_3 \vee \neg q)) \wedge ((\Box^{-1}\neg c_2 \vee (\Box^{-1}\neg c_3 \vee q)) \wedge ((c_1 \to \Diamond c_2) \wedge (c_1 \to \Diamond c_3))))))$

Splitting twice on conjunction gives us:

σ_{12}: $\neg \bigwedge((\bot \veebar (\neg c_1 \vee \Diamond\Box(\Box^{-1}\neg c_3 \vee \neg q))), (\bot \veebar (\Box^{-1}\neg c_2 \vee (\Box^{-1}\neg c_3 \vee q))), (\bot \veebar ((c_1 \to \Diamond c_2) \wedge (c_1 \to \Diamond c_3))))$

Now, applying 3.2 of *step* twice:

σ_{12}: $\neg \bigwedge((\bot \veebar (\neg c_1 \vee \Diamond\Box(\Box^{-1}\neg c_3 \vee \neg q))), (((\bot \vee \Box^{-1}\neg c_2) \vee \Box^{-1}\neg c_3) \veebar q), (\bot \veebar ((c_1 \to \Diamond c_2) \wedge (c_1 \to \Diamond c_3))))$

Finally, applying the Ackermann rule, translating and simplifying gives us:

$\forall y_1 \forall y_2(((x\ r_1\ y_1) \wedge (x\ r_1\ y_2)) \to \forall z_1((x\ r_1\ z_1) \to \exists z_2((y_1\ r_1\ z_2) \wedge (y_2\ r_1\ z_2) \wedge (z_1\ r_1\ z_2))))$

3.7 Sahlqvist Formulas

The Sahlqvist formulas are simpler to deal with than the inductive formulas, and here we examine them first to deal with a simpler case first.

In this section, $\Diamond$ stands for any diamond or reversed diamond, and $\Box$ means any box or reversed box.

We have adapted Definition 3.51 of [8], page 165.

We assume that a formula A is *positive* iff all occurrences of propositional variables in it are positive. Here, we disregard the occurrences of nominals. Again, we assume that a formula A is *negative* iff all occurrences of propositional variables in it are negative, again disregarding any occurrences of nominals.

Definition 84 (Sahlqvist Formulas) A *boxed atom* is a formula A, which is either a propositional variable p or $\Box A'$, where A' is a boxed atom. A *Sahlqvist antecedent* is a formula built up from $\top$, $\bot$, boxed atoms and negative formulas, using $\wedge$, $\vee$ and $\Diamond$. A *Sahlqvist implication* is of the form $(A' \to A'')$, where A' is a Sahlqvist antecedent and A'' is positive. A *Sahlqvist formula* (in the classical definition) is built up from Sahlqvist implications by using boxes and conjunctions, and by applying disjunctions only between formulas which do not share propositional variables. An *extended Sahlqvist formula* is built up from Sahlqvist implications by using boxes, conjunctions, and disjunctions, where the condition in [8] of using disjunctions is no longer necessary. From now on, we simply say *Sahlqvist formula* instead of *extended Sahlqvist formula*.

Consider the formula $(\neg\Box_1\Box_1 p_1 \vee \Diamond_1 p_1)$, or, as a formula of ML($\Box$), we may simply write $(\neg\Box\Box p \vee \Diamond p)$. By the definition of implication, this is the defined formula $(\Box\Box p \to \Diamond p)$. Clearly this is a Sahlqvist implication, where $\Box\Box p$ is a boxed atom, and $\Diamond p$ is positive. Thus, this formula is a Sahlqvist formula. Here we see why we may not apply the same algorithm invariant as used for the non-deterministic version SQEMA in [17]. In the original authors' article, the initial SQEMA equation for Sahlqvist formulas must be of the kind $(\neg c \veebar \beta)$, where c is a nominal and β is a Sahlqvist antecedent. However, due to box extraction, Deterministic SQEMA's initial equation is of the kind $(\neg c \veebar \Box(\Box p \wedge \neg p))$ and clearly the right-hand side is not syntactically a Sahlqvist antecedent. Thus, we may not apply the original proof that SQEMA succeeds on all Sahlqvist formulas when it comes to Deterministic SQEMA. We need a new invariant and a new proof.

A *boxed piece* is a formula A which is either p, $\Box A'$, *Neg*, $(\mathit{pure} \vee A')$, $(A' \vee \mathit{pure})$ or $(A'_1 \wedge A'_2)$, where A', A'_1, and A'_2 are boxed pieces, *Neg* is a negative formula, *pure* is a pure formula, the main rules 1, 2, 3 (see Definition 76) are not applicable to any subformula of A, the main rules 4 and 5 may only apply to negative or pure subformulas of A.

Examples of boxed pieces are: $\Box\Box p$, $\Box(c_1 \to \Diamond c_2)$, $(\neg q \vee \Diamond\neg p)$, $(\Box\Box p \vee \Box(c_1 \to \Diamond c_2))$, $(\neg q \vee \Diamond\neg p)$, and $\Box((\Box\Box p \wedge \Box q) \wedge (\neg q \vee \Diamond\neg p))$.

A *good piece* is a formula A which is built up from boxed pieces using $\wedge$ and $\Diamond$ such that the main rules 1, 2, 3 (see Definition 76) are not applicable to any subformula of A, the main rules 4 and 5 may only apply to negative or pure subformulas of A, and also the following diamond distribution rule — $\Diamond(\gamma_1 \vee \gamma_2) \Rightarrow (\Diamond\gamma_1 \vee \Diamond\gamma_2)$ — may only be applied to diamonds within negative or pure subformulas.

Examples of good pieces are $(\Box p \wedge \Diamond\Box p)$, and $(\Diamond\Box p \wedge (\Diamond\Box(\neg p \vee \neg q) \wedge \Diamond\Box q)))$.

We denote by δ a formula which is either a boxed piece, or of the form $(\neg c \vee A)$ where A is a good piece. We denote by δ' a formula which is either a δ, or of the kind $((\bot \vee \neg c) \vee A)$ where A is a good piece. We denote by δ'' a formula which is either a δ, or of the kind $(A \vee \neg c)$ where A is a good piece.

Proposition 85 If σ is a system of equations, where each equation χ of σ is such that either χ is a δ', or χ is of the form $(\bot \veebar \bigwedge(\delta''_1, \ldots, \delta''_n))$, then

1) Applying *step* gives a system of the same kind, and never *failure*.

2) The result of a system normalization procedure on σ is also a system of the same kind.

3) On Sahlqvist input formulas, Deterministic SQEMA only works on systems of the above kind.

Proof 1) Let σ be as described, and let p be the current variable to eliminate in Deterministic SQEMA's STEP 4. Let us examine how the *result* of *step* is obtained.

1. If *result* is obtained by 1., then clearly 1) holds.

2. If *result* is obtained by 2., the Ackermann rule, then we have replaced $\neg p$ with a conjunction of left-hand sides of equations of σ. But this conjunction

is clearly a pure formula. So, we have replaced occurrences of $\neg p$ with a pure formula. Clearly, $\neg p$ is a negative formula, and the conjunction is a negative formula and a pure formula. For each occurrence of $\neg p$, we have replaced $\neg p$ within a negative formula with a negative formula.

Clearly after the replacements, only negative subformulas on the right-hand side of equations may have changed, because the left-hand sides are pure. Then it is clear that in the resulting equations, rules 1, 2, 3 are not applicable to any subformula, while rules 4 and 5 may only be applicable on the right-hand sides within negative or pure subformulas.

Consider some occurrence of $\neg p$. Let χ be an equation of σ where $\neg p$ occurs. If χ is a boxed piece, then $\neg p$ is within a *Neg*, and after replacement we have that the conjunction occurs in another *Neg'*. If χ is $(\neg c \veebar A)$ where A is a good piece, then we have done the same as in the case where χ is a boxed piece. If χ is $((\bot \vee \neg c) \veebar A)$ where A is a good piece, then we have done the same as in the case where χ is a boxed piece. We can easily see by the same kind of reasoning that we have done the same when χ is of the kind $(\bot \veebar \bigwedge(\delta''_1, \ldots, \delta''_n))$.

Then 1) holds.

3. Now, let χ be the first equation of σ such that p occurs positively in χ and χ is not of the kind $(\alpha \veebar p)$ with $p \not\nearrow \alpha$.

3.1. If *result* is obtained by 3.1., then we have split χ on a conjunction. Let χ be $(A_1 \veebar (A_2 \wedge A_3))$ and let the resulting equations be $\chi_1 =_{def} (A_1 \veebar A_2)$ and $\chi_2 =_{def} (A_1 \veebar A_3)$. First suppose that χ is a boxed piece. Then clearly both χ_1 and χ_2 are boxed pieces. Now suppose $(A_2 \wedge A_3)$ is a good piece or a boxed piece. Then A_2 is a good piece or a boxed piece and A_3 is a good piece or a boxed piece. Now suppose that χ is $(\bot \veebar \bigwedge(\delta''_1, \ldots, \delta''_n))$. Suppose $n > 1$, then χ_1 is $(\bot \veebar \bigwedge(\delta''_1))$ and χ_2 is $(\bot \veebar \bigwedge(\delta''_2, \ldots, \delta''_n))$. Suppose $n = 1$. Then δ''_1 is a conjunction, so it must be a boxed piece, then χ is a boxed piece, which we have already covered. Suppose $n = 0$. Then χ is a pure formula, therefore it is a negative formula, which makes it a boxed piece, which we have covered above. Then 1) holds.

3.2. If *result* is obtained by 3.2., χ is $(A_1 \veebar (A_2 \vee A_3))$. Because p occurs positively in χ, by Lemma 82, p occurs positively in $(A_2 \vee A_3)$. Suppose χ is a boxed piece. Then A_1 is pure. Also w.l.o.g. A_2 is pure and A_3 is a boxed piece, so $p \not\nearrow A_2$, so the resulting equation χ' is $((A_1 \vee A_2) \veebar A_3)$. Clearly χ' is a boxed piece. Now suppose $(A_2 \vee A_3)$ is a good piece while A_1 is either $\neg c$ or $(\bot \vee \neg c)$. Then $(A_2 \vee A_3)$ is a boxed piece and A_1 is a pure formula, so χ is a boxed piece, which we have discussed above. Now suppose that χ is $(\bot \veebar \bigwedge(\delta''_1, \ldots, \delta''_n))$. Clearly in this case we must have that $n = 1$. Suppose δ''_1 is a boxed piece. Then χ is a boxed piece, which we have covered before. Now suppose that δ''_1 is either $(\neg c \vee A)$ or $(A \vee \neg c)$, where A is a good piece. Then clearly $p \not\nearrow \neg c$ so the resulting equation χ' is $((\bot \vee \neg c) \veebar A)$ with A being a good piece. Then χ is a δ'. Therefore 1) holds.

3.3. If *result* is obtained by 3.3, χ is $(A_1 \veebar \Box A_2)$. Then the resulting equation χ' is $(\Box^{-1} A_1 \veebar A_2)$. First suppose that χ is a boxed piece. By Lemma 82 and the definition of a boxed piece, A_1 is pure and p occurs positively in $\Box A_2$. Then clearly $\Box^{-1} A_1$ is pure and A_2 is a boxed piece. Now suppose that χ is either $(\neg c \veebar A)$ or $((\bot \vee \neg c) \veebar A)$. Then A is $\Box A_2$, so A is a boxed piece, and this makes χ also a boxed piece because its left-hand side is pure. But this case has already been discussed above. Now suppose that χ is $(\bot \veebar \bigwedge(\delta''_1, \ldots, \delta''_n))$. Then clearly $n = 1$ and δ''_1 is $\Box A_2$ so δ''_1 is a boxed piece, thus χ is also a boxed piece. But we have seen this case above. Then 1) holds.

3.4. If *result* is obtained by 3.4, χ is $(A_1 \veebar \Diamond A_2)$ with A_1 being either $\neg c$ or $(\bot \vee \neg c)$. The resulting equations are χ_1 which is $(c \to \Diamond c')$ and χ_2 which is $(\neg c' \veebar A_2)$. Suppose for the sake of contradiction that χ is a boxed piece. Clearly then $\Diamond A_2$ is a pure formula. But then by Lemma 82, $p \not\nearrow \chi$, contradicts the fact that p occurs positively in χ. Clearly χ may not be of the kind $(\bot \veebar \bigwedge(\delta''_1, \ldots, \delta''_n))$ because the left-hand side of χ is not $\bot$. It follows that χ is a δ' with $\Diamond A_2$ being a good piece. But then A_2 is also a good piece. Consider χ_1, it is a pure formula, so it is a negative formula and this makes χ_1 a boxed piece. Also χ_2 is a formula of the kind δ' because A_2 is a good piece. Then 1) holds.

3.5. Suppose for the sake of contradiction that *failure* is obtained by 3.5. Then by Corollary 83 and the fact that *failure* is not obtained by 3.3., there is an equation χ' in σ which is $(A_1 \veebar \Diamond A_2)$ such that A_1 is not a negated nominal and is not $(\bot \vee \neg c)$. Suppose for the sake of contradiction that χ' is a boxed piece. Then $\Diamond A_2$ is a pure formula and by Lemma 82, $p \not\nearrow \chi'$, contradiction. So χ' is not a boxed piece. Then because of the above facts for χ', it may not be of the kind δ'. It follows that χ' is of the kind $(\bot \veebar \bigwedge(\delta''_1, \ldots, \delta''_n))$. Then clearly $n = 1$ and δ''_1 is $\Diamond A_2$. But by the definition of δ'', $\Diamond A_2$ must be either a boxed piece or a disjunction. Clearly $\Diamond A_2$ is not a disjunction, so it remains that it must be a boxed piece. Then $\Diamond A_2$ is a negative formula, contradicts the fact that p occurs positively in χ'. This shows that *failure* is never obtained by 3.5.

We have shown that *failure* may not be obtained by 3.3 or 3.5 of *step*. We have also shown that for any of the other cases, 1) holds.

2) Let us consider a system normalization procedure and its run on σ. Let σ be $\neg \bigwedge(\chi_1, \ldots, \chi_m)$. The first formula A_0 as in Definition 80 is $\bigwedge(\chi'_1, \ldots, \chi'_m)$ where it is trivial to see that each χ'_i, $1 \leq i \leq m$ is a δ''.

It remains to show that for every formula γ of the kind $\bigwedge(\delta''_1, \ldots, \delta''_m)$, applying any of the main rules (see Definition 76) or the rules from the two tables above to a subformula of γ converts γ to a formula of the same kind.

Clearly rules 1, 2, and 3 are not applicable. Suppose rule 4 or 5 has been applied. By the definition of δ'', rules 4 or 5 are only applicable in two cases.

First, within negative (or pure) subformulas - where clearly applying rule 4 or 5 gives also a negative (or pure) subformula. Second, to subformulas of the kind $(\neg c \vee \gamma)$ or $(\gamma \vee \neg c)$ which are of the kind δ'' and where γ is a good piece and a conjunction $(\gamma_1 \wedge \gamma_2)$ of good pieces. Then the result is a formula of the kind $((\neg c \vee \gamma_1) \wedge (\neg c \vee \gamma_2))$ or $((\gamma_1 \vee \neg c) \wedge (\gamma_2 \vee \neg c))$, which is a conjunction of two formulas of the kind δ''.

Now let us consider the rules for the universal modality, which are the rules in the first table.

$(c_1 \rightarrow \langle U \rangle c_2) \Rightarrow \bot$ converts a negative and pure subformula into another negative and pure one.

$U_1 U_2 \gamma \Rightarrow U_2 \gamma$ converts a pure formula into a pure formula, a negative formula into a negative formula, a boxed piece into a boxed piece, a good piece into a good piece.

The following rules convert a pure formula into a pure formula, a negative formula into a negative formula, and a boxed piece into a boxed piece:

$\Box U_1 \gamma \Rightarrow (U_1 \gamma \vee \Box \bot)$

$[U](U_1 \gamma_1 \overset{\wedge}{\vee} U_2 \gamma_2) \Rightarrow (U_1 \gamma_1 \overset{\wedge}{\vee} U_2 \gamma_2)$

$[U](U_1 \gamma_1 \overset{\wedge}{\vee} \gamma_2) \Rightarrow (U_1 \gamma_1 \overset{\wedge}{\vee} [U] \gamma_2)$

The following rule converts a pure and negative formula into another one:

$[U] \neg c \Rightarrow \bot$

The following rules convert a pure formula into a pure formula, a negative formula into a negative formula, a boxed formula into a boxed formula, and a good piece into a good piece:

$\Diamond U_1 \gamma \Rightarrow (U_1 \gamma \wedge \Diamond \top)$

$\langle U \rangle (U_1 \gamma_1 \overset{\wedge}{\vee} U_2 \gamma_2) \Rightarrow (U_1 \gamma_1 \overset{\wedge}{\vee} U_2 \gamma_2)$

$\langle U \rangle (U_1 \gamma_1 \wedge \gamma_2) \Rightarrow (U_1 \gamma_1 \wedge \langle U \rangle \gamma_2)$

The following rule converts a pure and negative formula into another one:

$\langle U \rangle c \Rightarrow \top$

Because the diamond distribution rule is not applicable outside of negative or pure subformulas, the rule $\langle U \rangle (U_1 \gamma_1 \vee \gamma_2) \Rightarrow (U_1 \gamma_1 \vee \langle U \rangle \gamma_2)$ may only be applied to a negative or pure subformula. Then this rule clearly converts a pure formula into a pure formula, a negative formula into a negative formula, a boxed formula into a boxed formula, and a good piece into a good piece.

The ten rules in the right-hand piece of the first table (the table of universal modality rules) as well as the top ten rules of the second table (five on each side) may remove a subformula of either a conjunction or a disjunction or may replace a subformula with $\top$ or $\bot$. Clearly then a δ'' remains a δ'' after replacement, and a conjunction of δ'' formulas remains a conjunction of δ'' formulas.

Clearly the rules $(\gamma_1 \wedge \gamma_2) \Rightarrow (\gamma_2 \wedge \gamma_1)$, $(\gamma_1 \vee \gamma_2) \Rightarrow (\gamma_2 \vee \gamma_1)$, and $(\gamma_1 \wedge (\gamma_2 \wedge \gamma_3)) \Rightarrow ((\gamma_1 \wedge \gamma_2) \wedge \gamma_3)$ maintain the invariant.

Consider the rule $(\gamma_1 \vee (\gamma_2 \vee \gamma_3)) \Rightarrow ((\gamma_1 \vee \gamma_2) \vee \gamma_3)$. If it is applied within a pure or a negative subformula, clearly the invariant is preserved. If it is applied to a subformula of a boxed piece, there are two cases. First let γ_1 be pure. Then one of γ_2 or γ_3 is pure, so the result is a boxed piece. Now, let γ_1 be a boxed piece and $(\gamma_2 \vee \gamma_3)$ be pure. Then the result is a boxed piece. If the rule is applied to a δ'' of the kind $(\neg c \vee \gamma)$ or $(\gamma \vee \neg c)$ where γ is a good piece, then γ is a disjunction, so γ is a boxed piece and the δ'' is also a boxed piece, which we have discussed above.

The rules $\Box\top \Rightarrow \top$ and $\Diamond\bot \Rightarrow \bot$ replace a negative and pure formula with another negative and pure formula so the invariant holds.

Clearly the rule $(\Box\gamma_1 \wedge \Box\gamma_2) \Rightarrow \Box(\gamma_1 \wedge \gamma_2)$ converts a boxed piece or a good piece into a boxed piece because any good piece that starts with a box is a boxed piece. It is clear what happens when the rule is applied within a pure or a negative subformula. Thus the rule converts a conjunction of δ'' formulas into another one.

Consider the rule $(\Diamond\gamma_1 \vee \Diamond\gamma_2) \Rightarrow \Diamond(\gamma_1 \vee \gamma_2)$. Clearly neither $\Diamond\gamma_1$ nor $\Diamond\gamma_2$ is a negated nominal because it is a diamond. Then the disjunction in the premise of the rule may only occur within a negative or a pure subformula and the rule converts it into another negative or pure subformula.

3) Let A be a Sahlqvist formula. In STEP 1 of SQEMA we rewrite A in negation normal form and then distribute all boxes and disjunctions over conjunctions as much as possible. Then consider a conjunct A_i in STEP 2, and consider A_i' which is the result of applying main rules 1., 2., and 3. (see Definition 76) as much as possible to $\neg A_i$, which is the first step of the eliminating normalization procedure.

The formula A_i' is built up from the negation normal form of Sahlqvist antecedents and negative formulas using diamonds, conjunctions and disjunctions. Therefore, A' is the negation normal form of a Sahlqvist antecedent, as also seen in [17]. Because A_i is a conjunct, obtained by applying the rules 1.1, 1.2, and 1.3 of SQEMA's STEP 1, there is no subformula outside the scope of a box which is a diamond of a disjunction, or a conjunction of a disjunction and another formula in A_i'. Any boxed atom is a boxed piece, and so is any negative formula. Suppose for the sake of contradiction that there is a disjunction of two formulas each of which is either a boxed atom or a negative formula in A_i'. Then its parent formula (if any) within A_i' may not be a diamond or a conjunction. But by the definition of Sahlqvist antecedent, this means that there is a top-level disjunction in A_i', which is a contradiction with the way that A_i and A_i' were obtained. Then A_i' is built up from boxed atoms and negative formulas using diamonds and conjunctions. Clearly the diamond distribution rule is not applicable to A_i' outside of negative or pure subformulas, so A_i' is a good part. This makes the initial equation of SQEMA a δ. $\Box$

Corollary 86 Deterministic SQEMA succeeds on every Sahlqvist formula at the first permutation of its variables, without backtracking. $\square$

3.8 Example Runs with Sahlqvist Formulas

Now, some examples. Here $\Diamond$ means $\Diamond_1$ as in ML($\square$) and $\square$ means $\square_1$.

3.8.1 $(\square\square p \to \Diamond p)$

Consider the extended Sahlqvist formula $(\square\square p \to \Diamond p)$, which is the same formula as in the first example of 3.6.5.

In STEP 1, we rewrite the formula in negation normal form: $(\Diamond\Diamond\neg p \vee \Diamond p)$.

No applications of rules 1.1, 1.2, or 1.3 are possible.

Thus we have a single conjunct, $A_1 = (\Diamond\Diamond\neg p \vee \Diamond p)$.

We reserve the nominal c_1 and proceed to STEP 2.

In STEP 2, we need to normalize $\neg A_1$. By Definition 79, first we obtain the negation normal form of $\neg A_1$, which is $(\square\square p \wedge \square\neg p)$, which is a Sahlqvist antecedent and also a boxed piece. No more of the main rules are applicable. But we may apply the box extraction rule, obtaining $(\square(\square p \wedge \neg p))$, which is a boxed piece and a good piece. This is the normalized form of $\neg A_1$.

The initial system is:

$\sigma_1 = \neg \bigwedge((\neg c_1 \veebar \square(\square p \wedge \neg p)))$, where the equation is both a boxed piece and of the form $(\neg c \vee A)$, where A is a good piece.

In STEP 3, we pick the elimination order $\langle p \rangle$, create an empty backtracking stack and proceed to STEP 4.

In STEP 4, we save the context $\langle p, \sigma_1 \rangle$ to the stack and start applying *step*.

$step(\sigma_1, p)$ applies the box rule:

$\sigma_2 = \neg \bigwedge((\square^{-1}\neg c_1 \veebar (\square p \wedge \neg p)))$, where the equation is a boxed piece.

$step(\sigma_2, p)$ splits on a conjunction:

$\sigma_3 = \neg \bigwedge((\square^{-1}\neg c_1 \veebar \square p), (\square^{-1}\neg c_1 \veebar \neg p))$, where every equation is a boxed piece.

$step(\sigma_3, p)$ applies the $\square$-rule:

$\sigma_4 = \neg \bigwedge((\square^{-1}\square^{-1}\neg c_1 \veebar p), (\square^{-1}\neg c_1 \veebar \neg p))$, where each equation is a boxed piece.

$step(\sigma_4, p)$ applies the Ackermann rule:

$\sigma_5 = \neg \bigwedge((\square^{-1}\neg c_1 \veebar \square^{-1}\square^{-1}\neg c_1))$, where the equation is a pure formula and thus a negative formula, which makes it a boxed piece.

By the above steps and explanations, the invariant holds before and after every application of the function *step*.

This completes STEP 4, which returns to STEP 3, which returns to STEP 2, which returns to STEP 1, which goes to STEP 5.

In STEP 5, we take the next available nominal, c_2, and construct the final result as $\psi'(x_1) \equiv \exists x_2 st(3, x_2, \sigma_5)$.

After some simplification and renaming of individual variables, the result is:

$\psi(x) \equiv \exists z_1((x\ r_1\ z_1) \wedge \exists z_2((x\ r_1\ z_2) \wedge (z_2\ r_1\ z_1)))$.

3.8.2 $(\Box(\Box p \to \bot) \lor ((\top \to \Box\Diamond(p \land q)) \lor \Box(\Box q \to \bot)))$

Consider the extended Sahlqvist formula

$(\Box(\Box p \to \bot) \lor ((\top \to \Box\Diamond(p \land q)) \lor \Box(\Box q \to \bot)))$.

Note that the same deterministic SQEMA steps below apply to the formula $(\Box\Diamond\neg p \lor (\Box\Diamond(p \land q) \lor \Box\Diamond\neg q))$, which is the negation normal form of the above with eliminated occurrences of $\bot$ and $\neg\top$ in disjunctions.

We do not describe the normalization here because it only converts to negation normal form and eliminates the boolean constants, unable to apply any more rules, except for the ones marked with CNF.

The initial system (after normalization) is:

σ_1: $\neg \bigwedge((\neg c_1 \veebar (\Diamond\Box p \land (\Diamond\Box(\neg p \lor \neg q) \land \Diamond\Box q))))$

In the above equation, $\Box p$, and $\Box q$ are boxed pieces, $\Box(\neg p \lor \neg q)$ is a negative formula and thus also a boxed piece, and the equation is $(\neg c_1 \veebar A)$ where A is a good piece.

Now, we split on conjunction:

σ_2: $\neg \bigwedge((\neg c_1 \veebar \Diamond\Box p), (\neg c_1 \veebar (\Diamond\Box(\neg p \lor \neg q) \land \Diamond\Box q)))$

The equations above are of the kind $(\neg c_1 \veebar A)$ where A is a good piece.

Applying the $\Diamond$-rule:

σ_3: $\neg \bigwedge((c_1 \to \Diamond c_2), (\neg c_2 \veebar \Box p), (\neg c_1 \veebar (\Diamond\Box(\neg p \lor \neg q) \land \Diamond\Box q)))$

Now the first equation is a pure formula which makes it a δ, the second and third equation are of the kind $(\neg c_1 \veebar A)$ where A is a good piece.

Applying the $\Box$-rule:

σ_4: $\neg \bigwedge((c_1 \to \Diamond c_2), (\Box^{-1}\neg c_2 \veebar p), (\neg c_1 \veebar (\Diamond\Box(\neg p \lor \neg q) \land \Diamond\Box q)))$

The first two equations are boxed pieces, and the last equation hasn't changed.

Now, we apply the Ackermann rule:

σ_5: $\neg \bigwedge((c_1 \to \Diamond c_2), (\neg c_1 \veebar (\Diamond\Box(\Box^{-1}\neg c_2 \lor \neg q) \land \Diamond\Box q)))$

Here the first equation is still a boxed piece, and the last equation is of the kind $(\neg c_1 \veebar A$ where A is a good piece.

Now, we apply the system normalization procedure. First, we create a formula which is the negation normal form of

$\bigwedge((c_1 \to \Diamond c_2), (\neg c_1 \lor (\Diamond\Box(\Box^{-1}\neg c_2 \lor \neg q) \land \Diamond\Box q))$.

As in the proof of Proposition 85, this is a conjunction of the kind $\bigwedge(\delta''_1, \ldots, \delta''_n)$ because each conjunct is either a boxed piece, or of the kind $(\neg c \lor A)$ where A is a good piece.

But this formula is already in negation normal form. Now, we try applying the main rules 1., 2., or 3., but this is impossible.

Now we continue by applying all the main rules and the rules from the two tables.

First, we apply the disjunction distribution rule:

$\bigwedge((c_1 \to \Diamond c_2), (\neg c_1 \lor \Diamond\Box(\Box^{-1}\neg c_2 \lor \neg q)), (\neg c_1 \lor \Diamond\Box q))$, which is still of the kind $\bigwedge(\delta''_1, \ldots, \delta''_n)$.

No more rule application is possible, except for the rules marked with CNF, but the implementation also arranges the subformulas using a form of lexicographic ordering, using the (in CNF)-marked rules in the second table, so we obtain:

$\bigwedge((\neg c_1 \lor \Diamond\Box(\Box^{-1}\neg c_2 \lor \neg q)), (c_1 \to \Diamond c_2), (\neg c_1 \lor \Diamond\Box q))$.

Then we get the system:

σ_6: $\neg \bigwedge((\bot \veebar ((\neg c_1 \lor \Diamond\Box(\Box^{-1}\neg c_2 \lor \neg q)) \land ((c_1 \to \Diamond c_2) \land (\neg c_1 \lor \Diamond\Box q)))))$, The above equation is $(\bot \veebar \bigwedge((\neg c_1 \vee \Diamond\Box(\Box^{-1}\neg c_2 \vee \neg q)), (c_1 \rightarrow \Diamond c_2), (\neg c_1 \vee \Diamond\Box q)))$, where the first two components of the conjunction are negative formulas, thus are of the kind δ'', and the last one is a boxed piece and therefore of the kind δ''.

