Relative Set Genericity

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Abstract

A set of natural numbers is generic relatively a set B if and only if it is the preimage of some set A using a B-generic B-regular enumeration such that both A and its complement are e-reducible to B.

Introduction

The genericity and set genericity, as defined by Copestake in [2], are widely explored, and have important role in studying the structure of the enumeration degrees.

In this paper we consider the genericity relative a set of natural numbers, which is in fact a *set n-genericity*. We refer to some well known facts in this area, most of which can be found in [2] and [1] and can be used to prove similar properties for the relative genericity.

Further we provide some results concerning *regular enumerations* of the set of the natural numbers that we use to prove a characterization theorem. Concerning the regular enumerations, the used notions and results are taken mostly from Soskov's course on Recursion Theory and the author's Master's Thesis.

Basic notions and definitions

By ω we denote the set of all natural numbers, 2ω denotes the set of all even and $2\omega+1$ - the set of all odd natural numbers; by $[0..n-1]$, where $n \in \omega$, we denote the set $\{x \in \omega | x < n\}$. We use N to denote an arbitrary denumerable set.

We use bijective recursive coding of pairs of natural numbers $\langle \cdot, \cdot \rangle$, the notation $\langle x_1, x_2, ..., x_k \rangle$ means $\langle x_1, \langle x_2, ..., x_k \rangle \rangle$, and of finite sets - D_v denotes the finite set with code v. By $\varphi, \psi ...$ we denote partial functions from ω into ω and let $Gr(\varphi) = \{\langle x, y \rangle \mid \varphi(x) = y\}$ be the graph of the function φ. The notation $\varphi(x)\downarrow$ means $x \in Dom(\varphi)$, and $\varphi(x)\uparrow$ means $x \notin Dom(\varphi)$. The notation $\subseteq$ is used to denote *inclusion* between sets, *extension* between functions, ω-strings or 0-1-strings, considered as finite functions.

By C_A we denote the semicharacteristic function of a set $A \subseteq \omega$, and its characteristic function - by χ_A, where

$$\chi_A(x) = \begin{cases} 0 & \text{, if } x \in A \\ 1 & \text{, if } x \notin A \end{cases}$$

If each of P and Q denotes some property of natural numbers we use the following abbreviation:

$$\mu y_{\in\omega}[Q(y)][P(y)] \simeq \begin{cases} \mu y_{\in\omega}\big[Q(y)\&P(y)\big] & \text{, if } \exists y\big(P(y)\&Q(y)\big) \\ \mu y_{\in\omega}\big[Q(y)\big] & \text{, if } \exists y\big(Q(y)\big) \text{ and } \neg\big(P(y)\&Q(y)\big) \\ \uparrow & \text{, if } \forall y(\neg Q(y)) \end{cases},$$

where $\mu y_{\in\omega}[Q(y)]$ is the least y having the property Q.

Let A, B and C... be sets of natural numbers. We use the following standard definitions and notations:

$A \leq_e B$ if and only if $A = \Psi_a(B)$ for some e-operator Ψ_a, defined as follows: $\Psi_a(B) = \{x \mid \exists v\big(\langle x, v\rangle \in W_a \,\&\, D_v \subseteq B\big)\}$, where W_a is the recursively enumerable set with Gödel code a. $A \equiv_e B$ if and only if $A \leq_e B$ and $B \leq_e A$. The enumeration degree (e-degree) of the set A is the equivalence class $Deg_e(A) = \{B \subseteq \omega \,|\, A \equiv_e B\}$. We denote the e-degrees by a, b, c...

We use the standard *join* operation of two sets $A \oplus B = \{2x | x \in A\} \cup \{2x+1 | x \in B\}$ having the property that $Deg_e(A \oplus B)$ is the least upper bound of $Deg_e(A)$ and $Deg_e(B)$.

