On the Abstract Computability in Two-sort Structures

V.Boutchkova
Sofia University "St.Kl.Ohridski", Part of the Master's Thesis

November, 1998

1 Computability in N^*

Basic notions and definitions

Let N be an arbitrary countable set. Take 0^* which is not an element of N and an operation *ordered pair*, denoted by $\langle\cdot,\cdot\rangle$ such that no element of $N \cup \{0^*\}$ is an ordered pair. By induction we define the sets $N_0 = N \cup \{0^*\}$ and $N_{n+1} = N_n \cup N_n^2$, where $N_n^2 = \{\langle a, b\rangle \mid a \in N_n \,\&\, b \in N_n\}$. Finally, define the Moskovakis' extension of N, $N^* = \bigcup\limits_{n=0}^{\infty} N_n$.

For $s_1 \dots s_{n+1} \in N^*$ we use $\langle s_1, \dots, s_{n+1}\rangle$ to denote $\langle s_1, \langle s_2, \dots, s_{n+1}\rangle\rangle$.

Recall that ω is the set of all natural numbers. By induction we define the set $\omega^* \subseteq N^*$ as follows:

$$\left| \begin{array}{l} 0^* \in \omega^* \\ (n+1)^* = \langle 0^*, n^*\rangle \in \omega^* \end{array} \right.$$

Define the *left* and *right* functions L and $R : N^* \multimap N*$ as follows:

$$\left| \begin{array}{ll} L(0^*) = R(0^*) = 0^* & \\ L(n) = R(n) = 1^* & \text{for } n \in N \\ L(\langle s, t\rangle) = s,\ R(\langle s, t\rangle) = t & \end{array} \right.$$

By F we denote the set of one argument partial functions $\varphi : N^* \multimap N^*$. Define the following operations on functions in F :

1. composition $(\varphi \circ \psi) \in F$

$(\forall s \in N^*)\ (\forall t \in N^*)$

$(\varphi \circ \psi)(s) = t \Leftrightarrow (\exists p \in N^*)(\psi(s) = p \,\&\, \varphi(p) = t)$

2. pairing $\Pi(\varphi, \psi) \in F$

$(\forall s \in N^*)\ (\forall t \in N^*)$

$\Pi(\varphi,\psi)(s) = t \Leftrightarrow (\exists p,q \in N^*)\ (\langle p,q\rangle = t\ \&\ \varphi(s) = p\ \&\ \psi(s) = q)$

3. iteration $[\varphi,\psi] \in F$

$(\forall s \in N^*)\ (\forall t \in N^*)$

$[\varphi,\psi](s) = t \Leftrightarrow \exists w_0,\dots,w_n \in N^* \big(w_0 = s\ \&\ w_n = t\ \&\ \psi(t) \in N_0\ \&$
$\forall i_{(0\le i<n)}(\varphi(w_i) = w_{i+1}\ \&\ \psi(w_i) \notin N_0)\big)$

We define the constant functions as follows: if $c \in N^*$, by $\check{c}$ we denote the function $\check{c} : N^* \multimap N^*$, such that $\check{c}(s) = c$ for all $s \in N^*$.

Definition 1.1

The function $\varphi \in F$ is said to be *prime computable using constants,* relatively the set $\{\psi_1,\dots,\psi_k\}$, where $\psi_i \in F$ (write $\varphi \in \mathbf{PC}(\psi_1 \dots \psi_k)$), if it can be expressed with functions from $\{L, R, \psi_1,\dots,\psi_k\} \cup \{\check{c}_1,\dots,\check{c}_n,\dots\}$, where $\check{c}_1,\dots,\check{c}_n,\dots$ is an enumeration of the elements of N, using composition, pairing and iteration. The function φ is said to be prime computable with constants (write $\varphi \in \mathbf{PC}$), if it is prime computable relatively the empty set of functions, i.e. $\varphi \in \mathbf{PC}(\varnothing)$.

The function $\varphi \in F$ is said to be *prime computable* relatively the set $\{\psi_1,\dots,\psi_k\}$ (write $\varphi \in PC(\psi_1 \dots \psi_k)$), if it can be expressed with functions from $\{L, R, \psi_1,\dots,\psi_k\}$ using composition, pairing and iteration. The definition of $\varphi \in PC$ is analogous.

Definition 1.2 The function $\varphi \in F$ is said to be *search computable using constants,* relatively $\psi_1,\dots,\psi_k \in F$, write $\varphi \in \mathbf{SC}(\psi_1 \dots \psi_k)$, if there exists a function $\psi \in \mathbf{PC}(\psi_1,\dots,\psi_k)$, such that

$$(\forall s \in N^*)\,(\forall t \in N^*)\,\big(\varphi(s) = t \iff \exists r \in N^*(\psi(\langle r,s\rangle) = t)\big).$$

In the same way we define $\varphi \in \mathbf{SC}$, $\varphi \in SC(\psi_1 \dots \psi_k)$ and $\varphi \in SC$.

Examples The following functions are PC :

1. $0(s) = 0^*$ for $s \in N^*$ $\quad \overline{0} = L^3 \circ [L, L]$
   $\overline{n}(s) = n^*$ $\quad \overline{(n+1)} = \Pi(\overline{0}, \overline{n})$
2. $I(s) = s$ for $s \in N^*$ $\quad I = [L, \overline{0}]$
3. The nowhere defined function $\varnothing^*$: $\quad \varnothing^* = [I, I] \circ \Pi(I, I)$
4. $\chi_{N_0}(s) = \begin{cases} 0^* & , s \in N_0 \\ 1^* & , s \in N^* \setminus N_0 \end{cases} \quad \chi_{N_0} = L \circ [\Pi(\overline{1}, \overline{0}), R] \circ \Pi(\overline{0}, I)$

5. $Sg(s) = \begin{cases} 0^* & , s = 0^* \\ 1^* & , s \in N^* \setminus \{0^*\} \end{cases}$ $\qquad Sg = L \circ [\Pi(\overline{1}, \overline{0}), R] \circ \Pi(L, I)$

6. $\overline{Sg}(s) = \begin{cases} 0^* & , Sg(s) = 1^* \\ 1^* & , Sg(s) = 0^* \end{cases}$ $\qquad \overline{Sg} = L \circ [\Pi(\overline{0}, \overline{0}), R] \circ \Pi(\overline{1}, Sg)$

7. $\chi_{\omega^*}(s) = \begin{cases} 0^* & , s \in \omega^* \\ 1^* & , s \in N^* \setminus \omega^* \end{cases}$ $\qquad \chi_{\omega^*} = L \circ [\varphi, R] \circ \Pi(L, I)$, where

$\varphi(\langle s, \langle s, t \rangle\rangle) = \begin{cases} \langle 1^*, 0^* \rangle & , s \neq 0^* \\ \langle L(t), t \rangle & , s = 0^* \end{cases}$ $\qquad \varphi = L \circ [\psi, \overline{Sg} \circ L \circ R] \circ \Pi(\Pi(\overline{1}, \overline{0}), I)$ and $\psi = \Pi(\Pi(L \circ R^3, R^3), L)$

8. $And(s) = \begin{cases} 0^* & L(s) = R(s) = 0^* \\ 1^* & otherwise \end{cases}$ $\qquad And = Sg \circ L \circ [\Pi(\overline{1}, \overline{0}), Sg \circ R]$

$Or(s) = \begin{cases} 0^* & L(s) = 0^* \text{ or } R(s) = 0^* \\ 1^* & otherwise \end{cases}$ $\qquad Or = \overline{Sg} \circ And \circ \Pi(\overline{Sg} \circ L, \overline{Sg} \circ R)$

9. $N^k = \{\langle s_1 \dots s_k \rangle \mid s_1, \dots, s_k \in N\}$
$N^1 = N$

$\chi_{n^k}(s) = \begin{cases} 0^* & , s \in N \\ 1^* & , s \notin N \end{cases}$ $\qquad \chi_N = \overline{Sg} \circ [\overline{0}, I]$, $\quad \chi_{N^{k+1}} = And \circ \Pi(\chi_N \circ L, \chi_{N^k} \circ R)$

10. If $\varphi \in PC(\psi_1, \dots, \psi_n)$, $\psi \in PC(\psi_1, \dots, \psi_n)$ and $\xi \in PC(\psi_1, \dots, \psi_n)$, then the function $If(\xi, \varphi, \psi)$ is $PC(\psi_1, \dots, \psi_n)$, where

$$If(\xi, \varphi, \psi)(s) = \begin{cases} \varphi(s) & , \text{if } \xi(s) \in N_0 \\ \psi(s) & , \text{if } \xi(s) \in N^* \setminus N_0 \end{cases}$$

$$If(\xi, \varphi, \psi) = L \circ [\Pi(\varphi \circ L, \Pi(\overline{1}, \overline{0})), \overline{Sg} \circ L \circ R] \circ [\Pi(\psi \circ L, \Pi(\overline{1}, \overline{0})), R^2] \circ \Pi(I, \Pi(I, I) \circ \chi_{N_0} \circ \xi)$$

11. If $\varphi \in PC(\psi_1, \dots, \psi_n)$ and $\psi \in PC(\psi_1, \dots, \psi_n)$, then there exists a function $\chi \in PC(\psi_1, \dots, \psi_n)$, such that:

$$\left| \begin{array}{lcl} \chi(\langle 0^*, s \rangle) & \simeq & \varphi(s) \\ \chi(\langle (x+1)^*, s \rangle) & \simeq & \psi(\langle x^*, s, \chi(\langle x^*, s \rangle) \rangle) \end{array} \right. ,$$

for all $s \in N^*$ and $x \in \omega^*$; in the other cases χ is not defined.
$\chi = If(\zeta, \xi, \varnothing^*)$, where $\zeta = And(\overline{(Sg)} \circ \chi_{N_0}, \chi_{\omega^*})$ and $\xi = R^3 \circ \xi_2 \circ \xi_1$, where $\xi_1 = \Pi(L, \Pi(\overline{0}, \Pi(I, \varphi) \circ R))$, $\xi_2 = [\Pi(R \circ L, \Pi(\Pi(\overline{0}, L), \Pi(L \circ R, \psi)) \circ R), L]$.