We proceed by trying to eliminate q:

We split on conjunction twice, maintaining the invariant because of the above reasoning:

σ_7: $\neg \bigwedge((\bot \veebar (\neg c_1 \vee \Diamond\Box(\Box^{-1}\neg c_2 \vee \neg q))), (\bot \veebar (c_1 \rightarrow \Diamond c_2)), (\bot \veebar (\neg c_1 \vee \Diamond\Box q)))$

We apply point 3.2 of *step*:

σ_8: $\neg \bigwedge((\bot \veebar (\neg c_1 \vee \Diamond\Box(\Box^{-1}\neg c_2 \vee \neg q))), (\bot \veebar (c_1 \rightarrow \Diamond c_2)), ((\bot \vee \neg c_1) \veebar \Diamond\Box q))$

Because $\Diamond\Box q$ is a good piece, the third equation above is a δ' thus maintaining the invariant.

Now, we apply 3.4 of *step*, the $\Diamond$-rule:

σ_9: $\neg \bigwedge((\bot \veebar (\neg c_1 \vee \Diamond\Box(\Box^{-1}\neg c_2 \vee \neg q))), (\bot \veebar (c_1 \rightarrow \Diamond c_2)), (c_1 \rightarrow \Diamond c_3), (\neg c_3 \veebar \Box q))$, maintaining the invariant because $(c_3 \vee \Box q)$ is a boxed piece.

Applying the $\Box$-rule:

σ_{10}: $\neg \bigwedge((\bot \veebar (\neg c_1 \vee \Diamond\Box(\Box^{-1}\neg c_2 \vee \neg q))), (\bot \veebar (c_1 \rightarrow \Diamond c_2)), (c_1 \rightarrow \Diamond c_3), (\Box^{-1}\neg c_3 \veebar q))$, where $(\Box^{-1}c_3 \vee q)$ is a boxed piece.

Finally, we apply the Ackermann rule:

σ_{10}: $\neg \bigwedge((\bot \veebar (\neg c_1 \vee \Diamond\Box(\Box^{-1}\neg c_2 \vee \Box^{-1}\neg c_3))), (\bot \veebar (c_1 \rightarrow \Diamond c_2)), (c_1 \rightarrow \Diamond c_3))$, which is a clean system and thus each equation is a boxed piece.

Finally, we apply *st* and simplify to obtain:

$\forall y_1 \forall y_2(((x\ r_1\ y_1) \wedge (x\ r_1\ y_2)) \rightarrow \forall z_1((x\ r_1\ z_1) \rightarrow \exists z_2((y_1\ r_1\ z_2) \wedge (y_2\ r_1\ z_2) \wedge (z_1\ r_1\ z_2))))$

3.8.3 $(\Diamond p \rightarrow \Box\Diamond p)$

Consider the extended Sahlqvist formula $(\Diamond p \rightarrow \Box\Diamond p)$.

The initial system (after normalization) is:

σ_1: $\neg \bigwedge((\neg c_1 \veebar (\Diamond\Box\neg p \wedge \Diamond p)))$, where $(\Diamond\Box\neg p \wedge \Diamond p)$ is a good piece because $\Diamond\Box\neg p$ is a negative formula and p is a boxed piece.

Now, we split on conjunction:

σ_2: $\neg \bigwedge((\neg c_1 \veebar \Diamond\Box\neg p), (\neg c_1 \veebar \Diamond p))$

Where the invariant is maintained due to the above reasoning.

Now, we apply the $\Diamond$-rule:

σ_3: $\neg \bigwedge((\neg c_1 \veebar \Diamond\Box\neg p), (c_1 \rightarrow \Diamond c_2), (\neg c_2 \veebar p))$, maintaining the invariant.

After applying the Ackermann rule:

σ_4: $\neg \bigwedge((\neg c_1 \veebar \Diamond\Box\neg c_2), (c_1 \rightarrow \Diamond c_2))$, maintaining the invariant because it is a pure system thus every equation is a boxed piece.

Translating and simplifying, we obtain:

$\forall y_1((x\ r_1\ y_1) \rightarrow \forall z_1((x\ r_1\ z_1) \rightarrow (z_1\ r_1\ y_1)))$

3.8.4 $(\Box\Box p \rightarrow \Box p)$

Consider the extended Sahlqvist formula $(\Box\Box p \rightarrow \Box p)$. The initial system (after normalization) is:

σ_1: $\neg \bigwedge((\neg c_1 \veebar (\Diamond\neg p \wedge \Box\Box p)))$

Clearly the invariant holds because $(\Diamond\neg p \wedge \Box\Box p)$ is a boxed piece.
First, we split on conjunction:
σ_2: $\neg \bigwedge((\neg c_1 \veebar \Diamond\neg p), (\neg c_1 \veebar \Box\Box p))$
The invariant holds for the above because both equations are boxed pieces.
Now, we apply the $\Box$-rule twice, while each equation remains a boxed piece.
σ_3: $\neg \bigwedge((\neg c_1 \veebar \Diamond\neg p), (\Box^{-1}\Box^{-1}\neg c_1 \veebar p))$
Now, we apply the Ackermann rule:
σ_4: $\neg \bigwedge((\neg c_1 \veebar \Diamond\Box^{-1}\Box^{-1}\neg c_1))$, where the equation is a boxed piece.
We translate and simplify to obtain:
$\forall z_1((x\ r_1\ z_1) \rightarrow \exists z_2((x\ r_1\ z_2) \wedge (z_2\ r_1\ z_1)))$

3.9 Inductive Formulas

In this section, $\Diamond$ stands for any diamond or reversed diamond, and $\Box$ means any box or reversed box.

We assume that a formula A is *positive* iff all occurrences of propositional variables in it are positive. Here, we disregard the occurrences of nominals. Again, we assume that a formula A is *negative* iff all occurrences of propositional variables in it are negative, again disregarding any occurrences of nominals.

Definition 87 (Inductive Formulas) Let $\#$ be a symbol, which is not in the alphabet of any of the input modal languages. $\#$ is a *box-form* of $\#$. If $B(\#)$ is a box-form of $\#$, then $\Box B(\#)$ is a box-form of $\#$ for any $\Box$, and $(A \rightarrow B(\#))$ is a box-form of $\#$ for any *positive* formula A. Replacing all occurrences of $\#$ in $B(\#)$ with p, we get $B(p)$, a *box-formula* of p. The only positive occurrence of p in $B(p)$ is the *head of* $B(p)$, and any other occurrence of a propositional variable in $B(p)$ is *inessential*. For convenience, we also say that p is the *head of* $B(p)$ and the variables which have inessential occurrences in $B(p)$ are *inessential*. A *monadic regular formula (MRF)* is a modal formula built up from $\top$, $\bot$, positive formulas and negated box-formulas by applying $\wedge$, $\vee$ and $\Box$. The *dependency graph* of a set of box-formulas $\mathfrak{B} = \{B_1(p_1), \ldots, B_n(p_n)\}$ is a directed graph $G(\mathfrak{B}) = \langle V, E\rangle$ where $V = \{p_1, \ldots, p_n\}$ is the set of heads in $\mathfrak{B}$ and E is the set of edges, such that $\langle p_i, p_j\rangle \in E$ iff p_i occurs as an inessential variable in a box-formula from $\mathfrak{B}$ with head p_j. A directed graph is *acyclic* iff it does not contain directed cycles. The dependency graph of an MRF A is the dependency graph of the set of box-formulas which occur in the construction of A as an MRF. A *monadic inductive formula (MIF)* is a monadic regular formula with an acyclic dependency graph. We say that a conjunction of MIFs is an *inductive formula*.

We extend the definitions to negation normal forms of the above.

Consider the same ML($\Box$) formula that we took as an example of a Sahlqvist formula, $(\neg\Box\Box p \vee \Diamond p)$. We can see that it is an inductive formula because $\Box\Box p$ is a boxed formula of p with an empty graph and $\Diamond p$ is positive. Here we see why we may not apply the same algorithm invariant as used for the non-deterministic version SQEMA in [17]. In the original authors' article, the initial SQEMA equation for inductive formulas must be of the kind $(\neg c \veebar NegMIF*)$, where c is a nominal and $NegMIF*$ is the negation normal form of a negation of an inductive formula, built up from $\top$, $\bot$, positive formulas and negated box-formulas using disjunctions and boxes with an acyclic dependency graph, or, in other words, among other requirements, a formula built up from $\top$, $\bot$, negative formulas and box-formulas using conjunctions and diamonds. However, due to box extraction, Deterministic SQEMA's initial equation is of the kind $(\neg c \veebar \Box(\Box p \wedge \neg p))$ and clearly the right-hand side is not syntactically a $NegMIF*$. Thus, we may not apply the original proof that SQEMA succeeds on all Inductive formulas when it comes to Deterministic SQEMA. We need a new invariant and a new proof.

We define an *extended box-formula of p* thusly: p is an $EB(p)$, $\Box EB(p)$ is an $EB(p)$, $(EB_1(p) \wedge EB_2(p))$ is an $EB(p)$, if Neg' and Neg'' are negative formulas, then each of $(Neg' \vee EB(p))$, $(EB(p) \vee Neg')$, $(Neg'' \wedge EB(p))$ and $(EB(p) \wedge Neg'')$ is an $EB(p)$, the main rules 1, 2, 3 (see Definition 76) are not applicable to any subformula. Here, p is the *head* of the extended box-formula, any occurrences of propositional variables in any of the Neg' formulas is *inessential*. The dependency graph of $EB(p)$ is defined analogously to the above, but note that the variables of any Neg'' do not count as inessential.

Examples of $BF(p)$ are the following formulas: p, $\Box p$, $(\Box p \vee \neg p)$, which has a cycle in its dependency graph, $(\Box p \wedge \neg p)$, which has an empty dependency graph, $\Box(\Box p \wedge (\Box\Box(p \vee \neg q) \wedge \neg p))$, which has an edge from q to p in its dependency graph.

PureBox is a pure formula built up from negated nominals, $\bot$, $\vee$ and $\Box$.

We say that a formula A is a *Good* formula if it is such that the main rules 1, 2, 3 (see Definition 76) are not applicable to any subformula of A, and A is either $EB(p)$, Neg, $(A_1 \wedge A_2)$, $\Diamond A'$ outside the scope of boxes and disjunctions, $\Box A'$, $(A' \vee PureBox)$, or $(PureBox \vee A')$, where Neg is a negative formula, A', A_1 and A_2 are *Good* formulas, and also the following diamond distribution rule — $\Diamond(\gamma_1 \vee \gamma_2) \Rightarrow (\Diamond\gamma_1 \vee \Diamond\gamma_2)$ — may only be applied to diamonds within negative or pure subformulas. The *dependency graph of Good* is the union of the dependency graphs of the occurring formulas of kind $EB(p)$, and we require that all *Good* formulas have an acyclic dependency graph.

Examples of *Good* formulas are: p, $\Box p$, $(\Diamond\Box p \wedge (\Diamond\Box(\neg p \vee \neg q) \wedge \Diamond\Box q))$.

A *good system* is a system of equations $\sigma = \neg \bigwedge(\chi_1, \ldots, \chi_n)$, such that every χ_i is a *good equation* with an acyclic *dependency graph* $G(\chi_i)$ defined below, $G(\sigma) = \bigcup\{G(\chi_1), \ldots, G(\chi_n)\}$, $G(\sigma)$ is acyclic, where exactly one of the following holds for each χ_i:

good.1. χ_i is either $(Neg_{i_1} \veebar Neg_{i_2})$ or $(c'_i \to \Diamond c''_i)$, with $G(\chi_i) = \langle\emptyset, \emptyset\rangle$,

good.2.1. χ_i is not of kind good.1, but is either $(\neg c_i \veebar Good_i)$ or $((\bot \vee \neg c_i) \veebar$

Good$_i$), with $G(\chi_i) = G(\mathit{Good}_i)$,

good.2.2. χ_i is not of the kind good.1 or good.2.1., but is $(\mathit{PureBox} \veebar \mathit{Good'}_i)$, such that 1. there are no diamonds in $\mathit{Good'}_i$ outside of box-formulas or negative subformulas, 2. $G(\chi_i) = G(\mathit{Good'}_i)$,

good.3. χ_i is not of the above kinds, but χ_i is $(\mathit{Neg}'_i \veebar \mathit{EB}'_i(p_i))$, such that χ_i is some $\mathit{EB}_i(p_i)$ with an acyclic graph, and $G(\chi_i) = G(\mathit{EB}_i(p_i))$,

good.4. χ_i is not of the above kinds, but χ_i is $(\bot \veebar \bigwedge(\delta_1, \ldots, \delta_m))$, where each δ_j is either 1. negative with $G(\delta_j) = \langle\emptyset, \emptyset\rangle$, 2. $(\neg c \vee \mathit{Good})$ or $(\mathit{Good} \vee \neg c)$ with $G(\delta_j) = G(\mathit{Good})$, 3. $(\mathit{PureBox} \vee \mathit{Good}')$ or $(\mathit{Good}' \vee \mathit{PureBox})$ with $G(\delta_j) = G(\mathit{Good}')$, such that there are no diamonds in Good' outside of box-formulas or negative formulas, or 4. an EB with $G(\delta_j) = G(\mathit{EB})$ - an acyclic graph. The graph $G(\chi_i) = \bigcup\{G(\delta_1), \ldots, G(\delta_m)\}$ is acyclic.

Claim 88 Every output of *step*, where the input is a good system, is a good system.

Proof Consider *result*, which is $\mathit{step}(\sigma, p)$, where for σ the invariant holds. We show that *result* is not *failure* and that the invariant holds for *result*.

If *result* is obtained from (1), then the invariant holds.

If *result* is obtained from (2), the Ackermann rule, then *result* is σ', σ is $\neg \bigwedge((\alpha_1 \veebar p), \ldots, (\alpha_{n_a} \veebar p), \beta_1, \ldots, \beta_{n_b}, \theta_1, \ldots, \theta_{n_t})$, such that $p \not\nearrow \alpha_1, \ldots, \alpha_{n_a}, \theta_1, \ldots, \theta_{n_t}$. Then, each $(\alpha \veebar p)$ is of the form good.2.1, good.2.2, good.3, or good.4, so each α is a *Neg*, and the occurrences of $\neg p$ within every β are in occurrences of a *Neg* within β.

It remains to prove that $G(\sigma')$ is acyclic. It would follow that the graph of every resulting equation is acyclic and that each of the resulting equations are in some of the good equation forms.

Because of the replacement, for every edge $\langle q_1, q_2\rangle$ of $G(\sigma')$ either $\langle q_1, q_2\rangle$ is an edge of $G(\sigma)$ or there are edges $\langle q_1, p\rangle$ and $\langle p, q_2\rangle$ of $G(\sigma)$. Then for every cycle in $G(\sigma')$ there is a corresponding cycle in $G(\sigma)$. Hence $G(\sigma')$ is acyclic.

Then σ' is a good system.

If *result* is obtained from (3.1), then *result* is σ'. We have split on $\wedge$ an equation of type good.2.1, good.2.2, good.3, or good.4. Equations of type good.2.1 split into two equations of type good.2.1, or one of type good.1 and one of type good.2.1. Equations of type good.2.2 split into two equations of type good.2.2, or one of type good.1 and one of type good.2.2. Equations of type good.3 split into two equations of the same kind, or one of kind good.1 and one of kind good.3. Equations of kind good.4 split into two equations, each of them of type either good.1, good.2.2, good.3, or good.4. All resulting equations are good equations, because the resulting equations have graphs that are subgraphs of the original ones. Hence σ' is a good system.

If *result* is obtained from (3.2), then let the changed equation of σ be χ, which is $(A' \veebar (A_2 \vee A_3))$. We have that χ is not negative, and because of the invariant for σ and the definition of *Good*, we have that χ is either of type good.2.1, good.2.2, good.3, or good.4 with $m = 1$.

First, let χ be of type good.2.1, good.2.2 or good.3. Then A' is negative. Because the graph of χ is acyclic, either $p \not\nearrow A_2$, with A_2 negative or pure and A_3 a *Good* formula, or vice versa. So *result* is σ', not *failure*, and the invariant holds for σ' because we have converted χ to an equation of type good 2.1, good 2.2 or good.3 with a graph that is the same.

Now, let χ be of type good.4 with $m = 1$. Then A' is $\bot$. Then, because p occurs positively in χ, there are three cases for $(A_2 \vee A_3)$. If $(A_2 \vee A_3)$ is $(\neg c \vee \mathit{Good})$ or $(\mathit{PureBox} \vee \mathit{Good}')$, then $p \not\nearrow A_2$. If $(A_2 \vee A_3)$ is $(\mathit{Good} \vee \neg c)$ or $(\mathit{Good}' \vee \mathit{PureBox})$, then $p \not\nearrow A_3$. In these two cases we have converted χ into an equation of type good.2.1 or good.2.2. If $(A_2 \vee A_3)$ is an $\mathit{EB}(p')$ with an acyclic graph, then clearly p is p'. Either A_2 is negative and $p \not\nearrow A_2$ or A_3 is negative and $p \not\nearrow A_3$. In this case we have converted χ into an equation of type good.3.

In either case, *result* is σ' and the invariant holds for σ'.

If *result* is obtained from (3.3), then *result* is σ'. Suppose for the sake of contradiction that we have changed an equation χ of kind good.4. Then either χ is a negative formula, which contradicts the fact that χ is not of kind good.1, or the right-hand side of χ is a box, which contradicts the fact that χ is not of kind good.3. Now, because p occurs positively in the changed equation of σ, there are three cases. First, an equation of type good.3 was changed, then we have converted the equation into another one of type good.3 with a graph that is the same. Second, we have converted an equation of type good.2.2 into another one of the same kind, with a graph that is the same. Third, we have converted an equation of type good.2.1 into an equation of type good.2.2 with a graph that is the same. Therefore, the invariant holds.

If *result* is obtained from (3.4) or from (3.5), let the first equation of σ where p occurs positively and which is not of kind $(\alpha \veebar p)$ such that $p \not\nearrow \alpha$, be χ, which is $(A' \veebar \Diamond A_2)$ by Corollary 83. Because χ is not negative, χ can only be of type good.2.1, and the result can only have been obtained from (3.4). The invariant holds because we have converted χ into an equation of type good.1 and an equation of type good.2.1. $\Box$

Claim 89 Every output result of a system normalization procedure, where the input is a good system, is a good system.

Proof Consider a system normalization procedure run on a good system σ. Let σ be $\neg \bigwedge(\chi_1, \ldots, \chi_m)$. The first formula A_0 as in Definition 80 is $\bigwedge(\chi'_1, \ldots, \chi'_m)$ where it is trivial to see that each χ'_i, $1 \leq i \leq m$ is a δ as in the definition of good.4. above.

It remains to check that for every formula γ of the kind $\bigwedge(\delta_1, \ldots, \delta_m)$, applying any of the main rules (see Definition 76) or the rules from the two tables above to a subformula of γ converts γ to a formula of the same kind. This would guarantee that the resulting system has a single equation which is of kind good.4 if it is not any of the kinds good.1, good.2.1, good.2.2, or good.3.

Clearly the main rules 1, 2, and 3 (see Definition 76) do not apply.

Suppose rule 4 or 5 has been applied. Then the rule is applied to a subformula of some δ_i of γ.

If the rule is applied inside a negative formula, then we convert a negative formula to another one with the same propositional variables, so the invariant holds.

Suppose we apply the rule to the disjunction of a negative formula and an *EB*. There are two cases. Either the formula to which the rule has been applied is another *EB*, then the result is also an *EB* with the same graph as the original. Or, the rule has been applied to a disjunction of a *PureBox* and a formula of kind *Good* or *Good'*, which we discuss below.

Suppose the rule is applied to a disjunction of a *PureBox* formula and a *Good* or *Good'* formula. Then the result is a formula of the same kind, with the same graph as the original.

Now suppose the rule is applied directly to a δ.

If applied to a negative formula δ, we have already seen this case above.

If applied to a formula δ of the kind $(\neg c \vee \mathit{Good})$ or of the kind $(\mathit{Good} \vee \neg c)$, then we have split δ into two formulas of the same kind, with a subgraph of the original, so the invariant holds.

If applied to a formula δ of the kind $(\mathit{PureBox} \vee \mathit{Good}')$ or of the kind $(\mathit{Good}' \vee \mathit{PureBox})$, then we have clearly split δ into two formulas of the same kind, with a subgraph of the original, so the invariant holds.

If applied to a formula δ which is an *EB* with an acyclic graph, then by what we have discussed above, the invariant holds.

Now consider the rules in the first table above.

$(c_1 \to \langle U \rangle c_2) \Rightarrow \bot$ converts a negative and pure subformula into another negative and pure one, so it maintains the invariant.

The following rules convert a negative formula into another one, a *PureBox* formula into another one, a negative formula into another one with the same set of variables, an *EB* into another one with the same graph, a *Good'* into another one with the same graph, or a *Good* into another one with the same graph, so they maintain the invariant:

$U_1 U_2 \gamma \Rightarrow U_2 \gamma$

$\Box U_1 \gamma \Rightarrow (U_1 \gamma \vee \Box\bot)$

$[U](U_1\gamma_1 \overset{\wedge}{\vee} U_2\gamma_2) \Rightarrow (U_1\gamma_1 \overset{\wedge}{\vee} U_2\gamma_2)$

$[U](U_1\gamma_1 \overset{\wedge}{\vee} \gamma_2) \Rightarrow (U_1\gamma_1 \overset{\wedge}{\vee} [U]\gamma_2)$

$[U]\neg c \Rightarrow \bot$
$\Diamond U_1\gamma \Rightarrow (U_1\gamma \wedge \Diamond\top)$
$\langle U\rangle(U_1\gamma_1 \overset{\wedge}{\vee} U_2\gamma_2) \Rightarrow (U_1\gamma_1 \overset{\wedge}{\vee} U_2\gamma_2)$
$\langle U\rangle(U_1\gamma_1 \wedge \gamma_2) \Rightarrow (U_1\gamma_1 \wedge \langle U\rangle\gamma_2)$
$\langle U\rangle c \Rightarrow \top$

Because the diamond distribution rule is not applicable outside of negative or pure subformulas in a *Good* formula, the rule $\langle U\rangle(U_1\gamma_1 \vee \gamma_2) \Rightarrow (U_1\gamma_1 \vee \langle U\rangle\gamma_2)$ may only be applied to a negative or pure subformula within a *PureBox* formula, a negative formula, an *EB*, a *Good′* formula or a *Good* formula. Then this rule clearly converts a negative or pure subformula into another negative or pure subformula whose variables are the same as the variables of the original, thus preserving the invariant.

The ten rules in the right-hand part of the first table (the table of universal modality rules) as well as the top ten rules of the second table (five on each side) may replace a subformula with either $\top$ or $\bot$ or may remove a subformula of either a conjunction or a disjunction. Clearly then a δ remains a δ after replacement with its graph being a subgraph of the original, and a conjunction of δ formulas remains a conjunction of δ formulas with a graph which is a subgraph of the original.

Clearly the rules $(\gamma_1 \wedge \gamma_2) \Rightarrow (\gamma_2 \wedge \gamma_1)$, $(\gamma_1 \vee \gamma_2) \Rightarrow (\gamma_2 \vee \gamma_1)$, and $(\gamma_1 \wedge (\gamma_2 \wedge \gamma_3)) \Rightarrow ((\gamma_1 \wedge \gamma_2) \wedge \gamma_3)$ maintain the invariant.

Consider the rule $(\gamma_1 \vee (\gamma_2 \vee \gamma_3)) \Rightarrow ((\gamma_1 \vee \gamma_2) \vee \gamma_3)$. If it is applied to a negative or a *PureBox* formula, then the result is of the same kind and the same set of occurring variables. If it is applied to an *EB* formula, then one of γ_1 and $(\gamma_2 \vee \gamma_3)$ is a negative formula and the other one is an *EB*, so the invariant holds because the graph is the same. If the rule is applied to a *Good′* formula, the result is also a *Good′* formula with the same graph. If it is applied to a *Good* formula, the result is also a *Good* formula with the same graph. Suppose the rule is applied to a δ formula. If δ is negative, the result is also negative and the invariant is preserved. If δ is $(\neg c \vee \mathit{Good})$ or $(\mathit{Good} \vee \neg c)$, then *Good* is a disjunction and by the definition of *Good* there are no diamonds in the scope of a disjunction in *Good* except inside a pure or a negative subformula. So the result is a δ of the kind $(\mathit{PureBox} \vee \mathit{Good}')$ or $(\mathit{Good}' \vee \mathit{PureBox})$ with the same graph. Now suppose that the rule is applied to a δ of the kind $(\mathit{PureBox} \vee \mathit{Good}')$ or $(\mathit{Good}' \vee \mathit{PureBox})$, then the result is a formula of the same kind with the same graph, so the invariant holds. The case of *EB* has already been discussed above.

The rule $\Box\top \Rightarrow \top$ replaces a *PureBox* formula with another one and the rule $\Diamond\bot \Rightarrow \bot$ replaces a pure formula with another pure formula so the invariant holds.

The rule $(\Box\gamma_1 \wedge \Box\gamma_2) \Rightarrow \Box(\gamma_1 \wedge \gamma_2)$ converts a negative formula into another one with the same set of variables, a *PureBox* formula into another one, an

EB into another one with the same graph, a *Good'* formula into another one with the same graph, a *Good* formula into another one with the same graph, and a conjunction of δ formulas into another one.

Consider the rule $(\Diamond\gamma_1 \vee \Diamond\gamma_2) \Rightarrow \Diamond(\gamma_1 \vee \gamma_2)$. Clearly neither $\Diamond\gamma_1$ nor $\Diamond\gamma_2$ is a negated nominal because it is a diamond. The rule converts a negative formula into another one with the same set of variables, a *PureBox* formula into another one, a *Good'* formula into another one with the same graph because the diamonds only occur in negative or pure subformulas, a *Good* formula into another one with the same graph because diamonds do not occur in *PureBox*. It may only be applied to a δ formula if the δ formula is a negative formula, which we have already discussed above. $\square$

Claim 90 On inductive input formulas, Deterministic SQEMA only works on good systems, with the starting equation being either of kind good.1 or of kind good.2.1.

Proof By Claim 88 and Claim 89, it is enough to show that every initial equation is one of the kinds good.1, good.2.1, good.2.2, good.3, or good.4.

Let A be an inductive formula. In STEP 1 of SQEMA we rewrite A in negation normal form and then distribute all boxes and disjunctions over conjunctions as much as possible. Consider a conjunct A_i in STEP 2, and consider A_i' which is the result of applying main rules 1., 2., and 3. (see Definition 76) as much as possible to $\neg A_i$, which is the first step of the eliminating normalization procedure.

If γ is the negation normal form of a box-formula of the variable p, then γ is built up as follows: $p \mid (Neg \vee \gamma) \mid \Box\gamma$. Clearly then γ is an extended box-formula of p ($EB(p)$) with the same graph as γ.

A_i' is the negation normal form of a negated MRF formula with an acyclic graph, so it is built up from negative formulas and negation normal forms of box-formulas by applying $\vee$, $\wedge$, and $\Diamond$. We must prove that A_i' is a *Good* formula. To see that this is the case, first note that there are no occurrences of $\Diamond$ in A_i' in the scope of a disjunction or a box, except within negative subformulas. Suppose for the sake of contradiction that A_i' contains an occurrence of a disjunction outside any negative or EB subformulas. Because of STEP 1, this is impossible. So the only occurrences of disjunctions in A_i' must be within extended box-formulas or negative formulas. So A_i' is built up from negative formulas and negation normal forms of box-formulas (which are extended box-formulas) by applying $\wedge$ and $\Diamond$. Because of how A_i' was obtained, none of the main rules 1., 2., and 3. (see Definition 76) is applicable to A_i' and the diamond distribution rule is not applicable to any subformula of A_i' except within negative or pure subformulas. Then A_i' is a *Good* formula.

Similarly to the proof of Claim 89, we can see that the result of an eliminating normalization procedure on a *Good* formula is a *Good* formula. Therefore the initial equation is an equation of the kind $(\neg c \veebar \mathit{Good})$, so it is either of kind good.1 or of kind good.2.1. $\Box$

Corollary 91 Deterministic SQEMA succeeds on every inductive formula at the first permutation of its variables, without backtracking. $\Box$

3.10 Example Runs with Inductive Formulas

Now, some examples. Here $\Diamond$ means $\Diamond_1$ as in ML($\Box$) and $\Box$ means $\Box_1$.

3.10.1 $(p_1 \vee \Box\neg p_2 \vee \Diamond(\neg p_1 \wedge p_2))$

Consider the inductive formula $(p_1 \vee \Box\neg p_2 \vee \Diamond(\neg p_1 \wedge p_2))$. This is a good example because the execution of Deterministic SQEMA goes through all kinds of good equations, good.1, good.2.1, good.2.2, good.3, and good.4.

In STEP 1, the formula is already in negation normal form, and it is impossible to apply rules 1.1, 1.2, or 1.3 any more. Thus we have a single conjunct:

$A_1 = (p_1 \vee \Box\neg p_2 \vee \Diamond(\neg p_1 \wedge p_2))$.

In STEP 2, we need to normalize $\neg A = \neg(p_1 \vee \Box\neg p_2 \vee \Diamond(\neg p_1 \wedge p_2))$.

First, we obtain a negation normal form of $\neg A$, which is: $(\neg p_1 \wedge \Diamond p_2 \wedge \Box(p_1 \vee \neg p_2))$. None of the main rules are applicable. However, the implementation re-arranges the formulas using a form of lexicographical ordering, by the rules in the second table marked as CNF, to obtain $(\Diamond p_2 \wedge (\Box(p_1 \vee \neg p_2) \wedge \neg p_1))$.