A set of natural numbers C is said to be *total* if its complement is e-reducible to C, i.e. $\overline{C} \leq_e C$, (which is equivalent to $C \equiv_e C^+$, where we define $C^+ = C \oplus \overline{C}$, and thus for every set $C^+ \equiv_e Gr(\chi_C)$).

1 B-Generic sets

Definition 1.1 *ω-string* is a finite function from ω into ω, with domain an initial segment of ω. $\emptyset_\omega$ denotes the nowhere defined function, considered as *empty ω-string*; note that *length* of σ_ω is $lh(\sigma_\omega) = \mu x\big[\neg\exists y(\sigma_\omega(x) = y)\big]$;

0-1-string, (or 2-valued string) is an ω-string α_ω, such that $Rng(\alpha_\omega) \subseteq \{0, 1\}$. For every 0-1-string α_ω we define the set $\alpha_\omega^+ = \{x \mid \alpha_\omega(x) \simeq 0\}$.

Definition 1.2 The set A is *B-generic*, for $B \subseteq \omega$, if and only if for every set S, such that S is a set of 0-1-strings and $S \leq_e B$

$$\exists \alpha_\omega \subseteq \chi_A\big(\alpha_\omega \in S \vee \forall \beta_\omega \supseteq \alpha_\omega(\beta_\omega \notin S)\big).$$

The set A is *quasi-minimal over B*, if and only if

(1) $B \leq_e A$, but $A \not\leq_e B$; and (2) If C is a total set such that $C \leq_e A$, then $C \leq_e B$.

The set A is *minimal-like over B*, if and only if

(1) $B \leq_e A$, but $A \not\leq_e B$; and (2) For every partial function φ, such that $\varphi \leq_e A$, there exists partial function ψ, such that $\varphi \subseteq \psi$ and $\psi \leq_e B$.

In analogue to the definitions in [1], an e-degree containing such set is said to be strongly minimal-like over B.

Here we mention some of the properties of the B-generic sets, that we will need later: A is B-generic if and only if $\overline{A}$ is B-generic; if A is B-generic, there is no infinite e-reducible to B, subset of A; every B-generic set A is infinite and not e-reducible to B.

Concerning the existence of a B-generic set, a minimal like set over any set B and the existence of a quasi-minimal set over any set B, see [1], [2], it is proven that for an arbitrary B-generic set A, the set $A \oplus B$ is minimal like and quasi-minimal over B.

Theorem 1.3

Let $B_0, B_1, \dots, B_n, \dots$ be a sequence of sets of natural numbers. There exists a set of natural numbers A, which is *minimal-like over this sequence*, i.e. such that the next two conditions hold:

1) $\forall n (B_n \leq_e A)$;

2) For every partial function φ, such that $\varphi \leq_e A$, there exist a partial function ψ and natural number n, such that $\varphi \subseteq \psi$ and $\psi \leq_e B_0 \oplus \dots \oplus B_n$.

PROOF:

In the following proof the notation $\overset{\infty}{\forall} x\ P(x)$ is equivalent to $\exists y \forall x (x \geq y \Rightarrow P(x))$. We define a set A, satisfying two requirements:

(a) $\forall n \overset{\infty}{\forall} x \big(\langle x, n \rangle \in A \Leftrightarrow x \in B_n \big)$, and

(b) $\forall e \Big(\Psi_e(A)$ is a function $\Rightarrow \exists \psi \big(\Psi_e(A) \subseteq \psi \,\&\, \psi \leq_e B_0 \oplus \dots \oplus B_{2e+1} \big) \Big)$,

building finite sets $A_0 \subseteq \dots \subseteq A_s \subseteq \dots$ ..., having the next property:

$\forall s \big(\langle x, m \rangle \in A_{s+1} \setminus A_s \,\&\, m \leq s \Rightarrow x \in B \big)$, for all x and m.

Stage 0 : Let $A_0 = \emptyset$.