12. If $\psi \in PC(\psi_1, \dots, \psi_n)$, then there exists a function $\varphi \in PC(\psi_1, \dots, \psi_n)$, such that:

$$\varphi(s) \simeq \mu^* x^*_{x^* \in \omega^*}[\psi(\langle x^*, s\rangle) \in N_0],$$

i.e. $Rng(\varphi) \subseteq \omega^*$ and $\varphi(s) = x^* \Leftrightarrow \psi(\langle x^*, s\rangle) \in N_0 \;\&\; \forall y < x\; (\psi(\langle y^*, s\rangle)\downarrow \;\&\; (\psi(\langle y^*, s\rangle) \notin N_0)$, for all $s \in N^*$. Indeed $\varphi = L \circ [\Pi(\Pi(\overline{0}, L), I), \psi] \circ \Pi(\overline{0}, I)$.

The proof of 10, 11 and 12 for **PC** is similar.

Definition 1.3 Having the partial function $\rho : \omega^n \multimap\!\!\rightarrow \omega$, we can define its corresponding function $\rho^* \in F$ as follows:

$$\begin{array}{lll} \text{for } n \geq 2 \;:\; \rho^*(s) = t & \Leftrightarrow & \exists x_1 \dots x_n, y \in \omega \big(s = \langle x_1^* \dots x_n^*\rangle \;\& \\ & & \qquad y^* = t \;\&\; \rho(x_1, \dots, x_n) = y\big); \\ \text{for } n = 1 \;:\; \rho^*(s) = t & \Leftrightarrow & \exists x, y \in \omega \big(s = x^* \;\&\; y^* = t \;\&\; \rho(x) = y\big), \end{array}$$

for all s and t in N^*.

Proposition 1.4
If the function $\rho : \omega^n \multimap\!\!\rightarrow \omega$ is partial recursive relatively $\psi_1, \dots, \psi_k$, where each $\psi_i : \omega^{n_i} \multimap\!\!\rightarrow \omega$, then $\rho^* \in PC(\psi_1^*, \dots, \psi_k^*)$.

PROOF:
Since ρ is partial recursive relatively $\psi_1, \dots, \psi_k$, it can be expressed with functions from $\{O, S, I_1^m, \dots I_m^m\} \cup \{\psi_1, \dots, \psi_k\}$, using *superposition, prime recursion* and *minimization.* We will prove that $\rho^* \in PC(\psi_1^*, \dots, \psi_k^*)$ using induction of the structure of ρ :

Let $\rho = O$. Define $\rho^{*'} = \overline{0}$. Then $\rho^* = If(\chi_{\omega^*}, \rho^{*'}, \varnothing^*) \in PC$;

Let $\rho = S$ and define $\rho^{*'} = \Pi(\overline{0}, I)$. Then $\rho^* = If(\chi_{\omega^*}, \rho^{*'}, \varnothing^*) \in PC$;

Let $\rho = I_i^m$ and define $\rho^* = L \circ R^{i-1} \in PC$ for $m > 1$, and for $m = 1$ define $\rho^* = I \in PC$.

Superposition: Let $\rho(x_1 \dots x_n) \simeq f(g_1(x_1 \dots x_n) \dots g_s(x_1 \dots x_n))$ and f, $g_1, \dots, g_s$ are partial recursive relatively $\psi_1, \dots, \psi_k$. By the induction hypothesis it follows that $f^*, g_1^*, \dots, g_s^* \in PC(\psi_1^*, \dots, \psi_m^*)$. But $\rho^* = f^* \circ \Pi_s(g_1^* \dots g_s^*)$, where $\Pi_1(g^*) = g^*$, and $\Pi_{s+1}(g_1^* \dots g_{s+1}^*) = \Pi(g_1^*, \Pi_s(g_2^* \dots g_{s+1}^*))$ are $PC(\psi_1^*, \dots, \psi_k^*)$ and therefore so is ρ^*.

Prime recursion: Let

$$\left| \begin{array}{lll} \rho(0, \overline{y}) & \simeq & \varphi(\overline{y}) \\ \rho(x+1, \overline{y}) & \simeq & \psi(x, \overline{y}, \rho(x, \overline{y})) \end{array} \right.$$

where φ and ψ are partial recursive relatively $\psi_1, \dots, \psi_k$ and then $\varphi^*, \psi^* \in PC(\psi_1^*, \dots, \psi_k^*)$. But (see ex.11) there exists $\chi \in PC(\psi_1^*, \dots, \psi_k^*)$, such that $\chi = Pr(\varphi^*, \psi^*)$, i.e.

$$\left| \begin{array}{lcl} \chi(\langle 0^*, s\rangle) & \simeq & \varphi^*(s) \\ \chi(\langle x+1, s\rangle) & \simeq & \psi^*(\langle x^*, s, \chi(\langle x^*, s\rangle)\rangle) \end{array} \right.$$

But $\rho^* = \chi = Pr(\varphi^*, \psi^*)$ and therefore $\rho^* \in PC(\psi_1^*, \dots, \psi_k^*)$.

Minimization: Let $\rho(x_1 \dots x_n) \simeq \mu y[\psi(y, x_1 \dots x_n) = 0]$ and ψ is partial recursive relatively $\psi_1, \dots, \psi_k$. Then $\psi^* \in PC(\psi_1^*, \dots, \psi_k^*)$, but (see ex.12) there exists $\varphi \in PC(\psi_1^*, \dots, \psi_k^*)$ and s.t. $\varphi(s) \simeq \mu^* x^*_{x_* \in \omega^*}[\psi^*(\langle x^*, s\rangle) \in N_0]$, i.e. $\varphi(s) = x^* \Leftrightarrow \psi^*(\langle x^*, s\rangle) \in N_0$ & $\forall y < x$ $(\psi^*(\langle y^*, s\rangle) \downarrow\notin N_0)$, for all $s \in N^*$. Then $\varphi(s) = t \Leftrightarrow \exists z, x_1 \dots x_n \in \omega$ $\big(s = \langle x_1 \dots x_n\rangle$ & $t = z^*$ & $\psi^*(\langle z^*, s\rangle) = 0^*$ & $\forall y < z$ $(\psi^*(\langle y^*, s\rangle) \downarrow\neq 0^*)\big)$, i.e. $\varphi(s) = t$ if and only if $\exists z, x_1 \dots x_n \in \omega$ $\big(s = \langle x_1 \dots x_n\rangle$ & $t = z^*$ & $\psi(z, x_1 \dots x_n) = 0$ & $\forall y < z$ $\psi(y, x_1 \dots x_n) \Downarrow\neq 0\big)$, i.e. $\rho^* = \varphi = \mu^* x^*[\psi^*(\langle x^*, \cdot\rangle) \in N_0] \in PC(\psi_1^* \dots \psi_k^*)$. □

We have the following: $PC \subseteq SC$, $\mathbf{PC} \subseteq \mathbf{SC}$, $PC \subseteq \mathbf{PC}$ and $SC \subseteq \mathbf{SC}$.

In the general case the constant functions are prime computable using constants: $\check{c} \in \mathbf{PC}$, for all $c \in N^*$. And $\check{c} \in PC$, for all $c \in \omega^*$.

2 Abstract structures

Definition 2.1

Let ω be the set of all the natural numbers and N be a countable set. A *structure* we will call an abstract partial two-sort structure

$$\mathfrak{A} = \langle N, \omega; =_N, \neq_N; \Sigma_1, \dots, \Sigma_k\rangle,$$

with two fixed (basic) predicates in N^2 - equality in N $(=_N)$ and inequality in N $(\neq_N)$. $\Sigma_1, \dots \Sigma_k$ are partial predicates, $\Sigma_i \subseteq N^{a_i} \times \omega^{b_i}$, $a_i, b_i \geq 0$ and are not both zero, for all $i, 1 \leq i \leq k$. This kind of structure will be denoted as $\mathfrak{A}(\Sigma_1 \dots \Sigma_k)$.

Definition 2.2

For any predicate $\Sigma \subseteq N^a \times \omega^b$ we define its semicharacteristic function $\widehat{\Sigma} : N^* \multimap N^*$, $\widehat{\Sigma} \in F$, as follows: $Rng(\widehat{\Sigma}) \subseteq \{0^*\}$ and for all $s \in N^*$

$\widehat{\Sigma}(s) = 0^*$ if and only if

$\exists s_1 \dots s_a \in N$ $\exists x_1, \dots x_b \in \omega$

$$\big(s = \langle s_1 \dots s_a, x_1^* \dots x_b^*\rangle \ \&\ (s_1 \dots s_a, x_1 \dots x_b) \in \Sigma\big).$$

Examples:

$=_N = \{(n, n) \,|\, n \in N\}$, $\qquad \widehat{=}_N(s) = 0^* \leftrightarrow \exists n \in N$ $(s = \langle n, n\rangle)$

$\neq_N = \overline{=_N} = \{(n, m) \,|\, n, m \in N$ & $n \neq m\}$

$\widehat{\neq}_N(s) = 0^* \leftrightarrow \exists n \in N \exists m \in N$ $(s = \langle n, m\rangle)$ & $n \neq m)$.

Definition 2.3

Let $\mathfrak{A} = \langle \Sigma_1^{\mathfrak{A}}, \ldots, \Sigma_m^{\mathfrak{A}} \rangle$ and $\mathfrak{B} = \langle \Sigma_1^{\mathfrak{B}}, \ldots, \Sigma_n^{\mathfrak{B}} \rangle$ be structures and Σ_0, $\Sigma_1, \ldots, \Sigma_k$ be predicates.