Now, we reserve the nominal c_1 and form the initial system:

$\sigma_1 = \neg \bigwedge((\neg c_1 \veebar (\Diamond p_2 \wedge (\Box(p_1 \vee \neg p_2) \wedge \neg p_1))))$.

Here, the equation is of kind good.2.1, with a graph containing two vertices, p_1 and p_2, with an edge from p_2 to p_1, and thus acyclic.

In STEP 3, we pick the elimination order $\langle p_1, p_2\rangle$, create an empty backtracking stack and proceed to STEP 4, attempting to eliminate p_1.

In STEP 4, we save a backtracking context $\langle p_1, \sigma\rangle$, which we will not have to use.

We now apply the Deterministic SQEMA strategy, which is to call *step* as many times as possible, until obtaining either *failure* or a system, which is solved for p_1:

$\mathit{step}(\sigma_1, p_1)$ splits on conjunction:

$\sigma_2 = \neg \bigwedge((\neg c_1 \veebar \Diamond p_2), (\neg c_1 \veebar (\Box(p_1 \vee \neg p_2) \wedge \neg p_1)))$,

where the first equation is of kind good.2.1 with a graph with a single vertex p_2 and no edges, and the second equation is of kind good.2.1 with a graph with two vertices, p_1 and p_2, with an edge from p_2 to p_1, and thus acyclic. The combined graph is the same as the graph for the second equation, thus acyclic.

$\mathit{step}(\sigma_2, p_1)$ splits on conjunction:

$\sigma_3 = \neg \bigwedge((\neg c_1 \veebar \Diamond p_2), (\neg c_1 \veebar \Box(p_1 \vee \neg p_2)), (\neg c_1 \veebar \neg p_1))$.

The first equation does not change. The second one is of kind good.2.1, with the same graph as the second equation in σ_2, thus acyclic. The third equation is of kind good.1, with an empty graph. Thus the graph of σ_3 is the same as the graph of σ_2, thus acyclic.

$\mathit{step}(\sigma_3, p_1)$ applies the $\Box$-rule:

$\sigma_4 = \neg \bigwedge((\neg c_1 \veebar \Diamond p_2), (\Box^{-1}\neg c_1 \veebar (p_1 \vee \neg p_2)), (\neg c_1 \veebar \neg p_1))$.

The first and the third equations are the same as in σ_3. The second equation is of kind good.2.2, with the same graph as the second equation of σ_3, thus acyclic. The combined graph of σ_4 is the same as the graph of σ_3, thus acyclic.

step(σ_4, p_1) applies case 3.2:

$\sigma_5 = \neg \bigwedge((\neg c_1 \veebar \Diamond p_2), ((\Box^{-1}\neg c_1 \vee \neg p_2) \veebar p_1), (\neg c_1 \veebar \neg p_1))$.

The first and the third equations are unchanged. The second equation is of kind good.3, with the same graph as the second equation of σ_4, thus acyclic. The graph of σ_5 is the same.

step(σ_5, p_1) applies the Ackermann rule:

$\sigma_6 = \neg \bigwedge((\neg c_1 \veebar \Diamond p_2), (\neg c_1 \veebar (\Box^{-1}\neg c_1 \vee \neg p_2)))$.

The first equation is still the same as in σ_5, but the second one is of kind good.1 with an empty graph. Thus the combined graph for σ_6 has single vertex p_2 and no edges.

The system normalization procedure is very simple here, because there isn't much to do but endlessly apply the CNF-marked rules, which it does not do.

Thus the conjunction that is obtained is $\bigwedge((\neg c_1 \vee \Diamond p_2), (\neg c_1 \vee (\Box^{-1}\neg c_1 \vee \neg p_2)))$, where both conjuncts are of the kind δ. The first conjunct's graph is the same as the graph of the first equation of σ_6, and the second one's graph is the empty graph.

Thus, after normalization, the system is:

$\sigma_7 = \neg \bigwedge((\bot \veebar ((\neg c_1 \vee \Diamond p_2) \wedge (\neg c_1 \vee (\Box^{-1}\neg c_1 \vee \neg p_2)))))$

The equation here is of kind good.4. Because of the above reasoning about the conjuncts, the combined graph of σ_7 is the same as the graph of σ_6 with a single vertex and no edges, thus acyclic.

Then STEP 4 returns to STEP 3, which calls STEP 4 again with σ_7, p_2 and the existing backtracking context.

In STEP 4, we save a backtracking context $\langle p_2, \sigma_7\rangle$ and start calling *step*.

step(σ_7, p_2) splits on conjunction:

$\sigma_8 = \neg \bigwedge((\bot \veebar (\neg c_1 \vee \Diamond p_2)), (\bot \veebar (\neg c_1 \vee (\Box^{-1}\neg c_1 \vee \neg p_2))))$

Now the first equation is of kind good.4 with the same graph as σ_7, and the second equation is of kind good.1 with the empty graph. Thus the graph of σ_8 is the same as the graph of σ_7.

step(σ_8, p_2) applies case 3.2:

$\sigma_9 = \neg \bigwedge(((\bot \vee \neg c_1) \veebar \Diamond p_2), (\bot \veebar (\neg c_1 \vee (\Box^{-1}\neg c_1 \vee \neg p_2))))$

The first equation is of kind good.2.1 with the same graph as σ_8, and the second one is of kind good.1 with the empty graph. Thus the graph of σ_9 is the same as the graph of σ_8.

step(σ_9, p_2) applies the $\Diamond$-rule, case 3.4:

$\sigma_{10} = \neg \bigwedge((c_1 \to \Diamond c_2), (\neg c_2 \veebar p_2), (\bot \veebar (\neg c_1 \vee (\Box^{-1}\neg c_1 \vee \neg p_2))))$

The first equation is of kind good.1 with the empty graph. The second one is of kind good.2.1 with the same graph as σ_9. The last equation is of kind good.1 with the empty graph. So the graph of σ_{10} is the same as the graph of σ_9.

step(σ_{10}, p_2) applies the Ackermann rule:

$\sigma_{11} = \neg \bigwedge((c_1 \to \Diamond c_2), (\bot \veebar (\neg c_1 \vee (\Box^{-1}\neg c_1 \vee \neg c_2))))$, which is a pure system, so both equations are of kind good.1 with the empty graph.

After translating and simplifying, we arrive at the formula:

$\psi(x) \equiv \forall y_1((x\ r_1\ y_1) \to ((x = y_1) \wedge (x\ r_1\ x)))$

3.10.2 $(\Box\Diamond\neg p \vee (\Box\Diamond(p \wedge q) \vee \Box\Diamond\neg q))$

Consider the inductive formula $(\Box\Diamond\neg p \vee (\Box\Diamond(p \wedge q) \vee \Box\Diamond\neg q))$.

The initial system (after normalization) is:

σ_1: $\neg \bigwedge((\neg c_1 \veebar (\Diamond\Box p \wedge (\Diamond\Box(\neg p \vee \neg q) \wedge \Diamond\Box q))))$

Above, $\Box p$, and $\Box q$ are *EB*-formulas with graphs which have singletons as the sets of vertices, and no edges. $\Box(\neg p \vee \neg q)$ is a negative formula, and the equation is of kind good.2.1 with a graph with vertices $\{p, q\}$ and no edges.

Now, we split on conjunction:

σ_2: $\neg \bigwedge((\neg c_1 \veebar \Diamond\Box p), (\neg c_1 \veebar (\Diamond\Box(\neg p \vee \neg q) \wedge \Diamond\Box q)))$

Each equation above is of kind good.2.1. The graph of the system is the same as the graph of σ_1.

Applying the $\Diamond$-rule:

σ_3: $\neg \bigwedge((c_1 \to \Diamond c_2), (\neg c_2 \veebar \Box p), (\neg c_1 \veebar (\Diamond\Box(\neg p \vee \neg q) \wedge \Diamond\Box q)))$

The first equation is of kind good.1, the other two are still of kind good.2.1 with graphs without edges. The graph of σ_3 is the same as the graph of σ_2.

Applying the $\Box$-rule:

σ_4: $\neg \bigwedge((c_1 \to \Diamond c_2), (\Box^{-1}\neg c_2 \veebar p), (\neg c_1 \veebar (\Diamond\Box(\neg p \vee \neg q) \wedge \Diamond\Box q)))$

We have changed the second equation from good.2.1 into good.2.2 preserving the graphs.

Now, we apply the Ackermann rule:

σ_5: $\neg \bigwedge((c_1 \to \Diamond c_2), (\neg c_1 \veebar (\Diamond\Box(\Box^{-1}\neg c_2 \vee \neg q) \wedge \Diamond\Box q)))$

Above, the first equation is of kind good.1 with the empty graph, and the second one is of kind good.2.1 with the same graph as σ_4, because $\Diamond\Box(\Box^{-1}\neg c_2 \vee \neg q)$ is a negative formula and $\Diamond\Box q$ is a *Good* formula.

The normalization process is trivial, only splitting on conjunction once by applying the distributive rule for $\vee$ and rearranging the conjunction using a form of lexicographic ordering.

After normalization, we obtain:

σ_6: $\neg \bigwedge((\bot \veebar ((\neg c_1 \vee \Diamond\Box(\Box^{-1}\neg c_2 \vee \neg q)) \wedge ((c_1 \to \Diamond c_2) \wedge (\neg c_1 \vee \Diamond\Box q)))))$

The above equation is $(\bot \veebar \bigwedge((\neg c_1 \vee \Diamond\Box(\Box^{-1}\neg c_2 \vee \neg q)), (c_1 \to \Diamond c_2), (\neg c_1 \vee \Diamond\Box q)))$, where the first two components of the conjunction are negative formulas, and the last one is of kind $(\neg c \vee \textit{Good})$ with the same graph as the one of σ_5. Thus, the equation is of kind good.4 because, due to $\Diamond\Box q$, it is not of kind good.2.2 or good.3.

Now, we split on conjunction twice, maintaining the invariant because of the above reasoning, finally obtaining two equations of kind good.1 and one of kind good.4:

σ_7: $\neg \bigwedge((\bot \veebar (\neg c_1 \vee \Diamond\Box(\Box^{-1}\neg c_2 \vee \neg q))), (\bot \veebar (c_1 \to \Diamond c_2)), (\bot \veebar (\neg c_1 \vee \Diamond\Box q)))$

We apply point 3.2 of *step*:

σ_8: $\neg \bigwedge((\bot \veebar (\neg c_1 \vee \Diamond\Box(\Box^{-1}\neg c_2 \vee \neg q))), (\bot \veebar (c_1 \to \Diamond c_2)), ((\bot \vee \neg c_1) \veebar \Diamond\Box q))$

Because $\Diamond\Box q$ is a *Good* formula with a graph without edges and a single vertex, the third equation has changed into an equation of kind good.2.1 with the same graph.

Now, we apply 3.4 of *step*, the $\Diamond$-rule:

σ_9: $\neg \bigwedge((\bot \veebar (\neg c_1 \vee \Diamond\Box(\Box^{-1}\neg c_2 \vee \neg q))), (\bot \veebar (c_1 \to \Diamond c_2)), (c_1 \to \Diamond c_3), (\neg c_3 \veebar \Box q))$, producing a new equation of kind good.1, and converting the last equation into kind good.2.1, preserving the graph without edges.

Applying the $\Box$-rule:

σ_{10}: $\neg\bigwedge((\bot \veebar (\neg c_1 \vee \Diamond\Box(\Box^{-1}\neg c_2 \vee \neg q))), (\bot \veebar (c_1 \to \Diamond c_2)), (c_1 \to \Diamond c_3), (\Box^{-1}\neg c_3 \veebar q))$, where $(\Box^{-1}\neg c_3 \veebar q)$ is of kind good.2.2 with the same graph as before.

Finally, we apply the Ackermann rule:

σ_{11}: $\neg\bigwedge((\bot \veebar (\neg c_1 \vee \Diamond\Box(\Box^{-1}\neg c_2 \vee \Box^{-1}\neg c_3))), (\bot \veebar (c_1 \to \Diamond c_2)), (c_1 \to \Diamond c_3))$, which is a pure system so all equations are of kind good.1 with the empty graph.

We translate and simplify to obtain:

$\forall y_1 \forall y_2(((x\ r_1\ y_1) \wedge (x\ r_1\ y_2)) \to \forall z_1((x\ r_1\ z_1) \to \exists z_2((y_1\ r_1\ z_2) \wedge (y_2\ r_1\ z_2) \wedge (z_1\ r_1\ z_2))))$

3.10.3 More Examples

1. Consider the inductive formula $(\Diamond p \to \Box\Diamond p)$.

The initial system (after normalization) is:

σ_1: $\neg\bigwedge((\neg c_1 \veebar (\Diamond\Box\neg p \wedge \Diamond p)))$, where $(\Diamond\Box\neg p \wedge \Diamond p)$ is a good piece because $\Diamond\Box\neg p$ is a negative formula and p is an $EB(p)$ with a graph with a single vertex p and no edges, and thus the equation is of type good.2.1.

Now, we split on conjunction:

σ_2: $\neg\bigwedge((\neg c_1 \veebar \Diamond\Box\neg p), (\neg c_1 \veebar \Diamond p))$

Where the invariant is maintained due to the above reasoning, the first equation being of kind good.1 and the second one being of kind good.2.1.

Now, we apply the $\Diamond$-rule:

σ_3: $\neg\bigwedge((\neg c_1 \veebar \Diamond\Box\neg p), (c_1 \to \Diamond c_2), (\neg c_2 \veebar p))$, maintaining the invariant by converting an equation of type good.2.1 into one of type good.1 and one of type good.2.1.

After applying the Ackermann rule, translating and simplifying, we obtain:

$\forall y_1((x\ r_1\ y_1) \to \forall z_1((x\ r_1\ z_1) \to (z_1\ r_1\ y_1)))$

2. Consider the inductive formula $(\Box\Box p \to \Box p)$.

The initial system (after normalization) is:

σ_1: $\neg\bigwedge((\neg c_1 \veebar (\Diamond\neg p \wedge \Box\Box p)))$

Clearly the invariant holds because $\Diamond\neg p$ is a negative formula and $\Box\Box p$ is an $EB(p)$ with a graph with a single vertex p and no edges, and thus the equation is of type good.2.1.

First, we split on conjunction into one equation of kind good.1 and one of type good.2.1:

σ_2: $\neg\bigwedge((\neg c_1 \veebar \Diamond\neg p), (\neg c_1 \veebar \Box\Box p))$

Now, we apply the $\Box$-rule twice, first converting an equation of type good.2.1 into an equation of kind good.2.2, and then keeping an equation of kind good.2.2.

σ_3: $\neg\bigwedge((\neg c_1 \veebar \Diamond\neg p), (\Box^{-1}\Box^{-1}\neg c_1 \veebar p))$

Now, we apply the Ackermann rule, translate and simplify to obtain:

$\forall z_1((x\ r_1\ z_1) \to \exists z_2((x\ r_1\ z_2) \wedge (z_2\ r_1\ z_1)))$

3.11 Pre-Contact Logics

The language of pre-contact logics (PCL) is a first-order language with equality (=) and without quantifiers. It is intended to be a propositional language for point-free theories of space, as outlined in [5].

Definition 92 *Boolean terms* of PCL are: $\tau ::= p \mid 0 \mid 1 \mid -\tau \mid (\tau \cup \tau) \mid (\tau \cap \tau)$, where 0 and 1 are *boolean constants*, and $p \in PROP$ is a *boolean variable* (but note we use the same set of symbols as our propositional variables in modal languages).

Atomic formulas are: $\alpha ::= \bot \mid \top \mid (\tau = \tau) \mid (\tau \leq \tau) \mid C(\tau, \tau)$ where part-of ($\leq$) and contact (C) are binary predicate symbols.

Pre-Contact formulas are: $\psi ::= \alpha \mid \neg\psi \mid (\psi \vee \psi) \mid (\psi \wedge \psi)$. We may use $\to$ and $\leftrightarrow$ as defined symbols with their usual meaning.

We use Kripke frames and Kripke models for the basic modal language $\mathrm{ML}(\Box)$, which are also frames and models for the language $\mathrm{ML}(\Box, [U])$.

If $\mathrm{M} = \langle \mathrm{F}, V \rangle$ is a Kripke model, where $\mathrm{F} = \langle W, R \rangle$, then the valuation V, which is a valuation of propositional variables to subsets of W, can be extended to all boolean terms by evaluating the boolean variables of PCL the same way as the propositional variables of modal languages, in the following way:

$V(p) \subseteq W$

$V(0) = \emptyset$, $V(1) = W$

$V(-\tau) = W \setminus V(\tau)$

$V((\tau_1 \cup \tau_2)) = V(\tau_1) \cup V(\tau_2)$

$V((\tau_1 \cap \tau_2)) = V(\tau_1) \cap V(\tau_2)$

The definition of truth of atomic formulas in a Kripke model M is as follows:

$\mathrm{M} \vDash (\tau_1 = \tau_2)$ iff $V(\tau_1) = V(\tau_2)$

$\mathrm{M} \vDash (\tau_1 \leq \tau_2)$ iff $V(\tau_1) \subseteq V(\tau_2)$

$\mathrm{M} \vDash C(\tau_1, \tau_2)$ iff $\exists x \exists y (x \in V(\tau_1)\ \&\ y \in V(\tau_2)\ \&\ \langle x, y \rangle \in R)$

Truth of pre-contact formulas in M is defined as follows:

$\mathrm{M} \vDash \neg\psi_1$ iff $\mathrm{M} \nvDash \psi_1$

$\mathrm{M} \vDash ((\psi_1 \vee \psi_2))$ iff $\mathrm{M} \vDash \psi_1$ or $\mathrm{M} \vDash \psi_2$

$\mathrm{M} \vDash ((\psi_1 \wedge \psi_2))$ iff $\mathrm{M} \vDash \psi_1$ and $\mathrm{M} \vDash \psi_2$

We say that ψ is *valid* in a frame F, $\mathrm{F} \vDash \psi$, iff ψ is true in all models over F.

It is shown in [5] that pre-contact formulas can be represented as formulas of $\mathrm{ML}(\Box, [U])$. More precisely, there is a translation $\mathtt{t} : \mathrm{PCL} \mapsto \mathrm{ML}(\Box, [U])$ with the property that for every PCL formula ψ and every Kripke model M, $\mathrm{M} \vDash \psi$ iff $\mathrm{M} \Vdash \mathtt{t}(\psi)$.

This translation $\mathtt{t}$ maps variables to propositional variables. Function symbols map to the corresponding boolean connectives. $\mathtt{t}(0) = \bot \in \mathrm{ML}(\Box, [U])$, $\mathtt{t}(1) = \top \in \mathrm{ML}(\Box, [U])$. Let τ_1, τ_2 be terms. The predicate symbols translate as follows:

$\mathtt{t}((\tau_1 = \tau_2)) = [U](\mathtt{t}(\tau_1) \leftrightarrow \mathtt{t}(\tau_2))$

$\mathtt{t}((\tau_1 \leq \tau_2)) = [U](\mathtt{t}(\tau_1) \to \mathtt{t}(\tau_2))$

$\mathtt{t}(C(\tau_1, \tau_2)) = \langle U \rangle(\mathtt{t}(\tau_1) \wedge \Diamond_1 \mathtt{t}(\tau_2))$

The boolean connectives translate to themselves.

Now, we discuss Sahlqvist PCL formulas, as defined in [4].

A *positive* term is built up from variables, -0 and 1, using only $\cup$ and $\cap$.

A *negation-free* formula is built up from $\neg(\tau_1 = 0)$ and $C(\tau_1, \tau_2)$, where τ_1 and τ_2 are positive terms, using only $\top$, $\vee$, and $\wedge$.

A *positive* formula is built up from $\neg(\tau_1 = 0)$, $(-\tau_1 = 0)$, $(\tau_1 = 1)$, $C(\tau_1, \tau_2)$, and $\neg C(-\tau_1, -\tau_2)$, where τ_1 and τ_2 are positive terms, using only $\top$, $\vee$, and $\wedge$.

A *Sahlqvist* formula ψ is an implication $(\psi_1 \to \psi_2)$, where ψ_1 is negation-free, and ψ_2 is positive.

To translate Sahlqvist PCL formulas, as defined in [4], into Sahlqvist formulas in $\mathrm{ML}(\Box, [U])$, we define a *modified translation* $\mathtt{t}'$ as follows:

$\mathtt{t}'(p) =_{def} p \in \mathrm{ML}(\Box, [U])$

$\mathtt{t}'(0) =_{def} \bot \in \mathrm{ML}(\Box, [U])$

$\mathtt{t}'(1) =_{def} \top \in \mathrm{ML}(\Box, [U])$

$\mathtt{t}'(-\tau) =_{def} \neg\mathtt{t}'(\tau)$ where τ is any term

$\mathtt{t}'((\tau_1 \cup \tau_2)) =_{def} (\mathtt{t}'(\tau_1) \vee \mathtt{t}'(\tau_2))$, where τ_1 and τ_2 are any terms

$\mathtt{t}'((\tau_1 \cap \tau_2)) =_{def} (\mathtt{t}'(\tau_1) \wedge \mathtt{t}'(\tau_2))$, where τ_1 and τ_2 are any terms

$\mathtt{t}'((-\tau = 0)) =_{def} [U]\mathtt{t}'(\tau)$, where τ is any term.

$\mathtt{t}'((\tau = 1)) =_{def} [U]\mathtt{t}'(\tau)$, where τ is any term.

$\mathtt{t}'((\tau_1 = \tau_2)) =_{def} [U](\mathtt{t}'(\tau_1) \leftrightarrow \mathtt{t}'(\tau_2))$, where $(\tau_1 = \tau_2)$ is not as in the above two cases

$\mathtt{t}'((\tau_1 \leq \tau_2)) =_{def} [U](\mathtt{t}'(\tau_1) \to \mathtt{t}'(\tau_2))$, where τ_1 and τ_2 are any terms

$\mathtt{t}'(C(\tau_1, \tau_2)) =_{def} \langle U\rangle(\mathtt{t}'(\tau_1) \wedge \Diamond_1\mathtt{t}'(\tau_2))$, where τ_1 and τ_2 are any terms

$\mathtt{t}'(\neg C(-\tau_1, -\tau_2)) =_{def} [U](\mathtt{t}'(\tau_1) \vee \Box_1\mathtt{t}'(\tau_2))$, where τ_1 and τ_2 are any terms

$\mathtt{t}'(\neg(\tau = 0)) =_{def} \langle U\rangle\mathtt{t}'(\tau)$, where τ is any term

$\mathtt{t}'(\neg\psi) =_{def} \neg\mathtt{t}'(\psi)$, where $\neg\psi$ is not as in the above two cases

$\mathtt{t}'((\psi_1 \vee \psi_2)) =_{def} (\mathtt{t}'(\psi_1) \vee \mathtt{t}'(\psi_2))$ for any ψ_1 and ψ_2

$\mathtt{t}'((\psi_1 \wedge \psi_2)) =_{def} (\mathtt{t}'(\psi_1) \wedge \mathtt{t}'(\psi_2))$ for any ψ_1 and ψ_2

It is easy to see, by induction on PCL terms and PCL formulas, that for any PCL formula ψ, ψ and $\mathtt{t}'(\psi)$ are true in the same models.

We show now how to derive a result from [4] that Sahlqvist formulas have a first-order correspondent as a corollary to the fact that Deterministic SQEMA succeeds on all Sahlqvist $\mathrm{ML}(\Box, [U])$ formulas.

Theorem 93 *The modified translation maps Sahlqvist PCL formulas to Sahlqvist implications from* $\mathrm{ML}(\Box, [U])$.

Proof An easy induction on PCL terms shows that $\mathtt{t}'(\tau)$ for a positive term τ is a positive $\mathrm{ML}(\Box, [U])$ formula. Similarly, it is simple to show that $\mathtt{t}'(\psi)$ for a positive ψ is a positive $\mathrm{ML}(\Box, [U])$ formula. It remains to show that $\mathtt{t}'$ maps negation-free PCL formulas to $\mathrm{ML}(\Box, [U])$ Sahlqvist antecedents. This again follows from an easy induction, using the definition of $\mathtt{t}'$. $\Box$

We use Deterministic SQEMA for the language of Pre-Contact Logic, by translating a pre-contact formula to a formula of $\mathrm{ML}(\Box, [U])$, using $\mathtt{t}'$, and running Deterministic SQEMA on the translation. It immediately follows that Deterministic SQEMA succeeds on the modified translation of any Sahlqvist PCL formula.

It was proved in [5] that: Every pre-contact formula is complete with respect to the class of finite frames defined by it. Hence, every pre-contact formula is complete.

Theorem 94 *Every PCL formula ψ, on whose modified translation Deterministic SQEMA succeeds and produces a* FOL *formula ψ', is complete on the class of frames defined by ψ'.*

Proof By the properties of Deterministic SQEMA, $\mathtt{t}'(\psi)$ and ψ' are locally correspondent, therefore globally correspondent. By the properties of $\mathtt{t}'$, ψ and $\mathtt{t}'(\psi)$ define the same class of frames, so ψ and ψ' define the same class of frames, therefore they define the same class of finite frames. By the above-mentioned result in [5], ψ is complete in the class of finite frames, defined by ψ', and therefore is complete in the class of all frames, defined by ψ'. $\Box$

3.12 Example Runs with PCL Formulas

3.12.1 $((0 \neq p) \to C(p, 1))$

Consider the PCL formula $((0 \neq p) \to C(p, 1))$.

The modified translation $\mathtt{t}'(((0 \neq p) \to C(p, 1)))$ produces the formula:
$(\langle U \rangle p \to \langle U \rangle (\Diamond \top \wedge p))$.

This is a Sahlqvist formula of the language $\mathrm{ML}(\Box, [U])$, so we expect Deterministic SQEMA to succeed on the first try, without backtracking.

In STEP 1, we rewrite the formula in negation normal form, obtaining:
$([U]\neg p \vee \langle U \rangle (\Diamond \top \wedge p))$.

We have a single conjunct, $A_1 = ([U]\neg p \vee \langle U \rangle (\Diamond \top \wedge p))$.

We allocate the nominal c_1 and proceed to STEP 2.

In STEP 2, we need to normalize $\neg A_1 = \neg([U]\neg p \vee \langle U \rangle (\Diamond \top \wedge p))$.

First, we take the negation normal form, which is $(\langle U \rangle p \wedge [U](\Box \bot \vee \neg p))$. Then we check to see if we can apply any more rules, except for the ones marked with CNF. We cannot, so we form the initial system:

$\sigma_1 = \neg \bigwedge((\neg c_1 \veebar (\langle U \rangle p \wedge [U](\Box \bot \vee \neg p))))$.

In STEP 3, we set the variable elimination order to $\langle p \rangle$, and create a new backtracking context. Then we proceed to STEP 4.

In STEP 4, we save a backtracking context $\langle p, \sigma_1 \rangle$, and start applying *step*.

step(σ_1, p) splits on a conjunction:

$\sigma_2 = \neg \bigwedge((\neg c_1 \veebar \langle U \rangle p), (\neg c_1 \veebar [U](\Box \bot \vee \neg p)))$.

step(σ_2, p) applies the $\Diamond$-rule:

$\sigma_3 = \neg \bigwedge((c_1 \to \langle U \rangle c_2), (\neg c_2 \veebar p), (\neg c_1 \veebar [U](\Box \bot \vee \neg p)))$.

$step(\sigma_3, p)$ applies the Ackermann rule:
$\sigma_4 = \neg \bigwedge((c_1 \to \langle U \rangle c_2), (\neg c_1 \veebar [U](\Box\bot \vee \neg c_2)))$.
Thus we proceed to STEP 5.
In STEP 5, we take c_3 as the next available nominal, and set:
$fol_1 = \forall x_2 \exists x_3 st(4, x_3, \sigma_4)$.
The final result is a conjunction with a single element fol_1. After simplification, we obtain the sentence:
$\forall y_1 \exists z_1 (y_1\ r_1\ z_1)$.

3.12.2 $(C(p, q) \to (C(p, r) \vee C(-r, q)))$

Consider the PCL formula $(C(p, q) \to (C(p, r) \vee C(-r, q)))$.

First, we translate into ML($\Box, [U]$), $\mathtt{t}'((C(p, q) \to (C(p, r) \vee C(-r, q))))$:
$(\langle U \rangle(p \wedge \Diamond q) \to (\langle U \rangle(p \wedge \Diamond r) \vee \langle U \rangle(\neg r \wedge \Diamond q)))$.

This is an inductive formula of the language ML($\Box, [U]$), so we expect Deterministic SQEMA to succeed on the first try.

In the future, the author would like to explore a matching definition of inductive formulas in the PCL language.