Stage 2e+1 : A_s is built, where $s = 2e$. We have two cases:

Case 1: There exists $\langle x, n \rangle$, such that $x \in B_n$ and $\langle x, n \rangle \notin A_s$. Then we can define $A_{s+1} = A_s \cup \{\langle x, n \rangle\}$, for the first such $\langle x, n \rangle = \mu \langle x, n \rangle$.

Case 2: Otherwise, define $A_{s+1} = A_s$.

Stage 2e+2 : A_s is built, where $s = 2e + 1$. Again we have two cases:

Case 1: There exists a finite set D_v, such that $A_s \subseteq D_v$ and $\Psi_e(D_v)$ is not a function (i.e. $\exists x \exists y \exists z$ such that $y \neq z \,\&\, \langle x, y \rangle \in \Psi_e(D_v) \,\&\, \langle x, z \rangle \in \Psi_e(D_v)$) and such that $\forall t \forall m \big(\langle t, m \rangle \in D_v \setminus A_s \,\&\, m \leq s \Rightarrow t \in B_m \big)$?

Define A_{s+1} to be the least D_v (i.e. having the least code v), with this property.

Case 2: Otherwise, define $A_{s+1} = A_s$.

End.

Finally define $A = \bigcup_{s=0}^{\infty} A_s$.

For this set we can prove the properties (a) and (b), from which our theorem follows.

The interesting direction of the proof of (a) is $(\Rightarrow)$. We can prove that $\forall n \overset{\infty}{\forall} x\ \big(\langle x, n \rangle \in A \Rightarrow x \in B_n \big)$. Assume it is not true, i.e. there exist n and infinitely many $x_0 < \dots < x_i < \dots$, such that $\langle x_i, n \rangle \in A$ and $x_i \notin B_n$. Therefore $\forall x_i \exists s_i \big(\langle x_i, n \rangle \in A_{s_i+1} \setminus A_{s_i} \big)$. But at every stage s the set $A_{s+1} \setminus A_s$ is finite, then there exist infinitely many $x_{s_0}, \dots, x_{s_i}, \dots$ from this sequence, such that at stages $s_0 < \dots < s_i < \dots$ we have $\langle x_{s_i}, n \rangle \in A_{s_i+1} \setminus A_{s_i}$. But $x_{s_i} \notin B_n$ and then the stages

s_i+1 must be even (i.e. $s_i+1=2e_i+2$), and we have *Case 1*, i.e. $A_{s_i+1}=D_v$, where $D_v \supseteq A_{s_i}$ and $\forall t \forall m(\langle t,m\rangle \in D_v \setminus A_{s_i} \,\&\, m \le s_i \Rightarrow t \in B_m)$. Therefore for every $s_i \ge n$ if $\langle x_{s_i}, n\rangle \in A_{s_i+1} \setminus A_{s_i}$, then $x_{s_i} \in B_n$, which is a contradiction.

The proof of (b) consists in the following: supposing $\Psi_e(A)$ to be a graph of some function, at *Stage 2e+2*, for $s=2e+1$ we have *Case2*. Define the set $G_\psi=\{\langle x,y\rangle \mid \exists D_v(D_v \supseteq A_s \,\&\, \langle x,y\rangle \in \Psi_e(D_v) \,\&\, \forall\langle t,m\rangle(\langle t,m\rangle \in D_v \setminus A_s \,\&\, m \le s \Rightarrow t \in B_m))\}$. Therefore the following conditions hold:

• $G_\psi \le_e B_0 \oplus \ldots \oplus B_s$;

• $G_\psi = Gr(\psi)$, i.e. G_ψ is a graph of some function ψ, since assuming it not true, there exist x and $y_1 \ne y_2$, such that $\langle x,y_1\rangle \in G_\psi$ and $\langle x,y_1\rangle \in G_\psi$. Therefore there exist finite sets D_{v_1} and D_{v_2}, both extending A, s.t. $\langle x,y_1\rangle \in \Psi_e(D_{v_i})$ and $\forall\langle t,m\rangle(\langle t,m\rangle \in D_{v_i} \setminus A_s \,\&\, m \le s \Rightarrow t \in B_m)$. Then for $D_v = D_{v_1} \cup D_{v_2}$, $\Psi_e(D_v)$ is not a function and $\forall\langle t,m\rangle(\langle t,m\rangle \in D_v \setminus A_s \,\&\, m \le s \Rightarrow t \in B_m)$, which is a contradiction with *Case 2*.