1. $\Sigma_0 \leq_{\mathbf{SC}} \{\Sigma_1, \ldots, \Sigma_k\}$ if and only if $\widehat{\Sigma}_0 \in \mathbf{SC}(\widehat{\Sigma}_1, \ldots, \widehat{\Sigma}_k)$;

2. $\Sigma_0 \leq_{\mathbf{SC}} \mathfrak{A}$ if and only if $\Sigma_0 \leq_{\mathbf{SC}} \{=_N, \neq_N, \Sigma_1^{\mathfrak{A}}, \ldots, \Sigma_m^{\mathfrak{A}}\}$;

3. $\mathfrak{A} \leq_{\mathbf{SC}} \mathfrak{B}$ if and only if $\forall i_{1 \leq i \leq m}\ \Sigma_i^{\mathfrak{A}} \leq_{\mathbf{SC}} \mathfrak{B}$;

4. $\mathfrak{A} \oplus \mathfrak{B} = \langle N; \omega; =_N, \neq_N; \Sigma_1^{\mathfrak{A}} \ldots \Sigma_m^{\mathfrak{A}}, \Sigma_1^{\mathfrak{B}} \ldots \Sigma_n^{\mathfrak{B}} \rangle = \mathfrak{C}(\Sigma_1^{\mathfrak{A}} \ldots \Sigma_m^{\mathfrak{A}}, \Sigma_1^{\mathfrak{B}} \ldots \Sigma_n^{\mathfrak{B}})$.

5. $\mathfrak{A} \equiv_{\mathbf{SC}} \mathfrak{B}$ if and only if $\mathfrak{A} \leq_{\mathbf{SC}} \mathfrak{B}$ and $\mathfrak{B} \leq_{\mathbf{SC}} \mathfrak{A}$.

In the same way we define the relation $\leq_{SC}$ (without constants) but henceforth we consider the relation $\leq_{\mathbf{SC}}$ only.

The relation $\leq_{\mathbf{SC}}$ is a preorder (reflexive and transitive relation), and $\equiv_{\mathbf{SC}}$ is an equivalence relation.

S-degrees or *structure degrees* we call the equivalence classes, induced by the relation $\equiv_{\mathbf{SC}}$ between structures: given a structure $\mathfrak{A}$, an **S**-degree of $\mathfrak{A}$ we denote its structure degree by $\mathfrak{D}_S(\mathfrak{A})$ and mean the set $\{\mathfrak{B} \mid \mathfrak{B} \equiv_{\mathbf{SC}} \mathfrak{A}\}$.

The set $\{\mathfrak{D}_S(\mathfrak{A}) \mid \mathfrak{A}$ is an abstract structure$\}$ we denote by $\mathfrak{D}$. For $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathfrak{D}$, $\mathfrak{a} \leq_{\mathbf{sc}} \mathfrak{b}$ if and only if there exist structures $\mathfrak{A}$ in $\mathfrak{a}$ and $\mathfrak{B}$ in $\mathfrak{b}$, such that $\mathfrak{A} \leq_{\mathbf{SC}} \mathfrak{B}$. For $\mathfrak{a}$ and $\mathfrak{b}$ in $\mathfrak{D}$ and for $\mathfrak{A} \in \mathfrak{a}$ and $\mathfrak{B} \in \mathfrak{b}$ we define $\mathfrak{a} \cup \mathfrak{b} = \mathfrak{D}_s(\mathfrak{A} \oplus \mathfrak{B})$. So the structure $S = \langle \mathfrak{D}, \leq_{\mathbf{SC}}, \cup, \mathfrak{O} \rangle$ is an upper semilattice with least element the empty structure $\mathfrak{O} = \langle N; \omega; =_N, \neq_N \rangle$.

Lemma 2.4

The following functions are prime computable (PC):

1. The functions $\varphi_{n,m}$, defined as follows:
$\varphi_{0,1}(s) = \begin{cases} 0^* & \text{, if } s \in \omega^* \\ 1^* & \text{, otherwise} \end{cases}$ and $\varphi_{1,0}(s) = \begin{cases} 0^* & \text{, if } s \in N \\ 1^* & \text{, otherwise} \end{cases}$ and
for $n + m \geq 2$, $\varphi_{n,m}(s) = \begin{cases} 0^* & \text{, for } s = \langle s_1 \ldots s_n, t_1 \ldots t_m \rangle \text{ where} \\ & s_1, \ldots, s_n \in N \text{ and } t_1, \ldots, t_m \in \omega^* \\ 1^* & \text{, otherwise} \end{cases}$.

Indeed $\varphi_{0,1} = \chi_{\omega^*}$, $\varphi_{0,m+1} = And \circ \Pi(\overline{Sg} \circ \chi_{N_0}, And \circ \Pi(\chi_{\omega^*} \circ L, \varphi_{0,m} \circ R))$ for $m \geq 1$, $\varphi_{1,0} = \chi_N$, $\varphi_{n+1,m} = And \circ \Pi(\chi_N \circ L, \varphi_{n,m} \circ R)$, for $m \geq 0$ and $n \geq 0$ and $n + m \geq 1$.

2. The functions $\psi_{n,m}$, where for all u and s_j in N^*, $\psi_{n,m}(\langle u, \langle s_1 \dots s_n, \dots \rangle) = \langle s_1 \dots s_n, \underbrace{u \dots u}_{m\ times}, \dots \rangle$, $n \geq 0$, $m \geq 0$.

   Indeed $\psi_{0,m+1} = \psi_{0,m} \circ \Pi(L, I)$, for $m \geq 0$; and $\psi_{n+1,m} = \Pi(L \circ R, \psi_{n,m} \circ \Pi(L, R^2))$, for $n, m \geq 0$. Therefore $\psi_{n,m} \in PC$.

3. The functions γ_p^i, such that for all $s_1, \dots, s_p \in N^*$ and $p \geq 1, i \in \omega$, $\gamma_p^i(\langle s_1 \dots s_p \rangle) = \langle s_1 \dots s_p, i^* \rangle = \langle s_1, \langle s_2 \dots \langle s_p, i^* \rangle \dots \rangle \rangle$.

   Indeed $\gamma_1^i = \Pi(I, \overline{i^*})$, $\gamma_{p+1}^i = \Pi(L, \gamma_p^i \circ R)$, i.e. $\gamma_p^i = \psi_{p,i} \circ \Pi(\overline{i^*}, I)$. Therefore $\gamma_p^i \in PC$.

4. The functions Lt_n, $n \geq 1$, such that for u, s_j in N^*, $Lt_n(\langle s_1 \dots s_n \rangle) = s_n$.

   Indeed $Lt_1 = I$ and $Lt_{n+1} = Lt_n \circ R$, for $n \geq 1$.

5. The functions Fst_n, $n \geq 1$, such that for u, s_j in N^*, $Fst_n(\langle s_1 \dots s_n \dots \rangle) = \langle s_1 \dots s_n \rangle$.

   Indeed $Fst_1 = L$ and $Fst_{n+1} = \Pi(L, Fst_n \circ R)$, for $n \geq 1$.

□

Corollary 2.5

For every structure $\mathfrak{B}^k(\Sigma_1 \dots \Sigma_k)$ with predicates $\Sigma_i \subseteq N^{a_i} \times \omega^{b_i}$, $1 \leq i \leq k$, there exists a structure $\mathfrak{B}$ with one predicate $\mathfrak{B}(\Sigma)$, $\Sigma \subseteq N^a \times \omega^{b+1}$, where $a = \max\{a_i \mid 1 \leq i \leq k\}$ and $b = \max\{b_i \mid 1 \leq i \leq k\}$, such that $\mathfrak{B}^k \equiv_{\mathbf{SC}} \mathfrak{B}$.

PROOF:

Let $a = \max\{a_i \mid 1 \leq i \leq k\}$ and $b = \max\{b_i \mid 1 \leq i \leq k\}$ and define a predicate $\Sigma \subseteq N^a \times \omega^{b+1}$, as follows:

$$\Sigma = \{(s_1 \dots s_a, x_1 \dots x_b, i) \mid 1 \leq i \leq k \;\&\; s_1 \dots s_a \in N \;\&\; x_1 \dots x_b \in \omega \;\&\; (s_1 \dots s_a, x_1 \dots x_b) \in \Sigma_i\}.$$

Here we prove that fact, but we do not give such explicit proof in the rest of this paper.

1) $\mathfrak{B}^k \leq_{\mathbf{SC}} \mathfrak{B}$, proof: For every predicate Σ_i, $1 \leq i \leq k$, for all $s \in N^*$, $\hat{\Sigma}_i(s) = 0^*$ iff $s = \langle s_1 \dots s_{a_i}, x_1^* \dots x_{b_i}^* \rangle \;\&\; (s_1 \dots s_{a_i}, x_1 \dots x_{b_i}) \in \Sigma_i$, i.e. iff $s = \langle s_1 \dots s_{a_i}, x_1^* \dots x_{b_i}^* \rangle \;\&\; (\exists r \in N^*)\ (r = \langle r_1 \dots r_a, p_1 \dots p_b, c \rangle \;\&\; r_1 =_N s_1 \;\&\dots\&\; r_{a_i} =_N s_{a_i} \;\&\; p_1 =_N t_1 \;\&\dots\&\, p_{b_i} =_N t_{b_i} \;\&\; c =_N i^* \;\&\; \hat{\Sigma}(r) = 0^*)$, that is $\hat{\Sigma}_i = If(\varphi_{a_i,b_i}, \hat{\Sigma} \circ \psi_i \circ \gamma^i_{a_i+b_i}, \varnothing^*)$, where φ_{a_i,b_i} and $\gamma^i_{a_i+b_i}$ defined in Lemma2.4 are prime computable, and the function ψ_i is such that for all $s_1 \dots, s_{a_i}, t_1 \dots t_{b_i}, p \in N^*$,

$\psi_i(\langle s_1 \dots s_{a_i}, t_1 \dots t_{b_i}, p\rangle) = \langle s_1 \dots s_{a_i}, \underbrace{c \dots c}_{a-a_i \ times}, t_1 \dots t_{b_i}, \underbrace{d \dots d}_{b-b_i \ times}, p\rangle$, where $c \in N$ and $d \in \omega^*$ are constants, $\psi_i \in \mathbf{PC}$ is prime computable with constants c, since $\psi_i = \psi_{a_i,a-a_i} \circ \Pi(\hat{c}, I) \circ \psi_{a_i+b_i,b-b_i} \circ \Pi(\hat{d}, I)$, where the functions $\psi_{a_i,a-a_i}$ and $\psi_{a_i+b_i,b-b_i}$ defined in Lemma2.4(2) are prime computable (PC). Then $\hat{\Sigma}_i \in \mathbf{SC}(\hat{=}_N, \hat{\neq}_N, \hat{\Sigma})$, i.e. $\Sigma_i \leq_{\mathbf{SC}} (\mathfrak{B})$, i.e. $\mathfrak{B}^k = (\Sigma_1 \dots \Sigma_k) \leq_{\mathbf{SC}} \mathfrak{B}$.