In STEP 1, we rewrite the formula into negation normal form, obtaining:
$([U](\neg p \vee \Box\neg q) \vee (\langle U \rangle(p \wedge \Diamond r) \vee \langle U \rangle(\neg r \wedge \Diamond q)))$.
We have a single conjunct, $A_1 = ([U](\neg p \vee \Box\neg q) \vee (\langle U \rangle(p \wedge \Diamond r) \vee \langle U \rangle(\neg r \wedge \Diamond q)))$.
In STEP 2, we need to normalize $\neg A_1$. First, we rewrite the formula into negation normal form, obtaining:
$(\langle U \rangle(p \wedge \Diamond q) \wedge ([U](\neg p \vee \Box\neg r) \wedge [U](r \vee \Box\neg q)))$.
Now, the $\Box$-extraction rule is applied:
$(\langle U \rangle(p \wedge \Diamond q) \wedge [U]((\neg p \vee \Box\neg r) \wedge (r \vee \Box\neg q)))$.
Finally, we use the rules marked with CNF to re-arrange the subformulas in the conjunctive normal form in a kind of lexicographical order, obtaining:
$(\langle U \rangle(\Diamond q \wedge p) \wedge [U]((\Box\neg q \vee r) \wedge (\Box\neg r \vee \neg p)))$.
We reserve the nominal c_1 and form the initial system:
$\sigma_1 = \neg \bigwedge((\neg c_1 \veebar (\langle U \rangle(\Diamond q \wedge p) \wedge [U]((\Box\neg q \vee r) \wedge (\Box\neg r \vee \neg p)))))$.
In STEP 3, we choose the elimination order $\langle p, q, r \rangle$ and create a new backtracking context, then call STEP 4.
In STEP 4, we start solving σ_1 by calling *step*.
$step(\sigma_1, p)$ splits on conjunction:
$\sigma_2 = \neg \bigwedge((\neg c_1 \veebar \langle U \rangle(\Diamond q \wedge p)), (\neg c_1 \veebar [U]((\Box\neg q \vee r) \wedge (\Box\neg r \vee \neg p))))$.
$step(\sigma_2, p)$ applies the $\Diamond$-rule:
$\sigma_3 = \neg \bigwedge((c_1 \to \langle U \rangle c_2), (\neg c_2 \veebar (\Diamond q \wedge p)), (\neg c_1 \veebar [U]((\Box\neg q \vee r) \wedge (\Box\neg r \vee \neg p))))$.
$step(\sigma_3, p)$ splits on conjunction:
$\sigma_4 = \neg \bigwedge($
$(c_1 \to \langle U \rangle c_2)$,
$(\neg c_2 \veebar \Diamond q)$,
$(\neg c_2 \veebar p)$,
$(\neg c_1 \veebar [U]((\Box\neg q \vee r) \wedge (\Box\neg r \vee \neg p))))$.
$step(\sigma_4, p)$ applies the Ackermann rule:
$\sigma_5 = \neg \bigwedge($
$(c_1 \to \langle U \rangle c_2)$,

$(\neg c_2 \veebar \Diamond q)$,
$(\neg c_1 \veebar [U]((\Box\neg q \vee r) \wedge (\Box\neg r \vee \neg c_2))))$.

Now, we need to normalize the above system. In the normalization process, we remove the conjunct $(c_1 \to \langle U\rangle c_2)$ because it is equivalent to $\top$, obtaining the system:

$\sigma_6 = \neg \bigwedge((\bot \veebar ((\neg c_1 \vee [U]((\neg c_2 \vee \Box\neg r) \wedge (\Box\neg q \vee r))) \wedge (\neg c_2 \vee \Diamond q))))$.

Then STEP 4 is called again with the next variable to eliminate and the same backtracking stack, q. A backtracking context $\langle q, \sigma_6\rangle$ is saved to the backtracking stack.

$step(\sigma_6, p)$ splits on conjunction:

$\sigma_7 = \neg \bigwedge($
$(\bot \veebar (\neg c_1 \vee [U]((\neg c_2 \vee \Box\neg r) \wedge (\Box\neg q \vee r))))$,
$(\bot \veebar (\neg c_2 \vee \Diamond q)))$.

$step(\sigma_7, p)$ applies case 3.2:

$\sigma_8 = \neg \bigwedge($
$(\bot \veebar (\neg c_1 \vee [U]((\neg c_2 \vee \Box\neg r) \wedge (\Box\neg q \vee r))))$,
$((\bot \vee \neg c_2) \veebar \Diamond q))$.

$step(\sigma_8, p)$ applies the $\Diamond$-rule:

$\sigma_9 = \neg \bigwedge($
$(\bot \veebar (\neg c_1 \vee [U]((\neg c_2 \vee \Box\neg r) \wedge (\Box\neg q \vee r))))$,
$(c_2 \to \Diamond c_3)$,
$(\neg c_3 \veebar q))$.

$step(\sigma_9, p)$ applies the Ackermann rule:

$\sigma_{10} = \neg \bigwedge($
$(\bot \veebar (\neg c_1 \vee [U]((\neg c_2 \vee \Box\neg r) \wedge (\Box\neg c_3 \vee r))))$,
$(c_2 \to \Diamond c_3))$.

The system normalization procedure only removes the occurrence of $\bot$ in a disjunction. We obtain the system:

$\sigma_{11} = \neg \bigwedge((\bot \veebar ((c_2 \to \Diamond c_3) \wedge (\neg c_1 \vee [U]((\neg c_2 \vee \Box\neg r) \wedge (\Box\neg c_3 \vee r))))))$.

STEP 4 is called again with the same backtracking context and propositional variable to eliminate r.

$step(\sigma_{11}, r)$ splits on conjunction:

$\sigma_{12} = \neg \bigwedge($
$(\bot \veebar (c_2 \to \Diamond c_3))$,
$(\bot \veebar (\neg c_1 \vee [U]((\neg c_2 \vee \Box\neg r) \wedge (\Box\neg c_3 \vee r)))))$.

$step(\sigma_{12}, r)$ applies case 3.2:

$\sigma_{13} = \neg \bigwedge($
$(\bot \veebar (c_2 \to \Diamond c_3))$,
$((\bot \vee \neg c_1) \veebar [U]((\neg c_2 \vee \Box\neg r) \wedge (\Box\neg c_3 \vee r))))$.

$step(\sigma_{13}, r)$ applies the $\Box$-rule:

$\sigma_{14} = \neg \bigwedge($
$(\bot \veebar (c_2 \to \Diamond c_3))$,
$(\Box_0^{-1}(\bot \vee \neg c_1) \veebar ((\neg c_2 \vee \Box\neg r) \wedge (\Box\neg c_3 \vee r))))$.

$step(\sigma_{14}, r)$ splits on conjunction:

$\sigma_{15} = \neg \bigwedge($
$(\bot \veebar (c_2 \to \Diamond c_3))$,
$(\Box_0^{-1}(\bot \vee \neg c_1) \veebar (\neg c_2 \vee \Box\neg r))$,
$(\Box_0^{-1}(\bot \vee \neg c_1) \veebar (\Box\neg c_3 \vee r)))$.

$step(\sigma_{15}, r)$ applies case 3.2:
$\sigma_{16} = \neg \bigwedge($
$(\bot \veebar (c_2 \to \Diamond c_3)),$
$(\Box_0^{-1}(\bot \vee \neg c_1) \veebar (\neg c_2 \vee \Box \neg r)),$
$((\Box_0^{-1}(\bot \vee \neg c_1) \vee \Box \neg c_3) \veebar r)).$
$step(\sigma_{16}, r)$ applies the Ackermann rule:
$\sigma_{16} = \neg \bigwedge($
$(\bot \veebar (c_2 \to \Diamond c_3)),$
$(\Box_0^{-1}(\bot \vee \neg c_1) \veebar (\neg c_2 \vee \Box\ (\Box_0^{-1}(\bot \vee \neg c_1) \vee \Box \neg c_3))).$
We now go to STEP 5. We take the next available nominal, c_4, and set:
$fol_1(x_1) = \forall x_2 \forall x_3 \exists x_4 (5, x_4, \sigma_{16}).$
After simplification and suitable renaming of individual variables, we obtain the final result, which is a sentence:
$\forall y_1 \forall y_2 ((y_1\ r_1\ y_2) \to \exists z_1((y_1\ r_1\ z_1) \wedge (z_1\ r_1\ y_2)))$

3.13 Implementation in the Programming Language Java

A variant of the algorithm Deterministic SQEMA was implemented in the Java programming language back in 2006, as part of the author's master's thesis [25], and has been running at the website `http://www.fmi.uni-sofia.bg/fmi/logic/sqema` ever since. For this dissertation, several changes were made to the implementation:

- Support for the universal modality was finalized and theoretical work was done to show soundness of the variant of Deterministic SQEMA with the universal modality, i.e. that if successful, then the result is a local first-order correspondent, and the input formula is the proven to be d-persistent with respect to a language with the universal modality.

- The user interface of the implementation was completely changed from plain HTML with a backend running as a Java servlet on a Tomcat instance, to static HTML/CSS/Javascript files, served from a regular Apache installation. This was achieved by keeping the implementation in the Java language, but using the Google GWT compiler to translate the algorithm's implementation into Javascript and run it directly in the user's browser. This had several beneficial effects:

First, it removed the need for a servlet-capable backend, which helped a great deal when later the Tomcat instance was shut down and a migration to a plain Apache was inevitable.

Second, it relieved the machines of the Faculty of Mathematics and Informatics from the burden of running an algorithm whose first step is obtaining a potentially exponentially larger formula from the input by converting to a conjunctive normal form. Needless to say, that had caused quite a few problems over the years.

And finally, it helped the author create backup websites hosting the implementation, such as:

`http://debian.fmi.uni-sofia.bg/~dimitertg/sqema`,

`http://dimiter.slavi.biz/sqema`,

and

`http://geocities.ws/sqema`,

which are always kept up-to-date with the original website, and are sometimes used as testing environments for newer versions before the main website is updated.

- Some additional simplifications were implemented for the first-order result formula, improving the readability of the results.

- Support for the language of Pre-Contact Logics was added, by using a modified translation of PCL formulas into formulas of the modal language $\mathrm{ML}(\Box, [U])$ and thus ensuring that the implementation succeeds on all Sahlqvist PCL formulas, see details in 3.11.

- Support for polyadic modalities was added. The theoretical work on which this part of the implementation is based comes from [18], and no further proofs were made to ensure that the implementation succeeds on all polyadic Sahlqvist and polyadic Inductive formulas. This is considered out of scope for this dissertation. In the future, the author would like to explore this topic further.

- A recognizer for Sahlqvist and Inductive formulas was added to the user interface. Thus the user can quickly check if the input formula is a Sahlqvist or an Inductive formula. This recognizer does not yet support polyadic formulas and does not affect the work of the algorithm.

- An off-line testing system was implemented in Java, along with some formula generators, which were used to amass several million formulas as a test corpus. The tests are used for several reasons:

First, they are used as a regression check, after every major change in the algorithm, a test is run off-line by the author to see if there are any differences in the results between versions of the algorithm.

Second, they were used to test the hypotheses that the implementation always succeeds on all Sahlqvist and all Inductive formulas, using the recognizer mentioned above and checking whether the implementation indeed succeeds on all such formulas from the test corpus.

Last but not least, the tests were of great help when the author was looking for new invariants for Sahlqvist and Inductive formulas.

4 ML($\Box$) and $\mathcal{C}_{\mathrm{KD45}}$

We are using the basic modal language ML($\Box$) and the standard predicate calculus with equality and a single binary predicate symbol r FOL. The standard definitions of *Kripke frame* and *Kripke model* apply. We use the well-known constructions *bounded morphic image* (*p-morphic image*), *generated subframe*, and *disjoint union* ($\biguplus$) (see definitions 5, 6, and 7) and their properties as in [8].

We are interested in the class of all KD45-frames, and this class is axiomatized with the popular KD45 normal modal logic. The *axioms of* KD45 are the following formulas:
(D) ($\Box p \to \Diamond p$) (right-unboundedness axiom)
(4) ($\Box p \to \Box\Box p$) (transitivity axiom)
(5) ($\Diamond p \to \Box\Diamond p$) (Euclidean axiom)

We say that a frame F is a KD45-frame iff F validates the axioms (4), (5), and (D).

The first-order correspondents of the KD45 axioms are:
(D') $\forall x \exists z_1 (x \; r \; z_1)$
($4'$) $\forall x \forall z_1 ((x \; r \; z_1) \to \forall z_2 ((z_1 \; r \; z_2) \to (x \; r \; z_2)))$
($5'$) $\forall x \forall y_1 ((x \; r \; y_1) \to \forall z_1 ((x \; r \; z_1) \to (z_1 \; r \; y_1)))$

We say that a Kripke frame $\mathrm{F} = \langle W, R\rangle$ is a *daisy* iff $W = P(\mathrm{F}) \cup S(\mathrm{F})$, where $P(\mathrm{F}) \cap S(\mathrm{F}) = \emptyset$, $S(\mathrm{F}) \neq \emptyset$, $P(\mathrm{F})$ is the set of *petals*, $S(\mathrm{F})$ is the set of *stamens*, and the following hold:
(Daisy 1). $\forall x \in P(\mathrm{F}) \neg \exists y \in W(\langle y, x\rangle \in R)$
(Daisy 2). $\forall x \in P(\mathrm{F}) \forall y \in S(\mathrm{F})(\langle x, y\rangle \in R)$
(Daisy 3). $\forall x \in S(\mathrm{F}) \forall y \in S(\mathrm{F})(\langle x, y\rangle \in R)$

It is also easy to see that each daisy is a KD45 frame ($\mathrm{F} \vDash (D') \wedge (4') \wedge (5')$).

Proposition 95 Let F be a KD45 frame. Then there is an index set I and a set of daisies $D = \{\mathrm{F}_i \mid i \in I\}$, such that $\mathrm{F} = \biguplus D$.

Proof Let $\mathrm{F} = \langle W, R\rangle$ be a KD45 frame. Let $\mathrm{F}' =_{def} \langle W, R'\rangle$, where $R' =_{def} \{\langle x, y\rangle \mid \exists z(\langle x, z\rangle \in R \;\&\; \langle y, z\rangle \in R)\}$.

We show that R' is an equivalence relation over W. By the definition of R', it is symmetric. By $\mathrm{F} \vDash (D')$, R' is reflexive. Let us see that R' is transitive. Suppose $\langle a, b\rangle \in R' \;\&\; \langle b, c\rangle \in R'$. Then a and b have a common R-successor x, and b and c have a common R-successor y. Since R is Euclidean, $\langle x, y\rangle \in R$ because they are both descendants of b. But R is also transitive, therefore $\langle a, y\rangle \in R$, thus a and c have y as a common R-successor. Thus $\langle a, c\rangle \in R'$ and R' is transitive. Therefore, R' is an equivalence relation.

We denote by $|w|$ the R'-equivalence class of $w \in W$. Let $I =_{def} \{|w| \mid w \in W\}$, let $\mathrm{F}_{|w|} =_{def} \langle |w|, R \cap (|w| \times |w|)\rangle$, and let $D = \{\mathrm{F}_i \mid i \in I\}$. Clearly, I is a non-empty set of disjoint subsets of W. To see that $\mathrm{F} = \biguplus D$, suppose for the sake of contradiction that there are two R'-equivalence classes $|w_1| \neq |w_2|$, such that for some $x \in |w_1|$ and some $y \in |w_2|$ we have that $\langle x, y\rangle \in R$. Because R is euclidean, y is reflexive and thus it is a common R-successor of both x and y, obtaining the contradiction $|w_1| = |w_2|$. We conclude that $\mathrm{F} = \biguplus D$.

It remains to show that for all $w \in W$, $\mathrm{F}_{|w|}$ is a daisy. Let $w \in W$ and let $\mathrm{F}_{|w|} = \langle |w|, R_{|w|}\rangle$. Let $P =_{def} \{x \in |w| \mid \langle x, x\rangle \notin R\}$, and let $S =_{def} \{x \in |w| \mid \langle x, x\rangle \in R\}$. Clearly, P and S are disjoint, and $P \cup S = |w|$. Also, because $F \Vdash (D') \wedge (5')$, $S \neq \emptyset$.

For (Daisy 1), suppose for the sake of contradiction that for some $x \in P$, there is a $y \in W$, such that $\langle y, x\rangle \in R_{|w|}$. Then $\langle y, x\rangle \in R$, and because $F \vDash (5')$, $\langle x, x\rangle \in R$, contradiction.

For (Daisy 3), let $x \in S$ and $y \in S$. Because both x and y are in $|w|$, they have a common R-successor z. Because R is Euclidean, z is reflexive and $z \in |w|$. But y is also reflexive, so by the fact that R is Euclidean, $\langle z, y\rangle \in R$ and by transitivity $\langle x, y\rangle \in R$. Because $\mathrm{F} = \biguplus D$, $\langle x, y\rangle \in R_{|w|}$

For (Daisy 2), let $x \in P$ and $y \in S$. Because both x and y are in $|w|$, then they have a common R-successor z. Because R is Euclidean, z is reflexive and $z \in |w|$. Moreover, $z \in S$. By (Daisy 3), $\langle z, y\rangle \in R$. By transitivity, $\langle x, y\rangle \in R$ and thus $\langle x, y\rangle \in R_{|w|}$. $\square$

4.1 First-order Definability

Now, some definitions. The class of all KD45-frames is denoted by $\mathcal{C}_{\mathrm{KD45}}$. The class of all S5 frames is denoted by $\mathcal{C}_{\mathrm{S5}}$. Easily, $\mathcal{C}_{\mathrm{S5}} \subsetneq \mathcal{C}_{\mathrm{KD45}}$. We denote by C_0 the class of finite daisies without petals, and C_1 for the class of finite daisies with a single petal. We denote by D_i the finite daisy with i stamens and no petals, and D'_i for the finite daisy with i stamens and a single petal. Clearly, D_i and D'_i are well-defined up to ismorphism of Kripke structures.

Lemma 96 Let A be a modal formula. Then $\mathcal{C}_{\mathrm{S5}} \Vdash A$ iff $C_0 \Vdash A$. Also $\mathcal{C}_{\mathrm{KD45}} \Vdash A$ iff $C_1 \Vdash A$.

Proof Let $\mathcal{C}_{\mathrm{S5}} \Vdash A$. $C_0 \subseteq \mathcal{C}_{\mathrm{S5}}$, so $C_0 \Vdash A$. Let $C_0 \Vdash A$. Let $\mathrm{F} \in \mathcal{C}_{\mathrm{S5}}$ and suppose that $\mathrm{F} \nVdash A$. Then $\mathcal{C}_{\mathrm{S5}} \nVdash A$. Because of the finite model property of $\mathcal{C}_{\mathrm{S5}}$, there is a finite frame $\mathrm{F}' \in \mathcal{C}_{\mathrm{S5}}$, such that $\mathrm{F}' \nVdash A$. Therefore, there is a state w from F', such that $\mathrm{F}', w \nVdash A$. Let F'' be the generated subframe (see Definition 7) of F' at w. Then $\mathrm{F}'', w \nVdash A$ (see [8]), so $\mathrm{F}'' \nVdash A$, but F'' is the equivalence class of w in F', so $\mathrm{F}'' \in C_0$, and thus $C_0 \nVdash A$, contradicts $C_0 \Vdash A$.

Let $\mathcal{C}_{\mathrm{KD45}} \Vdash A$. $C_1 \subseteq \mathcal{C}_{\mathrm{KD45}}$, so $C_1 \Vdash A$. Let $C_1 \Vdash A$. Suppose there is some $\mathrm{F} \in \mathcal{C}_{\mathrm{KD45}}$, such that $\mathrm{F} \nVdash A$. Because of the finite model property of KD45, there is a finite frame $\mathrm{F}' \in \mathcal{C}_{\mathrm{KD45}}$, such that $\mathrm{F}' \nVdash A$. Then there is a state w from F', such that $\mathrm{F}', w \nVdash A$. Let F'' be the generated subframe of

F' at w. Then $\mathrm{F}'', w \nVdash A$, so $\mathrm{F}'' \nVdash A$. F'' is a finite daisy with at most one petal. So there is a number $n > 1$, such that either w is reflexive, and then $\mathrm{F}'' = D_n \in C_0$, or w is not reflexive, and then $\mathrm{F}'' = D'_n \in C_1$. If F'' is D_n, then F'' is a p-morphic image of $D'_n \in C_1$. But $C_1 \Vdash A$, so $D'_n \Vdash A$, therefore $\mathrm{F}'' \Vdash A$, contradicts $\mathrm{F}'' \nVdash A$. Otherwise, F'' is $D'_n \in C_1$, and because $C_1 \Vdash A$, $\mathrm{F}'' \Vdash A$, contradicts $\mathrm{F}'' \nVdash A$. □

Lemma 97 Let A be a modal formula. Exactly one of the following three holds: either $\mathcal{C}_{\mathrm{S5}} \Vdash A$, $D_1 \nVdash A$, or there is a number $n > 1$, such that for all i: $D_i \Vdash A \Leftrightarrow i < n$. Exactly one of the following three holds: either $\mathcal{C}_{\mathrm{KD45}} \Vdash A$, $D'_1 \nVdash A$, or there is a number $n' > 1$, such that for all i: $D'_i \Vdash A \Leftrightarrow i < n'$. If such n and n' exist, then $n' \leq n$.

Proof For the first condition, consider the validity of A in the class C_0. There are three cases. First, let $C_0 \Vdash A$. By Lemma 96, this happens if and only if $\mathcal{C}_{\mathrm{S5}} \Vdash A$. Second, let for all $\mathrm{F} \in C_0 : \mathrm{F} \nVdash A$. Then because $D_1 \in C_0$, we have that $D_1 \nVdash A$. Now let $D_1 \nVdash A$ and suppose that for some $\mathrm{F} \in C_0 : \mathrm{F} \Vdash A$. But D_1 is a p-morphic image of F, so $D_1 \Vdash A$, contradicts $D_1 \nVdash A$. Third, let there be some frame $\mathrm{F}' \in C_0$, such that $\mathrm{F}' \Vdash A$, and let there be some frame $\mathrm{F}'' \in C_0$, such that $\mathrm{F}'' \nVdash A$. Let $n \geq 1$ be the first number, such that $D_n \nVdash A$. Because D_1 is a p-morphic image of F', $D_1 \Vdash A$, so $n \neq 1$, therefore $n > 1$. Moreover, if $i \geq n$, D_n is a p-morphic image of D_i, so $D_i \nVdash A$, because suppose otherwise. Then $D_n \Vdash A$, contradicts $D_n \nVdash A$. Now, let $i < n$. Because n is the first number, such that $D_n \nVdash A$, it follows that $D_i \Vdash A$. Therefore, there is a number $n > 1$ and for all i: $D_i \Vdash A \Leftrightarrow i < n$. Now, suppose that there is a number $n > 1$, such that for all i: $D_i \Vdash A \Leftrightarrow i < n$. Then $D_1 \Vdash A$ and $D_n \nVdash A$.

For the second condition, consider the validity of A in the class C_1. There are again three cases. First, let $C_1 \Vdash A$. By Lemma 96, this happens if and only if $\mathcal{C}_{\mathrm{KD45}} \Vdash A$. Second, let for all $\mathrm{F} \in C_1 : \mathrm{F} \nVdash A$. Then because $D'_1 \in C_1$, we have that $D'_1 \nVdash A$. Now let $D'_1 \nVdash A$ and suppose that for some $\mathrm{F} \in C_1 : \mathrm{F} \Vdash A$. But D'_1 is a p-morphic image of F, so $D'_1 \Vdash A$, contradicts $D'_1 \nVdash A$. Third let there be some frame $\mathrm{F}' \in C_1$, such that $\mathrm{F}' \Vdash A$, and let there be some frame $\mathrm{F}'' \in C_1$, such that $\mathrm{F}'' \nVdash A$. Let $n \geq 1$ be the first number, such that $D'_n \nVdash A$. Because D'_1 is a p-morphic image of F', $D'_1 \Vdash A$, so $n \neq 1$, thus $n > 1$. Now, if $i \geq n$, then D'_n is a p-morphic image of D'_i, and suppose that $D'_i \Vdash A$, then $D'_n \Vdash A$, contradicts $D'_n \nVdash A$. Let $i < n$. Then because n is the first number, such that $D'_n \nVdash A$, we have that $D'_i \Vdash A$. Therefore, there is a number $n > 1$ and for all i: $D'_i \Vdash A \Leftrightarrow i < n$. Now, suppose that there is a number $n > 1$ and for all i: $D'_i \Vdash A \Leftrightarrow i < n$. Then $D_1 \Vdash A$ and $D_n \nVdash A$.

Finally, let such n and n' exist. Then $n' \leq n$ by the properties of p-morphic images. □

For $n \geq 1$, we denote $\psi_n(x) =_{def} \forall y_1 \ldots \forall y_n (\bigwedge\{(x\ r\ y_k) \mid 1 \leq k \leq n\} \rightarrow$

$\bigvee\{(y_k = y_l) \mid 1 \leq k < l \leq n\})$. Clearly, for any KD45 frame F, F $\vDash \psi_n(x)$ iff every daisy from F has less than n stamens.

Theorem 98 *Let A be a modal formula. Then there is a first-order formula ψ, such that A and ψ are globally correspondent over the class of frames $\mathcal{C}_{\mathrm{KD45}}$. Also, ψ can be effectively computed.*

Proof We note here the following: Because KD45 is finitely (hence recursively) axiomatizable, canonical (hence complete) and because KD45 has the finite model property, the problem of $\mathcal{C}_{\mathrm{KD45}} \Vdash A$ is decidable. The same applies to the problem of $\mathcal{C}_{\mathrm{S5}} \Vdash A$.

Now, we show an algorithm for finding the first-order correspondent of A over $\mathcal{C}_{\mathrm{KD45}}$, thus proving both points of the theorem.

First we show first-order definitions of A over the classes C_0 and C_1 separately, then we use these definitions to form the definition of A over the class $\mathcal{C}_{\mathrm{KD45}}$.

First, the definition of A over C_0. Because of Lemma 97, there are 3 cases:

1. If $\mathcal{C}_{\mathrm{S5}} \Vdash A$, then $\psi_{\mathcal{C}_0} =_{def} \top$ is a definition of A over C_0 and let $i =_{def} 1$.

2. If $D_1 \nVdash A$, then $\psi_{\mathcal{C}_0} =_{def} \bot$ is a definition of A over C_0 and let $i =_{def} 2$.

3. Otherwise, there is a number $n > 1$, such that for all m: $D_m \Vdash A \Leftrightarrow m < n$. Therefore, it is easy to check that the formula $\psi_{\mathcal{C}_0} =_{def} \psi_n$ is a definition of A over C_0 and let $i =_{def} 3$.

Now, the definition of A over C_1. Because of Lemma 97, there are 3 cases:

1. If $\mathcal{C}_{\mathrm{KD45}} \Vdash A$, then $\psi_{\mathcal{C}_1} =_{def} \top$ is a definition of A over $\mathcal{C}_1$ and let $j =_{def} 1$.

2. If $D'_1 \nVdash A$, then $\psi_{\mathcal{C}_1} =_{def} \bot$ is a definition of A over $\mathcal{C}_1$ and let $j =_{def} 2$.

3. Otherwise, there is a number $n' > 1$, such that for all m: $D'_m \Vdash A \Leftrightarrow m < n'$. Therefore, it is easy to check that the formula $\psi_{\mathcal{C}_1} =_{def} \psi_{n'}$ is a definition of A over $\mathcal{C}_1$ and let $j =_{def} 3$.

Let $\psi =_{def} \forall x(((x\ r\ x) \wedge \psi_{\mathcal{C}_0}) \vee (\neg(x\ r\ x) \wedge \psi_{\mathcal{C}_1}))$.

Let F $\in \mathcal{C}_{\mathrm{KD45}}$, we now show that F $\vDash \psi$ iff F $\Vdash A$, by examining the cases of the pair i, j:

1. $i = 1, j = 1$. $\psi \equiv \top$, so F $\vDash \psi$. Because $j = 1$, $\mathcal{C}_{\mathrm{KD45}} \Vdash A$, so F $\Vdash A$.

2. $i = 1, j = 2$. $\psi \equiv \forall x(x\ r\ x)$.

2.1. F $\in \mathcal{C}_{\mathrm{S5}}$. Then, F $\vDash \psi$ and because $i = 1$, F $\Vdash A$.

2.2. F $\notin \mathcal{C}_{\mathrm{S5}}$. Then F $\nvDash \psi$. Suppose F $\Vdash A$. But D'_1 is a p-morphic image of F, so $D'_1 \Vdash A$, contradicts $j = 2$.

3. $i = 1, j = 3$. $\psi \equiv \forall x((x\ r\ x) \vee (\neg(x\ r\ x) \wedge \psi_{n'}))$.

3.1. F $\in \mathcal{C}_{\mathrm{S5}}$. Then F $\vDash \forall x(x\ r\ x)$, so F $\vDash \psi$. But $i = 1$, so $\mathcal{C}_{\mathrm{S5}} \Vdash A$, so F $\Vdash A$.

3.2. F $\notin \mathcal{C}_{\mathrm{S5}}$.

3.2.1. F $\vDash \psi$. Suppose F $\nVdash A$. Then there is a state w from F, such that F$, w \nVdash A$. Let F$'$ be the generated subframe F$'$ of F at w, we have that F$', w \nVdash$

A. Suppose $\mathrm{F}' \in \mathcal{C}_{\mathrm{S5}}$, then because $i = 1$, $\mathrm{F}' \Vdash A$, contradiction. So $\mathrm{F}' \notin \mathcal{C}_{\mathrm{S5}}$, so because $\mathrm{F} \vDash \psi$, F' is a daisy with $< n'$ stamens and exactly 1 petal. Therefore there is a number $m < n'$, such that F' is isomorphic to D'_m, and because of $j = 3$, $D'_m \Vdash A$, so $\mathrm{F}' \Vdash A$, contradiction.