• $\Psi_e(A) \subseteq G_\psi$, since assuming there is $\langle x,y\rangle \in \Psi_e(A) \setminus G_\psi$, there exists $A_{s+p} \supseteq A_s$, such that $\langle x,y\rangle \in \Psi_e(A_{s+p})$ and $\exists\langle t,m\rangle\ (\langle t,m\rangle \in A_{s+p} \setminus A_s \,\&\, m \le s \,\&\, t \notin B_m)$. It follows that there is i, such that $0 \le i < p$ and $\langle t,m\rangle \in A_{s+i+1} \setminus A_{s+i}$, and therefore $m \le s+i$. Since $A_{s+i+1} \setminus A_{s+i} \ne \emptyset$, we have *Case 1* at *Stage* $s+i=2e_i+1$ or *Case 1* at *Stage* $s+i=2e_i$. But in both cases it follows that $t \in B_m$, which is a contradiction.

This proves our proposition. □

As a corollary of the above theorem we obtain the existence of strongly minimal-like e-degree over an infinite ascending sequence of e-degrees.

2 B-Generic regular enumerations

In this paragraph we illustrate briefly some results obtained using the relative generic regular enumerations and many of the proofs will be only sketched.

Definition 2.1 Let $B \subseteq \omega$ be a non-empty set of natural numbers.

1) The total and surjective function $f:\omega \to \omega$, is called B-regular ω-enumeration, if $f(2\omega)=B$, where $f(2\omega)=\{f(2x) \mid x \in \omega\}$.

2) An ω-string τ_ω is B-regular, if $\tau_\omega(2\omega) \subseteq B$, where $\tau_\omega(2\omega)=\{y \mid \exists x\ (\tau_\omega(2x)=y)\}$.

3) The B-regular ω-enumeration f is called B-generic if for every e-reducible to B set of ω-strings F, the following holds:

$$\exists \sigma_\omega \subseteq f(\sigma_\omega \in F \vee \forall \tau_\omega \supseteq \sigma_\omega(\tau_\omega \notin F)).$$

For every non-empty set B one can iteratively build a B-generic B-regular enumeration f at stages, using ω-strings to satisfy the requirements in the definition of f.

It is true that $f \not\le_e B$, for every B-generic B-regular enumeration f. This can be proved assuming $f \le_e B$, and defining the e-reducible to B set of ω-strings $S=\{\tau_\omega \mid \tau_\omega(2\omega) \subseteq B \,\&\, \tau_\omega \not\subseteq f\}$, that will lead to the contradiction.

Proposition 2.2

For every B-generic B-regular enumeration f, for every set R, such that $R \leq_e B$, $\overline{R} \leq_e B$, $R \cap B \neq \emptyset$ and $\overline{R} \cap B \neq \emptyset$, the set $f^{-1}(R)$ is B-generic.

PROOF:

Since $f^{-1}(R) = \{x \mid f(x) \in R\}$, we have that $\chi_{f^{-1}(R)} = \chi_R \circ f$. Assume $f^{-1}(R)$ is not B-generic, i.e. there is e-reducible to B set of ω-strings, such that
(1) $\forall \alpha_\omega \big(\alpha_\omega \subseteq \chi_{f^{-1}(R)} \Rightarrow \alpha_\omega \notin F \,\&\, \exists \beta_\omega (\beta_\omega \supseteq \alpha_\omega \,\&\, \beta_\omega \in F)\big)$.