2) $\mathfrak{B} \leq_{\mathbf{SC}} \mathfrak{B}^k$, proof:
$\hat{\Sigma}(r) = 0^*$ iff $s = \langle s_1 \dots s_a, x_1^* \dots x_b^*, i^*\rangle$ & $(s_1 \dots s_a, x_1 \dots x_b, i) \in \Sigma$ iff $s = \langle s_1 \dots s_a, x_1^* \dots x_b^*, i^*\rangle$ & $1 \leq i \leq k$ & $s_1 \dots s_a \in N$ & $x_1 \dots x_b \in \omega$ & $(s_1 \dots s_{a_i}, x_1 \dots x_{b_i}) \in \Sigma$. $\hat{\Sigma} = If(And(\varphi_{a_i,b+1}, \varphi_k \circ Lt_{a+b+1}), \psi_k, \varnothing^*)$, where $\varphi_{a,b+1}$ and Lt_{a+b+1} are defined in Lemma2.4 and φ_k is such that for $Dom(\varphi_k) = \omega^*$ and $\varphi_k(x^*) = 0^*$ iff $1 \leq x$ & $x \leq k$, i.e. $\varphi_k = If(\chi_{\omega^*}, And(\delta^*, \rho_k^*), \varnothing^*)$, where δ and ρ_k are functions from ω into ω, $\delta(x) = 0 \Leftrightarrow 1 \leq x$, and $\rho_k(x) = 0$ iff $x \leq k$, and δ^* and ρ^* are prime computable, see *Proposition 1.4.* The functions ψ_k is such that $\psi_k(\langle s_1 \dots s_a, x_1^* \dots x_b^*, i^*\rangle) = \hat{\Sigma}_i(\langle s_1 \dots s_{a_i}, x_1^* \dots x_{b_i}^*\rangle)$, for $1 \leq i \leq k$. Indeed $\psi_1 = If(\rho_1^*, \hat{\Sigma}_1 \circ Fst_{a_1+b_1}, \varnothing^*)$, that is $\psi_1 \in PC(\hat{\Sigma}_1)$; and $\psi_k = If(\rho_k^*, \hat{\Sigma}_k \circ Fst_{a_k+b_k}, \psi_{k-1}) \in PC(\hat{\Sigma}_1, \dots, \hat{\Sigma}_k)$. Therefore $\hat{\Sigma} \in PC(\hat{\Sigma}_1 \dots \hat{\Sigma}_k)$ and $\mathfrak{B} \leq_{\mathbf{SC}} \mathfrak{B}^k$.

□

3 N-enumerations

Definition 3.1

An *N-enumeration* is a total and surjective function $f_N : \omega \to N$. Define the following sets:

1. $f_N^{-1}(\Sigma) := \{\langle x_1 \dots x_a, y_1 \dots y_b\rangle \in \omega \mid (f_N(x_1) \dots f_n(x_a), y_1 \dots y_b) \in \Sigma\}$.
2. $E_f := \{\langle x, y\rangle \in \omega \mid f_N(x) =_N f_N(y)\} = f_N^{-1}(=_N)$;
   $\overline{E_f} := \{\langle x, y\rangle \in \omega \mid f_N(x) \neq_N f_N(y)\} = f_N^{-1}(\neq_N)$; and $E_f^+ := E_f \oplus \overline{E_f}$.

Definition 3.2

Let $f^* : \omega \to N^*$. For the functions $\varphi \in \mathcal{F}$ and the predicate $\Sigma \subseteq N^a \times \omega^b$, define:

1. $E^* := \{\langle x, y\rangle \in \omega \mid f^*(x) = f^*(y)\}$.
2. ${f^*}^{-1}(\varphi) := \{\langle x, y\rangle \in \omega \mid \varphi(f^*(x))\} = {f^*}^{-1}(Gr_\varphi)$.
3. ${f^*}^{-1}(\Sigma) := \{x \in \omega \mid \hat{\Sigma}(f^*(x)) = 0^*\} =$
   $\{x \mid f^*(x) = \langle s_1 \dots s_a, y_1^* \dots y_b^*\rangle$ & $(s_1 \dots s_a, y_1 \dots y_b) \in \Sigma\}$.

Lemma 3.3

The function $f^* : \omega \to N^*$ is total and surjective; there exist a recursive function $J : \omega^2 \to \omega$, such that for all $x, y \in \omega$, $f^*(J(x,y)) = \langle f^*(x), f^*(y) \rangle$, and there exists a recursively enumerable set X, such that $f^*[X] = N$. Then for all $\psi, \varphi_1 \ldots \varphi_k \in \mathcal{F}$, if $\psi \in \mathbf{SC}(\varphi_1 \ldots \varphi_k)$, then $f^{*-1}(\psi) \leq_e f^{*-1}(\varphi_1) \oplus \ldots \oplus f^{*-1}(\varphi_k) \oplus E^*$.

Proof:

Since $\psi \in \mathbf{SC}(\varphi_1 \ldots \varphi_k)$, there exists $\varphi \in \mathbf{PC}(\varphi_1 \ldots \varphi_n)$, such that $\psi(s) = t \Leftrightarrow (\exists u \in N^*)\ (\varphi\langle u, s \rangle) = t)$. Therefore $\langle x, y \rangle \in f^{*-1}(\psi) \Leftrightarrow \psi(f^*(x)) = f^*(y) \Leftrightarrow (\exists u \in N^*)\ (\varphi(\langle u, f^*(x) \rangle) = f^*(y))$ /since f is surjective/ $\Leftrightarrow (\exists t \in \omega)\ \varphi(\langle f^*(t), f^*(x) \rangle) = f^*(y)) \Leftrightarrow (\exists t \in \omega)\ (\varphi(f^*(J(t,x))) = f^*(y)) \Leftrightarrow (\exists t, z \in \omega)\ (J(t,x) = z\ \&\ \varphi(f^*(z)) = f^*(y))$, therefore $f^{*-1} \leq_e f^{*-1}(\varphi)$.

By induction on the definition of $\mathbf{PC}$, we are going to prove that for $\varphi \in \mathbf{PC}(\varphi_1 \ldots \varphi_k)$, $f^*(\varphi) \leq_e f^{*-1}(\varphi_1) \oplus \ldots \oplus f^{*-1}(\varphi_k) \oplus E^*$, from which follows that $f^*(\psi) \leq_e f^{*-1}(\varphi_1) \oplus \ldots \oplus f^{*-1}(\varphi_k) \oplus E^*$.

Let c_0 and $c_1 \in \omega$ be such that

$f^*(c_0) = 0^*$ and $f^*(c_1) = 1^*$. Such c_0 and c_1 exist, since f^* is a surjective function.

- If $\varphi = L$
  $\langle z, y \rangle f^{*-1}(L)$ iff $L(f^*(z)) = f^*(y)$ iff $f^*(z) = f^*(y) = 0^* \vee f^*(z) \in N$ & $f^*(y) = 1^* \vee f^*(z) \in N^* \backslash N - 0\ \&\ \exists x \in \omega\ \langle f^*(y), f^*(x) \rangle f^*(z))$ iff $\langle z, c_0 \rangle, \langle y, c_0 \rangle \in E^* \vee \exists x \in X(\langle z, x \rangle, \langle y, c_1 \rangle \in E^*) \vee \exists x \in \omega(f^*(J(y,x)) = f^*(z))$ iff $\langle z, c_0 \rangle, \langle y, c_0 \rangle \in E^* \vee \exists x \in X(\langle z, x \rangle, \langle y, c_1 \rangle \in E^*) \vee \exists x, t \in \omega(J(y,x) = t\ \&\ (f^*(t) = f^*(z))$ iff $\langle z, c_0 \rangle, \langle y, c_0 \rangle \in E^* \vee \exists x \in X\langle z, x \rangle, \langle y, c_1 \rangle \in E^*) \vee \exists x, t \in \omega(J(y,x) = t\ \&\ \langle t, z \rangle \in E^*)$, i.e. $f^{*-1}(L) \leq_e E^*$, and it follows that $f^{*-1}(L) \leq_e f^{*-1}(\varphi_1) \oplus \ldots \oplus f^{*-1}(\varphi_k) \oplus E^*$.

- For $\varphi = L$ the proof is similar.

- For $\varphi = \varphi_i$, $1 \leq i \leq k$.
  $f^{*-1}(\varphi) = f^{*-1}(\varphi_i) \leq_e f^{*-1}(\varphi_1) \oplus \ldots \oplus f^{*-1}(\varphi_k) \oplus E^*$.

- For $\varphi = \hat{c}$ the constant function, for $c \in N$.
  Therefore $f^{*-1}(\varphi) = f^{*-1}(\hat{c}) = f^{*-1}(Gr_{\hat{c}}) = \{\langle x, y \rangle \in \omega \mid \hat{c} f^*(x) =$

$f^*(y)\} = \{\langle x,y\rangle \in \omega \mid c = f^*(y)\}$. Since f^* is surjective, there exists $d \in \omega$, such that $f^*(d) = c$. Then $\langle x,y\rangle \in {f^*}^{-1}(\hat{c})$ iff $\langle x,y\rangle \in E^*$.

The induction hypothesis for ψ_1 and $\psi_2 \in \mathbf{PC}(\varphi_1 \ldots \varphi_k)$, ${f^*}^{-1}(\psi_j) \leq_e {f^*}^{-1}(\varphi_1) \oplus \ldots \oplus {f^*}^{-1}(\varphi_k) \oplus E^*$.

- Composition $\varphi = \psi_1 \circ \psi_2$.
  We have the following equivalences $\langle z,y\rangle \in {f^*}^{-1}(\varphi) \Leftrightarrow \varphi(f^*(z)) = f^*(y) \Leftrightarrow \exists s \in N^*\ (\psi_2(f^*(z)) = s\ \&\ \psi_1(s) = f^*(y))$ and since f^* is surjective, this is equivalent to $\exists x \in \omega\ \psi(f^*(z)) = f^*(x)\ \&\ \psi_1(f^*(x)) = f^*(y))$, i.e. $\exists x\ (\langle z,x\rangle \in {f^*}^{-1}(\psi_2)\ \&\ \langle x,y\rangle \in {f^*}^{-1}(\psi_1))$, therefore ${f^*}^{-1}(\varphi) \leq_e {f^*}^{-1}(\psi_1) \oplus {f^*}^{-1}(\psi_2)$, i.e. ${f^*}^{-1}(\varphi) \leq_e {f^*}^{-1}(\varphi_1) \oplus \ldots \oplus {f^*}^{-1}(\varphi_k) \oplus E^*$.