3.2.2. $\mathrm{F} \Vdash A$. Suppose $\mathrm{F} \nvDash \psi$. Then $\mathrm{F} \vDash \exists x(\neg(x\ r\ x) \wedge ((x\ r\ x) \vee \neg\psi_{n'}))$. Then there is a state w from F, such that w is a petal and w has $\geq n'$ R-successors. Then $D'_{n'}$ is a p-morphic image of the generated subframe of F at w, so $D'_{n'} \Vdash A$, however by $j = 3$, $D'_{n'} \nVdash A$, contradiction.

4. $i = 2, j = 1$. Impossible.

5. $i = 2, j = 2$. $\psi \equiv \bot$, so $\mathrm{F} \nvDash \psi$. Suppose $\mathrm{F} \Vdash A$. But D_1 is a p-morphic image of F, so $D_1 \Vdash A$, however by $i = 2$, $D_1 \nVdash A$, contradiction, therefore $\mathrm{F} \nVdash A$.

6. $i = 2, j = 3$. Impossible.

7. $i = 3, j = 1$. Impossible.

8. $i = 3, j = 2$. $\psi \equiv \forall x((x\ r\ x) \wedge \psi_n)$.

8.1. $\mathrm{F} \vDash \psi$. Then $\mathrm{F} \in \mathcal{C}_{\mathrm{S5}}$. Suppose $\mathrm{F} \nVdash A$. Then there is a state w from F, such that $\mathrm{F}, w \nVdash A$. Let F' be the generated subframe of F at w, we have that $\mathrm{F}' \nVdash A$. Because $\mathrm{F} \vDash \psi$, $\mathrm{F} \vDash \psi_n$ and therefore F' is an equivalence class with $< n$ stamens. Therefore, $\mathrm{F}' \in C_0$ and there is some $m < n$, such that F' is isomorphic to D_m. Therefore, $D_m \nVdash A$, but $i = 3$, so $D_m \Vdash A$, contradiction.

8.2. $\mathrm{F} \Vdash A$. Suppose $\mathrm{F} \nvDash \psi$. Then $\mathrm{F} \vDash \exists x(\neg(x\ r\ x) \vee \neg\psi_n)$.

8.2.1. There is some w from F, such that w is not reflexive. Then D'_1 is a p-morphic image of the generated subframe of F at w, so $D'_1 \Vdash A$, however by $j = 2$, $D'_1 \nVdash A$, contradiction.

8.2.2. All states of F are reflexive. Then $\mathrm{F} \vDash \exists x(\neg\psi_n)$, so there is some state w from F, such that w has $\geq n$ R-descendants. Then D_n is a p-morphic image of the generated subframe of F at w, therefore $D_n \Vdash A$, but by $i = 3$, $D_n \nVdash A$, contradiction.

9. $i = 3, j = 3$.

9.1. $\mathrm{F} \vDash \psi$. Suppose $\mathrm{F} \nVdash A$. Then there is a state w of F, such that $\mathrm{F}, w \nVdash A$. Let F' be the generated subframe of F at w, we have that $\mathrm{F}' \nVdash A$. Because $\mathrm{F} \vDash \psi$, F' is a finite daisy with at most one petal.

9.1.1. w is a petal of F'. Because $\mathrm{F} \vDash \psi$, we have that $\mathrm{F} \vDash \psi_{n'}[w]$, so w has $< n'$ R-descendants. So there is a number $m < n'$, such that F' is isomorphic to D'_m, so $D'_m \nVdash A$, but by $j = 3$, $D'_m \vDash A$, contradiction.

9.1.2. w is a stamen of F'. Because $\mathrm{F} \vDash \psi$, we have that $\mathrm{F} \vDash \psi_n[w]$, so w has $< n$ R-descendants. So there is a number $m < n$, such that F' is isomorphic to D_m, so $D_m \nVdash A$, but by $i = 3$, $D_m \Vdash A$, contradiction.

9.2. $\mathrm{F} \Vdash A$. Suppose $\mathrm{F} \nvDash \psi$. Then $\mathrm{F} \vDash \exists x((\neg(x\ r\ x) \vee \neg\psi_n) \wedge ((x\ r\ x) \vee \neg\psi_{n'}))$, so there is a state w of F, such that $\mathrm{F} \vDash ((\neg(x\ r\ x) \vee \neg\psi_n) \wedge ((x\ r\ x) \vee \neg\psi_{n'}))[w]$. Let F' be the generated subframe of F at w, then $\mathrm{F}' \Vdash A$.

9.2.1. $F \models (\neg(x\ r\ x) \wedge \neg\psi_{n'})[w]$. Then F' is a daisy with one petal and $\geq n'$ stamens. Therefore, $D'_{n'}$ is a p-morphic image of F', so $D'_{n'} \Vdash A$. But by $j = 3$, $D'_{n'} \nVdash A$, contradiction.

9.2.2. $F \models ((x\ r\ x) \wedge \neg\psi_n)[w]$. Then F' is a daisy with no petals and $\geq n$ stamens. Therefore, D_n is a p-morphic image of F', so $D_n \Vdash A$. But by $i = 3$, $D_n \nVdash A$, contradiction.

9.2.3. $F \models (\neg\psi_n \wedge \neg\psi_{n'})[w]$. Then there are two cases. Either $F \models (x\ r\ x)[w]$, in which case the proof is the same as in the case 9.2.2 above, or $F \models \neg(x\ r\ x)[w]$, where the proof is the same as in the case 9.2.1. above. $\square$

4.2 Modal Definability

Let $F \in \mathcal{C}_{\text{KD45}}$. Then there is a set of daisies D, such that $F = \biguplus D$. Let $Q =_{def} \{Card(S(x)) \mid x \in D\}$ and let $s(F) =_{def} sup(Q)$. Let $Q_0 =_{def} \{Card(S(x)) \mid x \in D \ \&\ P(x) = \emptyset\}$ and let $s_0(F) =_{def} sup(Q_0)$. Let $Q_1 =_{def} \{Card(S(x)) \mid x \in D \ \&\ P(x) \neq \emptyset\}$ and let $s_1(F) =_{def} sup(Q_1)$. We denote the class of frames $F \in \mathcal{C}_{\text{KD45}}$ such that $1 \leq s(F) < \omega$ by C^b.

Definition 99 (Restriction of a Daisy) Let F and F' be daisies and let $k > 0$. We say that F' is a *restriction of* F *to at most* k *petals and at most* k *stamens* iff F' has either k petals if F has at least k petals, or has the same number of petals as F otherwise, and F' has either k stamens if F has at least k stamens, or has the same number of stamens as F otherwise. Clearly, up to isomorphism, for a given daisy F and $k > 0$, there is a single daisy, which is the restriction of F to at most k petals and at most k stamens. We denote this daisy by $F \upharpoonright_1 k$ and we say that it is the *restriction of* F *to* k

Definition 100 (Restriction) Let F be a KD45-frame and let $k > 0$. Let D be the unique set of daisies (up to isomorphism), such that $F = \biguplus D$. We say that the following frame is the *restriction of* F *to* k: $F \upharpoonright_1 k =_{def} \biguplus\{F' \upharpoonright_1 k \mid F' \in D\}$.

Lemma 101 Let $F \in \mathcal{C}_{\text{KD45}}$ and let $n > 0$. Then there is a frame F' in C^b, such that: 1. for all FOL sentences ψ with quantifier rank n, $F \models \psi$ iff $F' \models \psi$. 2. $s_0(F') \leq min(s_0(F), n)$ and $s_1(F') \leq min(s_1(F), n)$. 3. F' is a p-morphic image of F.

Proof Let $F' =_{def} F \upharpoonright_1 n$ (see Definition 100), then $s(F') \leq n$ and $F' \in C^b$. We use the Ehrenfeucht-Fraïssé method, see chapter 1 of [22]. Clearly, the duplicator wins the Ehrenfeucht-Fraïssé game $G_n(F, F')$. By Ehrenfeucht's theorem, $F \equiv_n F'$ (for every sentence $\psi \in$ FOL with quantifier rank n, $F \models \psi$ iff $F' \models \psi$). By the definition of F', $s_0(F') \leq s_0(F)$, $s_0(F') \leq n$, $s_1(F') \leq s_1(F)$, $s_1(F') \leq n$, and F' is a p-morphic image of F. $\square$

Lemma 102 Let $\mathrm{F}_0 \in C^b, \mathrm{F}_1 \in C^b, \mathrm{F}' \in C^b$. If $s_0(\mathrm{F}_0) \geq s_0(\mathrm{F}')$ and $s_1(\mathrm{F}_1) \geq s_1(\mathrm{F}')$, then for every sentence $\psi \in \mathrm{FOL}$, if ψ is modally definable in $\mathrm{ML}(\Box)$ over $\mathcal{C}_{\mathrm{KD45}}$, $\mathrm{F}_0 \vDash \psi$, $\mathrm{F}_1 \vDash \psi$, then $\mathrm{F}' \vDash \psi$.

Proof Suppose $\mathrm{F}_0 \vDash \psi$ and $\mathrm{F}_1 \vDash \psi$, but $\mathrm{F}' \nvDash \psi$. Let A be a modal definition of ψ over $\mathcal{C}_{\mathrm{KD45}}$. Then $\mathrm{F}_0 \Vdash A$, $\mathrm{F}_1 \Vdash A$, and $\mathrm{F}' \nVdash A$, so there is some state w of F', such that $\mathrm{F}', w \nVdash A$, so let F'' be the generated subframe of F' at w. Then $\mathrm{F}'', w \nVdash A$, so $\mathrm{F}'' \nVdash A$. There are two cases for F''. First, let w be reflexive. Then $s_0(\mathrm{F}'') \leq s_0(\mathrm{F}') \leq s_0(\mathrm{F}_0)$, so F'' is a p-morphic image of a daisy of F_0, then $\mathrm{F}'' \Vdash A$, contradiction. Now, let w be a petal. Then $s_1(\mathrm{F}'') \leq s_1(\mathrm{F}') \leq s_1(\mathrm{F}_1)$. Then F'' is a p-morphic image of a daisy of F_1, so $\mathrm{F}'' \Vdash A$, contradiction. $\Box$

For $n > 0$, we denote $A_n =_{def} \bigwedge\{\Diamond p_k \mid 1 \leq k \leq n\} \to \bigvee\{\Diamond(p_i \wedge p_j) \mid 1 \leq i < j \leq n\}$. It can easily be verified that for any KD45 frame F and any state w from F, $\mathrm{F}, w \Vdash A_n$ iff w has less than n R-descendants. Therefore, for all $n > 0$, $\psi_n(x)$ and A_n are locally correspondent with respect to $\mathcal{C}_{\mathrm{KD45}}$. This implies that $\forall x \psi_n(x)$ and A_n are globally correspondent with respect to $\mathcal{C}_{\mathrm{KD45}}$.

We assume that q is a variable, which does not occur in any A_n.

Lemma 103 For all i and j such that $1 \leq i \leq j$, $\forall x(((x\ r\ x) \wedge \psi_j) \vee (\neg(x\ r\ x) \wedge \psi_i))$ is globally correspondent to $((q \to \Diamond q) \wedge A_j) \vee A_i$ with respect to $C_0 \cup C_1$.

Proof Let $\mathrm{F} = \langle W, R\rangle \in C_0 \cup C_1$.

First, we show that $\mathrm{F} \vDash \forall x(((x\ r\ x) \wedge \psi_j) \vee (\neg(x\ r\ x) \wedge \psi_i))$ iff $\mathrm{F} \Vdash A_i$ or $\mathrm{F} \Vdash ((q \to \Diamond q) \wedge A_j)$.

1. Let $\mathrm{F} \vDash \forall x(((x\ r\ x) \wedge \psi_j) \vee (\neg(x\ r\ x) \wedge \psi_i))$.

1.1. F has a petal w. Then $\mathrm{F} \vDash \psi_i[w]$, so F has $< i$ stamens. Then $\mathrm{F} \vDash \forall x \psi_i$, so $\mathrm{F} \Vdash A_i$.

1.2. F has no petals. Let w be any state of F. Then $\mathrm{F} \vDash \psi_j[w]$, so F has $< j$ stamens. Then $\mathrm{F} \vDash \forall x \psi_j$, so $\mathrm{F} \Vdash A_j$, but F has no petals ($\mathrm{F} \Vdash (q \to \Diamond q)$), so $\mathrm{F} \Vdash ((q \to \Diamond q) \wedge A_j)$.

2. Let $\mathrm{F} \Vdash A_i$. Then F has $< i$ stamens, so $\mathrm{F} \vDash \psi_i$. But $i \leq j$, so $\mathrm{F} \vDash \psi_j$. Then $\mathrm{F} \vDash \forall x(((x\ r\ x) \wedge \psi_j) \vee (\neg(x\ r\ x) \wedge \psi_i))$.

3. Let $\mathrm{F} \Vdash ((q \to \Diamond q) \wedge A_j)$. Then $\mathrm{F} \Vdash (q \to \Diamond q)$ and $\mathrm{F} \Vdash A_j$. Then $\mathrm{F} \vDash \forall x(x\ r\ x)$ and $\mathrm{F} \Vdash \forall x \psi_j$. So $\mathrm{F} \vDash \forall x(((x\ r\ x) \wedge \psi_j) \vee (\neg(x\ r\ x) \wedge \psi_i))$.

It remains to show that for $\mathrm{F} \Vdash A_i$ or $\mathrm{F} \Vdash ((q \to \Diamond q) \wedge A_j)$ iff $\mathrm{F} \Vdash ((q \to \Diamond q) \wedge A_j) \vee A_i$.

The left-to-right direction is obvious. Let $\mathrm{F} \Vdash ((q \to \Diamond q) \wedge A_j) \vee A_i$ and suppose that $\mathrm{F} \nVdash A_i$ and $\mathrm{F} \nVdash ((q \to \Diamond q) \wedge A_j)$. Then there are states w_1 and w_2 from F, such that $\mathrm{F}, w_1 \nVdash A_i$ and $\mathrm{F}, w_2 \nVdash ((q \to \Diamond q) \wedge A_j)$. Therefore, w_1 has at least i R-descendants, and w_2 is not reflexive or has at least j R-descendants.

Now, we have that $\mathrm{F} \Vdash A_j \vee A_i$ and $\mathrm{F} \Vdash (q \to \Diamond q) \vee A_i$, but this means that $\mathrm{F} \Vdash A_j$ (because $i \leq j$), so w_2 has less than j R-descendents, therefore w_2 is not reflexive. First, $\mathrm{F}, w_2 \nVdash (q \to \Diamond q)$. Second, $\mathrm{F}, w_2 \nVdash A_i$ because the stamens of F are at least i. Because q does not occur in A_i, $\mathrm{F}, w_2 \nVdash (q \to \Diamond q) \vee A_i$. This contradicts $\mathrm{F} \Vdash (q \to \Diamond q) \vee A_i$. □

Lemma 104 Let for some $1 \leq i \leq j$, $\forall x(((x\ r\ x) \wedge \psi_j) \vee (\neg(x\ r\ x) \wedge \psi_i))$ be globally correspondent to some modal formula A with respect to $C_0 \cup C_1$. Then $\forall x(((x\ r\ x) \wedge \psi_j) \vee (\neg(x\ r\ x) \wedge \psi_i))$ is globally correspondent to A with respect to $\mathcal{C}_{\mathrm{KD45}}$.

Proof Analogous to case 9 in the proof of Theorem 98. □

Lemma 105 For all $1 \leq i \leq j$, $\forall x(((x\ r\ x) \wedge \psi_j) \vee (\neg(x\ r\ x) \wedge \psi_i))$ is globally correspondent to $((q \to \Diamond q) \wedge A_j) \vee A_i$ with respect to $\mathcal{C}_{\mathrm{KD45}}$.

Proof Follows by Lemmas 103 and 104. □

We denote $A_0 =_{def} \bot$, $A_\omega =_{def} \top$, $\psi_0 =_{def} \bot$, $\psi_\omega =_{def} \top$.

Lemma 106 Let ψ be a FOL sentence. Then ψ is modally definable in $\mathrm{ML}(\Box)$ over $\mathcal{C}_{\mathrm{KD45}}$ iff there are ordinals σ_0, σ_1 such that $0 \leq \sigma_1 \leq \sigma_0 \leq \omega$ and for every $\mathrm{F} \in C^b$: $\mathrm{F} \vDash \psi$ iff $s_0(\mathrm{F}) \leq \sigma_0$ and $s_1(\mathrm{F}) \leq \sigma_1$.

Proof ("only if") Let ψ be modally definable in $\mathrm{ML}(\Box)$ over $\mathcal{C}_{\mathrm{KD45}}$. Let $S_0 =_{def} \{s_0(\mathrm{F}) \mid \mathrm{F} \in C^b \;\&\; \mathrm{F} \vDash \psi\}$, $S_1 =_{def} \{s_1(\mathrm{F}) \mid \mathrm{F} \in C^b \;\&\; \mathrm{F} \vDash \psi\}$. Let $\sigma_0 =_{def} sup(S_0)$ which is an ordinal even when $S_0 = \emptyset$. Let $\sigma_1 =_{def} sup(S_1)$ which is an ordinal even when $S_1 = \emptyset$. By the definition of C^b, $0 \leq \sigma_0 \leq \omega$ and $0 \leq \sigma_1 \leq \omega$.

Suppose $\sigma_1 > \sigma_0$. Then there is some frame $\mathrm{F}' \in C^b$, such that $\mathrm{F}' \in S_1$, and $s_1(\mathrm{F}') > \sigma_0$, because suppose otherwise. Then $\sigma_1 \leq \sigma_0$, contradicts $\sigma_1 > \sigma_0$. Also, $\mathrm{F}' \vDash \psi$. Let F'' be F' without its petals. Then F'' is a p-morphic image of F'. Let A be a modal definition of ψ, then $\mathrm{F}' \Vdash A$, so $\mathrm{F}'' \Vdash A$ and $\mathrm{F}'' \vDash \psi$. But $s(\mathrm{F}'') = s(\mathrm{F}')$, so $\mathrm{F}'' \in C^b$, so $s_0(\mathrm{F}'') \leq \sigma_0$. But $s_0(\mathrm{F}'') = s_1(\mathrm{F}')$, so $\sigma_0 < s_1(\mathrm{F}') = s_0(\mathrm{F}'') \leq \sigma_0$, contradiction. We conclude that $\sigma_0 \leq \sigma_1$.

Let $\mathrm{F} \in C^b$. First, let $\mathrm{F} \vDash \psi$. By the definition of S_0 and S_1, $s_0(\mathrm{F}) \leq \sigma_0$ and $s_1(\mathrm{F}) \leq \sigma_1$. Now let $s_0(\mathrm{F}) \leq \sigma_0$ and $s_1(\mathrm{F}) \leq \sigma_1$. Then there are frames $\mathrm{F}_0 \in S_0$, such that $s_0(\mathrm{F}) \leq s_0(\mathrm{F}_0)$ and $\mathrm{F}_1 \in S_1$, such that $s_1(\mathrm{F}) \leq s_1(\mathrm{F}_1)$. By Lemma 102, $\mathrm{F} \vDash \psi$.

("if") Let there be ordinals σ_0, σ_1, such that $0 \leq \sigma_1 \leq \sigma_0 \leq \omega$ and for every $\mathrm{F} \in C^b$: $\mathrm{F} \vDash \psi$ iff $s_0(\mathrm{F}) \leq \sigma_0$ and $s_1(\mathrm{F}) \leq \sigma_1$.

If $\sigma_0 < \omega$, let $\alpha_0 =_{def} \sigma_0 + 1$, otherwise let $\alpha_0 =_{def} \sigma_0$. If $\sigma_1 < \omega$, let $\alpha_1 =_{def} \sigma_1 + 1$, otherwise let $\alpha_1 =_{def} \sigma_1$. Let $A =_{def} ((q \to \Diamond q) \wedge A_{\alpha_0}) \vee A_{\alpha_1}$. We show that for all $\mathrm{F} \in \mathcal{C}_{\mathrm{KD45}}$, $\mathrm{F} \vDash \psi$ iff $\mathrm{F} \Vdash A$.

1. $\sigma_0 = \omega, \sigma_1 = \omega$. Then $A \equiv \top$. Suppose that $\mathrm{F} \not\vDash \psi$. Then $\mathrm{F} \vDash \neg\psi$ and by Lemma 101, there is a frame $\mathrm{F}' \in C^b$, such that $\mathrm{F}' \vDash \neg\psi$, so $\mathrm{F}' \not\vDash \psi$. But because $\mathrm{F}' \in C^b$, $s_0(\mathrm{F}') < \omega$ and $s_1(\mathrm{F}') < \omega$, so $\mathrm{F}' \vDash \psi$, contradiction.

2. $\sigma_0 = \omega, \sigma_1 = 0$. Then $A \equiv (q \to \Diamond q)$.

2.1. $\mathrm{F} \in \mathcal{C}_{\mathrm{S5}}$. Then $\mathrm{F} \Vdash A$. Suppose $\mathrm{F} \not\vDash \psi$. Then $\mathrm{F} \vDash \neg\psi$ and by Lemma 101, there is a frame $\mathrm{F}' \in C^b$, such that $\mathrm{F}' \vDash \neg\psi$, $s_0(\mathrm{F}') < \omega$ and $s_1(\mathrm{F}') \leq s_1(\mathrm{F}) = 0$. So $\mathrm{F}' \vDash \psi$, contradicts $\mathrm{F}' \vDash \neg\psi$.

2.2. $\mathrm{F} \notin \mathcal{C}_{\mathrm{S5}}$. Then $\mathrm{F} \not\Vdash A$. Suppose $\mathrm{F} \vDash \psi$. Then $\mathrm{F} \vDash \exists x \neg(x\ r\ x) \wedge \psi$. By Lemma 101, there is a frame $\mathrm{F}' \in C^b$, such that $\mathrm{F}' \vDash \exists x \neg(x\ r\ x) \wedge \psi$. Then $\mathrm{F}' \vDash \psi$ and $s_1(\mathrm{F}') > 0$, so $\mathrm{F}' \not\vDash \psi$, contradiction.

3. $\sigma_0 = \omega, 0 < \sigma_1 < \omega$. Then $A \equiv (q \to \Diamond q) \vee A_{\alpha_1}$.

3.1. $\mathrm{F} \in \mathcal{C}_{\mathrm{S5}}$. Then $\mathrm{F} \Vdash (q \to \Diamond q)$, so $\mathrm{F} \Vdash A$. Suppose $\mathrm{F} \not\vDash \psi$. Then $\mathrm{F} \vDash \neg\psi$ and by Lemma 101, there is a frame $\mathrm{F}' \in C^b$, such that $\mathrm{F}' \vDash \neg\psi$, $s_0(\mathrm{F}') < \omega$ and $s_1(\mathrm{F}') \leq s_1(\mathrm{F}) \leq \sigma_1$. So $\mathrm{F}' \vDash \psi$, contradicts $\mathrm{F}' \vDash \neg\psi$.

3.2. $\mathrm{F} \notin \mathcal{C}_{\mathrm{S5}}$.

3.2.1. $\mathrm{F} \Vdash A$. Suppose $\mathrm{F} \not\vDash \psi$, so $\mathrm{F} \vDash \neg\psi$. By Lemma 101, there is a p-morphic image of F, F′, such that $\mathrm{F}' \in C^b$, $s_0(\mathrm{F}') < \omega$, and $\mathrm{F}' \vDash \neg\psi$, so $\mathrm{F}' \not\vDash \psi$. Then $\mathrm{F}' \Vdash A$, but $s_1(\mathrm{F}') \geq \alpha_1 > \sigma_1 > 0$. By the definition of $s_1(\mathrm{F}')$, F′ has a petal, w, which has $\geq \alpha_1$ R-descendants. Then $\mathrm{F}', w \not\Vdash (q \to \Diamond q)$ and $\mathrm{F}', w \not\Vdash A_{\alpha_1}$. Because q does not occur in A_{α_1}, $\mathrm{F}', w \not\Vdash A$.

3.2.2. $\mathrm{F} \vDash \psi$. Suppose $\mathrm{F} \not\Vdash A$. So there is some state w of F, such that $\mathrm{F}, w \not\Vdash A$. So $\mathrm{F}, w \not\Vdash (q \to \Diamond q)$ and $\mathrm{F}, w \not\Vdash A_{\alpha_1}$. Then w is not reflexive and w has $> \sigma_1$ R-descendants. So $\mathrm{F} \vDash \psi \wedge \exists x(\neg(x\ r\ x) \wedge \psi_{\alpha_1})$. By Lemma 101, there is a frame $\mathrm{F}' \in C^b$, such that $\mathrm{F}' \vDash \psi \wedge \exists x(\neg(x\ r\ x) \wedge \psi_{\alpha_1})$. Then, $\mathrm{F}' \vDash \psi$, but $s_1(\mathrm{F}') > \sigma_1$, so $\mathrm{F}' \not\vDash \psi$, contradiction.

4. $\sigma_0 = 0, \sigma_1 = \omega$, contradicts $\sigma_1 \leq \sigma_0$.

5. $\sigma_0 = 0, \sigma_1 = 0$. Then $A \equiv \bot$, so $\mathrm{F} \not\Vdash A$. Suppose $\mathrm{F} \vDash \psi$. By Lemma 101, there is a frame $\mathrm{F}' \in C^b$, such that $\mathrm{F}' \vDash \psi$, but then both $s_0(\mathrm{F}') \leq 0$ and $s_1(\mathrm{F}') \leq 0$, so $s(\mathrm{F}') = 0$, impossible.

6. $\sigma_0 = 0, 0 < \sigma_1 < \omega$. contradicts $\sigma_1 \leq \sigma_0$.

7. $0 < \sigma_0 < \omega, \sigma_1 = \omega$. contradicts $\sigma_1 \leq \sigma_0$.

8. $0 < \sigma_0 < \omega, \sigma_1 = 0$. Then $A \equiv (q \to \Diamond q) \wedge A_{\alpha_0}$.

8.1. $\mathrm{F} \Vdash A$. Then $\mathrm{F} \Vdash (q \to \Diamond q)$, so $\mathrm{F} \in \mathcal{C}_{\mathrm{S5}}$, so $s_1(\mathrm{F}) = 0$. Suppose $\mathrm{F} \not\vDash \psi$, then $\mathrm{F} \vDash \neg\psi$. By Lemma 101, there is a p-morphic image F′ of F, such that $\mathrm{F}' \in C^b$, $s_1(\mathrm{F}') \leq s_1(\mathrm{F}) = 0$, and $\mathrm{F} \vDash \neg\psi$, so $s_0(\mathrm{F}) > \sigma_0$ and therefore $s(\mathrm{F}) > \sigma_0$. But $\mathrm{F}' \Vdash A_{\alpha_0}$, so every state in F′ has at most σ_0 successors, contradiction.

8.2. $\mathrm{F} \vDash \psi$. Suppose $\mathrm{F} \not\Vdash A$.

8.2.1. $\mathrm{F} \not\Vdash (q \to \Diamond q)$. Then F has a non-reflexive state, so $\mathrm{F} \vDash \exists x \neg(x\ r\ x) \wedge \psi$. By Lemma 101, there is a frame $\mathrm{F}' \in C^b$, such that $\mathrm{F}' \vDash \exists x \neg(x\ r\ x) \wedge \psi$. Then $\mathrm{F}' \vDash \psi$ and $s_1(\mathrm{F}') > 0$, so $\mathrm{F}' \not\vDash \psi$, contradiction.

8.2.2. F $\not\Vdash A_{\alpha_0}$. Then F has some state with more than σ_0 successors. So F $\vDash \psi \wedge \exists x(\neg\psi_{\alpha_0})$. By Lemma 101, there is a frame $\mathrm{F}' \in C^b$, such that $\mathrm{F}' \vDash \psi \wedge \exists x(\neg\psi_{\alpha_0})$, so $s_0(\mathrm{F}') \leq \sigma_0$ and $s_1(\mathrm{F}') \leq \sigma_1 = 0$. Then $s(\mathrm{F}') > \sigma_0$. Then either $s_0(\mathrm{F}') > \sigma_0$, contradicts $s_0(\mathrm{F}') \leq \sigma_0$, or $s_1(\mathrm{F}') > \sigma_0 > 0$, contradicts $s_1(\mathrm{F}') = 0$.

9. $0 < \sigma_0 < \omega, 0 < \sigma_1 < \omega$. Then $A \equiv ((q \to \Diamond q) \wedge A_{\alpha_0}) \vee A_{\alpha_1}$

9.1. Let F $\Vdash A$. Suppose F $\nvDash \psi$, so F $\vDash \neg\psi$. By Lemma 101, there is a p-morphic image F$'$ of F, such that $\mathrm{F}' \in C^b$ and $\mathrm{F}' \vDash \neg\psi$, so $\mathrm{F}' \nvDash \psi$ and $\mathrm{F}' \Vdash A$.

9.1.1. $s_0(\mathrm{F}') > \sigma_0$. So there is a finite daisy without petals in F$'$, D, which has at least α_0 stamens. Because D is a generated subframe of F$'$, $D \Vdash A$. Let $M = \langle D, V\rangle$ be a model defined as follows: $V(q) = \emptyset$, and for all k, such that $1 < k \leq \alpha_0$, $V(p_k)$ are distinct singletons from D. Such a valuation is possible because D has at least α_0 stamens. Clearly, $M \not\Vdash A$, contradiction.

9.1.2. $s_1(\mathrm{F}') > \sigma_1$. So there is a finite daisy D' in F at least one petal w and at least α_1 stamens. Let D be the generated subframe of F$'$ at w. Then $D \Vdash A$. Let $M = \langle D, V\rangle$ be a model defined as follows: $V(q) = \{w\}$, and for all k, such that $1 < k \leq \alpha_1$, $V(p_k)$ are distinct singletons of stamens of D. Such a valuation is possible because D has at least α_1 stamens. Clearly, $M \not\Vdash A$, contradiction.