Define $S = \{\sigma_\omega \mid \exists \alpha_\omega (\alpha_\omega \in F \,\&\, \chi_R \circ \sigma_\omega = \alpha_\omega)\}$, where $\chi_R \circ \sigma_\omega = \alpha_\omega$ if and only if $\big(lh(\alpha_\omega) = lh(\sigma_\omega) \,\&\, \forall x < lh(\alpha_\omega)\big(\alpha_\omega(x) = 0 \Leftrightarrow \sigma_\omega(x) \in R\big)\big)$, therefore S is a set of B-regular ω-strings and $S \leq_e B$. But f is B-generic B-regular enumeration, so there is $\sigma_\omega \subseteq f$, such that either $\sigma_\omega \in S$, either $\forall \tau_\omega \supseteq \sigma_\omega (\tau_\omega \notin S)$.

Assuming $\sigma_\omega \in S$, there is $\alpha_\omega \in F$, such that $\chi_R \circ \sigma_\omega = \alpha_\omega$, but $\sigma_\omega \subseteq f$ and then $\chi_R \circ f \supseteq \alpha_\omega$, i.e. $\alpha_\omega \subseteq \chi_{f^{-1}(R)}$, which is a contradiction with (1). Therefore for that σ_ω the following holds:
(2) $\forall \tau_\omega \supseteq \sigma_\omega (\tau_\omega \notin S)$.

Define $\alpha_\omega = \chi_R \circ \sigma_\omega$. Since $\sigma_\omega \subseteq f$, then $\alpha_\omega \subseteq \chi_R \circ f = \chi_{f^{-1}(R)}$, and from (1) it follows that there exists β_ω, such that $\beta_\omega \supseteq \alpha_\omega$ and $\beta_\omega \in F$. Therefore $\beta_\omega \supseteq \chi_R \circ \sigma_\omega = \alpha_\omega$ and $lh(\beta_\omega) \geq lh(\alpha_\omega)$. If we fix two elements of B - $a \in R \cap B$ and $b \in \overline{R} \cap B$, we can define an ω-string τ_ω, such that $\tau_\omega \supseteq \sigma_\omega$, $lh(\tau_\omega) = lh(\beta_\omega)$ and $\forall x \big(lh(\sigma_\omega) \leq x \leq lh(\tau_\omega) \Rightarrow (\beta_\omega(x) = 0 \Leftrightarrow \tau_\omega(x) \in R)\big)$, i.e. $\beta_\omega = \chi_R \circ \tau_\omega \supseteq \chi_R \circ \sigma_\omega = \alpha_\omega$. Since $\beta_\omega \in F$ and $\chi_R \circ \tau_\omega = \beta_\omega$, then $\tau_\omega \in S$, which is a contradiction with (b). Therefore $f^{-1}(R)$ is not B-generic set.

□

The following corollary follows directly from *Proposition 2.2* and from the properties of relative generic sets in §1.

Corollary 2.3

For every B-generic B-regular enumeration f, for every set R, such that $R \leq_e B$, $\overline{R} \leq_e B$, $R \cap B \neq \emptyset$ and $\overline{R} \cap B \neq \emptyset$, the set $f^{-1}(R) \oplus B$ is quasi-minimal over B.

Lemma 2.4

Let A be B-generic. Let $R \subseteq \omega$, such that $R \leq_e B$, $\overline{R} \leq_e B$, $R \cap B \neq \emptyset$ and $\overline{R} \cap B \neq \emptyset$. Let δ_ω be an ω-string, having the properties (1) and (2):

(1) δ_ω is B-regular;
(2) $\forall x < lh(\delta_\omega)$ $(x \in A \Leftrightarrow \delta_\omega(x) \in R)$.