- Pairing $\varphi = \Pi(\psi_1, \psi_2)$.
  Using that f^* is surjective, we obtain the following equivalences $\langle z,y\rangle \in {f^*}^{-1}(\varphi) \Leftrightarrow \varphi(f^*(z)) = f^*(y) \Leftrightarrow \exists s_1, s_2 \in N^*\ (\langle s_1, s_2\rangle = f^*(y)\ \&\ \psi(f^*(z)) = s-1\ \&\ \psi_2(f^*(z)) = s_2) \Leftrightarrow \exists x_1, x_2 \in \omega\ (\langle f^*(x_1), f^*(x_2)\rangle = f^*(y)\ \&\ \psi_1(f^*(z)) = f^*(x_1)\ \&\ \psi_2(f^*(z)) = f^*(x_2)) \Leftrightarrow \exists x_1, x_2 \in \omega\ (f^*(J(x_1,x_2)) = f^*(y)\ \&\ \psi_2(f^*(z)) = {f^*}^{-1}(x_1)\ \&\ \psi_2(f^*(z)) = f^*(x_2)) \Leftrightarrow \exists x_1, x_2, t\ (J(x_1,x_2) = t\ \&\ \langle t,y\rangle \in E^*\ \&\ \langle z,x_1\rangle \in {f^*}^{-1}(\psi_1)\ \&\ \langle z,x_2\rangle \in {f^*}^{-1}(\psi_2))$, i.e. ${f^*}^{-1}(\varphi) \leq_e {f^*}^{-1}(\psi_1) \oplus {f^*}^{-1}(\psi_2) \oplus E^*$, therefore ${f^*}^{-1}(\varphi \leq_e {f^*}^{-1}(\varphi_1) \oplus \ldots \oplus {f^*}^{-1}(\varphi_k) \oplus E^*$.

- Iteration $\varphi = [\psi_1, \psi_2]$.
  We have that $\langle z,y\rangle \in {f^*}^{-1}(\varphi) \Leftrightarrow \varphi(f^*(z)) = f^*(y) \Leftrightarrow [\psi_1,\psi_2](f^*(z)) = f^*(y) \Leftrightarrow \psi_2(f^*(y)) \downarrow\in N_0\ \&\ \exists w_0, \ldots, w_n \in N^*\ (w_0 = f^*(z)\ \&\ w_n = f^*(y)\ \&\ \forall i\ (0 \leq i < n)\ \big(\psi_1(w_i) = w_{i+1}\ \&\ \psi_2(w_i) \notin N_0)\big)$, and since f^* is surjective we obtain that the last part is equivalent to $\exists x_0, \ldots x_n \in \omega\ \big(f^*(x_0) = f^*(z)\ \&\ f^*(x_n) = f^*(y)\ \&\ \forall i\ (0 \leq i < n)\ \big(\psi_1(f^*(x_i)) = f^*(x_{i+1})\ \&\ \psi_2(f^*(x_i)) \notin N_0)\big)$. Therefore $\langle z,y\rangle \in {f^*}^{-1}(\varphi) \Leftrightarrow \big(\psi_2(f^*(y)) = f^*(c_0) = 0^* \vee \psi_2(f^*(y)) \in N\big)\ \&\ \exists x_0 \ldots x_n \in \omega\ \big(\langle x_0, z\rangle, \langle x_n, y\rangle \in E^*\ \&\ \forall i_{(0 \leq i < n)}\ (\langle x_i, x_{i+1}\rangle \in {f^*}^{-1}(\psi_1)\ \&\ \psi_2(f^*(x_i)) \downarrow\in N^* \backslash N_0)\big) \Leftrightarrow$
  $\big(\langle y, c_0\rangle \in {f^*}^{-1}(\psi_2) \vee \exists t \in X(\psi_2(f^*(y)) = f^*(t))\big)\ \&\ \exists x_0 \ldots x_n \in \omega\ \big(\langle x_0, z\rangle, \langle x_n, y\rangle \in E^*\ \&\ \forall i_{(0 \leq i < n)}\ (\langle x_i, x_{i+1}\rangle \in {f^*}^{-1}(\psi_1)\ \&\ \exists t_{i_1}, t_{i_2}, t_i \in \omega\ (J(t_{i_1}, t_{i_2}) = t_i\ \&\ \psi_2(f^*(x_i)) = f^* t_i))\big) \Leftrightarrow \exists t \in X \cup \{c_0\}\ \big(\langle y,t\rangle \in {f^*}^{-1}(\psi_2)\big)\ \&\ \exists x_0, \ldots, x_n \in \omega\ \big(\langle x_0, z\langle \in E^*\ \&\ \langle x_n, y\langle \in E^*\ \&\ \forall i_{(0 \leq i < n)}$

$\big(\langle x_i, x_{i+1}\rangle \in f^{*-1}(\psi_1)\ \&\ \exists t_i \in Range(J)\ (\langle x_i, t_i\rangle \in f^{*-1}(\psi_2))\big)$, i.e. $f^{*-1}(\varphi) \leq_e f^{*-1}(\psi_1) \oplus f^{*-1}(\psi_2) \oplus E^*$, therefore $f^{*-1}(\varphi) = leq_e f^{*-1}(\varphi_1) \oplus \ldots \oplus f^{*-1}(\varphi_k) \oplus E^*$.

□

Corollary 3.4

Let $f^* : \omega \to N^*$ be a total surjective function and $J : \omega^2 \to \omega$, such that for all $x, y \in \omega$, $f^*(J(x,y)) = \langle f^*(x), f^*(y)\rangle$ and there is a recursively enumerable set X, such that $f^*(X) = N$, $\Sigma_i \subseteq N^{a_i} \times \omega^{b_i}$, $0 \leq i \leq k$, are predicates, such that $\Sigma_0 \leq (\Sigma_1 \ldots \Sigma_k)$. Therefore

$$f^{*-1}(\Sigma) \leq_e f^{*-1}(\Sigma_1) \oplus \ldots \oplus f^{*-1}(\Sigma_k) \oplus E^*.$$

Proposition 3.5

Let $f_N : \omega \to N$ be an N-enumeration and $\Sigma_i \subseteq N^{a_i} \times \omega^{b_i}$, $0 \leq i \leq k$, such that $\Sigma_0 \leq_{\mathbf{SC}} (\Sigma_1 \ldots \Sigma_k)$.

Therefore $f^{*-1}(\Sigma_0) \leq_e f^{*-1}(\Sigma_1) \oplus \ldots \oplus f^{*-1}(\Sigma_k) \oplus E_f$.

PROOF:

Let $J(x,y) := 2\langle x,y\rangle + 2$, where the pairing is recursive injective functions, for example $\langle x,y\rangle := 2^x.(2y+1) - 1$. Then for all t and z, $t < J(t,z)$ and $z < J(t,z)$.

Having $f_N : \omega \to N$, define a total surjective function $f^* : \omega \to N^*$, as follows:

$$\left|\begin{array}{lcl} f^*(J(x,y)) & = & \langle f^*(x), f^*(y)\rangle \\ f^*(2z+1) & = & f_N(z) \\ f^*(0) & = & 0^* \end{array}\right.$$

The set $X = \{2z+1 \mid z \in \omega\}$ is recursive and $f^*(X) = N$. Therefore, using Lemma3.3, $f^{*-1}(\Sigma_0) \leq_e f^{*-1}(\Sigma_1) \oplus \ldots \oplus f^{*-1}(\Sigma_k) \oplus E^*$.

There exists a recursive function g, such that $y^* = f^*(g(y))$, that we define as follow:

$$\left|\begin{array}{lcl} g(0) & = & 0 \\ g(y+1) & = & J(0, g(y)) \end{array}\right.$$

Therefore there exists a recursive function $h_{a,b}$, such that $\langle f_N(z_1) \ldots f_N(z_a), y_1^* \ldots y_b^*\rangle = \langle f^*(2z_a+1), f^*(g * y_1)) \ldots f^*(g(y_b))\rangle = f^*(J(2z_1+1, J(\ldots, J(g(y_b))\ldots))) = f^*(h_{a,b}(\langle z_1 \ldots z_a, y_1 \ldots y_b\rangle))$.

1) We can prove that for every predicate $\Sigma \subseteq N^a \times \omega^b$, $f^{*-1}(\Sigma) \leq E^* \oplus {f_N}^{-1}(\Sigma)$, from which follows that $f^{*-1}(\Sigma_0) \leq_e {f_N}^{-1}(\Sigma_1) \oplus \ldots \oplus {f_N}^{-1}(\Sigma_k) \oplus E^*$. Indeed

$x \in f^{*-1}(\Sigma) \iff \hat{\Sigma}(f^*(x)) = 0^* \iff$

$\exists s_1 \ldots s_a \in N\ \exists y_1 \ldots y_b \in \omega$

$\big(f^*(x) = \langle s_1 \dots s_a, y_1^* \dots y_b^* \rangle \ \& \ (s_1 \dots s_a, y_1 \dots y_b) \in \Sigma\big) \iff$
$\exists z_1 z_a, y_1 \dots y_b \in \omega$
$\big(f^*(x) = \langle f_N(z_1) \langle f_N(z_a), y_1^* \dots y_b^* \rangle \ \& \ (f_N(z_1) \dots f_N(z_a), y_1 \dots y_b) \in \Sigma) \Leftrightarrow$
$\exists z_1 \dots z_a, y_1 \dots y_b$
$\big(f^*(x) = f^*(h_{a,b}(\langle z_1 \dots z_a, y_1 \dots y_b \rangle)) \ \& \ \langle z_1 \dots z_a, y_1 \dots y_b \rangle \in {f_N}^{-1}(\Sigma)\big) \Leftrightarrow$
$\exists z_1 \dots z_a, y_1 \dots y_b \big(\langle x, \langle z_1 \dots z_a, y_1 \langle y_b \rangle \rangle \in E^* \ \& \ \langle z_1 \dots z_a, y_1 \dots y_b \rangle \in f_N^{-1}(\Sigma)\big)$.