9.2. Let F $\vDash \psi$. Suppose F $\not\Vdash A$. Then by Lemma 105, F $\nvDash \forall x(((x\ r\ x) \wedge \psi_{\alpha_0}) \vee (\neg(x\ r\ x) \wedge \psi_{\alpha_1}))$. Then F $\vDash \psi \wedge \exists x(\neg((x\ r\ x) \wedge \psi_{\alpha_0}) \wedge \neg(\neg(x\ r\ x) \wedge \psi_{\alpha_1}))$. By Lemma 101, there is a frame $\mathrm{F}' \in C^b$, such that $\mathrm{F}' \vDash \psi \wedge \exists x(\neg((x\ r\ x) \wedge \psi_{\alpha_0}) \wedge \neg(\neg(x\ r\ x) \wedge \psi_{\alpha_1}))$. Then $\mathrm{F}' \vDash \psi$ and there is some world w from F$'$ such that $\mathrm{F}' \vDash \neg((x\ r\ x) \wedge \psi_{\alpha_0})[w]$ and $\mathrm{F}' \vDash \neg(\neg(x\ r\ x) \wedge \psi_{\alpha_1})[w]$, so $\mathrm{F}' \Vdash ((x\ r\ x) \to \neg\psi_{\alpha_0})[w]$ and $\mathrm{F}' \Vdash (\neg(x\ r\ x) \to \neg\psi_{\alpha_1})[w]$.

9.2.1. w is reflexive. Then $\mathrm{F}' \vDash \neg\psi_{\alpha_0}[w]$. If w is part of a daisy with a petal, then $s_1(\mathrm{F}') > \sigma_0$. Otherwise, $s_0(\mathrm{F}') > \sigma_0$. Because $s_1(\mathrm{F}') \leq s_0(\mathrm{F}')$, in both cases we have $s_0(\mathrm{F}') > \sigma_0$, contradicts $\mathrm{F}' \vDash \psi$.

9.2.2. w is not reflexive. Then $\mathrm{F}' \vDash \neg\psi_{\alpha_1}[w]$. Then $s_1(\mathrm{F}') > \sigma_1$, contradicts $\mathrm{F}' \vDash \psi$. □

Definition 107 (Restriction to at most k Daisies of the Same Kind)

Let F and F$'$ be KD45-frames and let $k > 0$. Let D be the unique set of daisies, up to isomorphism, such that $\mathrm{F} = \biguplus D$, and let D' be the unique set of daisies, up to isomorphism, such that $\mathrm{F}' = \biguplus D'$. We say that F$'$ is the *restriction of* F *to at most k daisies of the same kind* iff for any daisy $x \in D$, if D has at least k isomorphic copies of x then D' contains k isomorphic copies of x, otherwise D' contains the same number of isomorphic copies of x as D. Clearly, up to isomorphism, there is a single KD45-frame which is the restriction of F to at most k daisies of the same kind. We denote this frame by $\mathrm{F} \upharpoonright_2 k$.

Let C_{fin} be the class of all finite KD45 frames.

Let $k > 0$. We denote by C^k_{fin} the class of finite KD45-frames F such that every daisy in F has at most k petals and k stamens and F contains at most k isomorphic copies of every daisy.

Proposition 108 Let F be a $\mathcal{C}_{\mathrm{KD45}}$-frame and let $k \geq 0$. Then there is a frame $\mathrm{F}' \in C^k_{fin}$, such that: 1. for all FOL sentences ψ with quantifier rank k: $\mathrm{F} \models \psi$ iff $\mathrm{F}' \models \psi$. 2. F' is a p-morphic image of F.

Proof Let $\mathrm{F}'' =_{def} \mathrm{F} \upharpoonright_1 k$ (see Definition 100). This is similar to the proof of the weaker statement in Lemma 101. Let $\mathrm{F}' =_{def} \mathrm{F}'' \upharpoonright_2 k$ (see Definition 107). Clearly, F' is finite and $\mathrm{F}' \in C_{fin}$. Also, clearly F' is a p-morphic image of F'', and F'' is a p-morphic image of F, so F' is a p-morphic image of F. Clearly, the duplicator wins the Ehrenfeucht-Fraïssé game $G_n(\mathrm{F}, \mathrm{F}')$. By Ehrenfeucht's theorem, $\mathrm{F}' \equiv_k \mathrm{F}$. $\square$

Proposition 109 Let ψ be a sentence with quantifier rank k. Then $\mathcal{C}_{\mathrm{KD45}} \models \psi$ iff $C^k_{fin} \models \psi$.

Proof Similar to the proof of Proposition 108. $\square$

Corollary 110 Let $\psi \in$ FOL be a sentence. The problem of deciding $\mathcal{C}_{\mathrm{KD45}} \models \psi$ is PSPACE-complete.

Proof A theorem by Stockmeyer in [51] shows that the complexity of deciding the validity of a first-order formula with equality as its only predicate symbol is PSPACE-complete. We also use the fact that the first-order theory we are considering is a conservative extension of the theory for $=$, to conclude that the problem is PSPACE-hard. To see that the problem is in PSPACE, we use Proposition 109. $\square$

Theorem 111 *The problem of modal definability of a* FOL *sentence* ψ *in* ML($\Box$) *over the class* $\mathcal{C}_{\mathrm{KD45}}$ *is in PSPACE.*

Proof Let m' be the quantifier rank of ψ and let $m =_{def} m' + 1$. Let $Q =_{def} \{\psi_0, \psi_1, \ldots, \psi_m, \psi_\omega\}$, which has $m + 2$ elements.

If there are $\alpha_0, \alpha_1 \in \{0, 1, \ldots, m, \omega\}$, such that $\psi_{\alpha_0} \in Q$, $\psi_{\alpha_1} \in Q$, $0 \leq \alpha_1 \leq \alpha_0 \leq \omega$ and $\mathcal{C}_{\mathrm{KD45}} \models \psi \leftrightarrow \forall x(((x\ r\ x) \wedge \psi_{\alpha_0}) \vee (\neg(x\ r\ x) \wedge \psi_{\alpha_1}))$, then for all $\mathrm{F} \in C^b$, $\mathrm{F} \models \psi$ iff $s_0(\mathrm{F}) < \alpha_0$ and $s_1(\mathrm{F}) < \alpha_1$. Then there are ordinals σ_0 and σ_1, such that $0 \leq \sigma_0 \leq \sigma_1 \leq \omega$, and for all $\mathrm{F} \in C^b$, $\mathrm{F} \models \psi$ iff $s_0(\mathrm{F}) \leq \sigma_0$ and $s_1(\mathrm{F}) \leq \sigma_1$. Then by Lemma 106, ψ is modally definable in ML($\Box$) over $\mathcal{C}_{\mathrm{KD45}}$.

Now, let there be no $\alpha_0, \alpha_1 \in \{0, 1, \ldots, m, \omega\}$, such that $\psi_{\alpha_0} \in Q$, $\psi_{\alpha_1} \in Q$, $0 \leq \alpha_1 \leq \alpha_0 \leq \omega$ and $\mathcal{C}_{\mathrm{KD45}} \models \psi \leftrightarrow \forall x(((x\ r\ x) \wedge \psi_{\alpha_0}) \vee (\neg(x\ r\ x) \wedge \psi_{\alpha_1}))$.

Suppose ψ is modally definable in ML($\Box$) over $\mathcal{C}_{\mathrm{KD45}}$. Then by Lemma 106, there are some ordinals σ_0 and σ_1, such that $0 \leq \sigma_1 \leq \sigma_0 \leq \omega$ and for all $\mathrm{F} \in C^b$: $\mathrm{F} \models \psi$ iff $s_0(\mathrm{F}) \leq \sigma_0$ and $s_1(\mathrm{F}) \leq \sigma_1$. Then there are ordinals α_0 and α_1, such that $0 \leq \alpha_1 \leq \alpha_0 \leq \omega$ and for all $\mathrm{F} \in C^b$: $\mathrm{F} \models \psi$ iff $s_0(\mathrm{F}) < \alpha_0$ and $s_1(\mathrm{F}) < \alpha_1$. Then $\mathcal{C}_{\mathrm{KD45}} \models \psi \leftrightarrow \forall x(((x\ r\ x) \wedge \psi_{\alpha_0}) \vee (\neg(x\ r\ x) \wedge \psi_{\alpha_1}))$. But $\alpha_0, \alpha_1 \notin \{0, \omega\} \subseteq \{0, 1, \ldots, m, \omega\}$, so therefore $\mathcal{C}_{\mathrm{KD45}} \not\models \psi \leftrightarrow \bot$ and $\mathcal{C}_{\mathrm{KD45}} \not\models \psi \leftrightarrow \top$. Let $\mathrm{F} \in \mathcal{C}_{\mathrm{KD45}}$ be such that $\mathrm{F} \not\models \psi$, then $\mathrm{F} \models \neg\psi$. Let $\mathrm{F}' =_{def} \mathrm{F} \upharpoonright_1 m$ (see Definition 100). Then the duplicator wins the Ehrenfeucht-Fraïssé game $G_m(\mathrm{F}, \mathrm{F}')$, so $\mathrm{F} \equiv_m \mathrm{F}'$. Because the quantifier rank of $\neg\psi$ is $< m$, $\mathrm{F}' \not\models \psi$. Clearly $\mathrm{F}' \in C^b$, and also $s_0(\mathrm{F}') \leq m < \alpha_0$ and $s_1(\mathrm{F}') \leq m < \alpha_1$, which implies $\mathrm{F}' \models \psi$, contradicts $\mathrm{F}' \not\models \psi$.

By Corollary 110, the problem of $C_{\mathrm{KD45}} \models \gamma$ for FOL formulas γ is in PSPACE. Therefore, the problem of modal definability of FOL sentences in ML($\Box$) over the class C_{KD45} is also in PSPACE. $\Box$

Definition 112 (Relativized Reduct) Let F, F_0 be structures for some first-order language L. We say that F_0 is *a relativized reduct of* F if there exist a formula $\psi(\bar{x}, x) \in L$ and a list $\bar{s}$ of worlds in F such that F_0 is the restriction of F to the set of all worlds s in F such that $\mathrm{F} \models \psi(\bar{x}, x)[\bar{s}, s]$. In this case, F_0 is called *the relativized reduct of* F *with respect to* $\psi(\bar{x}, x)$ *and* $\bar{s}$.

Definition 113 (Stable Classes of Frames) Let $\mathcal{C}$ be a class of frames. We say that $\mathcal{C}$ is *stable* iff there is a FOL formula $\gamma_1(\bar{x}, x)$ and a FOL sentence γ_2, such that:

(a) for all frames F in $\mathcal{C}$, for all lists $\bar{s}$ of worlds in F, and for all frames F', if F' is the relativized reduct of F with respect to $\gamma_1(\bar{x}, x)$ and $\bar{s}$, then F' is in $\mathcal{C}$,

(b) for all frames F_0 in $\mathcal{C}$, there are frames F, F' in $\mathcal{C}$ and there is a list $\bar{s}$ of worlds from F, such that F_0 is the relativized reduct of F with respect to $\gamma_1(\bar{x}, x)$ and $\bar{s}$, $\mathrm{F} \models \gamma_2$, $\mathrm{F}' \not\models \gamma_2$, and for all ML($\Box$) formulas A: if $\mathrm{F} \Vdash A$, then $\mathrm{F}' \Vdash A$.

Theorem 114 *If $\mathcal{C}$ is a stable class of frames, then the problem of deciding the validity of* FOL *sentences in $\mathcal{C}$ is reducible to the problem of deciding the modal definability of* FOL *sentences in* ML($\Box$) *with respect to $\mathcal{C}$.*

Proof See the proof in [3]. $\Box$

Note that by examining the proof of the above theorem in [3], it becomes clear that in these cases, the validity problem is polynomially reducible to the modal definability problem.

Proposition 115 $\mathcal{C}_{\mathrm{KD45}}$ is a stable class.

Proof Let $\gamma_1(x_1, x_2, x) =_{def}$
$(\exists z((z\ r\ z) \wedge (z \neq x_1) \wedge (z \neq x_2))$
$\wedge \forall z((z\ r\ x_1) \vee (x_1\ r\ z) \vee (z\ r\ x_2) \vee (x_2\ r\ z) \to (z = x_1) \vee (z = x_2)))$
$\to ((x \neq x_1) \wedge (x \neq x_2))$.

It is clear that for any frame $\mathrm{F} \in \mathcal{C}_{\mathrm{KD45}}$ and any states w_1, w_2 from F, there is a relativized reduct of F with respect to $\gamma_1(x_1, x_2, x)$ and w_1, w_2 and it is in $\mathcal{C}_{\mathrm{KD45}}$, because we can only remove a whole daisy.

Let $\gamma_2 =_{def} \exists x \neg(x\ r\ x)$.

Now, let $\mathrm{F}_0 \in \mathcal{C}_{\mathrm{KD45}}$, we find $\mathrm{F} \in \mathcal{C}_{\mathrm{KD45}}$, states w_0 and w_1 from F, and $\mathrm{F}' \in \mathcal{C}_{\mathrm{KD45}}$, such that F_0 is the relativized reduct of F with respect to $\gamma_1(x_1, x_2, x)$ and w_1, w_2, F′ is a p-morphic image of F, $\mathrm{F} \vDash \gamma_2$, and $\mathrm{F}' \nvDash \gamma_2$, which proves that $\mathcal{C}_{\mathrm{KD45}}$ is a stable class.

Let D be an isomorphic copy of D'_1, which is disjoint with F_0, and let $\mathrm{F} =_{def} \mathrm{F}_0 \uplus D$. Let w_1 and w_2 be the two states of D in any order. Clearly, F_0 is the relativized reduct of F with respect to $\gamma_1(x_1, x_2, x)$ and w_1, w_2. Let F′ be the restriction of F to all its stamens, removing the petals. Clearly, F′ is a p-morphic image of F, $\mathrm{F} \vDash \gamma_2$ because of the petal of D, and $\mathrm{F}' \nvDash \gamma_2$. $\square$

Corollary 116 The problem of modal definability of FOL formulas in ML($\Box$) over $\mathcal{C}_{\mathrm{KD45}}$ is PSPACE-hard.

Proof A theorem by Stockmeyer in [51] shows that the complexity of deciding the validity of a first-order formula with equality as its only predicate symbol is PSPACE-complete. Clearly, this problem is polynomially reducible to validity of FOL sentences in the class of all KD45-frames. By Proposition 115 and Theorem 114 (with the remark after it), the latter problem is polynomially reducible to the problem of modal definability of FOL sentences in the language ML($\Box$) over C_{KD45}. Therefore, problem of modal definability of FOL sentences in ML($\Box$) over C_{KD45} is PSPACE-hard.

5 ML($\Box, [U]$) and $\mathcal{C}_{\mathrm{KD45}}$

With the language ML($\Box, [U]$), the KD45 axioms and frames are the same as with ML($\Box$). In the context of the language ML($\Box, [U]$), the operation disjoint union loses its meaning and well-known properties. Still, using the symbol $\uplus$ for disjoint union as in the context of ML($\Box$), we can see that Proposition 95 holds in the context of ML($\Box, [U]$).

5.1 Modal Definability

Let $\mathrm{F} \in C_{\mathit{fin}}$, m be the maximal number of petals in a daisy in F, and n be the maximal number of stamens in a daisy in F. The *pattern* of F is the matrix: $\begin{bmatrix} x_{01} & x_{02} & x_{03} & \dots & x_{0n} \\ x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & x_{m3} & \dots & x_{mn} \end{bmatrix}$, where for all i, j such that $0 \leq i \leq m$ and $1 \leq j \leq n$, x_{ij} is the number of daisies in F with i petals and j stamens.

Let $\mathrm{F} \in C_{\mathit{fin}}$, m be the maximal number of petals in a daisy in F, and n be the maximal number of stamens in a daisy in F and $\mathcal{P}$ be the pattern of F. Let i, j be such that $0 \leq i \leq m$ and $1 \leq j \leq n$. Let x be x_{ij} of $\mathcal{P}$. The *Jankov-Fine* formula for $\langle i, j, x\rangle$, $A(d, p, t)_{\langle i,j,x\rangle}$, similarly to [8], pp. 144–145, and [2], is:

$$\bigwedge_{\substack{1\leq q\leq i \\ 1\leq r\leq x}} \langle U\rangle p_q^r \wedge \bigwedge_{\substack{1\leq q\leq j \\ 1\leq r\leq x}} \langle U\rangle t_q^r \wedge \bigwedge_{1\leq r\leq x} \langle U\rangle d_r \wedge \bigwedge_{1\leq q<r\leq x} [U]\neg(d_q \wedge d_r)\wedge$$

$$\bigwedge_{\substack{1\leq k<q\leq i \\ 1\leq r\leq x}} [U]\neg(p_k^r \wedge p_q^r) \wedge \bigwedge_{\substack{1\leq k<q\leq j \\ 1\leq r\leq x}} [U]\neg(t_k^r \wedge t_q^r) \wedge \bigwedge_{\substack{1\leq k\leq i \\ 1\leq q\leq j \\ 1\leq r\leq x}} [U]\neg(p_k^r \wedge t_q^r)\wedge$$

$$\bigwedge_{1\leq r\leq x} [U](d_r \leftrightarrow p_1^r \vee \dots \vee p_i^r \vee t_1^r \vee \dots \vee t_j^r)\wedge$$

$$\bigwedge_{\substack{1\leq q\leq i \\ 1\leq r\leq x}} [U](p_q^r \to \Box\neg p_q^r) \wedge \bigwedge_{\substack{1\leq k\leq i \\ 1\leq q\leq j \\ 1\leq r\leq x}} [U](p_k^r \to \Diamond t_q^r) \wedge \bigwedge_{\substack{1\leq k\leq j \\ 1\leq q\leq j \\ 1\leq r\leq x}} [U](t_k^r \to \Diamond t_q^r)\wedge$$

$$\bigwedge_{1\leq r\leq x} [U](d_r \leftrightarrow \Box(\bigvee_{1\leq q\leq j} t_q^r))$$

where $p_1^1, \dots, p_i^1, \dots, p_1^x, \dots, p_i^x, t_1^1, \dots, t_j^1, \dots, t_1^x, \dots, t_j^x, d_1, \dots, d_x$ be pairwise distinct parametrized variables.

Proposition 117 Let $i \geq 0$, $j \geq 1$, $x \geq 0$. Let $A =_{def} A(d, p, t)_{\langle i,j,x\rangle}$ be the Jankov-Fine formula for $\langle i, j, x\rangle$. Then for all frames $\mathrm{F} \in \mathcal{C}_{\mathrm{KD45}}$: A is satisfiable in F iff F contains $\geq x$ daisies with $\geq i$ petals and $\geq j$ stamens.

Proof First, let $\mathrm{M} = \langle F, V\rangle$ and let $\mathrm{M} \Vdash A$.

If $x = 0$, then the proposition is trivially true, so let $x > 0$.

Because $\mathrm{M} \Vdash \bigwedge_{1\leq r\leq x} \langle U\rangle d_r \wedge \bigwedge_{1\leq q<r\leq x} [U]\neg(d_q \wedge d_r)$, then $V(d_1), \dots, V(d_x)$ are pairwise non-intersecting non-empty sets.

Because $\mathrm{M} \Vdash \bigwedge_{\substack{1\leq q\leq i \\ 1\leq r\leq x}} \langle U\rangle p_q^r \wedge \bigwedge_{\substack{1\leq q\leq j \\ 1\leq r\leq x}} \langle U\rangle t_q^r$, then the valuations of all p-variables, if any, and all t-variables of A are non-empty sets. Because $\mathrm{M} \Vdash \bigwedge_{\substack{1\leq k<q\leq i \\ 1\leq r\leq x}} [U]\neg(p_k^r \wedge p_q^r)$, $\mathrm{M} \Vdash \bigwedge_{\substack{1\leq k<q\leq j \\ 1\leq r\leq x}} [U]\neg(t_k^r \wedge t_q^r)$, and $\mathrm{M} \Vdash \bigwedge_{\substack{1\leq k\leq i \\ 1\leq q\leq j \\ 1\leq r\leq x}} [U]\neg(p_k^r \wedge t_q^r)$, these valuations are also pairwise non-intersecting.

Because $\mathrm{M} \Vdash \bigwedge_{1 \le r \le x} [U](d_r \leftrightarrow p_1^r \vee \cdots \vee p_i^r \vee t_1^r \vee \cdots \vee t_j^r)$, for all r such that $1 \le r \le x$, $V(d_r) = V(p_1^r) \cup \cdots \cup V(p_i^r) \cup V(t_1^r) \cup \ldots V(t_j^r)$.

Because $\mathrm{M} \Vdash \bigwedge_{\substack{1 \le q \le i \\ 1 \le r \le x}} [U](p_q^r \to \Box\neg p_q^r)$, then there cannot be any reflexive states in any valuation of the p-variables of A, if any, so all these valuations contain only petals.

Because $\mathrm{M} \Vdash \bigwedge_{\substack{1 \le k \le i \\ 1 \le q \le j \\ 1 \le r \le x}} [U](p_k^r \to \Diamond t_q^r) \wedge \bigwedge_{\substack{1 \le k \le j \\ 1 \le q \le j \\ 1 \le r \le x}} [U](t_k^r \to \Diamond t_q^r)$, then the valuations of all t-variables of A contain descendant states, and because all the variables of A valuate to non-empty sets, the valuations of all t-variables of A contain at least one stamen each. This also implies that each of the valuations $V(d_r)$ contains at least i distinct petals and j distinct stamens, which are from the same daisy.

Finally, because of the last conjunct of A, for all r such that $1 \le r \le x$, all descendants of states in any valuation of the d_r-variables of A are in some valuation of a corresponding r-superscripted t-variable of A. Then clearly all descendants of states of each $V(d_r)$ are only among $V(d_r)$. Therefore suppose that there are two d-variables of A, $d_{r'}$ and $d_{r''}$, such that $V(d_{r'})$ and $V(d_{r''})$ both contain states from the same daisy D of F and let w be a stamen of D. Then clearly by the above, $w \in V(d_{r'})$ and $w \in V(d_{r''})$, but $V(d_{r'}) \cap V(d_{r''}) = \emptyset$, contradiction. Therefore, each $V(d_r)$ contain only states from different daisies.

Then F contains $\ge x$ daisies with $\ge i$ petals and $\ge j$ stamens.

Now, let F contain $\ge x$ daisies with $\ge i$ petals and $\ge j$ stamens. Let $D_1, \ldots, D_x$ be x such daisies, let for all r such that $1 \le r \le x$, $w_1^r, \ldots, w_i^r$ be i distinct petals of D_r (if they exist) and $v_1^r, \ldots, v_j^r$ be j distinct stamens of D_r. We form a valuation V in this way: for all $1 \le r \le x$, $1 \le q \le i$, $1 \le k \le j$: $V(d_r) =_{def} |D_r|$, $V(p_q^r) =_{def} \{w_q^r\}$, $V(t_k^r) =_{def} \{v_k^r\}$. Let $\mathrm{M} =_{def} \langle \mathrm{F}, V \rangle$, clearly $\mathrm{M} \Vdash A$. $\Box$

Let $i \ge 0$, $j > 0$, and $x \ge 0$. Denote $Q' =_{def} \{\langle i_0, x_0 \rangle \mid 0 \le i_0 \le i \ \& \ 0 \le x_0 \le i\}$ and let $Q =_{def} \{y \mid y \in \mathbb{P}(Q') \ \& \ \Sigma_{\langle i_0, x_0 \rangle \in y} i_0.x_0 \ge i\}$. The *extended Jankov-Fine formula* for $\langle i, j, x \rangle$, $\underline{A}(d, p, t)_{\langle i,j,x \rangle}$ is:

$$\bigwedge_{k=1}^{x} \bigvee_{y \in Q} \bigwedge_{\substack{\langle i_0, x_0 \rangle \in y \\ param = \langle k, y, i_0, x_0, i, j, x \rangle}} A(d_{param}, p_{param}, t_{param})_{\langle i_0, j, x_0 \rangle}$$

where for all finite sets of numbers y and for all numbers k, i_0, x_0, i, j, x, all possible parameters $param = \langle k, y, i_0, x_0, i, j, x \rangle, d_{param}, p_{param}, t_{param}$ are pairwise-distinct.

Let $\mathrm{G} \in C_{fin}$, let m be the maximum number of petals per daisy in G and n be the maximum number of stamens per daisy in G. Let $\mathcal{P}$ be the pattern of

G and let for all $\langle i, j, x\rangle$, such that x is x_{ij} of P, $A_{ij} =_{def} \underline{A}(d, p, t)_{\langle i,j,x\rangle}$ be the extended Jankov-Fine formula for $\langle i, j, x\rangle$, where the sets of variables of each A_{ij} are pairwise non-intersecting. Let $A' =_{def} \bigwedge_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} A_{ij}$, let Q_1 be the set of all d-variables of A', and let Q_2 be the set of all disjuncts of all A_{ij}.

$$A =_{def} A' \wedge$$

$$\bigwedge_{\substack{d_1,d_2\in Q_1\\ A_1,A_2\in Q_2\\ d_1 \not\nearrow d_2, d_1\nearrow A_1, d_2\nearrow A_2}} ((A_1 \wedge A_2) \rightarrow [U]\neg(d_1 \wedge d_2)) \wedge [U](\bigvee_{\substack{d\in Q_1, A\in Q_2\\ d\nearrow A}} (A \wedge d))$$

We say that A is the *Jankov-Fine* formula of G.

Proposition 118 Let $\mathrm{F} \in \mathcal{C}_{\mathrm{KD45}}$, $\mathrm{G} \in C_{\mathit{fin}}$. Let A be the Jankov-Fine formula of G. Then $\mathrm{F} \Vdash \neg A$ iff G is not a p-morphic image of F.

Proof We show that A is satisfiable in F iff G is a p-morphic image of F.

First, let A be satisfiable in F. So there is a valuation V and a model $\mathrm{M} = \langle \mathrm{F}, V\rangle$, such that $\mathrm{M} \Vdash A$. Let $\mathcal{P}$ be the pattern of G, let m be the maximal number of petals and n be the maximal number of stamens per daisy in G. Then for all x_{ij} from P, $\mathrm{M} \Vdash A_{ij}$, where A_{ij} is the extended Jankov-Fine formula for $\langle i, j, x_{ij}\rangle$. Each A_{ij} is a conjunction of disjunctions, so for all conjuncts of A_{ij}, there is some disjunct, which holds in M. Let A' be the conjunction of all such disjuncts from all A_{ij}. Clearly, $\mathrm{M} \Vdash A'$. By Proposition 117, and by the definition of A_{ij}, F contains a p-morphic pre-image for each daisy of G, and by the last two conjuncts of A, the parts of these pre-images (the V-valuations of all d-variables of A') are non-intersecting and contain all states of F. Therefore, G is a p-morphic image of F.

Now, let G be a p-morphic image of F and let f be one such p-morphism.

We need two easy to prove lemmas. First, let D be a daisy of G. Then $f^{-1}[D]$ is a union of daisies of F. The second lemma states that whenever D is a daisy of G, which is the image of a union of some daisies of F: $\{\mathrm{F}_i \mid i \in I\}$, then for all $i \in I$: $|S(D)| \leq |S(\mathrm{F}_i)|$ and $|P(D)| \leq \Sigma_{i\in I}|P(\mathrm{F}_i)|$.

Let $\mathrm{F}_1, \ldots, \mathrm{F}_n$ be all daisies of G. Let $\mathrm{F}'_1, \ldots, \mathrm{F}'_n$ be their corresponding pre-images from F. Note that these pre-images contain all states of F. By the lemmas, each F'_k is a union of daisies, where the cardinality of stamens per daisy of F'_k is $\geq |S(\mathrm{F}_k)|$, and where cardinality of all petals of all daisies in F'_k is $\geq |P(\mathrm{F}_k)|$. It is clear how to define a valuation over F which satisfies A. $\square$

For simplicity, when discussing patterns of frames from C^k_{fin}, we only consider $(k + 1) \times k$ matrices.

Proposition 119 Let ψ be a sentence with quantifier rank k and modally definable by a formula A from $\mathrm{ML}(\Box,[U])$ in $\mathcal{C}_{\mathrm{KD45}}$. If there is some $\mathrm{F} \in C^k_{\mathit{fin}}$ with pattern $\mathcal{P}_1 = \begin{bmatrix} x_{01} & \dots & x_{0(j-1)} & k & x_{0(j+1)} & & x_{0k} \\ x_{11} & \dots & & & & & x_{1k} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{k1} & \dots & & & & & x_{kk} \end{bmatrix}$, such that $1 \leq j \leq k$, and $\mathrm{F} \models \psi$, then there is a frame $\mathrm{F}' \in C^k_{\mathit{fin}}$ with a pattern $\mathcal{P}_2 = \begin{bmatrix} k & \dots & k & k & x_{0(j+1)} & & x_{0k} \\ x_{11} & \dots & & & & & x_{1k} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{k1} & \dots & & & & & x_{kk} \end{bmatrix}$, such that $\mathrm{F}' \models \psi$.