For every S, such that S is e-reducible to B set of ω-strings, there exists ω-string σ_ω, having the properties (a), (b), (c) and (d) :

(a) $\sigma_\omega \supseteq \delta_\omega$;
(b) σ_ω is B-regular;
(c) $\forall x < lh(\sigma_\omega)$ $(x \in A \Leftrightarrow \sigma_\omega(x) \in R)$;
(d) $\sigma_\omega \in S \vee \forall \tau_\omega (\tau_\omega \supseteq \sigma_\omega \Rightarrow \tau_\omega \notin S)$.

PROOF:

Let us denote by $\alpha_\omega \sim_R \sigma_\omega$ the property $\forall x \in Dom(\sigma_\omega)\big(\alpha_\omega(x) = 0 \Leftrightarrow \sigma_\omega(x) \in R\big)$, where α_ω is a 0-1-string, σ_ω is a ω-string and $R \subseteq \omega$.

Define the set $P = \{\alpha_\omega \mid \exists \sigma_\omega\big(\sigma_\omega \in S \;\&\; \sigma_\omega \supseteq \delta_\omega \;\&\; \sigma_\omega(2\omega) \subseteq B \;\&\; lh(\alpha_\omega) = lh(\sigma_\omega) \;\&\; \alpha_\omega \sim_R \sigma_\omega\big)\}$, that is e-reducible to B. Since A is B-generic, we have two possibilities:

Case 1. $\exists \alpha_\omega \subseteq \chi_A \ (\alpha_\omega \in P)$.

In this case there exists σ_ω - a B-regular extension of δ_ω in S with the same length as α_ω, such that $\alpha_\omega \sim_R \sigma_\omega$. But $\alpha_\omega \subseteq \chi_A$, then $\forall x < lh(\sigma_\omega) \ \big(x \in A \Leftrightarrow \sigma_\omega(x) \in R\big)$, i.e. σ_ω has the properties (a), (b), (c) and (d).

Case 2. $\exists \alpha_\omega \subseteq \chi_a \forall \beta_\omega \supseteq \alpha_\omega(\beta_\omega \notin P)$.

In this case $\exists \alpha_\omega \subseteq \chi_A\big(lh(\delta_\omega) \leq lh(\alpha_\omega) \;\&\; \forall \beta_\omega \supseteq \alpha_\omega(\beta_\omega \notin S)\big)$. Fix two elements a in $R \cap B \neq \emptyset$ and b in $\overline{R} \cap B \neq \emptyset$. Now we can define an ω-string σ_ω, such that $\sigma_\omega \supseteq \delta_\omega$ and $lh(\sigma_\omega) = lh(\alpha_\omega)$, such that for the arguments x, s.t. $lh(\delta_\omega) \leq x < lh(\alpha_\omega)$, $\sigma_\omega(x) \simeq a$ if $\alpha_\omega(x) = 0$; and $\sigma_\omega(x) \simeq b$ if $\alpha_\omega(x) = 1$. Since δ_ω is B-regular, σ_ω is B-regular too. And from (2) and $\alpha_\omega \subseteq \chi_A$ follows that $\forall x < lh(\sigma_\omega)$ $(x \in A \Leftrightarrow \sigma_\omega(x) \in R)$. So, σ_ω has the properties (a), (b) and (c). It remains to verify (d).

First, notice that $\alpha_\omega \sim_R \sigma_\omega$. Assume that there exists τ_ω, such that $\tau_\omega \supseteq \sigma_\omega \supseteq \delta_\omega$ and $\tau_\omega \in S$, (then τ_ω is B-regular). Therefore there exists 0-1-string β_ω, such that $\beta_\omega \supseteq \alpha_\omega$ and $lh(\beta_\omega) = lh(\tau_\omega)$, such that for the arguments $lh(\alpha_\omega) \leq x < lh(\tau_\omega)$, $\beta_\omega(x) \simeq 0$ if $\tau_\omega(x) \in R$; and $\beta_\omega(x) \simeq 1$ if $\tau_\omega(x) \notin R$. Since $\alpha_\omega \sim_R \sigma_\omega$ for this β follows that $\forall x < lh(\beta_\omega)$ $(\beta_\omega(x) = 0 \Leftrightarrow \tau_\omega(x) \in R)$, i.e. $\beta_\omega \sim_R \tau_\omega$ and therefore $\beta_\omega \in P$, which is a contradiction with *Case 2*, then the property (d) holds.