Therefore ${f^*}^{-1}(\Sigma_0) \leq_e {f^*}^{-1}(\Sigma_1) \oplus \dots \oplus {f^*}^{-1}(\Sigma_k) \oplus E^* \leq_e$ ${f_N}^{-1}(\Sigma_1) \oplus \dots \oplus {f_N}^{-1}(\Sigma_k) \oplus E^*$.

2) We can prove that for every predicate $\Sigma \subseteq N^a \times \omega^b$, $f_N^{-1}(\Sigma) \leq_e {f^*}^{-1}(\Sigma) \oplus E^*$. Indeed
$\langle x_1 \dots x_a, y_1 \dots y_b \rangle \in f_N^{-1}(\Sigma) \Leftrightarrow (f_N(x_1) \dots f_N(x_a), y_1 \dots y_b) \in \Sigma \Leftrightarrow$
$\exists s \in N^* \ \big(\hat{\Sigma}(s) = 0^* \ \& \ 4s = \langle f_N(x_1) \dots f_N(x_a), y_1^* \dots y_b^* \rangle\big) \Leftrightarrow$
$\exists x \in \omega \ \big(\hat{\Sigma}(f^*(x)) = 0^* \ \& \ f^*(x) = \langle f_N(x_1) \dots f_N(x_a), y_1^* \dots y_b^* \rangle\big) \Leftrightarrow$
$\exists x \ \big(x \in {f^*}^{-1}(\Sigma) \ \& \ f^*(x) = f^*(x) = f^*(h_{a,b}(\langle x_1 \dots x_a, y_1 \dots y_b \rangle))\big) \Leftrightarrow$
$\exists x, z \ \big(x \in {f^*}^{-1}(\Sigma) \ \& \ 4h(\langle x_1 \dots x_a, y_1 \langle y_b \rangle) = z \ \& \ \langle x, z \rangle \in E^*\big)$.

Therefore $f_N^{-1}(\Sigma_0) \leq_e {f^*}^{-1}(\Sigma_0) \oplus E^*$, and since ${f^*}^{-1}(\Sigma_0) \leq_e f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \oplus E^*$, we obtain that $f_n^{-1}(\Sigma_0) \leq_e f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \oplus E^*$.

3) We have $E^* \leq_e E_f$, from which follows that $f_N^{-1}(\Sigma_0) \leq_e f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \oplus E_f$.
For the proof of that statement is necessary to know that $\langle x, y \rangle \in E^* \Leftrightarrow f^*(x) = f^*(y) \Leftrightarrow x = y = 0 \vee \exists z, t \ \big(x = 2z{+}1 \ \& \ y = 2t{+}1 \ \& \ f_N(z) = f_N(t)\big)$ $\vee \ \exists x_1, x_2, y_1, y_2$ such that $(x_1 < x \ \& \ y_1 < y \ \& \ x_2 < x \ \& \ y_2 < y)$ and $\big(x = J(x_1, x_2) \ \& \ y = J(y_1, y_2) \ \& \ f^*(x_1) = f^*(y_1) \ \& \ f^*(x_2) = f^*(y_2)\big)$.

□

Definition 3.6

For an N-enumeration $f_N : \omega \to N$ and a structure $\mathfrak{A}(\Sigma_1 \dots \Sigma_k)$, define:

$$f_N^{-1}(\mathfrak{A}) := \{\langle i, z \rangle \mid z \in f_N^{-1}(\Sigma_i)\} \cup \{\langle 0, z \rangle \mid z \in E_f^+\}.$$

Remark: $f_N^{-1}(\mathfrak{A}) \equiv_e f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \oplus E_f \oplus \overline{E_f}$.

Definition 3.7

1) N-string τ_N is a function $\tau_N : [0 \dots n{-}1] \to N$, with length $lh(\tau_N) = n$.
2) $\tau_N \subseteq \sigma_N$ iff $\forall x (x < lh(\tau_N) \Rightarrow \tau_N(x) = \sigma_N(x))$.
3) Code of the N-string τ_N define to be $\ulcorner \tau_N \urcorner = \langle n^*, \tau_N(0), \dots, \tau_N(n-1) \rangle$.

Remark: The functions h_0, h_1, h_2, such that

$\forall s \in N^*(h_0(s)\downarrow \Leftrightarrow \exists \tau_N(s = \ulcorner \tau_N \urcorner))$,
$h_1(\langle \ulcorner \tau_N \urcorner, \ulcorner \sigma_N \urcorner \langle) = 0^* \Leftrightarrow \tau_N \subseteq \sigma_N$,
$h_2(\langle \ulcorner \tau_N \urcorner, x^*, y\rangle) = 0^* \Leftrightarrow \tau_N(x) = y)$
are prime computable.

Definition 3.8

For a structure $\mathfrak{A} = \mathfrak{A}(\Sigma_1 \dots \Sigma_k)$ with predicates $\Sigma_i \subseteq N^{a_i} \times \omega^{b_i}$, a N-string τ_N and a formula $F_e(z)$ with $e, z \in \omega$, define $\tau_n \Vdash_{\mathfrak{A}} F_e(z)$ as follows:

$\tau_n \Vdash_{\mathfrak{A}} F_e(z)$ iff $\exists v(\langle v, z\rangle \in W_e \;\&\; \tau_n \Vdash_{\mathfrak{A}} D_v)$
and
$\tau_n \Vdash_{\mathfrak{A}} D_v$ iff $\forall u \in D_v(u = \langle i, \langle x_1 \dots x_{a_i}, y_1 \dots y_{b_i}\rangle\rangle \;\&$
$1 \leq i \leq k \;\&\; x_1 \dots x_{a_i} \in Dom(\tau_N) \;\&(\tau_N(x_1) \dots \tau_N(x_{a_i}), y_1 \dots y_{b_i}) \in \Sigma_i \vee$
$u = \langle 0, 2.\langle x, y\rangle\rangle \;\&\; x, y \in Dom(\tau_N) \;\&$
$\tau_N(x) = \tau_N(y) \;\&\; u = \langle 0, 2.\langle x, y\rangle + 1\rangle \;\&\; x, y \in Dom(\tau_N) \;\&\; \tau_N(x) \neq_N \tau_N(y))$.

Properties:
1) If $\tau_n \subseteq \sigma_N$ and $\tau_N \Vdash_{\mathfrak{A}} F_e(z)$, then $\sigma_N \Vdash_{\mathfrak{A}} F_e(z)$.
2) The function h_3, such that $h_3(\langle \ulcorner \tau_N \urcorner, e^*, z^*\rangle) = 0* \Leftrightarrow \tau_N \Vdash_{\mathfrak{A}} F_e(z)$ is $\mathbf{SC}(\hat{\Sigma}_1 \dots \hat{\Sigma}_k, \hat{=}_N, \hat{\neq}_N)$.

Definition 3.9

For an N-enumeration $f_N : \omega \to N$ and a structure $\mathfrak{A} = \mathfrak{A}(\Sigma_1 \dots \Sigma_k)$ with predicates $\Sigma_i \subseteq N^{a_i} \times \omega^{b_i}$, define

$$f_n \models_{\mathfrak{A}} F_e(z) \text{ if and only if } z \in \Psi_e(f_N^{-1}(\mathfrak{A})).$$

Proposition 3.10 $f_N \models_{\mathfrak{A}} F_e(z)$ iff $\exists \tau_N \subseteq f_N(\tau_N \Vdash_{\mathfrak{A}} F_e(z))$.

Definition 3.11

The predicate $\Sigma \subseteq N^a \times \omega^b$ has *normal form* in the structure $\mathfrak{A} = \mathfrak{A}(\Sigma_1 \dots \Sigma_k)$, when there exist $e \in \omega$, an N-string δ_N and $x_1 \dots x_a \notin Dom(\delta_N)$, such that
$(\forall s_1 \dots s_a \in N, \forall y_1 \dots y_b \in \omega)$
$((s_1 \dots s_a, y_1 \dots y_b) \in \Sigma$ iff $\exists \tau_N \supseteq \delta_N$ such that
$(\tau_N(x_1) = s_1 \;\&\; \dots \&\; \tau_N(x_a) = s_a \;\&\; \tau_N \Vdash_{\mathfrak{A}} F_e(\langle x_1 \dots x_a, y_1 \dots y_b\rangle)))$.

Theorem 3.12 (The Normal form theorem)

Let $\mathfrak{A}(\Sigma_1 \dots \Sigma_k)$ be a structure, with predicates $\Sigma_i \leq N^{a_i} \times \omega^{b_i}$. Then for every predicate $\Sigma \subseteq N^a \times \omega^b$ the followings are equivalent:

1) Σ has in normal form in $\mathfrak{A}$.

2) $\Sigma \leq_{\mathbf{SC}} \mathfrak{A}$.

3) $f_N^{-1}(\Sigma) \leq_e f_N^{-1}(\mathfrak{A})$ where f_N is an arbitrary N-enumeration.

Proof:

1) (3) follows from (2).

Let $\Sigma \leq_{\mathbf{SC}} \mathfrak{A}$, i.e. $\Sigma \leq_{\mathbf{SC}} (\Sigma_1 \dots \Sigma_k, =_N, \neq_N)$. Therefore, using Corollary 7.6, for every N-enumeration f_N, we have that $f_N^{-1}(\Sigma) \leq_e f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \oplus f_N^{-1}(=_N) \oplus f_N^{-1}(\hat{\neq}_N) \oplus E_f$, but $f_N^{-1}(=_N) = E_f$ and $f_N^{-1}(\hat{\neq}_N) = \overline{E}_f$, therefore $f_N^{-1}(\Sigma) \leq_e f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \oplus E_f^+ \equiv_e f_N^{-1}(\mathfrak{A})$.

2) (1) follows from(3).