Proof Let $\mathrm{F}' \in C^k_{\mathit{fin}}$ be with pattern $\mathcal{P}_2$. Let F'' be obtained from F by adding $(k+2)^2$ new daisies, each of them with 0 petals and j stamens. By Ehrenfeucht's theorem, $\mathrm{F}'' \equiv_k \mathrm{F}$, so $\mathrm{F}'' \models \psi$. So $\mathrm{F}'' \Vdash A$. But F' is a p-morphic image of F'', so $\mathrm{F}' \Vdash A$ and $\mathrm{F}' \models \psi$. $\Box$

Proposition 120 Let ψ be a sentence with quantifier rank k and modally definable by a formula A from $\mathrm{ML}(\Box,[U])$ in $\mathcal{C}_{\mathrm{KD45}}$. If there is some $\mathrm{F} \in C^k_{\mathit{fin}}$ with pattern $\mathcal{P}_1 = \begin{bmatrix} x_{01} & \dots & \dots & x_{0j} & x_{0(j+1)} & \dots x_{0k} \\ \vdots & \dots & \ddots & & & \dots \vdots \\ \vdots & \dots & & x_{(i-1)j} & x_{(i-1)(j+1)} & \dots \vdots \\ \vdots & \dots & x_{i(j-1)} & k & x_{i(j+1)} & \dots \vdots \\ \vdots & \dots & & x_{(i+1)j} & x_{(i+1)(j+1)} & \dots \vdots \\ \vdots & \dots & & & \vdots & \dots \vdots \\ x_{k1} & \dots & \dots & x_{kj} & x_{k(j+1)} & \dots x_{kk} \end{bmatrix}$, such that $1 \leq i \leq k$, $1 \leq j \leq k$, and $\mathrm{F} \models \psi$, then there is a frame $\mathrm{F}' \in C^k_{\mathit{fin}}$ with a pattern $\mathcal{P}_2 = \begin{bmatrix} k & \dots & \dots & k & x_{0(j+1)} & \dots x_{0k} \\ \vdots & \dots & \ddots & \vdots & & \dots \vdots \\ \vdots & \dots & & k & x_{(i-1)(j+1)} & \dots \vdots \\ \vdots & \dots & k & k & x_{i(j+1)} & \dots \vdots \\ \vdots & \dots & & k & x_{(i+1)(j+1)} & \dots \vdots \\ \vdots & \dots & & \vdots & \vdots & \dots \vdots \\ k & \dots & k & k & x_{k(j+1)} & \dots x_{kk} \end{bmatrix}$, such that $\mathrm{F}' \models \psi$.

Proof Let $\mathrm{F}' \in C^k_{\mathit{fin}}$ be with pattern P_2. Let F'' be obtained from F by adding $(k+2)^4$ new daisies, each of them with i petals and j stamens. By Ehrenfeucht's theorem, $F'' \equiv_k F$, so $F'' \vDash \psi$. So $F'' \Vdash A$. But F' is a p-morphic image of F'', so $F' \Vdash A$ and $F' \vDash \psi$. $\square$

Let ψ be a sentence and $k > 0$. We denote by $C^k_{fin}(\psi)$ the class of all frames $F \in C^k_{fin}$, such that $F \vDash \psi$.

Theorem 121 *Let ψ be a sentence with quantifier rank k, such that $\mathcal{C}_{\mathrm{KD45}} \nvDash \psi$ and $\mathcal{C}_{\mathrm{KD45}} \nvDash \neg\psi$. Then ψ is modally definable over $\mathcal{C}_{\mathrm{KD45}}$ with a formula of $\mathrm{ML}(\Box, [U])$ iff $C^k_{fin}(\psi)$ satisfies the following conditions: (1) $\emptyset \neq C^k_{fin}(\psi) \neq C^k_{fin}$ and $C^k_{fin}(\psi)$ is closed under p-morphisms; (2) $C^k_{fin}(\psi)$ is closed under the pattern transformations described in Proposition 119 and Proposition 120.*

Proof Let ψ be modally definable over $\mathcal{C}_{\mathrm{KD45}}$ with a formula of $\mathrm{ML}(\Box, [U])$. Then (1) holds by Proposition 109, and (2) follows by Proposition 119 and Proposition 120.

Now, let (1) and (2) hold. Because of (1), $\emptyset \subsetneq C^k_{fin}(\psi) \subsetneq C^k_{fin}$, so there is a frame $F_0 \in C^k_{fin}$, such that $F_0 \notin C^k_{fin}(\psi)$, so $F_0 \in C^k_{fin}(\neg\psi)$, so $C^k_{fin}(\neg\psi) \neq \emptyset$. For all frames $F_0 \in C^k_{fin}(\neg\psi)$, take their Jankov-Fine formulas A_{F_0}, and take A_ψ to be the conjunction of the negations of all such formulas. We show that for all frames $F \in \mathcal{C}_{\mathrm{KD45}}$:

$$F \vDash \psi \Leftrightarrow F \Vdash A_\psi$$

i.e. that A_ψ modally defines ψ over $\mathcal{C}_{\mathrm{KD45}}$.

First, let $F \in \mathcal{C}_{\mathrm{KD45}}$ and let $F \Vdash A_\psi$. Let $F'' =_{def} F \restriction_1 k$ (see Definition 100). Let $F' =_{def} F'' \restriction_2 k$ (see Definition 107). By Ehrenfeucht's theorem, $F' \equiv_k F$, and F' is finite, so $F' \in C^k_{fin}$. Also, F' is an p-morphic image of F, so $F' \Vdash A_\psi$. By the construction of A_ψ, and by Proposition 118, no frame of $C^k_{fin}(\neg\psi)$ is a p-morphic image of F', so $F' \in C^k_{fin}(\psi)$. Because $F' \equiv_k F$, $F \vDash \psi$.

Now, let $F \in \mathcal{C}_{\mathrm{KD45}}$ and let $F \vDash \psi$. Suppose for the sake of contradiction that $F \nVdash A_\psi$. So there is a valuation V over F and a model $M = \langle F, V \rangle$, such that $M \nVdash A_\psi$. Then we can find a frame $G \in C^k_{fin}(\neg\psi)$, such that $M \nVdash \neg A_G$, so $M \Vdash A_G$. By Proposition 118, G is a p-morphic image of F.

Let $F' =_{def} F \restriction_1 k$ (see Definition 100). Then by Ehrenfeucht's theorem, $F' \vDash \psi$, and also it is not hard to see that G is a p-morphic image of F'.

Now, let $F'' =_{def} F' \restriction_2 k$ (see Definition 107). Clearly, $F'' \equiv_k F'$, $F'' \vDash \psi$, $F'' \in C^k_{fin}(\psi)$.

It remains to show that G is a p-morphic image of some frame $F_0 \in C^k_{fin}(\psi)$. Then by (1), $G \in C^k_{fin}(\psi)$, contradicts $G \in C^k_{fin}(\neg\psi)$.

Let $\mathcal{P}$ be the pattern of F''.

If F'' is F', then we are done.

Otherwise, clearly there are indices i_0, j_0, such that $x_{i_0 j_0}$ of $\mathcal{P}$ is k. Let i_1, j_1 be such that $x_{i_1 j_1} = k$, j_1 is the largest such second index, and i_1 is the largest first index, such that the second index is j_1. If there is such a pair of indices i_0, j_0, such that $x_{i_0 j_0} = k$ and $i_0 > 0$, then let i_2, j_2 be one such pair, where j_2 is the largest second index, such that $i_2 > 0$; otherwise, let j_2 be j_1 and i_2 be i_1.

Case 1. $i_1 = 0$.

Case 1.1. $j_2 = j_1$. Then $i_2 = 0$. Let $j =_{def} j_1$. Then we have the case as in Proposition 119. Let F_1 be obtained from F'' by applying the transformation described in Proposition 119 for ψ and the indices $0, j$. By (2), $\mathrm{F}_1 \in C^k_{\mathit{fin}}(\psi)$. Also, it is easy to see that G is a p-morphic image of F_1.

Case 1.2. $j_2 < j_1$. Let F_1 be obtained from F'' by applying the transformation from Proposition 119 for i_1, j_1, then let F_2 be obtained by F_1 by the transformation from Proposition 120 for i_2, j_2. Then $\mathrm{F}_2 \in C^k_{\mathit{fin}}(\psi)$ and again it is easy to see that G is a p-morphic image of F_2.

Case 2. $i_1 > 0$. Then $j_1 \geq j_2$. Let F_2 be obtained from F'' by the transformation from Proposition 120 for i_1, j_1. Then $\mathrm{F}_2 \in C^k_{\mathit{fin}}(\psi)$ and again it is easy to see that G is a p-morphic image of F_2. □

Proposition 122 Let ψ be a sentence with quantifier depth k. Let τ_k be the sentence which says 'there are at least $(k+2)^4$ daisies, each with at least $k+1$ petals and $k+1$ stamens'. Then $\psi \vee \tau_k$ is modally definable in $\mathrm{ML}(\Box, [U])$ over $\mathcal{C}_{\mathrm{KD45}}$ iff $\mathcal{C}_{\mathrm{KD45}} \vDash \psi$.

Proof The right-to-left direction is obvious, so suppose the sentence $\psi \vee \tau_k$ is modally definable by the formula A from $\mathrm{ML}(\Box, [U])$ over $\mathcal{C}_{\mathrm{KD45}}$. Let $\mathrm{F}_{k+1} \in \mathcal{C}_{\mathrm{KD45}}$ be a structure with exactly $(k+2)^4$ daisies with $k+1$ petals and $k+1$ stamens. Then $\mathrm{F}_{k+1} \vDash \psi \vee \tau_k$ because $\mathrm{F}_{k+1} \vDash \tau_k$. Therefore $\mathrm{F}_{k+1} \Vdash A$. Let $\mathrm{F} \in C^k_{\mathit{fin}}$. Because F is a p-morphic image of F_{k+1}, $\mathrm{F} \Vdash A$, so $\mathrm{F} \vDash \psi \vee \tau_k$. But $\mathrm{F} \nvDash \tau_k$, so $\mathrm{F} \vDash \psi$. Thus $C^k_{\mathit{fin}} \vDash \psi$. By Proposition 109, $\mathcal{C}_{\mathrm{KD45}} \vDash \psi$. □

Theorem 123 *The problem of the modal definability of sentences in* $\mathrm{ML}(\Box, [U])$ *over* $\mathcal{C}_{\mathrm{KD45}}$ *is PSPACE-complete.*

Proof We have that the problem is PSPACE-hard by Corollary 110 and Proposition 122. Theorem 121 guarantees that the problem is in PSPACE, because it is possible to create a polynomial-space algorithm, which, given a sentence ψ, calculates k and checks whether $C^k_{\mathit{fin}}(\psi)$ satisfies the conditions (1) and (2) of the theorem. □

5.2 First-order Definability

Now, we are going to apply the main idea of [2], which is to apply the properties of Ehrenfeucht-Fraïssé games and Ehrenfeucht's theorem to the standard translation of modal formulas, in order to restrict the Kripke frames that we are working with. Thus, as we see below in Theorem 133, it is possible to show that every modal formula of ML($\Box$, $[U]$) has a first-order definition over $\mathcal{C}_{\mathrm{KD45}}$.

Definition 124 ((n, d)-sort and non-null (n, d)-sort) Let $d > 0$ and $n > 0$. We say that a sequence of numbers $\sigma_1, \ldots, \sigma_{2^n}$ is an *(n, d)-sort* iff for all i, such that $1 \leq i \leq 2^n$, $0 \leq \sigma_i \leq d$. Clearly, the number of all (n, d)-sorts is $(d+1)^{2^n}$. Additionally, we may require that at least one of the σ_i is positive. Thus we define a *non-null (n, d)-sort*, and the number of all non-null (n, d)-sorts is $< (d+1)^{2^n}$.

Let the variables of $PROP$ be ordered as $p_1, p_2, \ldots$. Let $\{P_1, P_2, \ldots\}$ be a countably infinite set of unary predicate symbols.

Definition 125 ($L(=, R, P_1, \ldots, P_n)$) Let $n > 0$. We extend FOL to $L(=, R, P_1, \ldots, P_n)$ by adding the new predicate symbols $P_1, \ldots, P_n$. Clearly, any Kripke model M is also a structure for $L(=, R, P_1, \ldots, P_n)$. In this case, for all i such that $1 \leq i \leq n$, we will use $V(P_i)$ interchangeably with $V(p_i)$.

For any $n > 0$, let $\epsilon^1, \ldots, \epsilon^{2^n}$ be a fixed liner order of all sequences of zeroes and ones of length n, for example, the lexicographical order. Let the i^{th} element of a sequence ϵ^j for $1 \leq j \leq 2^n$ and $1 \leq i \leq n$ be denoted by ϵ^j_i.

Definition 126 ((n, d)-sort of a set) Let $\mathrm{M} = \langle W, R, V\rangle$ be a Kripke model, let X be subset of W, let $d > 0$, $n > 0$. Let $\sigma_1, \ldots, \sigma_{2^n}$ be an (n, d)-sort. For each j such that $1 \leq j \leq 2^n$, we define the set $X_{\epsilon^j} =_{def} X \cap V(P_1)^{\epsilon^j_1} \cap \cdots \cap V(P_n)^{\epsilon^j_n}$, where for all i, such that $1 \leq i \leq n$, $V(P_i)^0 = V(P_i)$ and $V(P_i)^1 = W \setminus V(P_i)$. We say that X is *of the given (n, d)-sort $\sigma_1, \ldots, \sigma_{2^n}$* iff for any j such that $1 \leq j \leq 2^n$, the set X_{ϵ^j} has σ_j elements if $\sigma_j < d$, and at least d elements if $\sigma_j = d$.

We say that a Kripke model is a KD45-model iff its frame is a KD45-frame.

Definition 127 ((n, d)-sort of a daisy) Let M a KD45-model, and let D be a daisy from M. Let D_{pet} be the set of petals of D, and let D_{sta} be the set of stamens of D. Let $d > 0$, $n > 0$. Let Σ_{pet} be an (n, d)-sort. Let Σ_{sta} be a non-null (n, d)-sort. We say that any such tuple $\langle \Sigma_{\mathrm{pet}}, \Sigma_{\mathrm{sta}}\rangle$ is an *(n, d)-sort of a daisy*. We say that *D is of the sort $\langle \Sigma_{\mathrm{pet}}, \Sigma_{\mathrm{sta}}\rangle$* iff D_{pet} is of sort Σ_{pet} and D_{sta} is of sort Σ_{sta}. Clearly, for fixed KD45-model M, d and n, the number of all sorts of daisies is $\leq ((d+1)^{2^n})^2$.

Definition 128 ((n, d)-similar KD45-models) Let M_1 and M_2 be two KD45-models. Let $d > 0$, $n > 0$. We say that M_1 and M_2 are *(n, d)-similar* iff for any (n, d) sort of a daisy $\langle \Sigma_{\mathrm{pet}}, \Sigma_{\mathrm{sta}}\rangle$ either M_1 and M_2 contain the same number of daisies having the sort $\langle \Sigma_{\mathrm{pet}}, \Sigma_{\mathrm{sta}}\rangle$ whenever this number is $< d$, or each of them contains at least d daisies of sort $\langle \Sigma_{\mathrm{pet}}, \Sigma_{\mathrm{sta}}\rangle$.

Proposition 129 Let $d > 0$, $n > 0$. Let M_1 and M_2 be two (n, d)-similar KD45-models. Then they validate the same sentences from $L(=, R, P_1, \ldots, P_n)$ with quantifier depth $\leq d$.

Proof Clearly, the duplicator wins the Ehrenfeucht-Fraïssé d-rounds game over M_1 and M_2. Therefore they validate the same sentences from $L(=, R, P_1, \ldots, P_n)$ with quantifier depth $\leq d$. $\Box$

Let $A \in \mathrm{ML}(\Box, [U])$. Let $n > 0$ be such that all variables occurring in A be among $p_1, \ldots, p_n$. It is a well-known fact that the *standard translation* (see 2.7) $\mathrm{ST}(A, x)$ of the formula A, which gives a formula $\psi(x) \in L(=, R, P_1, \ldots, P_n)$, has the property that for any given Kripke model M, $\mathrm{M} \Vdash A$ iff $\mathrm{M} \models \psi(x)$. Also if $d > 0$ and the modal depth of A is $< d$, then the quantifier depth of $\forall x \psi$ is $\leq d$.

Therefore, by Proposition 129 and by the above, we get the following lemma:

Lemma 130 Let $d > 0$, $n > 0$. Let M_1 and M_2 be two (n, d)-similar KD45-models. Then for any modal formula $A \in \mathrm{ML}(\Box, [U])$ with modal depth $< d$ and variables among $p_1, \ldots, p_n$: $\mathrm{M}_1 \Vdash A$ iff $\mathrm{M}_2 \Vdash A$. $\Box$

Definition 131 (Restriction of a Kripke Model) Let $\mathrm{M} = \langle W, R, V\rangle$ be a Kripke model, let $\mathrm{F}' = \langle W', R'\rangle$ be a Kripke frame where $W' \subseteq W$, $R' = R \cap (W' \times W')$. Then we say that the model $M' = \langle W', R', V'\rangle$ is the *restriction of* M *to* F', denoted by $\mathrm{M} \restriction \mathrm{F}'$, iff for all propositional variables p, $V'(p) = V(p) \cap W'$.

Proposition 132 Let $d > 0$, $n > 0$. Let $k > d.((d+1)^{2^n})^2$. Let $\mathrm{F} \in \mathcal{C}_{\mathrm{KD45}}$. Let $\mathrm{F}_1 =_{def} \mathrm{F} \restriction_1 k$ (see Definition 100). Let $\mathrm{F}_2 =_{def} \mathrm{F}_1 \restriction_2 k$ (see Definition 107). Then for any modal formula $A \in \mathrm{ML}(\Box, [U])$ with variables among $p_1, \ldots, p_n$ and modal depth m such that $(m + 1) < d$ the following are equivalent:
(1) $\mathrm{F} \Vdash A$
(2) $\mathrm{F}_1 \Vdash A$
(3) $\mathrm{F}_2 \Vdash A$.

Proof First, we show the equivalence of (1) and (2). Clearly, F_1 is a p-morphic image of F, so (1) $\Rightarrow$ (2). Now, let $\mathrm{F}_1 \Vdash A$ and suppose $\mathrm{F} \nVdash A$. Then there is a model M over F and a state w in F, such that $\mathrm{M}, w \nVdash A$. Then $\mathrm{M} \Vdash \langle U\rangle\neg A$. Let $\mathrm{M}_1 =_{def} \mathrm{M} \restriction \mathrm{F}_1$ (see Definition 131). We now show that M and M_1 are (n, d)-similar and by Lemma 130, this shows the contraposition of (2) $\Rightarrow$ (1).

Clearly by Definition 100, there is a bijection f between the daisies of F and F_1, such that for every daisy D_1 of F_1, $D_1 = f^{-1}[D_1] \restriction_1 k$.

Let D_1 be a daisy from F_1 and let D be $f^{-1}[D_1]$. It is enough to show that for every daisy type $\langle \Sigma_{\mathrm{pet}}, \Sigma_{\mathrm{sta}} \rangle$, D_1 is of the type $\langle \Sigma_{\mathrm{pet}}, \Sigma_{\mathrm{sta}} \rangle$ iff D is of the type $\langle \Sigma_{\mathrm{pet}}, \Sigma_{\mathrm{sta}} \rangle$. If $D_1 = D$, then there is nothing to prove, so suppose $D_1 \neq D$.

Let $\langle \Sigma_{\mathrm{pet}}, \Sigma_{\mathrm{sta}} \rangle$ be a daisy type, let $\Sigma_{\mathrm{pet}} = \{\sigma_1^{\mathrm{pet}}, \ldots, \sigma_{2^n}^{\mathrm{pet}}\}$, let $\Sigma_{\mathrm{sta}} = \{\sigma_1^{\mathrm{sta}}, \ldots, \sigma_{2^n}^{\mathrm{sta}}\}$. Denote the set of petals of D_1 by D_1^{pet}, the set of stamens of D_1 by D_1^{sta}, the set of petals of D by D^{pet}, the set of stamens of D by D^{sta}.

First, let D_1 be of type $\langle \Sigma_{\mathrm{pet}}, \Sigma_{\mathrm{sta}} \rangle$. Then easily, D is also of type $\langle \Sigma_{\mathrm{pet}}, \Sigma_{\mathrm{sta}} \rangle$.

Now, let D be of type $\langle \Sigma_{\mathrm{pet}}, \Sigma_{\mathrm{sta}} \rangle$. Then, because D_1^{pet} is a subframe of D^{pet}, D_1^{sta} is a subframe of D^{sta}, $\sigma_1^{\mathrm{pet}} + \cdots + \sigma_{2^n}^{\mathrm{pet}} \leq d.2^n < k$, and $\sigma_1^{\mathrm{sta}} + \cdots + \sigma_{2^n}^{\mathrm{sta}} \leq d.2^n < k$, we have that D_1 is also of type $\langle \Sigma_{\mathrm{pet}}, \Sigma_{\mathrm{sta}} \rangle$.

Now, we show the equivalence of (2) and (3). Because F_2 is a p-morphic image of F_1, we easily get the direction (2) $\Rightarrow$ (3). We need to show that for every model M_1 over F_1, we can find an (n, d)-similar model M_2 over F_2, then the result would follow by Lemma 130. Let M_1 be a model over F_1. By Definition 128 and the fact that $k > d.((d+1)^{2^n})^2$, we can see that we can find an (n, d)-similar model M_2 over F_2. $\square$

Theorem 133 *Any modal formula A of* $\mathrm{ML}(\Box, [U])$ *is first-order definable in* FOL *over the class $\mathcal{C}_{\mathrm{KD45}}$.*

Proof Let $A \in \mathrm{ML}(\Box, [U])$. Let the modal depth of A be m. Let $n > 0$ be such that the variables of A are among $p_1, \ldots, p_n$. Let $d > 0$ be such that $(m+1) < d$. Let k be such that $k > d.((d+1)^{2^n})^2$. Let $C_{\mathit{fin}}^k(A)$ be the class of all frames in C_{fin}^k which validate A.

If $C_{\mathit{fin}}^k(A) = \emptyset$, then by Proposition 132, $\bot$ is a FOL definition of A over $\mathcal{C}_{\mathrm{KD45}}$.

If $C_{\mathit{fin}}^k(A) = C_{\mathit{fin}}^k$, then by Proposition 132, $\top$ is a FOL definition of A over $\mathcal{C}_{\mathrm{KD45}}$.

Otherwise, let $\mathrm{F} \in C_{\mathit{fin}}^k(A)$. by Proposition 132, F is a witness of a class of frames from $\mathcal{C}_{\mathrm{KD45}}$, which validate A. Namely, this is the class of all frames F_0 which produce F as $\mathrm{F} = (\mathrm{F}_0 \upharpoonright_1 k) \upharpoonright_2 k$. Clearly, this class of frames can be described by a single FOL sentence, because F is a finite frame. Denote this formula by ψ_{F}.

Let $\psi =_{def} \bigvee(\psi_{\mathrm{F}} \mid \mathrm{F} \in C_{\mathit{fin}}^k(A))$.

Suppose $\mathrm{F} \in \mathcal{C}_{\mathrm{KD45}}$ and suppose $\mathrm{F} \Vdash A$. Let $\mathrm{F}' =_{def} (\mathrm{F} \upharpoonright_1 k) \upharpoonright_2 k$. By the Proposition 132, $\mathrm{F}' \Vdash A$, so $\mathrm{F}' \in C_{\mathit{fin}}^k(A)$. Then by the definition of $\psi_{\mathrm{F}'}$, $\mathrm{F} \vDash \psi_{\mathrm{F}'}$, so $\mathrm{F} \vDash \psi$.

Now, let $\mathrm{F} \in \mathcal{C}_{\mathrm{KD45}}$ and let $\mathrm{F} \vDash \psi$. Then there is a disjunct of ψ, $\psi_{\mathrm{F}'}$, such that $\mathrm{F}' \in C_n^k(A)$ (and so, $\mathrm{F}' \Vdash A$), and $\mathrm{F} \vDash \psi_{\mathrm{F}'}$. Then by the definition of $\psi_{\mathrm{F}'}$, $\mathrm{F}' = (\mathrm{F} \upharpoonright_1 k) \upharpoonright_2 k$. By Proposition 132, $\mathrm{F} \Vdash A$. $\square$

6 ML($\Box$) and $\mathcal{C}_{\mathrm{K5}}$

We are using the basic modal language ML($\Box$) and the standard predicate calculus with individual variables VAR, equality, and a single binary predicate symbol r FOL. The standard definitions of *Kripke frame* and *Kripke model* apply. See 2 for more information.

The *axiom of* K5 is the following formula:
(5) $(\Diamond p \to \Box\Diamond p)$ (Euclidean axiom)

A first-order correspondent of the K5 axiom is:
(5′) $\forall x \forall y_1((x\ r\ y_1) \to \forall z_1((x\ r\ z_1) \to (z_1\ r\ y_1)))$

We say that a frame F is a K5 frame iff the axiom (5) is valid on F. Denote the class of all Euclidean frames by $\mathcal{C}_{\mathrm{K5}}$.

Let us examine the problems of modal definability of FOL formulas in ML($\Box$) over $\mathcal{C}_{\mathrm{K5}}$ and of first-order definability of ML($\Box$) formulas in FOL over $\mathcal{C}_{\mathrm{K5}}$.

6.1 ML($\Box$) formulas are FOL-definable over $\mathcal{C}_{\mathrm{K5}}$

Definition 134 (Simple K5 Frame) We say that an K5 frame $\mathrm{F} = \langle W, R\rangle$ is a *simple* K5 *frame* iff there are sets $P(\mathrm{F})$ (the set of petals) and $S(\mathrm{F})$ (the set of stamens), such that $W = P(\mathrm{F}) \cup S(\mathrm{F})$, $P(\mathrm{F}) \cap S(\mathrm{F}) = \emptyset$, and the following hold:
(SK5F 1). $\forall x \in P(\mathrm{F}) \neg\exists y \in W(\langle y, x\rangle \in R)$
(SK5F 2). $S(\mathrm{F}) \neq \emptyset \Rightarrow \forall x \in P(\mathrm{F}) \exists y \in S(\mathrm{F})(\langle x, y\rangle \in R)$
(SK5F 3). $\forall x \in S(\mathrm{F}) \forall y \in S(\mathrm{F})(\langle x, y\rangle \in R)$

It is easy to check that any K5-frame is a disjoint union of simple K5 frames.

Let the class of all generated subframes of K5-frames be denoted by C_{gen}. Clearly any $\mathrm{F} \in C_{gen}$ has zero or one irreflexive states and possibly a cluser of states, with the irreflexive state, if present, being related to some of the states in the cluster.

Denote the class of finite generated subframes of K5 by C_f. Clearly $C_f \subseteq C_{gen}$. We denote by $\mathrm{F}_{e,s,m}$ any frame of C_f, where $e \in \{0, 1\}$ is the number of irreflexive states, s is the number of descendents of the irreflexive state (if any), and m is the total number of states in the cluster. Clearly $m \geq s \geq 0$ and $m + e > 0$.

We say that a Kripke model M is a K5-model iff its frame is a K5-frame, denoted by $\mathrm{M} \in \mathcal{C}_{\mathrm{K5}}$. We say that a Kripke model M is a *gen*-model iff its frame is a *gen*-frame, denoted by $\mathrm{M} \in C_{gen}$. We say that a Kripke model M is an f-model iff its frame is an f-frame, denoted by $\mathrm{M} \in C_f$.

We must now remember definitions 124, 125, and 126.

Definition 135 ((n, d)-sort of a *gen*-model) Let Σ_{pet}, Σ_{sta_1}, and Σ_{sta_2} be (n, d)-sorts. We say that any such tuple $\langle\Sigma_{\text{pet}}, \Sigma_{\text{sta}_1}, \Sigma_{\text{sta}_2}\rangle$ is an *(n, d)-sort of a gen-model* iff there is at least one number in one of the three (n, d)-sorts, which is > 0, and also in Σ_{pet}, there is at most one number, which is equal to 1, and the rest are 0.

Let $\mathrm{M} \in C_{gen}$. Then its frame D is some $\mathrm{F}_{e,s,m}$. Let D_{pet} be the set of petals of D. Let D_{sta_1} be the set of stamens of D, which are descendants of the petal, if present. Let D_{sta_2} be the set of stamens of D, which are not descendants of a petal. Let $d > 0$, $n > 0$. We say that M is of the (n, d)-sort $\langle\Sigma_{\text{pet}}, \Sigma_{\text{sta}_1}, \Sigma_{\text{sta}_2}\rangle$ iff D_{pet} is of sort Σ_{pet}, D_{sta_1} is of sort Σ_{sta_1}, and D_{sta_2} is of sort Σ_{sta_2}.

Definition 136 ((n, d)-similar *gen*-models) Let $\mathrm{M}_1, \mathrm{M}_2 \in C_{gen}$. Let $d > 0$, $n > 0$. We say that M_1 and M_2 are *(n, d)-similar* iff they have the same (n, d)-sort.

Proposition 137 Let $d > 0$, $n > 0$. Let $\mathrm{M}_1 \in C_{gen}$ and $\mathrm{M}_2 \in C_{gen}$ be two (n, d)-similar models. Then they validate the same sentences from $L(=, R, P_1, \ldots, P_n)$ (see Definition 125) with quantifier depth $\leq d$.

Proof Clearly, the duplicator wins the Ehrenfeucht-Fraïssé d-rounds game over M_1 and M_2. Therefore they validate the same sentences from $L(=, R, P_1, \ldots, P_n)$ with quantifier depth $\leq d$. $\square$

Let $n > 0$. Let $A \in \mathrm{ML}(\Box)$ and let all variables occurring in A be among $p_1, \ldots, p_n$. It is a well-known fact that the *standard translation* (see 2.7) $\mathrm{ST}(A, x)$ of the formula A, which gives a formula $\psi(x) \in L(=, R, P_1, \ldots, P_n)$, has the property that for any given Kripke model M and a state w in M, $\mathrm{M}, w \Vdash A$ iff $\mathrm{M} \vDash \psi(x)[w]$. Also if $d > 0$ and the modal depth of A is $< d$, then the quantifier depth of $\exists x\psi$ is $\leq d$.