In both cases we found an ω-string satisfying (a), (b), (c) and (d).

□

Proposition 2.5

Let A be B-generic and R be such that $R \cap B \neq \emptyset$, $\overline{R} \cap B \neq \emptyset$, $R \leq_e B$ and $\overline{R} \leq_e B$. There exists B-generic B-regular enumeration f, such that $A = f^{-1}(R)$.

PROOF:

Since $f^{-1}(R) = \{x \mid f(x) \in R\}$, $A = f^{-1}(R)$ is equivalent to $\forall x(x \in A \Leftrightarrow f(x) \in R)$.

We build a sequence of ω-strings $\sigma^0_\omega \subseteq \sigma^1_\omega \subseteq \ldots \sigma^q_\omega \subseteq \ldots$, such that each σ^q_ω has the properties (1) and (2):

(1) σ^q_ω is B-regular, i.e. $\sigma^q_\omega(2\omega) \subseteq B$;

(2) $\forall x < lh(\sigma^q_\omega)$ $(x \in A \Leftrightarrow \sigma^q_\omega(x) \in R)$.

If (1) holds for all σ^q_ω, then $f(2\omega) \subseteq B$. If (2) for each σ^q_ω and from (3) it follows that $A = f^{-1}(R)$.

At *Stage* $(2e+1)$ we insure f to be total, surjective and $f(2\omega) \subseteq B$, i.e.

(3) $\forall q = 2e+1 \ \big(lh(\sigma^{q+1}_\omega) > lh(\sigma^q_\omega)\big)$;

(4) $\forall x \in \omega \ \exists q = 2e+1 \ \big(x \in Rng(\sigma^q_\omega)\big)$;

(5) $\forall x \in B \ \exists q = 2e+1 \ \big(x \in \sigma^q_\omega(2\omega)\big)$.

At *Stage* $(2e+2)$ we insure f to be B-generic, i.e.

(6) $\forall q = 2e+2 \Big($ If $\Psi_e(B)$ is a set of B-regular ω-strings, then

$$\big(\sigma^q_\omega \in \Psi_e(B) \vee \forall \tau_\omega \supseteq \sigma^q_\omega (\tau_\omega \notin \Psi_e(B))\big)\Big).$$

Stage 0: Define $\sigma^0_\omega = \emptyset_\omega$.

Stage 2e+1: At this stage σ^q_ω is built, with $q = 2e$.

Let x_0, x_1, x_2 and x_3 be the first numbers, greater or equal to $lh(\sigma^q_\omega)$, that belong to $2\omega \cap A$, $(2\omega+1) \cap A$, $2\omega \cap \overline{A}$ and $(2\omega+1) \cap \overline{A}$ respectively. Such x_i exist, because assuming for example $\forall x\ (x \geq lh(\sigma^q_\omega)\ \&\ x \in 2\omega \Rightarrow x \notin A)$, the set $C_0 = \{x \mid x \geq lh(\sigma^q_\omega)\ \&\ x \in 2\omega\}$ is infinite and recursively enumerable and $C_0 \subseteq \overline{A}$, which is a contradiction with the properties of the B-generic sets.