Let for every total surjective function $f_N : \omega \to N$, $f_N^{-1}(\Sigma) \leq_e f_N^{-1}(\mathfrak{A})$, i.e. there exists e, such that $f_N^{-1}(\Sigma) = \Psi_e(f_N^{-1}(\mathfrak{A}))$. Assume that Σ has no normal form in $\mathfrak{A}$, i.e. for all $e \in \omega$, for all N-string δ_N and $x_1 \dots x_a \in \omega \setminus Dom(\delta_N)$, there exist $s_1 \dots s_a \in N$ and $y_1 \dots y_b \in \omega$, such that the following holds true:

$$(*) \quad (\overline{s}, \overline{y}) \notin \Sigma \text{ iff } \exists \tau_N \supseteq \delta_N \big(\tau_N(x_1) = s_1 \,\&\, \dots \,\&\, \tau_N(x_a) = s_a \,\&\, \tau_N \Vdash_{\mathfrak{A}} F_e(\langle \overline{x}, \overline{y} \rangle)\big)$$

We can define an N-enumeration $f_N : \omega \to N$, such that $f_N^{-1} \not\leq_e f_N^{-1}(\mathfrak{A})$, i.e. such that for every e, $f_N^{-1}(\Sigma) \neq \Psi_e(f_N^{-1}(\mathfrak{A}))$. First we define by induction N-strings σ_N^q, such that $\sigma_N^0 \subseteq \dots \subseteq \sigma_N^q \subseteq \dots$, so that at stage $2e+1$ we insure f_N to be total and surjective,and at stage $2e+2$ we insure $f_N^{-1}(\Sigma) \neq \Psi_e(f_N^{-1}(\mathfrak{A}))$, for $f_n := \bigcup_q \sigma_N^q$, as follows:

Stage 0

$\sigma_N^0 = \varnothing$

Stage 2e+1

Suppose we have already defined σ_N^q, for $q = 2e$. Let $x = lh(\sigma_N^q)$, for the least $n \notin N$, such that $n \notin Range(\sigma_N^q)$.

Define $\sigma_N^{q+1} \supseteq \sigma_N^q$, $lh(\sigma_N^{q+1}) = lh(\sigma_N^q) + 1$ and $\sigma_N^{q+1}(x) = n$.

Stage 2e+2

Suppose we have already defined σ_N^q, where $q = 2e+1$.

Let $x_1 \dots x_a \in \omega$ are such that $x_j = lh(\sigma_N^q) + j - 1$, $1 \leq j \leq a$. Therefore $x_1 \dots x_a \notin dom(\sigma_N^q)$ are the first a elements, where σ_N^q is not defined.

Case 1. There exist $s_1 \dots s_a \in N$ and $y_1 \dots y_b \in \omega$, such that **(*)** holds true, i.e. $(\overline{s}, \overline{y}) \notin \Sigma$ iff $\exists \tau_N \supseteq \sigma_N^q \big(\tau_N(x_1) = s_1 \,\&\, \dots \,\&\, \tau_N(x_a) = s_a \,\&\, \tau_N \Vdash_{\mathfrak{A}} F_e(\langle \overline{x}, \overline{y} \rangle)\big)$, i.e. for $\delta_N \supseteq \sigma_N^q$, such that $\delta_N(x_1) = s_1 \,\&\, \dots \,\&\, \delta_N(x_a) = s_a \,\&$

$lh(\delta_N(x_a) = lh(\sigma_N^q) + a$, the following holds: $(\overline{s},\overline{y}) \notin \Sigma$ iff $\exists \tau_N \supseteq \delta_N$ $\big(\tau_N \Vdash_{\mathfrak{A}} F_e(\langle \overline{x},\overline{y}\rangle)\big)$.

Subcase 1.1. $\exists \tau_N \supseteq \delta_N$ $\big(\tau_N \Vdash_{\mathfrak{A}} F_e(\langle \overline{x},\overline{y}\rangle)\big)$. Define $\sigma_N^{q+1} := \tau_N$.

Subcase 1.2. $\forall \tau_N \supseteq \delta_N$ $\big(\tau_N \nVdash_{\mathfrak{A}} F_e(\langle \overline{x},\overline{y}\rangle)\big)$. Define $\sigma_N^{q+1} := \delta_N$.

Case 2. Otherwise, define $\sigma_N^{q+1} := \sigma_N^q$.

Define $f := \bigcup_q \sigma_N^q$, which is total and surjective, i.e. an N-enumeration.

Using the assumption that Σ has no normal form in $\mathfrak{A}$ it follows that at stage $2e+2$ the *Case 2* never happens, so we have the following two subcases only:

In *Subcase 1.1,* $\sigma_N^{q+1} \Vdash_{\mathfrak{A}} F_e(\langle \overline{x},\overline{y}\rangle)$ and $(\overline{s},\overline{y}) \notin \Sigma$, and $\sigma_N^{q+1}(x_j) = s_j$ for $1 \leq j \leq a$ (from the last one it follows that $f_N(d_j) = s_j$). From $\sigma_N^{q+1} \Vdash_{\mathfrak{A}} F_e(\langle \overline{x},\overline{y}\rangle)$, using *Proposition* 3.10, we obtain that $f_N \Vdash_{\mathfrak{A}} F_e(\langle \overline{x},\overline{y}\rangle)$, i.e. $\rangle \overline{x},\overline{y}) \in \Psi_e(f_N^{-1}(\mathfrak{A})))$. But $(f_N(x_1)\dots f_N(x_a), y_1 \dots y_b) = (\overline{s},\overline{y}) \notin \Sigma$, i.e. $\langle \overline{x},\overline{y}\rangle \notin f_N^{-1}(\Sigma)$ and therefore $f_N^{-1}(\Sigma) \neq \Psi_e(f_N^{-1}(\mathfrak{A}))$.

In *Subcase 1.2,* $\forall \tau_N \supseteq \sigma_N^{q+1}$ $\big(\tau_N \nVdash_{\mathfrak{A}} F_e(\langle \overline{x},\overline{y}\rangle)\big)$, $(\overline{s},\overline{y}) \in \Sigma$, and $\sigma_N^{q+1}(d_j) = s_j$ for $1 \leq j \leq a$ (then we have $f_N(x_j) = s_j$ and therefore $\langle \overline{x},\overline{y}\rangle \in f_N^{-1}(\Sigma)$). From $\forall \tau_N \supseteq \sigma_N^{q+1}$ $\big(\tau_N \nVdash_{\mathfrak{A}} F_e(\langle \overline{x},\overline{y}\rangle)\big)$, using *Proposition* 3.10, it follows that $f_N \nVdash_{\mathfrak{A}} F_e(\langle \overline{x},\overline{y}\rangle)$, i.e. $\langle \overline{x},\overline{y}\rangle \notin \Psi_e(f_N^{-1}(\mathfrak{A}))$. Therefore $f_N^{-1}(\Sigma) \neq \Psi_e(f_N^{-1}(\mathfrak{A}))$.

We have proven that for all e, $f_N^{-1}(\Sigma) \neq \Psi_e(f_N^{-1}(\mathfrak{A}))$, i.e. $f_N^{-1}(\Sigma) \not\leq_e f_N^{-1}(\mathfrak{A})$, which is a contradiction. Therefore our assumption is false, i.e. Σ has normal form in $\mathfrak{A}$.

2) (2) follows from(1).

Suppose Σ has normal form in $\mathfrak{A}$.

We can prove that $\hat{\Sigma} \in \mathbf{SC}(\hat{\Sigma}_1 \dots \hat{\Sigma}_k, \hat{=}_N, \hat{\neq}_N)$, which according to the definition means $\Sigma \leq_{\mathbf{SC}} (\Sigma_1 \dots \Sigma_k, =_N, \neq_N)$, i.e. that $\Sigma \leq_{\mathbf{SC}} \mathfrak{A}$. Since Σ has normal form in $\mathfrak{A}$,

$(\exists e \in \omega, \exists \delta$ and N-string, $\exists x_1 \dots x_a \notin Dom(\delta_N))$
$(\forall s_1 \dots s_a \in N, \forall y_1 \dots y_b \in \omega)$
$\big((\overline{s},\overline{y}) \in \Sigma$ iff

$$\exists \tau_N \supseteq \delta_N \ \big(\tau_N(x_1) = s_1 \,\&\dots\&\, \tau_N(x_a) = s_a \,\&\, \tau_N \Vdash_{\mathfrak{A}} F_e(\langle \overline{xy}\rangle)\big)\big),$$

therefore $\hat{\Sigma}(s) = 0^*$ iff $\exists s_1 \dots s_a \in N\ \exists y_1 \dots y_b \in \omega$ $\big(s = \langle s_1 \dots s_a, y_1^* \dots y_b^*\rangle \&$ $(\overline{s},\overline{y}) \in \Sigma\big)$ iff $\big(s = \langle \overline{s},\overline{y}^*\rangle \,\&\, s_j \in N \,\&\, y_l^* \in \omega^* \,\&\, \exists \tau \supseteq \delta$ $\big(\tau_N(x_1) = s_1 \,\&\dots$ $\&\, \tau_N(x_a) = s_a \,\&\, \exists v(\langle v, \langle \overline{x},\overline{y}\rangle\rangle \in W_e \,\&\, \tau_N \Vdash_{\mathfrak{A}} D_v)\big)\big)$.

Note that the functions $\chi_{\omega^*}, \chi_N, I_j^m$, (where $I_j^m(r) = r_j \Leftrightarrow r = \langle r_1 \dots r_m\rangle$), h_0, (where $h_0(s) = 0^* \Leftrightarrow s = \ulcorner \tau_N \urcorner$), h_1, (where $h_1(\langle \ulcorner \tau_N \urcorner, \ulcorner \sigma_N \urcorner\rangle) = 0^* \Leftrightarrow \tau_N \subseteq \sigma_N$, h_2, (where $h_2(\langle \ulcorner \tau_N \urcorner, x^*, s\rangle) = 0^* \Leftrightarrow \tau_N(x) = s)$ are **PC**. The function φ (the semicharacteristic function of the set W_e), is partial recursive and therefore (see Proposition 1.4) the function φ^* is $\mathbf{PC}$, for the same reason the function $\langle\cdot,\cdot\rangle^*$, the coding of pair of natural numbers (where $\langle x^*, y^*\rangle = z^* \Leftrightarrow \langle x, y\rangle = z$) is $\mathbf{PC}$.