Therefore, by Proposition 137 and by the above, we get the following lemma:

Lemma 138 Let $d > 0$, $n > 0$. Let M_1 and M_2 be two (n, d)-similar C_{gen}-models. Then for any modal formula $A \in \mathrm{ML}(\Box)$ with modal depth $< d$ and variables among $p_1, \ldots, p_n$: A is satisfied on M_1 iff A is satisfied on M_2.

Proof By Proposition 137, either both models validate the sentence $\exists x \mathrm{ST}(A, x)$, or both models validate its negation. The result follows by the properties of ST. $\square$

For $k > 0$, $C_f^k =_{def} \{\mathrm{F} \in C_f \mid \mathit{Card}(\mathrm{F}_{sta_1}) \leq k \ \& \ \mathit{Card}(\mathrm{F}_{sta_2}) \leq k\}$

Definition 139 (Restriction of a *gen*-frame) Let $\mathrm{F} \in C_{gen}$ and let $k > 0$. We say that the frame $\mathrm{F}' \in C_f^k$ is a *restriction of* F *to* k iff:

1. $\mathrm{F}_{pet} = \mathrm{F}'_{pet}$,

2. Either $Card(\mathrm{F}_{sta_1}) < k$ and $Card(\mathrm{F}_{sta_1}) = Card(\mathrm{F}'_{sta_1})$, or $Card(\mathrm{F}_{sta_1}) \geq k$ and $Card(\mathrm{F}'_{sta_1}) = k$,

3. Either $Card(\mathrm{F}_{sta_2}) < k$ and $Card(\mathrm{F}_{sta_2}) = Card(\mathrm{F}'_{sta_2})$, or $Card(\mathrm{F}_{sta_2}) \geq k$ and $Card(\mathrm{F}'_{sta_2}) = k$.

Clearly, up to isomorphism, for any $\mathrm{F} \in C_{gen}$, there is a single frame $\mathrm{F}' \in C_f^k$, which is the restriction of F to k. We denote this by $\mathrm{F}' =_{def} \mathrm{F} \lceil_3 k$. Up to isomorphism, we may consider F' to be a subframe of F.

Proposition 140 Let $A \in \mathrm{ML}(\Box)$. Let $n > 0$ be such that the variables of A be among $p_1, \ldots, p_n$. Let the modal depth of A be d_A. Let $d > 0$ be such that $(d_A + 1) < d$. Let k be such that $k > (d+1).2^n$. Let $\mathrm{F} \in C_{gen}$ and let $\mathrm{F}' = \mathrm{F} \lceil_3 k$. Then A is satisfiable on F iff A is satisfiable on F'.

Proof Let M be any model over F. Because $k > (d+1).2^n$, clearly we may construct a model M' over F', such that M and M' are (n, d)-similar. By Proposition 138, A is satisfiable on M iff A is satisfiable on M'.

Now, let M' be any model over F'. Because $k > (d+1).2^n$, and because F' is a subframe of F, we may extend M' to an (n, d)-similar model M over F. By Proposition 138, A is satisfiable on M iff A is satisfiable on M'. $\Box$

Let $A \in \mathrm{ML}(\Box)$ and let $k > 0$. Denote $C_f^k(A) =_{def} \{\mathrm{F} \in C_f^k \mid \mathrm{F} \Vdash A\}$.

Proposition 141 Let $A \in \mathrm{ML}(\Box)$. Then A has a definition in FOL over $\mathcal{C}_{\mathrm{K5}}$.

Proof Let $n > 0$ be such that the variables of A be among $p_1, \ldots, p_n$. Let the modal depth of A be d_A. Let $d > 0$ be such that $(d_A + 1) < d$. Let k be such that $k > (d+1).2^n$.

Let $\mathrm{F} \in C_f^k \setminus C_f^k(A)$. Then F is a finite frame and we construct a FOL sentence ψ_F, with the following property: for all $\mathrm{F}' \in \mathcal{C}_{\mathrm{K5}}$, $\mathrm{F}' \vDash \psi_\mathrm{F}$ iff there is a generated subframe F'' of F', such that, up to isomorphism, $\mathrm{F}'' \lceil_3 k = \mathrm{F}$. We construct the sentence in the following way. Because $\mathrm{F} \in C_f^k$, then there are numbers e, s, m such that $\mathrm{F} = \mathrm{F}_{e,s,m}$.

First, if $m = s = 0$, then $e = 1$ and because $k > 1$ the sentence is:
$\psi_\mathrm{F} =_{def} \exists x \forall x'(\neg(x\ r\ x') \wedge \neg(x'\ r\ x))$.

Second, if $e = s = 0$, then $m > 0$ and there are two cases. First, if $m = k$:
$\psi_\mathrm{F} =_{def} \exists z_1 \ldots \exists z_m \bigwedge_{1 \leq i,j \leq m}(z_i\ r\ z_j) \wedge \bigwedge_{1 \leq i \neq j \leq m}(z_i \neq z_j)$.

Else if $m < k$, the formula is:
$\psi_\mathrm{F} =_{def} \exists z_1 \ldots \exists z_m \bigwedge_{1 \leq i,j \leq m}(z_i\ r\ z_j) \wedge \bigwedge_{1 \leq i \neq j \leq m}(z_i \neq z_j) \wedge$
$\bigwedge_{1 \leq i \leq m} \forall z'((z_i\ r\ z') \rightarrow \bigvee_{1 \leq j \leq m}(z' = z_j))$.

Third, $e = 1$ and $s > 0$. Then $m \geq s$ and there are three cases.

Let $0 < s = m = k$

$\psi_{\mathrm{F}} =_{def} \exists x \exists y_1 \ldots \exists y_s$
$(\forall x'(\neg(x' \; r \; x))) \wedge \bigwedge_{1 \leq i \neq j \leq s}(y_i \neq y_j) \wedge \bigwedge_{1 \leq i \leq s}(x \; r \; y_i) \wedge \bigwedge_{1 \leq i,j \leq s}(y_i \; r \; y_j)$.

Now let $0 < s < m = k$.

$\psi_{\mathrm{F}} =_{def} \exists x \exists y_1 \ldots \exists y_s \exists z_1 \ldots \exists z_m$
$(\forall x'(\neg(x' \; r \; x))) \wedge \bigwedge_{1 \leq i \neq j \leq s}(y_i \neq y_j) \wedge \bigwedge_{1 \leq i \leq s}(x \; r \; y_i) \wedge$
$\bigwedge_{1 \leq i,j \leq m}(z_i \; r \; z_j) \wedge \bigwedge_{1 \leq i \neq j \leq m}(z_i \neq z_j) \wedge \bigwedge_{1 \leq i \leq s} \bigvee_{1 \leq j \leq m}(y_i = z_j) \wedge$
$\forall y'(\bigwedge_{1 \leq i \leq s}(y' \neq y_i) \rightarrow \neg(x \; r \; y'))$.

Now let $0 < s \leq m < k$.

$\psi_{\mathrm{F}} =_{def} \exists x \exists y_1 \ldots \exists y_s \exists z_1 \ldots \exists z_m$
$(\forall x'(\neg(x' \; r \; x))) \wedge \bigwedge_{1 \leq i \neq j \leq s}(y_i \neq y_j) \wedge \bigwedge_{1 \leq i \leq s}(x \; r \; y_i) \wedge$
$\bigwedge_{1 \leq i,j \leq m}(z_i \; r \; z_j) \wedge \bigwedge_{1 \leq i \neq j \leq m}(z_i \neq z_j) \wedge \bigwedge_{1 \leq i \leq s} \bigvee_{1 \leq j \leq m}(y_i = z_j) \wedge$
$\forall y'(\bigwedge_{1 \leq i \leq s}(y' \neq y_i) \rightarrow \neg(x \; r \; y')) \wedge$
$\forall z'(\bigwedge_{1 \leq i \leq m}(z' \neq z_i) \rightarrow \bigwedge_{1 \leq i \leq m} \neg(z_i \; r \; z'))$.

Because, up to isomorphism, the class $C_f^k \setminus C_f^k(A)$ is finite, let ψ be a sentence, which is the disjunction of all sentences ψ_{F} for all $\mathrm{F} \in C_f^k \setminus C_f^k(A)$. Let us see that $\neg\psi$ is a definition of A over $\mathcal{C}_{\mathrm{K5}}$.

First, let $\mathrm{F} \in \mathcal{C}_{\mathrm{K5}}$, $\mathrm{F} \vDash \neg\psi$, and suppose that $\mathrm{F} \nVdash A$. Then $\neg A$ is satisfiable in F at some state w. Let F$'$ be the generated subframe of F at w. Then $\neg A$ is satisfiable on F$'$. Now let $\mathrm{F}'' =_{def} \mathrm{F}' \upharpoonright_3 k$. By Proposition 140, $\neg A$ is satisfiable on F$''$, so $\mathrm{F}'' \in C_f^k \setminus C_f^k(A)$, therefore $\mathrm{F}'' \vDash \psi$. But then by the definition of ψ, $\mathrm{F} \vDash \psi$, contradiction.

Now, let $\mathrm{F} \in \mathcal{C}_{\mathrm{K5}}$, $\mathrm{F} \Vdash A$, and suppose that $\mathrm{F} \nvDash \neg\psi$, so $\mathrm{F} \vDash \psi$. Because ψ is a disjunction of sentences, there is a disjunct of ψ, ψ', such that $\mathrm{F} \vDash \psi'$. By the definition of ψ', there is a generated subframe F$'$ of F, such that, up to isomorphism, $\mathrm{F}'' = \mathrm{F}' \upharpoonright_3 k \in C_f^k \setminus C_f^k(A)$, so $\mathrm{F}'' \nVdash A$ i.e. $\neg A$ is satisfiable on F$''$. But $\mathrm{F}' \Vdash A$ and by Proposition 140, $\neg A$ is satisfiable on F$'$, contradiction.

We conclude that $\neg\psi$ is a definition of A over $\mathcal{C}_{\mathrm{K5}}$. $\square$

6.2 Undecidability of validity of FOL formulas in $\mathcal{C}_{\mathrm{K5}}$

In this section, we use the definitions from [50] for *first-order theory*, or just a *theory*. As in [50], if T is a theory, then $L(\mathrm{T})$ denotes the *language of* T.

We use a variant of the interpretation argument from [50] to show that satisfiability of FOL formulas in the class of all reflexive and symmetrical frames is reducible to satisfiability of FOL formulas in the class of all K5-frames. This shows that validity of FOL sentences in $\mathcal{C}_{\mathrm{K5}}$ is undecidable.

It is clear that by renaming bound variables while preserving semantic equivalence, we may obtain a variant of a first-order formula $U(x)$ where a given variable x' does not occur and x does not have any bound occurrences. Thus we denote by $U(x')$ the formula obtained by substituting x' for x in a formula obtained in this way.

Now, remember the definition of a relativized reduct of structure for a first-order language, Definition 112.

Definition 142 (Relativization of a Formula) Let L be a first-order language and let $\psi, U(x) \in L$. We define inductively $\tau(\psi, U)$, the *relativization of ψ with respect to $U(x)$*, in the following way:
$\tau(\alpha, U) = \alpha$ for atomic formulas α.
$\tau(\neg\psi, U) = \neg\tau(\psi, U)$.
$\tau(\psi_1 \vee \psi_2, U) = (\tau(\psi_1, U) \vee \tau(\psi_2, U))$.
$\tau(\psi_1 \wedge \psi_2, U) = (\tau(\psi_1, U) \wedge \tau(\psi_2, U))$.
$\tau(\exists x\psi, U) = \exists x(U(x) \wedge \tau(\psi, U))$.
$\tau(\forall x\psi, U) = \forall x(U(x) \rightarrow \tau(\psi, U))$.

Lemma 143 (Relativization Theorem) Let F, F' be structures for a first-order language L, let $U(x)$ be a formula from L. If F' is the relativized reduct of F with respect to $U(x)$ then for all first-order formulas $\psi(\bar{y}) \in L$ and for all lists $\bar{t}$ of worlds in F', $\mathrm{F} \vDash \tau(\psi(\bar{y}), U(x))[\bar{t}]$ iff $\mathrm{F}' \vDash \psi(\bar{y})[\bar{t}]$.

Proof See Theorem 5.1.1 in [38]. □

Proposition 144 Let T_1 and T_2 be theories with equality such that $L(\mathrm{T}_1) \subseteq L(\mathrm{T}_2)$. Let $U(x) \in L(\mathrm{T}_2)$ be a formula. Let the following two conditions hold:

(i). If for some structure F_1 for the language $L(\mathrm{T}_1)$, $\mathrm{F}_1 \vDash \mathrm{T}_1$, then there is a structure F_2 for $L(\mathrm{T}_2)$ such that $\mathrm{F}_2 \vDash \mathrm{T}_2$ and F_1 is the restriction to $L(\mathrm{T}_1)$ of the relativized reduct of F_2 with respect to $U(x)$.

(ii). If for some structure F_2 for the language $L(\mathrm{T}_2)$, $\mathrm{F}_2 \vDash \mathrm{T}_2$ and $\mathrm{F}_2 \vDash \exists x U(x)$, then F_2 has a relativized reduct with respect to $U(x)$ and its restriction to $L(\mathrm{T}_1)$, F_1, is such that $\mathrm{F}_1 \vDash \mathrm{T}_1$.

Then for any $\psi(x_1, \ldots, x_n) \in L(\mathrm{T}_1)$, the following two conditions are equivalent:

(1) There is some structure F_1 for $L(\mathrm{T}_1)$ such that $\mathrm{F}_1 \vDash \mathrm{T}_1$ and $\mathrm{F}_1 \vDash \exists x_1 \ldots \exists x_n \psi(x_1, \ldots, x_n)$

(2) There is some structure F_2 for $L(\mathrm{T}_2)$ such that $\mathrm{F}_2 \vDash \mathrm{T}_2$ and $\mathrm{F}_2 \vDash \exists x U(x) \wedge \exists x_1 \ldots \exists x_n (U(x_1) \wedge \cdots \wedge U(x_n) \wedge \tau(\psi))$, where $\tau(\psi)$ is the relativization of $\psi(x_1, \ldots, x_n)$ with respect to $U(x)$.

Proof Follows by Lemma 143. □

From now on, let T_1 be the first-order theory of symmetrical and reflexive relations, which is a first-order theory with equality and the non-logical axiom:
$\forall x(x \; r_1 \; x) \wedge \forall x \forall y((x \; r_1 \; y) \rightarrow (y \; r_1 \; x))$.

According to [47], the problem of deciding the validity of formulas in T_1 is undecidable.

Let us use the following theory as the first-order theory of all Euclidean Kripke frames: T_{K5}, which is a first-order theory with equality and the following non-logical axiom:

$(5'')$ $\forall x \forall y_1((x\ r_2\ y_1) \rightarrow \forall z_1((x\ r_2\ z_1) \rightarrow (z_1\ r_2\ y_1)))$

Let $U(x)$ be the formula $\neg(x\ r_2\ x) \wedge \exists z(x\ r_2\ z)$.

Now, extend $L(\mathrm{T}_{\mathrm{K5}})$ to L by adding the binary predicate symbol r_1 and defining it in the theory T_2, which is an extension of T_{K5}, with language L and with the following additional non-logical axiom:

$(\mathit{Interpr}')$ $\forall x \forall y((x\ r_1\ y) \leftrightarrow U(x) \wedge U(y) \wedge \exists z((x\ r_2\ z) \wedge (y\ r_2\ z)))$

Clearly, by using techniques from [50], the problem of deciding validity of formulas in T_2 is reducible to the problem of deciding validity of formulas in T_{K5}.

Proposition 145 The problem of deciding satisfiability of formulas in T_1 is reducible to the problem of deciding satisfiability of formulas in T_2.

Proof We use Proposition 144 to show the desired result.

First we prove (i). Let $\mathrm{F}_1 = \langle W_1, R_1 \rangle$ be a structure for $L(\mathrm{T}_1)$ such that $\mathrm{F}_1 \models \mathrm{T}_1$. Up to isomorphism, choose F_1 such that:
$W_1 \cap \{\{a,b\} \mid a \in W_1\ \&\ b \in W_1\} = \emptyset$.

Define W_2 and R_2 in the following way:
$W_2 =_{def} W_1 \cup \{\{a,b\} \mid \langle a,b \rangle \in R_1\}$ (this is a disjoint union).
$R_2 =_{def}$

$\{\langle a, \{a,b\} \rangle \mid a \in W_1, b \in W_1, \{a,b\} \in W_2\} \cup$

$\{\langle \{a,b\}, \{c,d\} \rangle \mid a,b,c,d \in W_1, \{a,b\} \in W_2, \{c,d\} \in W_2\}$.

Let $\mathrm{F}_2 =_{def} \langle W_2, R_1, R_2 \rangle$, which is a structure for $L(\mathrm{T}_2)$. Clearly, by the definition of F_2, $\mathrm{F}_2 \models (\mathit{Interpr}')$. Now, $\mathrm{F}_2 \models (5'')$ because every R_2-descendant state in F_2 is a pair of states in F_1 and every two such pairs are in relation R_2. Thus $\mathrm{F}_2 \models \mathrm{T}_2$. Also, $\mathrm{F}_2 \models \exists x U(x)$, because W_1 is non-empty and because R_1 is reflexive. This means that F_2 has a relativized reduct with respect to $U(x)$ and clearly the restriction of this relativized reduct to $L(\mathrm{T}_1)$ is F_1.

Now, let us prove (ii). Let $\mathrm{F}_2 = \langle W_2, R_1, R_2 \rangle$ be a structure for $L(\mathrm{T}_2)$ such that $\mathrm{F}_2 \models \mathrm{T}_2$ and $\mathrm{F}_2 \models \exists x U(x)$. Let $\mathrm{F}' = \langle W_1, R_1', R_2' \rangle$ be the relativized reduct of F_2 with respect to $U(x)$ and let $\mathrm{F}_1 = \langle W_1, R_1' \rangle$ be the restriction of F' to the language $L(\mathrm{T}_1)$. Because $\mathrm{F}_2 \models \mathrm{T}_2$, $\mathrm{F}_2 \models (\mathit{Interpr}')$, so R_1 is symmetrical, which makes R_1' also symmetrical. Because F' is the relativized reduct of F_1 with respect to $U(x)$, because R_2 is Euclidean, and because $\mathrm{F}_2 \models (\mathit{Interpr}')$, R_1' is reflexive. Thus $\mathrm{F}_1 \models \mathrm{T}_1$. $\square$

Corollary 146 The problem of deciding validity of formulas in T_1 is reducible to the problem of deciding validity of formulas in T_{K5}.

Proof Follows by Proposition 145. $\square$

6.3 Undecidability of modal definability over $\mathcal{C}_{\mathrm{K5}}$

Now, let us remember the definition for a stable class of frames (Definition 113) and the theorem that states that if $\mathcal{C}$ is a stable class of frames, then the first-order validity problem with respect to $\mathcal{C}$ is reducible to the problem of modal definability over $\mathcal{C}$ in ML($\Box$) of FOL formulas (Theorem 114).

Proposition 147 $\mathcal{C}_{\mathrm{K5}}$ is a stable class of frames.

Proof Let $\gamma_1(x_1, x_2, x) =_{def} x \neq x_1 \wedge x \neq x_2$.

Let $\gamma_2 =_{def} \exists x \exists y (x \neq y \wedge \forall z (\neg((z\ r\ x) \vee (z\ r\ y) \vee (x\ r\ z) \vee (y\ r\ z))))$.

Clearly, γ_2 is a sentence that states that there are at least two r-independent states.

Let $\mathrm{F}_0 = \langle W_0, R_0 \rangle \in \mathcal{C}_{\mathrm{K5}}$.

If F_1 is a relativized reduct of F_0 with respect to γ_1 and some states v_1, v_2 in F_0, then $\mathrm{F}_1 \in \mathcal{C}_{\mathrm{K5}}$.

Now, we show frames $\mathrm{F}, \mathrm{F}' \in \mathcal{C}_{\mathrm{K5}}$ with the property (b) from Definition 113. Let $\mathrm{F} =_{def} \langle W, R \rangle$ be the disjoint union of F_0 and two new independent states w_1 and w_2. Thus $W = W_0 \cup \{w_1, w_2\}$ and $R = R_0$. Clearly, F_0 is the relativized reduct of F with respect to γ_1 and w_1, w_2, and also $\mathrm{F} \in \mathcal{C}_{\mathrm{K5}}$. Let W_1 be the set of states in W, which do not have R-successors or R-predecessors (the independent states in F). Let $W_2 =_{def} W \setminus W_1$. We can see that $w_1, w_2 \in W_1$ and also $W_2 \subseteq W_0$. Thus $\mathrm{F} = \langle W_1 \setminus \{w_1\}, \emptyset \rangle \uplus \langle W_2 \cup \{w_1\}, R \rangle$. Now, let us define a new frame $\mathrm{F}' = \langle W_2 \cup \{w_1\}, R \rangle$. Clearly $\mathrm{F}' \in \mathcal{C}_{\mathrm{K5}}$, F' is a p-morphic image of F, $\mathrm{F} \models \gamma_2$, but $\mathrm{F}' \not\models \gamma_2$. $\Box$

Thus we have shown that the problem of modal definability of FOL sentences over the class of all K5-frames is undecidable.

7 Conclusion

In section 3, we have seen a deterministic version of the SQEMA algorithm which uses conjunctive normal form and subformula elimination. We have seen that this algorithm always terminates. We have seen the invariants for Sahlqvist and Inductive formulas. There were proofs that the algorithm always succeeds on Sahlqvist and Inductive formulas.

In section 3, we have also seen a new translation of PCL formulas into formulas of ML($\Box, [U]$). We have seen a proof that Sahlqvist PCL formulas translate with this new translation into Sahlqvist ML($\Box, [U]$) formulas. We have also seen how to use Deterministic SQEMA to find first-order correspondents of PCL formulas using this new translation. The results have been implemented in the programming language Java into the SQEMA website, `http://www.fmi.uni-sofia.bg/fmi/logic/sqema`.

In section 4, we have seen that every modal formula of ML($\Box$) has a first-order definition over $\mathcal{C}_{\mathrm{KD45}}$. Also, that deciding whether a first-order sentence has a modal definition in ML($\Box$) over $\mathcal{C}_{\mathrm{KD45}}$ is PSPACE-complete.

In section 5, we have seen that every modal formula of ML($\Box, [U]$) has a first-order definition over $\mathcal{C}_{\mathrm{KD45}}$. Also, the problem of deciding whether a first-order sentence has a modal definition which is in ML($\Box, [U]$) and is over $\mathcal{C}_{\mathrm{KD45}}$ is PSPACE-complete.

In section 6, we have seen that every modal formula of ML($\Box$) has a first-order definition over the class of all Euclidean frames, $\mathcal{C}_{\mathrm{K5}}$. Also, we have seen that deciding validity of first-order formulas in the class $\mathcal{C}_{\mathrm{K5}}$ is undecidable. Modal definability of first-order sentences over $\mathcal{C}_{\mathrm{K5}}$ is also undecidable.

7.1 Future work

It would be useful to modify Deterministic SQEMA with additional rules for the special cases of S5 and KD45 modalities.

The author speculates that in its current form, Deterministic SQEMA may be able to succeed on all modal formulas of modal depth 1, which do have a first-order correspondent, according to van Benthem in [56][57] - and this is because of the conjunctive normal form eliminating procedure that the algorithm uses. It may also be possible to show that by applying the conjunctive normal form eliminating procedure at the beginning of the algorithm three times (by using negation), that any formula containing only the universal modality is reduced to a formula of modal depth 1, similarly to Chapter Three of [39], thus making sure that Deterministic SQEMA succeeds on all such formulas. There is some experimental data to suggest that both of these conjectures hold, but a more formal proof is required.

It would also be interesting to see whether modal and first-order definability are decidable in some classes of frames, such as $\mathcal{C}_{\mathrm{S5}}$, $\mathcal{C}_{\mathrm{KD45}}$, or $\mathcal{C}_{\mathrm{K5}}$, when the language is the basic modal language with the added difference operator.

Authenticity Claims

The author declares that the following are original findings (items number 1, 2, and 3 below), or co-authored with Tinko Tinchev and Philippe Balbiani (item number 4 below).

Scientific Results

The author considers the following to be the main results of the dissertation:

The results in the dissertation can be grouped in the following groups.

1. **Results about the algorithm Deterministic SQEMA, Sahlqvist and Inductive formulas.**

These include:

- Defining a new deterministic version of the SQEMA algorithm with additional simplification rules for the universal modality.

- A proof of termination of Deterministic SQEMA.

- A new invariant for Deterministic SQEMA executions on Sahlqvist formulas.

- A proof that Deterministic SQEMA succeeds on all Sahlqvist formulas.

- A new invariant for Deterministic SQEMA executions on Inductive formulas.

- A proof that Deterministic SQEMA succeeds on all Inductive formulas.

2. **Results about applying Deterministic SQEMA to formulas of the Pre-Contact Logic language, and results about Sahlqvist PCL formulas**

- Defining a modified translation of PCL formulas into $\mathrm{ML}(\Box, [U])$.

- Proving that the above translation converts Sahlqvist PCL formulas into Sahlqvist $\mathrm{ML}(\Box, [U])$ formulas.

- Modifying the existing Deterministic SQEMA implementation at `http://www.fmi.uni-sofia.bg/fmi/logic/sqema` to accept PCL formulas and succeed on all Sahlqvist PCL formulas by using the modified translation.

3. **Computability and complexity results about the correspondence problems in the class of all** KD45 **Kripke frames**

The results of this group are:

- A proof that all modal formulas of the basic modal language are first-order definable in the class of all KD45 frames.

- A proof that the problem of deciding whether first-order formulas are modally definable in the basic modal language in the class of KD45 frames is PSPACE-complete.

- A proof that all modal formulas of the basic modal language extended with the universal modality are first-order definable in the class of all KD45 frames.

- A proof that the problem of deciding modal definability in the basic modal language extended with the universal modality of first-order formulas in the class of KD45 frames is PSPACE-complete.

4. **Computability and complexity results about the correspondence problems in the class of all Euclidean Kripke frames** - this group of results was examined in collaboration with Tinko Tinchev and Philippe Balbiani.

- A proof that all modal formulas of the basic modal language have a first-order definition in the class of all Euclidean Kripke frames.

- A proof that the problem of deciding whether a first-order formula is valid in the class of all Euclidean Kripke frames is undecidable.

- A proof that the problem of deciding whether a first-order formula is modally definable in the class of all Euclidean Kripke frames is undecidable.

Referred Publications

Some results of the dissertation have been published in the referred works:

[26]: Georgiev, D.: *SQEMA with Universal Modality.* proc. of the 10^{th} Panhellenic Logic Symposium, 2015, pp. 76–81.

[27]: Georgiev, D.: *Deterministic SQEMA and application for pre-contact logic*, Annual of Sofia University "St. Kliment Ohridski" Faculty of Mathematics and
Informatics, Sofia 2016, Volume **103**, 2016, pp. 149–176.

[28]: Georgiev, D.: *Computability of definability in the class of all KD45 frames*, 11th International Conference on Advances in Modal Logic, Short presentations, 2016, pp. 59–63.

And an extension of the above is currently under review:

[29]: Georgiev, D.: *Definability in the class of all KD45-frames - computability and complexity*, submitted in November 2016, under review, Journal of Applied Non-Classical Logics

Also, the author is co-authoring with Tinko Tinchev and Philippe Balbiani in the following work, which is still under development:

Balbiani, P., Georgiev, D., Tinchev, T.: *Definability in the Class of All Euclidean Kripke Frames.*

Citations

There are no known citations of the referred publications.

However, there have been references to the Deterministic SQEMA website, `http://www.fmi.uni-sofia.bg/fmi/logic/sqema`, in the following:

Dunin-Keplicz, B., Verbrugge, R.: *Teamwork in Multi-Agent Systems. A Formal Approach*, John Wiley, 2010 section 3.9.

Gabbay, D. M., Schmidt, R. A., Szalas, A.: *Second-order Quantifier Elimination: Foundations, Computational Aspects and Applications*, College Publications, 2008 - Computers.

Lokhorst, G.J., *Mens Rea, Logic and the Brain*, in: *Law and Neuroscience*, Freeman, M., F.B.A, editor, Oxford University Press, 2010, pp. 29–39, citation on page 33.

Presentations at Conferences and Seminars

Parts of the dissertation have been presented at the following presentations:

A) “The algorithm SQEMA for a modal language with the universal modality”, Spring science session of FMI, Sofia University, March 2015.

B) “SQEMA with Universal Modality”, 10th Panhellenic Logic Symposium, Samos, Greece, June 10^{th} 2015.

C) Seminar in IRIT, Toulouse, October 2015.

D) “Deterministic SQEMA and application for the language of pre-contact loggics”, Spring science session of FMI, Sofia University, March 2016.

E) Short presentation “Computability of definability in the class of all KD45 frames”, Advances in Modal Logic, Budapest, Hungary, 2016.

F) “Modal and first-order definability in the class of all KD45 Kripke frames”, Conference on Mathematical Logic, dedicated to the 80th anniversary of prof. Dimiter Skordev, October, Gyolechitsa, Bulgaria, 2016.

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