Let $m = \max\{x_0, x_1, x_2, x_3\}$ Define σ^{q+1}_ω, such that $\sigma^{q+1}_\omega \supseteq \sigma^q_\omega$ and $lh(\sigma^{q+1}_\omega) = m+1 > lh(\sigma^q_\omega)$, and for the arguments $lh(\sigma^q_\omega) \leq x \leq m$, define as follows:

$$\sigma^{q+1}_\omega(x) \simeq \begin{cases} \mu y[y \in R \cap B][y \notin Rng(\sigma^q_\omega)] & , x \in 2\omega\ \&\ x \in A \\ \mu y[y \in \overline{R} \cap B][y \notin Rng(\sigma^q_\omega)] & , x \in 2\omega\ \&\ x \notin A \\ \mu y[y \in R][y \notin Rng(\sigma^q_\omega)] & , x \notin 2\omega\ \&\ x \in A \\ \mu y[y \in \overline{R}][y \notin Rng(\sigma^q_\omega)] & , x \notin 2\omega\ \&\ x \notin A \end{cases}$$

Stage 2e+2: At this stage σ^q_ω is built, with $q = 2e+2$.

Define $G = \{\sigma_\omega \mid \sigma_\omega(2\omega) \subseteq B\ \&\ \forall x < lh(\sigma_\omega)\ (x \in A \Leftrightarrow \sigma_\omega(x) \in R)\}$, i.e. $G = \{\sigma_\omega \mid$ for σ_ω (1) and (2) hold true $\}$. We have two possibilities:

Case 1. $\exists \sigma_\omega \supseteq \sigma^q_\omega\ \Big(\sigma_\omega \in G\ \&\ \big(\sigma_\omega \in \Psi_e(B) \vee \forall \tau_\omega \supseteq \sigma_\omega(\tau_\omega \notin \Psi_e(B))\big)\Big)$. Define σ^{q+1}_ω to be the least such σ_ω.

Case 2. $\forall \sigma_\omega \supseteq \sigma^q_\omega\ \Big(\sigma_\omega \in G \Rightarrow \big(\sigma_\omega \notin \Psi_e(B)\ \&\ \exists \tau_\omega \supseteq \sigma_\omega\ (\tau_\omega \in \Psi_e(B))\big)\Big)$. Define $\sigma^{q+1}_\omega = \sigma^q_\omega$.

End.

Define $f = \bigcup_{q=0}^{\infty} \sigma^q_\omega$.

Using induction on q one can prove that for each σ^q_ω the conditions (1) and (2) holds. At *Stage 2e+1* we satisfy the requirements (3), (4) and (5). It follows that f is B-regular enumeration and $A = f^{-1}(R)$.

From (1) and (2) for σ_ω it follows, that for every $e \in \omega$, if $\Psi_e(B)$ is a set of B-regular ω-strings, then there exists σ_ω, having the properties $(a), (b), (c)$ and (d) of *Lemma 2.4*, i.e. $\sigma_\omega \supseteq \sigma^q_\omega$, σ_ω is B-regular, $\forall x < lh(\sigma_\omega)\ (x \in A \Leftrightarrow \sigma_\omega(x) \in R)$ and $\big(\sigma_\omega \in \Psi_e(B) \vee \forall \tau_\omega\ (\tau_\omega \supseteq \sigma_\omega \Rightarrow \tau_\omega \notin \Psi_e(B))\big)$. This means that if $\Psi_e(B)$ is a set of B-regular ω-strings, at *Stage 2e+1*, we never have Case 2, i.e the requirement (6) is satisfied.

Therefore our f is B-generic B-regular enumeration, such that $A = f^{-1}(R)$.

□

Theorem 2.6

Let B be a non-empty set of natural numbers. Any set $A \subseteq \omega$ is B-generic if and only if there exist a set R and B-generic B-regular enumeration f, such that $R \leq_e B$ and $\overline{R} \leq_e B$, and $A = f^{-1}(R)$.

PROOF:
($\Leftarrow$) The *Proposition 2.2*.
($\Rightarrow$) If A is B-generic and there exists at least two different elements in B (otherwise B is recursively enumerable and therefore e-equivalent to a set containing at least two different elements) $a \neq b$. Then for $R = \{a\}$ the conditions in *Proposition 2.5* hold and therefore there exists B-generic B-regular enumeration f, such that $A = f^{-1}(R)$, and for the existence of B-generic B-regular enumeration we need only $B \neq \emptyset$.

□

References

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