It is not difficult to check that the function g, such that $g(\ulcorner\tau_N\urcorner, v^*) = 0^* \Leftrightarrow \tau_N \Vdash_{\mathfrak{A}} D_v \Leftrightarrow \forall u \in D_v \big(u = \langle i, \langle x_1 \dots x_{a_i}, y_1 \dots y_{b_i}\rangle\rangle$ & $1 \le i \le k$ & $x_1 \dots x_{a_i} \in Dom(\tau_N)$ & $(\tau_N(x_1) \dots \tau_N(x_{a_i}), y_1 \dots y_{b_i}) \in \Sigma_i \vee u = \langle 0, 2\langle x, y\rangle\rangle$ & $x, y \in Dom(\tau_N)$ & $\tau_N(x) =_N \tau_N(y) \vee u = \langle 0, 2\langle x, y\rangle + 1\rangle$ & $x, y \in Dom(\tau_N)$ & $\tau_N(x) \ne \tau_N(y)\big)$, is $\mathbf{PC}(\hat{\Sigma}_1 \dots \hat{\Sigma}_k, \hat{=}_N, \hat{\ne})$.

Therefore $\Sigma \le_{\mathbf{SC}} \mathfrak{A}$.

□

4 Enumerations

Definition 4.1

An enumeration is a pair $\alpha = (f_N, R_\alpha)$, where $f_N : \omega \to N$ is a total surjective function and $R_\alpha \subseteq \omega$.

The structure of the enumeration α is the structure $\mathfrak{A}_\alpha = \mathfrak{A}(Gr(f_N), R_\alpha)$, where $Gr(f_N) = \{(s, x) \mid f_N(x) = s\} \subseteq N \times \omega$.

Definition 4.2

For every enumeration $\alpha = (f_N, R_\alpha)$ we define the following sets:

1. $E_\alpha := \{\langle x, y\rangle \mid f_N(x) =_N f(y)\} = E_f = f_N^{-1}(=_N) \subseteq \omega$.
2. $D(\alpha) = E_\alpha^+ \oplus R_\alpha$.

Proposition 4.3

Let a structure $\mathfrak{A} = \mathfrak{A}(\Sigma_1 \dots \Sigma_k)$ and an enumeration $\alpha = (f_N, R_\alpha)$ and a predicate $\Sigma \subseteq N^a \times \omega^b$ be given. Then $\Sigma \le_{\mathbf{SC}} \mathfrak{A} \oplus \mathfrak{A}_\alpha$ if and only if ${f_N}^{-1}(\Sigma) \le_e {f_N}^{-1}(\mathfrak{A}) \oplus R_\alpha$.

PROOF:

($\Rightarrow$) Let $\Sigma \le_{\mathbf{SC}} \mathfrak{A} \oplus \mathfrak{A}_\alpha$, where $\mathfrak{A} \oplus \mathfrak{A}_\alpha = (\Sigma_1 \dots \Sigma_k, Gr(f_N), R_\alpha)$. Then from *The Normal Form Theorem*(3.12) it follows that $f_N^{-1}(\Sigma) \le_e f_N^{-1}(\mathfrak{A} \oplus \mathfrak{A}_\alpha)$, i.e.

$f_N^{-1}(\Sigma) \le_e f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \oplus f_N^{-1}(Gr(f_N)) \oplus f_N^{-1}(R_\alpha) \oplus E_f^+$, but $f_N^{-1}(Gr(f_N)) = \{\langle x, y\rangle \mid (f_N(x), y) \in Gr(f_N)\} = \{\langle x, y\rangle \;\; f_N(y) =_N f_N(x)\}$ $= f_N^{-1}(=_N) = E_\alpha$. Then $f_N^{-1}(\Sigma) \le_e f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \oplus f_N^{-1}(R_\alpha) \oplus E_\alpha^+$, but $f_N^{-1}(R_\alpha) = R_\alpha \subseteq \omega$, and therefore $f_N^{-1}(\Sigma) \le_e f_N^{-1}(\mathfrak{A}) \oplus R_\alpha$.

($\Leftarrow$) Let

(1) $f_N^{-1}(\Sigma) \le_e f_N^{-1}(\mathfrak{A}) \oplus R_\alpha$.

Consider an arbitrary N-enumeration $g_N : \omega \to N$, for which we can prove that $g_N^{-1}(\Sigma) \leq_e g_N^{-1}(\mathfrak{A} \oplus \mathfrak{A}_\alpha)$, from which, using the *Normal form theorem* (3.12) follows that $\Sigma \leq_{\mathbf{SC}} \mathfrak{A} \oplus \mathfrak{A}_\alpha$.

We know that $g_N^{-1}(Gr(f_N)) = \{\langle x, y\rangle \mid (g_N(x), y) \in Gr(f_N)\} = \{\langle x, y\rangle \mid g_N(x) = f_N(y)\}$, and $g_N^{-1}(R_\alpha) = R_\alpha$.

(2) $g_N^{-1}(\Sigma) \leq_e f_N^{-1}(\Sigma) \oplus g_N^{-1}(Gr(f_N))$, since
$\langle \overline{x}, \overline{y}\rangle \in g_N^{-1}(\Sigma) \Leftrightarrow (g_N(x) \dots g_N(x_a), y_1 \dots y_b) \in \Sigma \Leftrightarrow \exists\ z_1 \dots z_a$ $(\langle x_1, z_1\rangle, \dots, \langle_a, z_a\rangle \in g_N^{-1}(Gr(f_N))\ \&\ (f_N(z_1), \dots, f_N(z_a), y_1 \dots y_b) \in \Sigma)$ $\Leftrightarrow \exists z_1 \dots z_a\ (\langle x_1, z_1\rangle, \dots, \langle x_a, z_a\rangle \in g_N^{-1}(Gr(f_N))\ \&\ \langle \overline{z}, \overline{y}\rangle \in f_N^{-1}(\Sigma))$.

(3) $f_N^{-1}(\Sigma_i) \leq_e g_N^{-1}(\Sigma_i) \oplus Gr(f_N)$, for all $1 \leq i \leq k$, since
$\langle \overline{x}, \overline{y}\rangle \in f_N^{-1}(\Sigma_i) \Leftrightarrow (f_N(x_1) \dots f_N(x_{a_i}), y_1 \dots y_{b_i}) \in \Sigma \Leftrightarrow \exists z_1 \dots z_{a_i}$ $(\langle z_1, x_1\rangle \dots \langle z_{a_i}, x_{a_i}\rangle \in g_N^{-1}(Gr(f_N))\ \&\ \langle \overline{z}, \overline{y}\rangle \in g_N^{-1}(\Sigma_i))$.

Using the same reasoning as in (3), we obtain $E_\alpha = E_f = f_N^{-1}(=_N) \leq g_N^{-1}(=_N) \oplus g_N^{-1}(Gr(f_N)) = E_g \oplus g_N^{-1}(Gr(f_N))$ and $\overline{E_\alpha} \leq_e \overline{E_g} \oplus g_N^{-1}(Gr(f_N))$.

From (1), (2) and (3) it follows that $g_N^{-1}(\Sigma) \leq_e g_N^{-1}(\mathfrak{A} \oplus \mathfrak{A}_\alpha)$.

□

Corollary 4.4

For any enumeration $\alpha = (f_N, R_\alpha)$ and predicate $\Sigma \subseteq N^a \times \omega^b$, $\Sigma \leq_{\mathbf{SC}} \mathfrak{A}_\alpha$ if and only if $f_N^{-1}(\Sigma) \leq_e D(\alpha)$.

PROOF:

Let $\varnothing = (N, \omega; =_N, \neq_N)$ be the empty structure. From *Proposition 4.3* if follows that $\Sigma \leq_{\mathbf{SC}} \varnothing \oplus \mathfrak{A}_\alpha \Leftrightarrow f_N^{-1}(\Sigma) \leq_e f_N^{-1}(\varnothing) \oplus R_\alpha$, but $f_N^{-1}(\varnothing) = E_\alpha^+$, i.e. $\Sigma \leq_{\mathbf{SC}} \varnothing \oplus \mathfrak{A}_\alpha \Leftrightarrow f_N^{-1}(\Sigma) \leq_e D(\alpha)$.

□

Corollary 4.5

For any structure $\mathfrak{A} = (\Sigma_1 \dots \Sigma_k)$ and enumeration $\alpha = (f_N, R_\alpha)$, $\mathfrak{A} \leq \mathfrak{A}_\alpha$ if and only if $f_N^{-1}(\mathfrak{A}) \leq_e D(\alpha)$.

PROOF:

Form *Corollary 4.4* it follows that $\mathfrak{A} \leq_{\mathbf{SC}} \mathfrak{A}_\alpha \Leftrightarrow \forall i_{1 \leq i \leq k}\ \Sigma_i \leq_{\mathbf{SC}} \mathfrak{A}_\alpha \Leftrightarrow \forall i_{1 \leq i \leq k}\ f_N^{-1}(\Sigma_i) \leq_e D(\alpha) \Leftrightarrow f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \leq_e f_N^{-1}(\Sigma_1) \oplus \dots \oplus f_N^{-1}(\Sigma_k) \oplus E_\alpha^+ \leq_e D(\alpha) \Leftrightarrow f_N^{-1}(\mathfrak{A}) \leq_e D(\alpha)$.

□

Definition 4.6

The pair $f = (f_N, f_\omega)$, where $f_N : \omega \to N$ and $f_\omega : \omega \to \omega$ are total functions, is called an *N-ω-enumeration*.

Notation: $Gr(f_\omega) = \{\langle x, y\rangle \mid f_\omega(x) = y\} \subseteq \omega$ is the graph of f_ω.

Remark: The N-ω-enumerations $f = (f_N, f_\omega)$ are enumerations $\alpha = (f_N, R_\alpha)$ with $R_\alpha = Gr(f_\omega)$, $D(\alpha) = D(f) = E_f^+ \oplus Gr(f_\omega)$;
$\mathfrak{A}_\alpha = \mathfrak{A}_f = \mathfrak{A}(Gr(f_N), Gr(f_\omega))$; $E_\alpha = E_f = f_N^{-1}(=_N)$.